Iran J Sci Technol Trans Sci (2019) 43:1755–1760
https://doi.org/10.1007/s40995-018-0624-7

RESEARCH PAPER

Cross-Gram Matrix Associated to Two Sequences in Hilbert Spaces

E. Osgooei[1] **· A. Rahimi**[2]

Received: 22 January 2017 / Accepted: 4 August 2018 / Published online: 28 August 2018

Abstract

The conditions for sequences $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ being Bessel sequences, frames or Riesz bases, can be expressed in terms of the so-called cross-Gram matrix. In this paper, we investigate the cross-Gram operator G, associated to the sequence $\{\langle f_k, g_j\rangle\}_{j,k=1}^{\infty}$ and sufficient and necessary conditions for boundedness, invertibility, compactness and positivity of this operator are determined depending on the associated sequences. We show that invertibility of G is not possible when the associated sequences are frames but not Riesz Bases or at most one of them is Riesz basis. In the special case, we prove that G is a positive operator when $\{g_k\}_{k=1}^{\infty}$ is the canonical dual of $\{f_k\}_{k=1}^{\infty}$.

Keywords Frames · Riesz bases · Dual frames · Cross-Gram matrix

Mathematics Subject Classification 42C15 · 42C40

1 Introduction

The fundamental operators in frame theory are the synthesis, analysis and frame operators associated with a given frame. The ability of combining these operators to make a sensitive operator is indeed essential in frame theory and its applications. Time-invariant filter's, i. e. convolution operators, are used frequently in applications. These operators can be called Fourier multipliers Benyi et al. (2005), Feichtinger and Narimani (2006). In the last decade Gabor filters which are beneficial tools to perform time-variant filters have very strong applications in psychoacoustics Balazs et al. (2010), computational auditory scene analysis Wang and Brown (2006), and seismic data analysis Margrave et al. (2005). For more information on these operators we refer to Balazs (2007), Faroughi et al. (2013), Stoeva and Balasz (2012).

In this paper for given two Bessel sequences $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$, the synthesis operator of the sequence $\{f_k\}_{k=1}^{\infty}$ with the analysis operator of the sequence $\{g_k\}_{k=1}^{\infty}$ is composed and a fundamental operator is generated. This operator is called the cross-Gram operator associated with the sequence $\{\langle f_k, g_j\rangle\}_{j,k=1}^{\infty}$ Balazs (2008), Pekalska and Duin (2005). This paper concerns this question that when can sequences $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ in a Hilbert space H generate a cross-Gram operator with the properties of boundedness, invertibility and positivity. Vise versa if the cross-Gram operator has the above properties what can be expected of the sequences $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$.

Let H be a complex Hilbert space. A frame for H is a sequence $\{f_k\}_{k=1}^{\infty} \subset H$ such that there are positive constants A and B satisfying

$$A\|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k\rangle|^2 \leq B\|f\|^2, \quad f \in H. \tag{1}$$

The constants A and B are called lower and upper frame bounds, respectively. We call $\{f_k\}_{k=1}^{\infty}$ a Bessel sequence with bound B, if we have only the second inequality in (1). Associated with each Bessel sequence $\{f_k\}_{k=1}^{\infty}$ we have three linear and bounded operators, the synthesis operator:

✉ E. Osgooei
e.osgooei@uut.ac.ir

A. Rahimi
rahimi@maragheh.ac.ir

[1] Faculty of Science, Urmia University of Technology, Urmia, Iran

[2] Department of Mathematics, University of Maragheh, Maragheh, Iran

$$T : \ell^2(N) \rightarrow H, \quad T(\{c_k\}_{k=1}^{\infty}) = \sum_{k=1}^{\infty} c_k f_k,$$

the analysis operator which is defined by:

$$T^* : H \rightarrow \ell^2(N); \quad T^*f = \{\langle f, f_k\rangle\}_{k=1}^{\infty},$$

and the frame operator:

$$S : H \rightarrow H; \quad Sf = TT^*f = \sum_{k=1}^{\infty} \langle f, f_k\rangle f_k.$$

If $\{f_k\}_{k=1}^{\infty}$ is a Bessel sequence, we can compose the synthesis operator T and its adjoint T^* to obtain the bounded operator

$$T^*T : \ell^2(N) \rightarrow \ell^2(N); \quad T^*T\{c_k\}_{k=1}^{\infty} = \left\{\left\langle \sum_{\ell=1}^{\infty} c_\ell f_\ell, f_k \right\rangle\right\}_{k=1}^{\infty}.$$

Therefore the matrix representation of T^*T is as follows:

$$T^*T = \{\langle f_k, f_j\rangle\}_{j,k=1}^{\infty}.$$

The matrix $\{\langle f_k, f_j\rangle\}_{j,k=1}^{\infty}$ is called the matrix associated with $\{f_k\}_{k=1}^{\infty}$ or Gram matrix and it defines a bounded operator on $\ell^2(N)$ when $\{f_k\}_{k=1}^{\infty}$ is a Bessel sequence.

