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RESEARCH PAPER

Recurrence Relations and Differential Equations of the Hermite–Sheffer and Related Hybrid Polynomial Sequences

Subuhi Khan[1] · Shakeel Ahmad Naikoo[1] · Mahvish Ali[1]

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Abstract
In this article, the recurrence relations and differential equation for the 3-variable Hermite–Sheffer polynomials are derived by using the properties of the Pascal functional and Wronskian matrices. The corresponding results for certain members belonging to the Hermite–Sheffer polynomials are also obtained.

Keywords Hermite–Sheffer polynomials · Generalized Pascal functional matrix · Wronskian matrix · Recurrence relations · Differential equations

Mathematics Subject Classification 15A15 · 15A24 · 33C45 · 65QXX

1 Introduction

The Sheffer sequences (Sheffer 1939) arise in numerous problems of applied mathematics, theoretical physics, approximation theory and several other mathematical branches. Properties of Sheffer sequences are naturally handled within the framework of modern classical umbral calculus by Roman (1984). We recall the following definition of the Sheffer sequences (Roman 1984).

Let $f(t)$ be a delta series and let $\mathsf{g}(t)$ be an invertible series. Then there exists a unique sequence $s_n(x)$ of polynomials satisfying the orthogonality conditions

$$\langle \mathsf{g}(t) f(t)^k | s_n(x) \rangle = n! \delta_{n,k}, \quad \forall n, k \geq 0. \tag{1}$$

We say that the sequence $s_n(x)$ is the Sheffer for the pair $(\mathsf{g}(t), f(t))$. The Sheffer sequence for the pair $(\mathsf{g}(t), t)$ reduces to the Appell sequence for $\mathsf{g}(t)$ (Roman 1984, p. 27).

✉ Mahvish Ali
mahvishali37@gmail.com

Subuhi Khan
subuhi2006@gmail.com

Shakeel Ahmad Naikoo
shakeelnaikoo21@gmail.com

[1] Department of Mathematics, Aligarh Muslim University, Aligarh, India

The exponential generating function of $s_n(x)$ is given by Roman (1984, p.18):

$$\frac{1}{\mathsf{g}\left(f^{-1}(t)\right)} \exp\left(x f^{-1}(t)\right) = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!}, \quad x \in \mathbb{C}, \tag{2}$$

where $f^{-1}(t)$ is the compositional inverse of $f(t)$.

The Sheffer class contains important sequences such as the Hermite, Laguerre, Bessel, Poisson–Charlier and factorial polynomials (Rainville 1971). These polynomials are important from the viewpoint of applications in physics and number theory.

For $f(t) = t$, the Sheffer sequence becomes the Appell sequence $A_n(x)$ (Roman 1984) defined by the following generating function:

$$\frac{1}{\mathsf{g}(t)} \exp(xt) = \sum_{n=0}^{\infty} A_n(x) \frac{t^n}{n!}. \tag{3}$$

The function $\mathsf{g}(t)$ may be called the determining function for the Appell polynomials $A_n(x)$. By properly choosing $\mathsf{g}(t)$, several classical polynomials can be obtained from the Hermite to the Euler ones.

Operational methods are useful to derive the properties of special functions of mathematical physics. Combining operational methods, integral transforms and the theory of special functions and orthogonal polynomials, even more powerful instrument is obtained for solving a wide spectrum of differential equations and physical problems relevant to them. By using operational techniques, many properties of ordinary and multi-variable special functions are simply derived and framed in a more general context, see for example Cesarano (2017) and Dattoli et al. (2006). Dattoli et al. (2004) introduced a family of hybrid polynomials exhibiting a nature lying between the Hermite and the Laguerre polynomials and studied their properties by means of appropriate operational rules. Certain new families of hybrid special polynomials related to the Sheffer sequences are introduced and studied by Khan et al. (2010) and Khan and Raza (2012).

We recall the generating function of the Hermite–Sheffer polynomials ${}_{H}s_n(x,y,z)$ (Khan et al. 2010) in the following form:

$$\frac{1}{\mathsf{g}(f^{-1}(t))}\exp\left(xf^{-1}(t)+y(f^{-1}(t))^2+z(f^{-1}(t))^3\right) = \sum_{n=0}^{\infty} {}_{H}s_n(x,y,z)\frac{t^n}{n!}. \tag{4}$$

For $f(t)=t$, the Hermite–Sheffer polynomials reduce to the Hermite–Appell polynomials ${}_{H}A_n(x,y,z)$ (Khan et al. 2009), which are defined by the following generating function:

$$\frac{1}{\mathsf{g}(t)}\exp\left(xt+yt^2+zt^3\right)=\sum_{n=0}^{\infty} {}_{H}A_n(x,y,z)\frac{t^n}{n!}, \tag{5}$$

which for $\mathsf{g}(t)=\frac{(e^t-1)}{t}$ and $\mathsf{g}(t)=\frac{(e^t+1)}{2}$, i.e., corresponding to the Bernoulli and Euler polynomials $B_n(x)$ and $E_n(x)$, respectively, yields the following generating functions for the Hermite–Bernoulli polynomials ${}_{H}B_n(x,y,z)$ and Hermite–Euler polynomials ${}_{H}E_n(x,y,z)$:

$$\frac{t}{e^t-1}\exp\left(xt+yt^2+zt^3\right)=\sum_{n=0}^{\infty} {}_{H}B_n(x,y,z)\frac{t^n}{n!}, \tag{6}$$

and

$$\frac{2}{e^t+1}\exp\left(xt+yt^2+zt^3\right)=\sum_{n=0}^{\infty} {}_{H}E_n(x,y,z)\frac{t^n}{n!}, \tag{7}$$

respectively.

The concepts and formalism associated with the monomiality treatment (Dattoli 1999) can be exploited in different ways. They can be used to introduce new families of special polynomials as well as to establish rules of operational nature, framing the special polynomials within the context of particular solutions of generalized forms of partial differential equations of evolution type. The study of differential equations is a wide field in pure and applied mathematics, physics, and engineering. The problems arising in different areas of science and engineering are usually expressed in terms of differential equations, which in most of the cases have special functions as their solutions. Srivastava et al. (2014) established the differential, integro-differential and partial differential equations for the Hermite–Appell polynomials family. The recurrence relations, differential equations and other results of these mixed type special polynomials can be used to solve the existing as well as new emerging problems in certain branches of science. To establish the determinantal forms for the mixed special polynomials is a new and recent investigation which can be helpful for computation purposes.

A unifying tool for studying polynomial sequences, namely the representation of Appell polynomials in matrix form has been studied in Aceto et al. (2015). Recently, a unified matrix representation for the Sheffer polynomials is proposed (Aceto and Cação 2017). The recurrence relations and differential equations for the Appell and Sheffer sequences are derived in Yang and Youn (2009) and Youn and Yang (2011), respectively, by using the generalized Pascal functional matrix of an analytic function and Wronskian matrix of several analytic functions. This approach is further used by Kim and Kim (2015) to find some identities of the Sheffer polynomials.

In this article, the method adopted in Youn and Yang (2011) and Kim and Kim (2015) is extended to derive certain properties of the Hermite–Sheffer polynomials ${}_{H}s_n(x,y,z)$. In Sect. 2, properties of the generalized Pascal functional and Wronskian matrices are recalled. Hermite–Sheffer vectors are introduced. In Sect. 3, certain recurrence relations and differential equations satisfied by these polynomials are derived. The corresponding results for certain members belonging to the Hermite–Sheffer family are obtained in Sect. 4.