In order to recognize that a sequence $\{f_k\}_{k=1}^{\infty}$ is a Bessel sequence or frame, we need to check (1) for all $f \in H$. But in practice this is not always so easy. The following two results give us a practical method to diagnose Bessel sequences or frames by the concept of Gram matrix or in other words just by calculating $\{\langle f_k, f_j\rangle\}_{j,k=1}^{\infty}$.

Lemma 1 Christensen (2016) *Suppose that* $\{f_k\}_{k=1}^{\infty} \subseteq H$. *Then the following statements are equivalent*:

1. $\{f_k\}_{k=1}^{\infty}$ *is a Bessel sequence with bound* B.
2. *The Gram matrix associated with* $\{f_k\}_{k=1}^{\infty}$ *defines a bounded operator on* $\ell^2(N)$ *with norm at most* B.

Definition 1 A Riesz basis $\{f_k\}_{k=1}^{\infty}$ for H is a family of the form $\{Ue_k\}_{k=1}^{\infty}$, where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for H and $U : H \rightarrow H$ is a bounded bijective operator.

Proposition 1 Christensen (2016) *A sequence* $\{f_k\}_{k=1}^{\infty}$ *is a Riesz basis for H if and only if it is an unconditional basis for H and*

$$0 < \inf \|f_k\| \leq \sup \|f_k\| < \infty.$$

Theorem 1 Christensen (2016) *Suppose that* $\{f_k\}_{k=1}^{\infty} \subseteq H$. *Then the following conditions are equivalent:*

1. $\{f_k\}_{k=1}^{\infty}$ *is a Riesz basis for H.*
2. $\{f_k\}_{k=1}^{\infty}$ *is complete and its Gram matrix* $\{\langle f_k, f_j\rangle\}_{j,k=1}^{\infty}$ *defines a bounded, invertible operator on* $\ell^2(N)$.

2 Cross-Gram Matrix

If $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are Bessel sequences, we compose the synthesis operator of the sequence $\{f_k\}_{k=1}^{\infty}$, T_{f_k}, and the analysis operator of the sequence $\{g_k\}_{k=1}^{\infty}$, $T^*_{g_k}$, to obtain a bounded operator on $\ell^2(N)$

$$T^*_{g_k}T_{f_k} : \ell^2(N) \rightarrow \ell^2(N); \quad T^*_{g_k}T_{f_k}\{c_k\}_{k=1}^{\infty} = \left\{\left\langle \sum_{\ell=1}^{\infty} c_\ell f_\ell, g_k \right\rangle\right\}_{k=1}^{\infty}.$$

This operator, $G = T^*_{g_k}T_{f_k}$, is called the cross-Gram operator associated to $\{\langle f_k, g_j\rangle\}_{j,k=1}^{\infty}$ Balazs (2008), Pekalska and Duin (2005).

If $\{e_k\}_{k=1}^{\infty}$ is the canonical orthonormal basis for $\ell^2(N)$, the *jk*-th entry in the matrix representation for $T^*_{g_k}T_{f_k}$ is

$$\langle T^*_{g_k}T_{f_k}e_k, e_j\rangle = \langle T_{f_k}e_k, T_{g_k}e_j\rangle = \langle f_k, g_j\rangle.$$

Therefore the matrix representation of $T^*_{g_k}T_{f_k}$ is as follows:

$$T^*_{g_k}T_{f_k} = \{\langle f_k, g_j\rangle\}_{j,k=1}^{\infty}.$$

The matrix $\{\langle f_k, g_j\rangle\}_{j,k=1}^{\infty}$ is called the cross-Gram matrix associated to $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ Balazs (2008), Pekalska and Duin (2005). In the special case that the sequences $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are biorthogonal, the cross-Gram matrix is the identity matrix.

The above discussion shows that the cross-Gram matrix is bounded above if $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are Bessel sequences. The following example shows that the inverse of the above assertion is not valid, in other words, the cross-Gram matrix associated to two sequences can be well-defined and bounded in the case that one of the sequences is not Bessel.