2 Preliminaries

We review certain definitions and concepts related to the Pascal and Wronskian matrices, which will be used in Sect. 3.

Let $\mathcal{F}=\{h(t)=\sum_{k=0}^{\infty}a_k\frac{t^k}{k!}\,|a_k\in\mathbb{C}\}$ be the $\mathbb{C}$-algebra of formal power series.

For $h(t)\in\mathcal{F}$, the generalized Pascal functional matrix (Yang and Micek 2007) of an analytic function $h(t)$ denoted by $\mathcal{P}_n[h(t)]$ is a square matrix of order $(n+1)$ defined as:

$$\mathcal{P}_n[h(t)]_{i,j}=\begin{cases}\binom{i}{j}h^{(i-j)}(t), & \text{if } i\geq j, i,j=0,1,2,\ldots,n\\ 0, & \text{otherwise.}\end{cases} \tag{8}$$

It should be noted that $h^{(k)}$ denotes the kth order derivative of h and h^k denotes the kth power of h throughout the article.

Also, the nth order Wronskian matrix of analytic functions $h_1(t), h_2(t), h_3(t), \ldots, h_m(t)$ is an $(n+1)\times m$ matrix and is defined as:

$$\mathcal{W}_n[h_1(t), h_2(t), h_3(t), \ldots, h_m(t)] = \begin{bmatrix} h_1(t) & h_2(t) & h_3(t) & \cdots & h_m(t) \\ h_1'(t) & h_2'(t) & h_3'(t) & \cdots & h_m'(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_1^{(n)}(t) & h_2^{(n)}(t) & h_3^{(n)}(t) & \cdots & h_m^{(n)}(t) \end{bmatrix}. \quad (9)$$

It is important to note that t is considered as working variable and x as a parameter in the expressions $\mathcal{P}_n[h(x,t)]_{t=0}$ and $\mathcal{W}_n[h(x,t)]_{t=0}$.

We recall that for $a, b \in \mathbb{C}$ and any analytic functions $h(t), l(t) \in \mathcal{F}$, the following properties hold true (Youn and Yang 2011):

$$\mathcal{P}_n[ah(t)+bl(t)] = a\mathcal{P}_n[h(t)] + b\mathcal{P}_n[l(t)], \quad (10)$$

$$\mathcal{W}_n[ah(t)+bl(t)] = a\mathcal{W}_n[h(t)] + b\mathcal{W}_n[l(t)], \quad (11)$$

$$\mathcal{P}_n[l(t)]\mathcal{P}_n[h(t)] = \mathcal{P}_n[h(t)]\mathcal{P}_n[l(t)] = \mathcal{P}_n[h(t)l(t)], \quad (12)$$

$$\mathcal{P}_n[l(t)]\mathcal{W}_n[h(t)] = \mathcal{P}_n[h(t)]\mathcal{W}_n[l(t)] = \mathcal{W}_n[h(t)l(t)], \quad (13)$$

$$\mathcal{W}_n[l(h(t))]_{t=0} = \mathcal{W}_n[1, h(t), h^2(t), h^3(t), \ldots, h^n(t)]_{t=0}\Lambda_n^{-1}\mathcal{W}_n[l(t)]_{t=0}, \quad (14)$$

where $\Lambda_n = \operatorname{diag}[0!, 1!, 2!, \ldots, n!]$ and $h(0)=0$ and $h'(0)\neq 0$.

Further, for any analytic functions $l(t)$ and $h_1(t), h_2(t), \ldots, h_m(t)$, the following property holds true:

$$\mathcal{P}_n[l(t)]\mathcal{W}_n[h_1(t), h_2(t), \ldots, h_m(t)] = \mathcal{W}_n[(lh_1)(t), (lh_2)(t), \ldots, (lh_m)(t)]. \quad (15)$$

In order to utilize the Wronskian matrices, the vector form of the Hermite–Sheffer sequence is required. The Hermite–Sheffer vector denoted by $\overline{\mathbf{{}_Hs_n}}(x,y,z)$ is defined as

$$\overline{\mathbf{{}_Hs_n}}(x,y,z) = [{}_Hs_0(x,y,z), {}_Hs_1(x,y,z), {}_Hs_2(x,y,z), \ldots, {}_Hs_n(x,y,z)]^T, \quad (16)$$

where $\{{}_Hs_n(x,y,z)\}$ is the Hermite–Sheffer sequence defined by Eq. (4).

Since

$$\frac{1}{g(f^{-1}(t))}\exp\left(xf^{-1}(t) + y(f^{-1}(t))^2 + z(f^{-1}(t))^3\right)$$

is analytic, therefore by Taylor's theorem, it follows that

$${}_Hs_k(x,y,z) = \left(\frac{d}{dt}\right)^{(k)} \times \left(\frac{1}{g(f^{-1}(t))}\exp\left(xf^{-1}(t) + y(f^{-1}(t))^2 + z(f^{-1}(t))^3\right)\right)\Bigg|_{t=0}, \quad k\geq 0. \quad (17)$$

In view of Eq. (17), the Hermite–Sheffer vector (16) can be expressed as

$$\overline{\mathbf{{}_Hs_n}}(x,y,z) = \mathcal{W}_n\left[\frac{1}{g(f^{-1}(t))}\exp\left(xf^{-1}(t) + y(f^{-1}(t))^2 + z(f^{-1}(t))^3\right)\right]_{t=0}. \quad (18)$$

It should be noted that in expression $(\mathcal{W}_n[{}_Hs_0(x,y,z), {}_Hs_1(x,y,z), \ldots, {}_Hs_n(x,y,z)])^T$, the partial derivatives of ${}_Hs_k(x,y,z)$, $k=0,1,2,\ldots,n$ are taken w.r.t. x, keeping y and z as constants.

In order to establish the properties of the Hermite–Sheffer polynomials, the following Lemma is required.

Lemma 2.1 *Let* $\{{}_Hs_k(x,y,z)\}$ *be the Hermite–Sheffer polynomials sequence. Then,*

$$(\mathcal{W}_n[{}_Hs_0(x,y,z), {}_Hs_1(x,y,z), {}_Hs_2(x,y,z), \ldots, {}_Hs_n(x,y,z)])^T\Lambda_n^{-1} = \mathcal{W}_n\left[1, f^{-1}(t), (f^{-1}(t))^2, \ldots, (f^{-1}(t))^n\right]_{t=0}\Lambda_n^{-1}\mathcal{P}_n \times \left[\frac{\exp(yt^2+zt^3)}{g(t)}\right]_{t=0}\mathcal{P}_n[\exp(xt)]_{t=0}. \quad (19)$$

Proof Use of property (14) in the r.h.s. of (18), gives

$$[{}_Hs_0(x,y,z), {}_Hs_1(x,y,z), {}_Hs_2(x,y,z), \ldots, {}_Hs_n(x,y,z)]^T = \mathcal{W}_n\left[\frac{1}{g(f^{-1}(t))}\exp\left(xf^{-1}(t) + y(f^{-1}(t))^2 + z(f^{-1}(t))^3\right)\right]_{t=0} = \mathcal{W}_n\left[1, f^{-1}(t), (f^{-1}(t))^2, \ldots, (f^{-1}(t))^n\right]_{t=0}\Lambda_n^{-1}\mathcal{W}_n \times \left[\frac{\exp(xt+yt^2+zt^3)}{g(t)}\right]_{t=0}. \quad (20)$$

Again using property (13) and in view of the fact that $\mathcal{W}_n[\exp(xt)]_{t=0} = [1xx^2\ldots x^n]^T$, the above equation takes the form

$$[{}_Hs_0(x,y,z), {}_Hs_1(x,y,z), {}_Hs_2(x,y,z), \ldots, {}_Hs_n(x,y,z)]^T = \mathcal{W}_n\left[1, f^{-1}(t), (f^{-1}(t))^2, \ldots, (f^{-1}(t))^n\right]_{t=0}\Lambda_n^{-1}\mathcal{P}_n \times \left[\frac{\exp(yt^2+zt^3)}{g(t)}\right]_{t=0}[1xx^2\cdots x^n]^T. \quad (21)$$

Taking the kth order partial derivative with respect to x on both sides of Eq. (21) and then dividing the resulting equation by $k!$, we find

$$\frac{1}{k!}\left[\frac{\partial^k}{\partial x^k}{}_Hs_0(x,y,z)\,\frac{\partial^k}{\partial x^k}{}_Hs_1(x,y,z)\ldots\frac{\partial^k}{\partial x^k}{}_Hs_n(x,y,z)\right]^T$$
$$=\mathcal{W}_n\left[1,f^{-1}(t),(f^{-1}(t))^2,\ldots,(f^{-1}(t))^n\right]_{t=0}\Lambda_n^{-1}\mathcal{P}_n$$
$$\times\left[\frac{\exp(yt^2+zt^3)}{\mathsf{g}(t)}\right]_{t=0}$$
$$\times\left[0\cdots01\binom{k+1}{k}x\binom{k+2}{k}x^2\cdots\binom{n}{k}x^{n-k}\right]^T. \tag{22}$$

The l.h.s. of Eq. (22) is the kth column of

$$(\mathcal{W}_n[{}_Hs_0(x,y,x),{}_Hs_1(x,y,z),{}_Hs_2(x,y,z),\ldots{}_Hs_n(x,y,z)])^T\Lambda_n^{-1}$$

and the r.h.s. of Eq. (21) is the kth column of

$$\mathcal{W}_n\left[1,f^{-1}(t),(f^{-1}(t))^2,\ldots,(f^{-1}(t))^n\right]_{t=0}\Lambda_n^{-1}\mathcal{P}_n$$
$$\left[\frac{\exp(yt^2+zt^3)}{\mathsf{g}(t)}\right]_{t=0}\mathcal{P}_n[\exp(xt)]_{t=0}.$$

Consequently, assertion (19) follows. □

In the next section, recurrence relations and differential equation for the Hermite–Sheffer polynomials are derived.

3 Recurrence Relations and Differential Equations

First, we derive a differential recurrence relation for the Hermite–Sheffer polynomials ${}_Hs_k(x,y,z)$ by proving the following result.

Theorem 3.1 *For the Hermite–Sheffer polynomials ${}_Hs_n(x,y,z)$, the following differential recurrence relation holds true*:

$$ {}_Hs_{n+1}(x,y,z)=\sum_{k=0}^{n}\frac{(xA_k+2yB_k+3zC_k+D_k)}{k!}\frac{\partial^k}{\partial x^k}{}_Hs_n(x,y,z), \tag{23}$$
$$n\geq 0;\quad {}_Hs_0(x,y,z)=\frac{1}{\mathsf{g}(0)},$$

where

$$A_k=\left(\frac{1}{f'(t)}\right)^{(k)}\Big|_{t=0};B_k=\left(\frac{t}{f'(t)}\right)^{(k)}\Big|_{t=0}C_k=\left(\frac{t^2}{f'(t)}\right)^{(k)}\Big|_{t=0}D_k=\left(-\frac{\mathsf{g}'(t)}{\mathsf{g}(t)f'(t)}\right)^{(k)}\Big|_{t=0}.$$

Proof In view of definition (9), it follows that

$$\mathcal{W}_n\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\exp\left(xf^{-1}(t)+y(f^{-1}(t))^2+z(f^{-1}(t))^3\right)}{\mathsf{g}(f^{-1}(t))}\right)\right]_{t=0}$$
$$=[{}_Hs_1(x,y,z)\,{}_Hs_2(x,y,z)\,{}_Hs_3(x,y,z)\ldots{}_Hs_{n+1}(x,y,z)]^T. \tag{24}$$

Performing the differentiation in expression

$$\mathcal{W}_n\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\exp\left(xf^{-1}(t)+y(f^{-1}(t))^2+z(f^{-1}(t))^3\right)}{\mathsf{g}(f^{-1}(t))}\right)\right]_{t=0}$$

and using properties (12)–(14) in a suitable manner, we find

$$\mathcal{W}_n\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\exp\left(xf^{-1}(t)+y(f^{-1}(t))^2+z(f^{-1}(t))^3\right)}{\mathsf{g}(f^{-1}(t))}\right)\right]_{t=0}$$
$$=\mathcal{W}_n[1,f^{-1}(t),(f^{-1}(t))^2,\ldots,(f^{-1}(t))^n]_{t=0}\Lambda_n^{-1}$$
$$\times\mathcal{W}_n\left[\left((x+2yt+3zt^2)-\frac{\mathsf{g}'(t)}{\mathsf{g}(t)}\right)\frac{\exp(xt+yt^2+zt^3)}{f'(t)\mathsf{g}(t)}\right]_{t=0}$$
$$=\mathcal{W}_n[1,f^{-1}(t),(f^{-1}(t))^2,\ldots,(f^{-1}(t))^n]_{t=0}\Lambda_n^{-1}$$
$$\times\mathcal{P}_n\left[\frac{\exp(yt^2+zt^3)}{\mathsf{g}(t)}\right]_{t=0}\mathcal{P}_n[\exp(xt)]_{t=0}\mathcal{W}_n$$
$$\left[\left((x+2yt+3zt^2)-\frac{\mathsf{g}'(t)}{\mathsf{g}(t)}\right)\frac{1}{f'(t)}\right]_{t=0}. \tag{25}$$

Further, in view of Lemma 2.1, we have

$$\mathcal{W}_n\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\exp\left(xf^{-1}(t)+y(f^{-1}(t))^2+z(f^{-1}(t))^3\right)}{\mathsf{g}(f^{-1}(t))}\right)\right]_{t=0}$$
$$=(\mathcal{W}_n[{}_Hs_0(x,y,z),{}_Hs_1(x,y,z),{}_Hs_2(x,y,z),\ldots,{}_Hs_n(x,y,z)])^T\Lambda_n^{-1}$$
$$\times\mathcal{W}_n\left[\frac{x}{f'(t)}+\frac{2yt}{f'(t)}+\frac{3zt^2}{f'(t)}-\frac{\mathsf{g}'(t)}{\mathsf{g}(t)f'(t)}\right]_{t=0}$$
$$=\begin{bmatrix} {}_Hs_0(x,y,z) & 0 & 0 & \cdots & 0\\ {}_Hs_1(x,y,z) & \frac{1}{1!}\frac{\partial}{\partial x}{}_Hs_1(x,y,z) & 0 & \cdots & 0\\ {}_Hs_2(x,y,z) & \frac{1}{1!}\frac{\partial}{\partial x}{}_Hs_2(x,y,z) & \frac{1}{2!}\frac{\partial^2}{\partial x^2}{}_Hs_2(x,y,z) & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ {}_Hs_n(x,y,z) & \frac{1}{1!}\frac{\partial}{\partial x}{}_Hs_n(x,y,z) & \frac{1}{2!}\frac{\partial^2}{\partial x^2}{}_Hs_n(x,y,z) & \cdots & \frac{1}{n!}\frac{\partial^n}{\partial x^n}{}_Hs_n(x,y,z)\end{bmatrix}$$
$$\times\begin{bmatrix} xA_0+2yB_0+3zC_0+D_0\\ xA_1+2yB_1+3zC_1+D_1\\ xA_2+2yB_2+3zC_2+D_2\\ \vdots\\ xA_n+2yB_n+3zC_n+D_n\end{bmatrix}. \tag{26}$$

Equating the last rows of Eqs. (24) and (26), assertion (23) follows.