Example 1 Suppose that $\{e_k\}_{k=1}^{\infty}$ is the orthonormal basis for a Hilbert space H. Consider $\{f_k\}_{k=1}^{\infty} = \{\frac{1}{k}e_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty} = \{ke_k\}_{k=1}^{\infty}$. A simple calculation shows that the cross-Gram matrix associated to these sequences is the identity matrix, but $\{g_k\}_{k=1}^{\infty}$ is not a Bessel sequence.

Definition 2 Let U be an operator on a Hilbert space H, and suppose that E is an orthonormal basis for H. We say that U is a Hilbert–Schmidt operator if

$$\|U\|_2 = \left(\sum_{x \in E} \|Ux\|^2\right)^{\frac{1}{2}} < \infty.$$

Theorem 2 *Suppose that* $\{f_k\}_{k=1}^{\infty}$ *and* $\{g_k\}_{k=1}^{\infty}$ *are sequences in H and* $\{g_k\}_{k=1}^{\infty}$ *is a Bessel sequence with bound* B'. *Assume that there exists* $M > 0$ *such that* $\sum_{k=1}^{\infty} \|f_k\|^2 \leq M$. *Then the cross-Gram operator associated to* $\{\langle f_k, g_j\rangle\}_{j,k=1}^{\infty}$ *is a well-defined, bounded and compact operator.*

Proof Suppose that $G=\{\langle f_k,g_j\rangle\}_{j,k=1}^{\infty}$. For a given sequence $\{c_k\}_{k=1}^{\infty}\in\ell^2(N)$ we have

$$\begin{aligned}\|G\{c_k\}_{k=1}^{\infty}\|^2&=\sum_{j=1}^{\infty}\left|\sum_{k=1}^{\infty}c_k\langle f_k,g_j\rangle\right|^2\\&\leq\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|c_k|^2\sum_{k=1}^{\infty}|\langle f_k,g_j\rangle|^2\\&=\sum_{k=1}^{\infty}|c_k|^2\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}|\langle f_k,g_j\rangle|^2\\&\leq B'\sum_{k=1}^{\infty}|c_k|^2\sum_{k=1}^{\infty}\|f_k\|^2\\&\leq B'M\sum_{k=1}^{\infty}|c_k|^2.\end{aligned}$$

By above assertion, $G\{c_k\}_{k=1}^{\infty}\in\ell^2(N)$ and therefore G is well-defined and bounded.

Now suppose that $\{e_k\}_{k=1}^{\infty}$ is the canonical orthonormal basis for $\ell^2(N)$. Then

$$\begin{aligned}\left(\sum_{k=1}^{\infty}\|G(e_k)\|^2\right)^{\frac{1}{2}}&=\left(\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}|\langle f_k,g_j\rangle|^2\right)^{\frac{1}{2}}\\&\leq\sqrt{B'}\left(\sum_{k=1}^{\infty}\|f_k\|^2\right)^{\frac{1}{2}}\leq\sqrt{B'M}.\end{aligned}$$

Therefore G is a Hilbert–Schmidt operator and so is compact Pedersen (1999).

Example 2 Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for H. Consider $\{f_k\}_{k=1}^{\infty}=\{e_1,\frac{1}{2}e_2,\frac{1}{3}e_3,\frac{1}{4}e_4,\ldots\}$ and $\{g_k\}_{k=1}^{\infty}=\{\frac{1}{2}e_1,e_2,\frac{1}{2^2}e_1,e_3,\ldots\}$. Suppose that G is the cross-Gram operator associated to $\{\langle f_k,g_j\rangle\}_{j,k=1}^{\infty}$. A simple calculation shows that

$$\sum_{k=1}^{\infty}\|G(e_k)\|^2=\sum_{k=1}^{\infty}\frac{1}{2^{2k}}+\sum_{k=1}^{\infty}\frac{1}{k^2}.$$

Therefore G is a Hilbert–Schmidt operator and so is compact.

If $\sup_k\|f_k\|<\infty$ (resp. $\inf_k\|f_k\|>0$), the sequence $\{f_k\}_{k=1}^{\infty}$ will be called norm-bounded above or NBA, (resp. norm-bounded below or NBB).

Theorem 3 *Let $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ be sequences for H and G be the cross-Gram operator associated to $\{\langle f_k,g_j\rangle\}_{j,k=1}^{\infty}$. Then the following statements are satisfied:*

1. *Assume that G is well-defined and bounded above and $\{g_k\}_{k=1}^{\infty}$ is a frame with lower bound A. Then $\{f_k\}_{k=1}^{\infty}$ is norm-bounded above.*
2. *Assume that G is well-defined and bounded below and $\{g_k\}_{k=1}^{\infty}$ is a Bessel sequence with upper bound B. Then $\{f_k\}_{k=1}^{\infty}$ is norm-bounded below.*
3. *Assume that G is well-defined and bounded above and $\{f_k\}_{k=1}^{\infty}$ is an orthonormal basis for H. Then $\{g_k\}_{k=1}^{\infty}$ is a Bessel sequence for H.*

Proof

1. Since G is well-defined and bounded, there exists a constant $M>0$ such that

$$\sum_{j=1}^{\infty}\left|\sum_{k=1}^{\infty}c_k\langle f_k,g_j\rangle|^2\leq M\sum_{k=1}^{\infty}|c_k\right|^2. \tag{2}$$

Via (2) applied to the elements of the canonical orthonormal basis of $\ell^2(N)$, we have

$$\sum_{j=1}^{\infty}|\langle f_k,g_j\rangle|^2\leq M,\quad k\in N. \tag{3}$$

Since $\{g_k\}_{k=1}^{\infty}$ is a frame for H, by (3) we have

$$A\|f_k\|^2\leq\sum_{j=1}^{\infty}|\langle f_k,g_j\rangle|^2\leq M,\quad \ell\in N. \tag{4}$$

Therefore

$$\|f_k\|^2\leq\frac{M}{A},\quad k\in N.$$

2. By assumption there exist $M'>0$, such that

$$M'\sum_{k=1}^{\infty}|c_k|^2\leq\sum_{j=1}^{\infty}\left|\sum_{k=1}^{\infty}c_k\langle f_k,g_j\rangle\right|^2. \tag{5}$$

Similar to above discussion, since $\{g_k\}_{k=1}^{\infty}$ is a Bessel sequence, for each $k\in N$, via (5) applied to the elements of the canonical orthonormal basis of $\ell^2(N)$, we have

$$M'\leq\sum_{j=1}^{\infty}|\langle f_k,g_j\rangle|^2\leq B\|f_k\|^2. \tag{6}$$

Therefore

$$\|f_k\|^2\geq\frac{M'}{B},\quad k\in N.$$

3. Since G is well-defined and bounded above there exists $M''>0$, such that

$$\sum_{j=1}^{\infty}\left|\sum_{k=1}^{\infty}c_k\langle f_k,g_j\rangle|^2\leq M''\sum_{k=1}^{\infty}\right|c_k|^2. \tag{7}$$

Since $\{f_k\}_{k=1}^{\infty}$ is an orthonormal basis for H, there exist a sequence $\{c_k\}\in\ell^2(N)$, such that for each $f\in H$ we have

$$\sum_{j=1}^{\infty} |\langle f, g_j\rangle|^2 = \sum_{j=1}^{\infty} \left| \left\langle \sum_{k=1}^{\infty} c_k f_k, g_j \right\rangle \right|^2.$$

Now by (7) we have

$$\sum_{j=1}^{\infty} \left| \sum_{k=1}^{\infty} c_k \langle f_k, g_j\rangle \right|^2 \leq M'' \sum_{k=1}^{\infty} |c_k|^2,$$

Therefore

$$\sum_{j=1}^{\infty} |\langle f, g_j\rangle|^2 \leq M'' \sum_{k=1}^{\infty} |c_k|^2 = M'' \|f\|^2.$$

Proposition 2 *Suppose that $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are Bessel sequences and G is the cross-Gram operator associated to $\{\langle f_k, g_j\rangle\}_{j,k=1}^{\infty}$. Then the following statements are satisfied*:

1. *If $\{f_k\}_{k=1}^{\infty}$ is a Riesz basis and $\{g_k\}_{k=1}^{\infty}$ is a frame for H, then G is a bounded injective operator.*
2. *If $\{f_k\}_{k=1}^{\infty}$ is a frame and $\{g_k\}_{k=1}^{\infty}$ is a Riesz basis for H, then G is a bounded surjective operator.*

Proof Since $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are Bessel sequences, we deduce that G is a well-defined and bounded operator.