Remark 3.1 Since $f(t)=t\Rightarrow A_0=1,\ A_k=0\ (k\neq 0)$, $B_1=1$, $B_k=0\ (k\neq 1)$, $C_2=2$, $C_k=0\ (k\neq 2)$, therefore for $f(t)=t$, the following consequence of Theorem 3.1 is obtained.

Corollary 3.1 *For the Hermite–Appell polynomials* ${}_HA_n(x,y,z)$, *the following differential recurrence relation holds true*:

$$ {}_HA_{n+1}(x,y,z) = x\,{}_HA_n(x,y,z) + 2y\frac{\partial}{\partial x}{}_HA_n(x,y,z) + 6z\frac{\partial^2}{\partial x^2}{}_HA_n(x,y,z) + \sum_{k=0}^{n}\mathcal{D}_k\frac{1}{k!}\frac{\partial^k}{\partial x^k}{}_HA_n(x,y,z), n\geq 0;{}_HA_0(x,y,z)=\frac{1}{\mathsf{g}(0)}, \tag{27} $$

where

$$ \mathcal{D}_k = \left(-\frac{\mathsf{g}'(t)}{\mathsf{g}(t)}\right)^k\Big|_{t=0}. $$

Next, a pure recurrence relation for ${}_Hs_n(x,y,z)$ is derived by proving the following result.

Theorem 3.2 *For the Hermite–Sheffer polynomials* ${}_Hs_n(x,y,z)$, *the following pure recurrence relation holds true*:

$$ E_0\,{}_Hs_{n+1}(x,y,z) = x\,{}_Hs_n(x,y,z) + \sum_{k=0}^{n}\binom{n}{k}(2yF_k+3zG_k+H_k)\,{}_Hs_{n-k}(x,y,z) - \sum_{k=1}^{n}\binom{n}{k}E_k\,{}_Hs_{n+1-k}(x,y,z), n\geq 0;{}_Hs_0(x,y,z)=\frac{1}{\mathsf{g}(0)}, \tag{28} $$

where

$$ E_k = (f'(f^{-1}(t)))^{(k)}\Big|_{t=0} = \left(\frac{1}{(f^{-1}(t))'}\right)^{(k)}\Big|_{t=0}; F_k = (f^{-1}(t))^{(k)}\Big|_{t=0}; G_k = \left((f^{-1}(t))^2\right)^{(k)}\Big|_{t=0} $$

$$ H_k = \left(-\frac{\mathsf{g}'(f^{-1}(t))}{\mathsf{g}(f^{-1}(t))}\right)^{(k)}\Big|_{t=0}. $$

Proof Using property (13) in expression $\mathcal{W}_n\left[f'(f^{-1}(t))\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\exp(x(f^{-1}(t))+y(f^{-1}(t))^2+z(f^{-1}(t))^3)}{\mathsf{g}(f^{-1}(t))}\right)\right]_{t=0}$, it follows that

$$ \mathcal{W}_n\left[f'(f^{-1}(t))\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\exp(x(f^{-1}(t))+y(f^{-1}(t))^2+z(f^{-1}(t))^3)}{\mathsf{g}(f^{-1}(t))}\right)\right]_{t=0} = \mathcal{P}_n\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\exp(x(f^{-1}(t))+y(f^{-1}(t))^2+z(f^{-1}(t))^3)}{\mathsf{g}(f^{-1}(t))}\right)\right]_{t=0}\mathcal{W}_n[f'(f^{-1}(t))]_{t=0} $$
$$ = \begin{bmatrix} {}_Hs_1(x,y,z) & 0 & 0 & \cdots & 0 \\ {}_Hs_2(x,y,z) & {}_Hs_1(x,y,z) & 0 & \cdots & 0 \\ {}_Hs_3(x,y,z) & \binom{2}{1}{}_Hs_2(x,y,z) & {}_Hs_1(x,y,z) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ {}_Hs_{n+1}(x,y,z) & \binom{n}{1}{}_Hs_n(x,y,z) & \binom{n}{2}{}_Hs_{n-1}(x,y,z) & \cdots & {}_Hs_1(x,y,z) \end{bmatrix}\begin{bmatrix} E_0 \\ E_1 \\ E_2 \\ \vdots \\ E_n \end{bmatrix}. \tag{29} $$

On the other hand, performing the differentiation in the same expression and using properties (11) and (13), it follows that

$$ \mathcal{W}_n\left[f'(f^{-1}(t))\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\exp(xf^{-1}(t)+y(f^{-1}(t))^2+z(f^{-1}(t))^3)}{\mathsf{g}(f^{-1}(t))}\right)\right]_{t=0} = x\mathcal{W}_n\left[\frac{\exp(xf^{-1}(t)+y(f^{-1}(t))^2+z(f^{-1}(t))^3)}{\mathsf{g}(f^{-1}(t))}\right]_{t=0} $$
$$ + \mathcal{P}_n\left[\frac{\exp(xf^{-1}(t)+y(f^{-1}(t))^2+z(f^{-1}(t))^3)}{\mathsf{g}(f^{-1}(t))}\right]_{t=0} \times \mathcal{W}_n\left[2yf^{-1}(t)+3z(f^{-1}(t))^2-\frac{\mathsf{g}'(f^{-1}(t))}{\mathsf{g}(f^{-1}(t))}\right]_{t=0} $$
$$ = x\begin{bmatrix} {}_Hs_0(x,y,z) \\ {}_Hs_1(x,y,z) \\ {}_Hs_2(x,y,z) \\ \vdots \\ {}_Hs_n(x,y,z) \end{bmatrix} + \begin{bmatrix} {}_Hs_0(x,y,z) & 0 & 0 & \cdots & 0 \\ {}_Hs_1(x,y,z) & {}_Hs_0(x,y,z) & 0 & \cdots & 0 \\ {}_Hs_2(x,y,z) & \binom{2}{1}{}_Hs_1(x,y,z) & {}_Hs_0(x,y,z) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ {}_Hs_n(x,y,z) & \binom{n}{1}{}_Hs_{n-1}(x,y,z) & \binom{n}{2}{}_Hs_{n-2}(x,y,z) & \cdots & {}_Hs_0(x,y,z) \end{bmatrix} \times \begin{bmatrix} 2yF_0+3zG_0+H_0 \\ 2yF_1+3zG_1+H_1 \\ 2yF_2+3zG_2+H_2 \\ \vdots \\ 2yF_n+3zG_n+H_n \end{bmatrix}. \tag{30} $$

□

Equating the last rows of Eqs. (29) and (30), we get assertion (28).

Remark 3.2 Since $f(t)=t\Rightarrow E_0=1, E_k=0\ (k\neq 0)$, $F_1=1, F_k=0\ (k\neq 1), G_2=2, G_k=0\ (k\neq 2)$, therefore for $f(t)=t$, the following consequence of Theorem 3.2 is obtained.

Corollary 3.2 *For the Hermite–Appell polynomials* ${}_HA_n(x,y,z)$, *the following pure recurrence relation holds true*:

$$\begin{aligned}{}_HA_{n+1}(x,y,z) &= x_HA_n(x,y,z) \\ &+ 2ny_HA_{n-1}(x,y,z) + 3n(n-1)z_HA_{n-2}(x,y,z) \\ &+ \sum_{k=0}^{n}\binom{n}{k}\mathcal{D}_{kH}A_{n-k}(x,y,z), n \geq 0;_H A_0(x,y,z) = \frac{1}{\mathsf{g}(0)},\end{aligned} \tag{31}$$

where

$$\mathcal{D}_k = \left(-\frac{\mathsf{g}'(t)}{\mathsf{g}(t)}\right)^k\Big|_{t=0}.$$

Finally, we derive a pure recurrence relation, which provides a representation of ${}_Hs_{n+1}(x,y,z)$ in terms of ${}_Hs_k(x,y,z)$ $(k = 0,1,2,\ldots n)$, by proving the following result.

Theorem 3.3 *For the Hermite–Sheffer polynomials* ${}_Hs_n(x,y,z)$, *the following pure recurrence relation holds true*:

$$\begin{aligned}{}_Hs_{n+1}(x,y,z) &= \sum_{k=0}^{n}\binom{n}{k}(xI_k + 2yJ_k + 3zL_k \\ &+ M_k)_Hs_{n-k}(x,y,z), n \geq 0;_H s_0(x,y,z) \\ &= \frac{1}{\mathsf{g}(0)},\end{aligned} \tag{32}$$

where

$$I_k = \left(\frac{1}{f'(f^{-1}(t))}\right)^{(k)}\Big|_{t=0}; J_k = \left(\frac{f^{-1}(t)}{f'(f^{-1}(t))}\right)^{(k)}\Big|_{t=0}; L_k = \left(\frac{(f^{-1}(t))^2}{f'(f^{-1}(t))}\right)^{(k)}\Big|_{t=0}$$

$$M_k = \left(-\frac{\mathsf{g}'(f^{-1}(t))}{\mathsf{g}(f^{-1}(t))}\frac{1}{f'(f^{-1}(t))}\right)^{(k)}\Big|_{t=0}.$$

Proof Performing the differentiation in expression

$$\mathcal{W}_n\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\exp\left(xf^{-1}(t) + y(f^{-1}(t))^2 + z(f^{-1}(t))^3\right)}{\mathsf{g}(f^{-1}(t))}\right)\right]_{t=0}$$

and then using property (13), we have

$$\begin{aligned}&\mathcal{W}_n\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\exp\left(xf^{-1}(t) + y(f^{-1}(t))^2 + z(f^{-1}(t))^3\right)}{\mathsf{g}(f^{-1}(t))}\right)\right]_{t=0} \\ &= \mathcal{P}_n\left[\left(\left(x + 2yf^{-1}(t) + 3z(f^{-1}(t))^2\right)\right.\right. \\ &\left.\left. - \frac{\mathsf{g}'(f^{-1}(t))}{\mathsf{g}(f^{-1}(t))}\right)\frac{1}{f'(f^{-1}(t))}\right]_{t=0} \\ &\times \mathcal{W}_n\left[\frac{\exp\left(xf^{-1}(t) + y(f^{-1}(t))^2 + z(f^{-1}(t))^3\right)}{\mathsf{g}(f^{-1}(t))}\right]_{t=0} \\ &= \begin{bmatrix} \lambda_0 & 0 & 0 & \cdots & 0 \\ \lambda_1 & \lambda_0 & 0 & \cdots & 0 \\ \lambda_2 & \binom{2}{1}\lambda_1 & \lambda_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_n & \binom{n}{1}\lambda_{n-1} & \binom{n}{2}\lambda_{n-2} & \cdots & \lambda_0 \end{bmatrix}\begin{bmatrix} {}_Hs_0(x,y,z) \\ {}_Hs_1(x,y,z) \\ {}_Hs_2(x,y,z) \\ \vdots \\ {}_Hs_n(x,y,z) \end{bmatrix}, \lambda_k \\ &= (xI_k + 2yJ_k + 3zL_k + M_k).\end{aligned} \tag{33}$$

Equating the last rows of equations (24) and (33), we get assertion (32). □

Remark 3.3 Since $f(t) = t \Rightarrow\ I_0 = 1,\ I_k = 0\ (k \neq 0)$, $J_1 = 1,\ J_k = 0\ (k \neq 1),\ L_2 = 2,\ L_k = 0\ (k \neq 2)$, therefore for $f(t) = t$, the following consequence of Theorem 3.3 is obtained.

Corollary 3.3 *For the Hermite–Appell polynomials* ${}_HA_n(x,y,z)$, *the following pure recurrence relation holds true*:

$$\begin{aligned}{}_HA_{n+1}(x,y,z) &= x_HA_n(x,y,z) + 2ny_HA_{n-1}(x,y,z) \\ &+ 3n(n-1)z_HA_{n-2}(x,y,z) \\ &+ \sum_{k=0}^{n}\binom{n}{k}\mathcal{D}_{kH}A_{n-k}(x,y,z), n \geq 0;_H A_0(x,y,z) = \frac{1}{\mathsf{g}(0)}.\end{aligned} \tag{34}$$

In order to derive the differential equation for the Hermite–Sheffer polynomial sequence ${}_Hs_n(x,y,z)$, we prove the following result.

Theorem 3.4 *The Hermite–Sheffer polynomials* ${}_Hs_n(x,y,z)$ *satisfy the following differential equation*:

$$\begin{aligned}&\sum_{k=0}^{n}\frac{(P_kx + 2yQ_k + 3zR_k + T_k)}{k!}\frac{\partial^k}{\partial x^k}{}_Hs_n(x,y,z) \\ &- n_Hs_n(x,y,z) \\ &= 0,\end{aligned} \tag{35}$$

where

$$P_k = \left(\frac{f(t)}{f'(t)}\right)^{(k)}\Big|_{t=0}; Q_k = \left(\frac{tf(t)}{f'(t)}\right)^{(k)}\Big|_{t=0};$$

$$R_k = \left(\frac{t^2 f(t)}{f'(t)}\right)^{(k)}\Big|_{t=0}; T_k = \left(-\frac{g'(t)(f(t))}{g(t)(f'(t))}\right)^{(k)}\Big|_{t=0}.$$