1. Suppose that

   $$G\{c_k\}_{k=1}^{\infty} = G\{b_k\}_{k=1}^{\infty}, \quad \{c_k\}_{k=1}^{\infty}, \{b_k\}_{k=1}^{\infty} \in \ell^2(N).$$

   Then $T_{g_k}^* T_{f_k}\{c_k\}_{k=1}^{\infty} = T_{g_k}^* T_{f_k}\{b_k\}_{k=1}^{\infty}$. Since $\{g_k\}_{k=1}^{\infty}$ is a frame for H, $T_{g_k}^*$ is an injective operator and we have $T_{f_k}\{c_k\}_{k=1}^{\infty} = T_{f_k}\{b_k\}_{k=1}^{\infty}$. Since $\{f_k\}_{k=1}^{\infty}$ is a Riesz basis, T_{f_k} is invertible. So $\{c_k\}_{k=1}^{\infty} = \{b_k\}_{k=1}^{\infty}$ and we get the result.
2. Since $\{g_k\}_{k=1}^{\infty}$ is a Riesz basis, $T_{g_k}^*$ is a bijective operator. Therefore for a given sequence $\{c_k\}_{k=1}^{\infty} \in \ell^2(N)$, there exist $h \in H$ such that $T_{g_k}^* h = \{c_k\}_{k=1}^{\infty}$. Also by assumption T_{f_k} is a surjective operator and there exists $\{b_k\}_{k=1}^{\infty} \in \ell^2(N)$ such that $T_{f_k}\{b_k\}_{k=1}^{\infty} = h$. Therefore we have $T_{g_k}^* T_{f_k}\{b_k\}_{k=1}^{\infty} = \{c_k\}_{k=1}^{\infty}$ and so $G\{b_k\}_{k=1}^{\infty} = \{c_k\}_{k=1}^{\infty}$.

Theorem 4 *Suppose that $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are Riesz bases for H. Then $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are complete and the cross-Gram matrix associated to $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ defines a bounded invertible operator on $\ell^2(N)$.*

Proof Since $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are Riesz bases, there exist bijective operators U and W such that $\{f_k\}_{k=1}^{\infty} = \{Ue_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty} = \{We_k\}_{k=1}^{\infty}$, where $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis of H. For every $k, j \in N$ we have

$$\langle f_k, g_j\rangle = \langle Ue_k, We_j\rangle = \langle W^* Ue_k, e_j\rangle.$$

i.e., the cross-Gram matrix associated to $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ representing the bounded invertible operator W^*U in the basis $\{e_k\}_{k=1}^{\infty}$.

If the cross-Gram matrix associated to the sequences $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ is invertible and the sequence $\{f_k\}_{k=1}^{\infty}$ is a Bessel sequence, then there is no need to sequence $\{g_k\}_{k=1}^{\infty}$ to be a Bessel sequence, see Example 1. Having in mind this result, now what can we say about the inverse of Theorem 4? By having the assumption of the invertibility of the cross-Gram matrix associated to $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ and completeness of the sequences $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$, by Example 1, we deduce that there is no need to sequences $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ to be Riesz bases. But what can we say in the case that $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are frames? In the following theorems we answer to this question by considering the assumption of being frame of both sequences.

Theorem 5 *Suppose that $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are frames for H and the cross-Gram matrix associated to these sequences is bounded and invertible. Then $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are Riesz bases for H.*

Proof Suppose that G is the cross-Gram operator associated to $\{\langle f_k, g_j\rangle\}_{j,k=1}^{\infty}$. So we have

$$G = T_{g_k}^* T_{f_k}.$$

Since $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are frames for H, T_{f_k} and T_{g_k} are bounded and surjective operators. Now we want to show that T_{f_k} is an injective operator. For the given sequences $\{c_k\}_{k=1}^{\infty}$, $\{b_k\}_{k=1}^{\infty} \in \ell^2(N)$, suppose that

$$T_{f_k}\{c_k\}_{k=1}^{\infty} = T_{f_k}\{b_k\}_{k=1}^{\infty}.$$

Then we have

$$T_{g_k}^* T_{f_k}\{c_k\}_{k=1}^{\infty} = T_{g_k}^* T_{f_k}\{b_k\}_{k=1}^{\infty}.$$

So

$$G\{c_k\}_{k=1}^{\infty} = G\{b_k\}_{k=1}^{\infty}.$$

Since G is an invertible operator, we deduce that $\{c_k\}_{k=1}^{\infty} = \{b_k\}_{k=1}^{\infty}$ and therefore T_{f_k} is an injective operator and so $\{f_k\}_{k=1}^{\infty}$ is a Riesz basis for H.

Now we want to show that T_{g_k} is also a bijective operator. Since $N(T_{g_k}) = R(T_{g_k}^*)^{\perp}$, it is enough to show that $T_{g_k}^* : H \to \ell^2(N)$ is a surjective operator. Since G is invertible, for a given sequence $\{c_k\}_{k=1}^{\infty} \in \ell^2(N)$ there exists a sequence $\{b_k\}_{k=1}^{\infty} \in \ell^2(N)$ such that

$$G\{b_k\}_{k=1}^{\infty} = \{c_k\}_{k=1}^{\infty}.$$

So

$T^*_{g_k} T_{f_k}\{b_k\}_{k=1}^{\infty} = \{c_k\}_{k=1}^{\infty}$,

which shows that $T^*_{g_k}$ is a surjective operator. Therefore $\{g_k\}_{k=1}^{\infty}$ is a Riesz basis for H.

Corollary 1 *Suppose that* $\{f_k\}_{k=1}^{\infty}$ *and* $\{g_k\}_{k=1}^{\infty}$ *are frames but not Riesz bases, then* G *cannot be invertible.*

Example 3 Suppose that $\{e_k\}_{k=1}^{\infty}$ is an orthonormal basis for H. Consider the sequences $\{f_k\}_{k=1}^{\infty} = \{e_1, e_1, e_2, e_3, e_4, \ldots\}$ and $\{g_k\}_{k=1}^{\infty} = \{e_1, e_1, e_2, e_2, e_3, e_3, \ldots\}$. A simple calculation shows that these sequences are frames but not Riesz bases. We get the cross-Gram matrix associated to sequences $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$:

$$G = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{bmatrix},$$

We obtain that $\det(G) = 0$ and so G is not invertible.

Theorem 6 *Suppose that* $\{f_k\}_{k=1}^{\infty}$ *and* $\{g_k\}_{k=1}^{\infty}$ *are Bessel sequences for* H *and the cross-Gram matrix, associated to these sequences is bounded and invertible. Then the following statements are satisfied:*

1. *If* $\{f_k\}_{k=1}^{\infty}$ *is a Riesz basis for* H*, then* $\{g_k\}_{k=1}^{\infty}$ *is a Riesz basis for* H*.*
2. *If* $\{f_k\}_{k=1}^{\infty}$ *is a frame for* H *and* $\{g_k\}_{k=1}^{\infty}$ *is complete in* H*, then* $\{g_k\}_{k=1}^{\infty}$ *is a frame for* H*.*

Proof

1. Suppose that G is the cross-Gram operator associated to $\{\langle f_k, g_j\rangle\}_{j,k=1}^{\infty}$. Then we have

   $G = T^*_{g_k} T_{f_k}$.

   Since $\{f_k\}_{k=1}^{\infty}$ is a Riesz basis, T_{f_k} has a bounded inverse and we can write

   $T^*_{g_k} = G(T^{-1}_{f_k})$.

   Therefore $T^*_{g_k}$ is an invertible operator and we deduce that $\{g_k\}_{k=1}^{\infty}$ is a Riesz basis for H.
2. In order to show that $\{g_k\}_{k=1}^{\infty}$ is a frame for H it is enough to prove that T_{g_k} is a surjective operator. Since $\{g_k\}_{k=1}^{\infty}$ is complete in H, we need to show that $T^*_{g_k}$ is injective. Suppose that

   $T^*_{g_k}(f_1) = T^*_{g_k}(f_2), \;\; f_1, f_2 \in H$.

   Since T_{f_k} is a surjective operator, there exist sequences $\{c_k\}_{k=1}^{\infty}$, $\{b_k\}_{k=1}^{\infty} \in \ell^2(N)$ such that $f_1 = T_{f_k}\{c_k\}_{k=1}^{\infty}$, $f_2 = T_{f_k}\{b_k\}_{k=1}^{\infty}$, and therefore we can write

   $T^*_{g_k} T_{f_k}\{c_k\}_{k=1}^{\infty} = T^*_{g_k} T_{f_k}\{b_k\}_{k=1}^{\infty}$.

   Now by the invertibility of G we deduce that $\{c_k\}_{k=1}^{\infty} = \{b_k\}_{k=1}^{\infty}$, and so $T_{f_k}\{c_k\}_{k=1}^{\infty} = T_{f_k}\{b_k\}_{k=1}^{\infty}$. Hence we get the proof.

By changing the role of the sequences $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ in above theorem we deduce the same results.