Proof In view of property (13), the expression

$$\mathcal{W}_n\left[t\frac{d}{dt}\left(\frac{\exp(xf^{-1}(t)+yf^{-1}(t)^2+zf^{-1}(t)^3)}{g(f^{-1}(t))}\right)\right]_{t=0}$$

can be written as

$$\mathcal{W}_n\left[t\frac{d}{dt}\left(\frac{\exp\left(xf^{-1}(t)+y(f^{-1}(t))^2+z(f^{-1}(t))^3\right)}{g(f^{-1}(t))}\right)\right]_{t=0}$$
$$=\mathcal{P}_n[t]_{t=0}\mathcal{W}_n\left[\frac{d}{dt}\left(\frac{\exp\left(xf^{-1}(t)+y(f^{-1}(t))^2+z(f^{-1}(t))^3\right)}{g(f^{-1}(t))}\right)\right]_{t=0}$$
$$=\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ & \ddots & \ddots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & n-1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & n & 0 \end{bmatrix}\begin{bmatrix} {}_Hs_1(x,y,z) \\ {}_Hs_2(x,y,z) \\ {}_Hs_3(x,y,z) \\ \vdots \\ {}_Hs_n(x,y,z) \\ {}_Hs_{n+1}(x,y,z) \end{bmatrix}. \tag{36}$$

On the other hand, performing the differentiation in the same expression and using properties (12)-(14) in a suitable manner, we find

$$\mathcal{W}_n\left[t\frac{d}{dt}\left(\frac{\exp\left(xf^{-1}(t)+y(f^{-1}(t))^2+z(f^{-1}(t))^3\right)}{g(f^{-1}(t))}\right)\right]_{t=0}$$
$$=\mathcal{W}_n\left[1,f^{-1}(t),(f^{-1}(t))^2,\ldots,(f^{-1}(t))^n\right]\Lambda_n^{-1}\mathcal{P}_n$$
$$\times\left[\frac{\exp(yt^2+zt^3)}{g(t)}\right]_{t=0}\mathcal{P}_n[\exp(xt)]_{t=0}$$
$$\times\mathcal{W}_n\left[(x+2yt+3zt^2)\frac{f(t)}{f'(t)}-\frac{g'(t)}{g(t)}\frac{f(t)}{f'(t)}\right]$$
$$=\mathcal{W}_n\left[1,f^{-1}(t),(f^{-1}(t))^2,\ldots,(f^{-1}(t))^n\right]\Lambda_n^{-1}\mathcal{P}_n$$
$$\times\left[\frac{\exp(yt^2+zt^3)}{g(t)}\right]_{t=0}\mathcal{P}_n[\exp(xt)]_{t=0}$$
$$\times\mathcal{W}_n\left[x\frac{f(t)}{f'(t)}+2y\frac{tf(t)}{f'(t)}+3z\frac{t^2f(t)}{f'(t)}-\frac{g'(t)}{g(t)}\frac{f(t)}{f'(t)}\right]_{t=0}. \tag{37}$$

Again, in view of Lemma 2.1, we have

$$\mathcal{W}_n\left[t\frac{d}{dt}\left(\frac{\exp\left(xf^{-1}(t)+y(f^{-1}(t))^2+z(f^{-1}(t))^3\right)}{g(f^{-1}(t))}\right)\right]_{t=0}$$
$$=\mathcal{W}_n[{}_Hs_0(x,y,x),{}_Hs_1(x,y,z),$$
$${}_Hs_2(x,y,z),\ldots{}_Hs_n(x,y,z)]^T\Lambda_n^{-1}$$
$$\times\mathcal{W}_n\left[x\frac{f(t)}{f'(t)}+2y\frac{tf(t)}{f'(t)}+3z\frac{t^2f(t)}{f'(t)}-\frac{g'(t)}{g(t)}\frac{f(t)}{f'(t)}\right]_{t=0}.$$
$$=\begin{bmatrix} {}_Hs_0(x,y,z) & 0 & 0 & \cdots & 0 \\ {}_Hs_1(x,y,z) & \frac{1}{1!}\frac{\partial}{\partial x}{}_Hs_1(x,y,z) & 0 & \cdots & 0 \\ {}_Hs_2(x,y,z) & \frac{1}{1!}\frac{\partial}{\partial x}{}_Hs_2(x,y,z) & \frac{1}{2!}\frac{\partial^2}{\partial x^2}{}_Hs_2(x,y,z) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ {}_Hs_n(x,y,z) & \frac{1}{1!}\frac{\partial}{\partial x}{}_Hs_n(x,y,z) & \frac{1}{2!}\frac{\partial^2}{\partial x^2}{}_Hs_n(x,y,z) & \cdots & \frac{1}{n!}\frac{\partial^n}{\partial x^n}{}_Hs_n(x,y,z) \end{bmatrix}$$
$$\times\begin{bmatrix} xP_0+2yQ_0+3zR_0+T_0 \\ xP_1+2yQ_1+3zR_1+T_1 \\ \vdots \\ xP_n+2yQ_n+3zR_n+T_n \end{bmatrix}. \tag{38}$$

Equating last two rows of (36) and (38), assertion (35) follows. □

Remark 3.4 Since $f(t)=t\Rightarrow\ P_0=1,\ P_k=0\ \ (k\neq 0)$, $Q_1=1, Q_k=0\ (k\neq 1), R_2=2, R_k=0\ (k\neq 2)$, therefore for $f(t)=t$, the following consequence of Theorem 3.4 is obtained.

Corollary 3.4 *The Hermite–Appell polynomials ${}_HA_n(x,y,z)$ satisfy the following differential equation*:

$$n{}_HA_n(x,y,z)=x\frac{\partial}{\partial x}{}_HA_n(x,y,z)$$
$$+4y^2\frac{\partial^2}{\partial x^2}{}_HA_n(x,y,z)+9z^2\frac{\partial^3}{\partial x^3}{}_HA_n(x,y,z) \tag{39}$$
$$+\sum_{k=0}^{n}\mathcal{T}_k\frac{1}{k!}\frac{\partial^k}{\partial x^k}{}_HA_n(x,y,z),$$

where

$$\mathcal{T}_k=\left(-\frac{g'(t)t}{g(t)}\right)^k\Big|_{t=0}.$$

In the next section, the recurrence relations and differential equations for some members belonging to the Hermite–Sheffer family are derived.

4 Examples

We derive the recurrence relations and differential equations for some members belonging to the Hermite–Sheffer family by applying Theorems 3.1–3.4.

Example 4.1 For $g(t)=e^{(\frac{t}{v})^m}, f(t)=\frac{t}{v}$ and $f^{-1}(t)=vt$, the Sheffer polynomials become the generalized Hermite polynomials $H_{n,m,v}(x)$. Therefore, for these values of $g(t)$, $f(t)$, the Hermite–Sheffer polynomials become the Hermite–generalized Hermite polynomials ${}_HH_{n,m,v}(x,y,z)$ defined by the following generating function:

$$\exp\left(vxt + v^2yt^2 + v^3zt^3 - t^m\right) = \sum_{n=0}^{\infty} {}_HH_{n,m,v}(x,y,z)\frac{t^n}{n!}. \tag{40}$$

From Theorem 3.1, it follows that

$$A_0 = v; A_k = 0\ (k \neq 0), B_1 = v; B_k = 0\ (k \neq 1), C_2 = 2v;$$
$$C_k = 0\ (k \neq 2), D_{m-1} = -\frac{m!}{v^{m-1}}; D_k = 0\ (k \neq m-1). \tag{41}$$