Corollary 2 *Suppose that* $\{f_k\}_{k=1}^{\infty}$ *is a Riesz basis and* $\{g_k\}_{k=1}^{\infty}$ *is a Bessel sequence. Then the following statements are satisfied:*

1. *If* $\{g_k\}_{k=1}^{\infty}$ *is not a frame, then* G *cannot be invertible.*
2. *If* $\{g_k\}_{k=1}^{\infty}$ *is not NBA or NBB, then* G *cannot be invertible.*

3 Dual Frames Associated to Cross-Gram Matrix

In this section, we investigate the cases when $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ are a pair of dual frames. Recall that for the Bessel sequences $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$, the pair $(\{f_k\}_{k=1}^{\infty}, \{g_k\}_{k=1}^{\infty})$ is a dual pair if for any $f \in H$ one of the following equivalent conditions holds:

1. $f = \sum_{k=1}^{\infty}\langle f, g_k\rangle f_k, f \in H$.
2. $f = \sum_{k=1}^{\infty}\langle f, f_k\rangle g_k, f \in H$.
3. $\langle f, g\rangle = \sum_{k=1}^{\infty}\langle f, f_k\rangle\langle g_k, g\rangle, \;\; f, g \in H$.

Theorem 7 *Suppose that* $\{f_k\}_{k=1}^{\infty}$ *and* $\{g_k\}_{k=1}^{\infty}$ *are Bessel sequences and* $\langle f, g_k\rangle \neq 0$ *for each* $k \in N$ *and* $f \in H$*. Assume that* G*, the cross-Gram operator associated to* $\{\langle f_k, g_j\rangle\}_{j,k=1}^{\infty}$*, is a well-defined and bounded operator with bound* M *such that* $0 < M < 1$*. Then* $\{f_k\}_{k=1}^{\infty}$ *and* $\{g_k\}_{k=1}^{\infty}$ *cannot be a pair of dual frames.*

Proof Suppose that $\{f_k\}_{k=1}^{\infty}$ is a dual frame of $\{g_k\}_{k=1}^{\infty}$. Then for every $f \in H$,

$$f = \sum_{k=1}^{\infty}\langle f, g_k\rangle f_k. \tag{8}$$

Since G is a bounded operator with bound M, for $\{c_k\}_{k=1}^{\infty} \in \ell^2(N)$ we have

$$\sum_{j=1}^{\infty}\left|\sum_{k=1}^{\infty}c_k\langle f_k, g_j\rangle\right|^2 \leq M\sum_{k=1}^{\infty}|c_k|^2, \quad (9)$$

Now for each $f \in H$, by (8) and (9) we have

$$\begin{aligned}\sum_{j=1}^{\infty}|\langle f, g_j\rangle|^2 &= \sum_{j=1}^{\infty}\left|\left\langle\sum_{k=1}^{\infty}\langle f, g_k\rangle f_k, g_j\right\rangle\right|^2\\ &= \sum_{j=1}^{\infty}\left|\sum_{k=1}^{\infty}\langle f, g_k\rangle\langle f_k, g_j\rangle\right|^2 \leq M\sum_{k=1}^{\infty}|\langle f, g_k\rangle|^2,\end{aligned}$$

which is a contradiction. Therefore we get the proof.

Example 4 Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for H. Consider $\{f_k\}_{k=1}^{\infty} = \{e_1, e_2, e_3, e_4, \ldots\}$ and $\{g_k\}_{k=1}^{\infty} = \{\frac{1}{2}e_1, \frac{1}{2}e_1, \frac{1}{3}e_1, \frac{1}{4}e_1, \ldots\}$. Suppose that G is the cross-Gram operator associated to $\{\langle f_k, g_j\rangle\}_{j,k=1}^{\infty}$. Then G is a well-defined and bounded operator with bound $\sqrt{\frac{89}{100}}$. Suppose that $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ be a pair of dual frames. Then we have $e_2 = \frac{1}{2}e_1$, which is a contradiction. Therefore $\{f_k\}_{k=1}^{\infty}$ and $\{g_k\}_{k=1}^{\infty}$ cannot be a dual pair.

Theorem 8 *Suppose that $\{f_k\}_{k=1}^{\infty}$ is a frame for H and $\{S^{-1}f_k\}_{k=1}^{\infty}$ is its canonical dual frame. Assume that G is the cross-Gram operator associated to $\{\langle f_k, S^{-1}f_j\rangle\}_{j,k=1}^{\infty}$. Then G is self-adjoint and positive operator.*

Proof Since G is the cross-Gram operator associated to $\{\langle f_k, S^{-1}f_j\rangle\}_{j,k=1}^{\infty}$, we have

$$G = T^*_{S^{-1}f_k}T_{f_k} = T^*_{f_k}S^{-1}T_{f_k}.$$

Therefore G is self-adjoint. Now we show that G is a positive operator. For $\{c_k\}_{k=1}^{\infty} \in \ell^2(N)$ we have

$$\begin{aligned}\langle G\{c_k\}_{k=1}^{\infty}, \{c_k\}_{k=1}^{\infty}\rangle &= \langle T^*_{f_k}S^{-1}T_{f_k}\{c_k\}_{k=1}^{\infty}, \{c_k\}_{k=1}^{\infty}\rangle\\ &= \langle S^{-1}T_{f_k}\{c_k\}_{k=1}^{\infty}, T_{f_k}\{c_k\}_{k=1}^{\infty}\rangle\\ &= \sum_{k=1}^{\infty}|\langle T_{f_k}\{c_k\}_{k=1}^{\infty}, S^{-1}f_k\rangle|^2.\end{aligned}$$

Corollary 3 *If $\{f_k\}_{k=1}^{\infty}$ is a Riesz basis in above theorem, then G is the identity operator.*

Example 5 Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis for H. Consider

$$\{f_k\}_{k=1}^{\infty} = \{e_1, e_1, e_2, e_3, \ldots\}.$$

The canonical dual frame is given by

$$\{S^{-1}f_k\}_{k=1}^{\infty} = \left\{\frac{1}{2}e_1, \frac{1}{2}e_1, e_2, e_3, \ldots\right\}.$$

The cross-Gram matrix associated to these sequences is as follows:

$$G = \begin{bmatrix}\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdots\\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdots\\ 0 & 0 & 1 & 0 & 0 & \cdots\\ 0 & 0 & 0 & 1 & 0 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots\end{bmatrix},$$

Then

$$\begin{aligned}&\langle G\{c_k\}_{k=1}^{\infty}, \{c_k\}_{k=1}^{\infty}\rangle =\\ &\left\langle\left\{\frac{1}{2}c_1 + \frac{1}{2}c_2, \frac{1}{2}c_1 + \frac{1}{2}c_2, c_3, c_4, \ldots\right\}, \{c_1, c_2, c_3, c_4, \ldots\}\right\rangle\\ &= \frac{1}{2}(c_1 + c_2)\overline{c_1} + \frac{1}{2}(c_1 + c_2)\overline{c_2} + \sum_{k=3}^{\infty}|c_k|^2,\end{aligned}$$

which shows that G is a positive operator and $G^2 = G$.

References

Balazs P (2007) Basic definition and properties of Bessel multipliers. J Math Anal Appl 325(1):571–585

Balazs P (2008) Hilbert–Schmidt operators and frames-classification, best approximation by multipliers and algorithms. Int J Wavelets Multiresolut Inf Process 6(2):315–330

Balazs P, Laback B, Eckel G, Deutsch WA (2010) Time-frequency sparsity by removing perceptually irrelevant components using a simple model of simultaneous masking. IEEE Trans Speech Audio Process 18(1):34–49

Benyi A, Grafakos L, Grochenig K, Okoudjou K (2005) A class of Fourier multipliers for modulation spaces. Appl Comput Harmon Anal 19(1):131–139

Christensen O (2016) An introduction to frames and Riesz bases. Birkhauser, Boston

Faroughi MH, Osgooei E, Rahimi A (2013) (X_d, X_d^*)-multipliers in Banach spaces. Banach J Math Anal 7:146–161

Feichtinger HG, Narimani G (2006) Fourier multipliers of classical modulation spaces. Appl Comput Harmon Anal 21(3):349–359

Margrave GF, Gibson PC, Grossman JP, Henley DC, Iliescu V, Lamoureux MP (2005) The Gabor transform, pseudodifferential operators, and seismic deconvolution. Integr Comput Aid E 12(1):43–55

Pedersen M (1999) Functional analysis in applied mathematics and engineering. CRC Press, New York

Pekalska E, Duin RPW (2005) The dissimilarity representation for pattern recognition: foundations and applications. World Scientific Publishing Co., Singapore

Stoeva DT, Balasz P (2012) Unconditional convergence and invertibility of multipliers. Appl Comput Harm Anal 33:292–299

Wang D, Brown GJ (2006) Computational auditory scene analysis: principles, algorithms, and applications. Wiley-IEEE Press, Hoboken