Substituting the values from Eq. (41) in Eq. (23), the following differential recurrence relation for the Hermite-generalized Hermite polynomials ${}_HH_{n,m,v}(x,y,z)$ is obtained:

$$\begin{aligned}{}_HH_{n+1,m,v}(x,y,z) &= xv{}_HH_{n,m,v}(x,y,z) + 2yv\frac{\partial}{\partial x}{}_HH_{n,m,v}(x,y,z)\\ &+ 3zv\frac{\partial^2}{\partial x^2}{}_HH_{n,m,v}(x,y,z)\\ &- \frac{m}{v^{m-1}}\frac{\partial^{m-1}}{\partial x^{m-1}}{}_HH_{n,m,v}(x,y,z), n \geq 0;\\ {}_HH_{0,m,v}(x,y,z) &= 1.\end{aligned} \tag{42}$$

From Theorem 3.2, it follows that

$$E_0 = \frac{1}{v}; E_k = 0\ (k \neq 0), F_1 = v; F_k = 0\ (k \neq 1), G_2 = 2v^2;$$
$$G_k = 0\ (k \neq 2), H_{m-1} = -\frac{m!}{v}; H_k = 0\ (k \neq m-1). \tag{43}$$

Substituting the values from Eq. (43) in Eq. (28), the following pure recurrence relation for the Hermite-generalized Hermite polynomials ${}_HH_{n,m,v}(x,y,z)$ is obtained:

$$\begin{aligned}{}_HH_{n+1,m,v}(x,y,z) &= xv{}_HH_{n,m,v}(x,y,z) + 2ynv^2_H H_{n-1,m,v}(x,y,z)\\ &+ 3zv^3\frac{n(n-1)}{2}{}_HH_{n-2,m,v}(x,y,z)\\ &- m!\binom{n}{m-1}{}_HH_{n-m+2,m,v}(x,y,z), n \geq 0.\end{aligned} \tag{44}$$

From Theorem 3.3, it follows that

$$I_0 = v; I_k = 0(k \neq 0), J_1 = v^2; J_k = 0(k \neq 1), L_2 = 2v^3;$$
$$L_k = 0(k \neq 2), M_{m-1} = -m!; M_k = 0(k \neq m-1). \tag{45}$$

Substituting the values from Eq. (45) in Eq. (32), the following pure recurrence relation for the Hermite-generalized Hermite polynomials ${}_HH_{n,m,v}(x,y,z)$ is obtained:

$$\begin{aligned}{}_HH_{n+1,m,v}(x,y,z) &= xv{}_HH_{n,m,v}(x,y,z) + 2ynv^2_H H_{n-1,m,v}(x,y,z)\\ &+ 3zv^3n(n-1){}_HH_{n-2,m,v}(x,y,z)\\ &- m!\binom{n}{m-1}{}_HH_{n-m+1,m,v}(x,y,z),\\ n &\geq 0.\end{aligned} \tag{46}$$

Further from Theorem 3.4, it follows that

$$P_1 = 1; P_k = 0\ (k \neq 1), Q_2 = 4y; Q_k = 0\ (k \neq 2),$$
$$R_3 = 18z; R_k = 0\ (k \neq 3), T_m = -\frac{m!m}{v^m}; M_k = 0\ (k \neq m). \tag{47}$$

Substituting the values from Eq. (47) in Eq. (35), we find the following differential equation for the Hermite-generalized Hermite polynomials ${}_HH_{n,m,v}(x,y,z)$:

$$\begin{aligned}n{}_HH_{n,m,v}(x,y,z) &= x\frac{\partial}{\partial x}{}_HH_{n,m,v}(x,y,z) + 4y^2\frac{\partial^2}{\partial x^2}{}_HH_{n,m,v}(x,y,z)\\ &+ 9z^2\frac{\partial^3}{\partial x^3}{}_HH_{n,m,v}(x,y,z) - \frac{m}{v^m}\frac{\partial^m}{\partial x^m}{}_HH_{n,m,v}(x,y,z).\end{aligned} \tag{48}$$

***Example* 4.2** For $g(t) = (1-t)^{-\alpha-1}$, $f(t) = \frac{t}{t-1}$ and $f^{-1}(t) = -\frac{t}{1-t}$, the Sheffer polynomials become the generalized Laguerre polynomials $L_n^{(\alpha)}(x)$. Therefore, for these values of $g(t)$, $f(t)$, the Hermite–Sheffer polynomials become the Hermite-generalized Laguerre polynomials ${}_HL_n^{(\alpha)}(x,y,z)$ defined by the following generating function:

$$\begin{aligned}&\frac{1}{(1-t)^{\alpha+1}}\exp\left(-\frac{xt}{1-t} + \frac{yt^2}{(1-t)^2} - \frac{zt^3}{(1-t)^3}\right)\\ &= \sum_{n=0}^{\infty} {}_HL_n^{(\alpha)}(x,y,z)\frac{t^n}{n!}.\end{aligned} \tag{49}$$

From Theorem 3.1, it follows that

$$\begin{aligned}&A_0 = -1, A_1 = 2, A_2 = -2, A_k = 0\ (k \geq 2); B_1 = -1,\\ &B_2 = 4, B_3 = -6, B_k = 0\ (k \neq 1,2,3)\\ &C_2 = -2, C_3 = 12, C_4 = -24, C_k = 0\ (k \neq 2,3,4);\\ &D_0 = \alpha+1, \alpha, D_1 = -\alpha-1, D_k = 0\ (k \geq 1).\end{aligned} \tag{50}$$

Substituting the values from Eq. (50) in Eq. (23), the following differential recurrence relation for the Hermite-generalized Laguerre polynomials ${}_HL_n^{(\alpha)}(x,y,z)$ is obtained:

$$ {}_HL_{n+1}^{(\alpha)}(x,y,z) = (-x+\alpha+1)\,{}_HL_n^{(\alpha)}(x,y,z) + (2x-2y-\alpha-1)\frac{\partial}{\partial x}{}_H L_n^{(\alpha)}(x,y,z) (-x+4y-3z)\frac{\partial^2}{\partial x^2}{}_H L_n^{(\alpha)}(x,y,z) + (-2y-6z)\frac{\partial^3}{\partial x^3}{}_H L_n^{(\alpha)}(x,y,z) - 3z\frac{\partial^4}{\partial x^4}{}_H L_n^{(\alpha)}(x,y,z), {}_H L_0^{(\alpha)}(x,y,z) = 1. \tag{51} $$

Similarly, from Theorem 3.2, the following pure recurrence relation for the Hermite-generalized Laguerre polynomials ${}_HL_n^{(\alpha)}(x,y,z)$ is obtained:

$$ {}_HL_{n+1}^{(\alpha)}(x,y,z) = (2n+\alpha+1-x)\,{}_HL_n^{(\alpha)}(x,y,z) - \sum_{k=0}^{n}\binom{n}{k}(2yF_k+3xG_k)\,{}_HL_{n-k}^{(\alpha)}(x,y,z) - (n(n-1)+\alpha+1)\,{}_HL_{n-1}^{(\alpha)}(x,y,z), n \geq 0, \tag{52} $$

where

$$ F_k = \left(\frac{t}{t-1}\right)^{(k)}\Big|_{t=0}; G_k = \left(\frac{t^2}{(t-1)^2}\right)^{(k)}\Big|_{t=0}. \tag{53} $$

Again, from Theorem 3.3, the following recurrence relation for the Hermite-generalized Laguerre polynomials ${}_HL_n^{(\alpha)}(x,y,z)$ is obtained:

$$ {}_HL_{n+1}^{(\alpha)}(x,y,z) = \sum_{k=0}^{n}\binom{n}{k}(xI_k+2yJ_k+3zL_k+M_k)_H L_{n-k}^{(\alpha)}(x,y,z), n \geq 0, \tag{54} $$

where

$$ I_k = (-(t-1)^{-2})^{(k)}\Big|_{t=0}; J_k = \left(-t(t-1)^{-3}\right)^{(k)}\Big|_{t=0}, L_k = \left(-t^2(t-1)^{-4}\right)^{(k)}\Big|_{t=0}; M_k = \left(-(\alpha+1)(t-1)^{-1}\right)^{(k)}\Big|_{t=0}. \tag{55} $$

Further, from Theorem 3.4, it follows that

$$ P_1 = 1, P_2 = -2; P_k = 0(k \neq 1,2); Q_2 = 4y, Q_3 = -12y; Q_k = 0(k \neq 2,3), R_3 = 18z, R_4 = -72z; R_k = 0(k \neq 3,4), T_1 = -\alpha-1; T_k = 0(k \neq 1). \tag{56} $$

Substituting the values from Eq. (56) in Eq. (35), the following differential equation for the Hermite-generalized Laguerre polynomials ${}_HL_n^{(\alpha)}(x,y,z)$ is obtained:

$$ n\,{}_HL_n^{(\alpha)}(x,y,z) = (x-\alpha-1)\frac{\partial}{\partial x}{}_H L_n^{(\alpha)}(x,y,z) + (-x+4y^2)\frac{\partial^2}{\partial x^2}{}_H L_n^{(\alpha)}(x,y,z) (-4y^2+9z^2)\frac{\partial^3}{\partial x^3}{}_H L_n^{(\alpha)}(x,y,z) - 9z^2\frac{\partial^4}{\partial x^4}{}_H L_n^{(\alpha)}(x,y,z). \tag{57} $$

Example 4.3 For $g(t) = (1-t)^{-\beta}$, $f(t) = \ln(1-t)$ and $f^{-1}(t) = 1-e^t$, the Sheffer polynomials become the Actuarial polynomials $a_n^{(\beta)}(x)$. Therefore, for these values of $g(t)$, $f(t)$, the Hermite–Sheffer polynomials become the Hermite–Actuarial polynomials ${}_Ha_n^{(\beta)}(x,y,z)$ defined by the following generating function:

$$ \exp\left(x(1-e^t)+y(1-e^t)^2+z(1-e^t)^3+\beta t\right) = \sum_{n=0}^{\infty}{}_Ha_n^{(\beta)}(x,y,z)\frac{t^n}{n!}. \tag{58} $$

From Theorem 3.1, it follows that

$$ A_0 = -1, A_1 = 1; A_k = 0(k \neq 0,1), B_1 = -1, B_2 = 2; B_k = 0(k \neq 1,2), C_2 = -2, C_3 = 6; C_k = 0(k \neq 2,3), D_0 = \beta; D_k = 0(k \neq 0). \tag{59} $$

Substituting the values from Eq. (59) in Eq. (23), the following differential recurrence relation for the Hermite–Actuarial polynomials ${}_Ha_n^{(\beta)}(x,y,z)$ is obtained:

$$ {}_Ha_{n+1}^{(\beta)}(x,y,z) = (-x+\beta)\,{}_Ha_n^{(\beta)}(x,y,z) + (x-2y)\frac{\partial}{\partial x}{}_H a_n^{(\beta)}(x,y,z) + (2y-3z)\frac{\partial^2}{\partial x^2}{}_H a_n^{(\beta)}(x,y,z) + 3z\frac{\partial^3}{\partial x^3}{}_H a_n^{(\beta)}(x,y,z), n \geq 0; {}_Ha_0^{(\beta)}(x,y,z) = 1. \tag{60} $$

Similarly, from Theorem 3.2, the following recurrence relation for the Hermite–Actuarial polynomials ${}_Ha_n^{(\beta)}(x,y,z)$ is obtained:

$$ {}_Ha_{n+1}^{(\beta)}(x,y,z) = -x\,{}_Ha_n^{(\beta)}(x,y,z) - \sum_{k=0}^{n}\binom{n}{k}(2yF_k+3zG_k+H_k)\,{}_Ha_{n-k}^{(\beta)}(x,y,z) + \sum_{k=1}^{n}\binom{n}{k}E_k Ha_{n+1-k}^{(\beta)}(x,y,z), n \geq 0, \tag{61} $$

where

$$E_k = (-\exp(-t))^{(k)}\Big|_{t=0}; F_k = ((1-\exp(t)))^{(k)}\Big|_{t=0},$$
$$G_k = \left((1-\exp(t))^2\right)^{(k)}\Big|_{t=0}; H_k = (-\beta\exp(-t))^{(k)}\Big|_{t=0} \quad (62)$$

Again, from Theorem 3.3, the following recurrence relation for the Hermite–Actuarial polynomials ${}_Ha_n^{(\beta)}(x,y,z)$ is obtained:

$$ {}_Ha_{n+1}^{(\beta)}(x,y,z) = \sum_{k=0}^{n}\binom{n}{k} (xI_k + 2yJ_k + 3zL_k)\,{}_Ha_{n-k}^{(\beta)}(x,y,z) + \beta\,{}_Ha_n^{(\beta)}(x,y,z), n \geq 0. \quad (63)$$

where

$$I_k = (-\exp(t))^{(k)}\Big|_{t=0}; J_k = (\exp(2t)-\exp(t))^{(k)}\Big|_{t=0},$$
$$L_k = (2\exp(2t)-\exp(3t)-\exp(t))^{(k)}\Big|_{t=0}; M_k = (\beta)^{(k)}\Big|_{t=0}. \quad (64)$$

Finally, from Theorem 3.4, the following differential equation satisfied by the Hermite–Actuarial polynomials ${}_Ha_n^{(\beta)}(x,y,z)$ is obtained:

$$\sum_{k=0}^{n}\frac{(xP_k + 2yQ_k + 3zR_k + T_k)}{k!}\frac{\partial^k}{\partial x^k}{}_Ha_n^{(\beta)}(x,y,z) - n\,{}_Ha_n^{(\beta)}(x,y,z) = 0, \quad (65)$$

where

$$P_k = ((t-1)\ln(1-t))^{(k)}\Big|_{t=0}; Q_k = \left(2y(t^2-t)\ln(1-t)\right)^{(k)}\Big|_{t=0},$$
$$R_k = \left(3z(t^3-t^2)\ln(1-t)\right)^{(k)}\Big|_{t=0}; T_k = (\beta\ln(1-t))^{(k)}\Big|_{t=0}. \quad (66)$$

It is to be noted that the differential equations and recurrence relations for other members belonging to the Hermite–Sheffer family can also be obtained in a similar manner by making suitable substitutions. Also, the recurrence relations and differential equations for the members belonging to the Hermite–Appell family can be obtained by applying Corollaries 3.1–3.4.

5 Concluding Remarks

In order to further stress the importance of the approach adopted in previous sections, we establish the following result connecting two different Sheffer sequences.

Theorem 5.1 *Let* ${}_Hs_n^1(x,y,z)$ and ${}_Hs_n^2(x,y,z)$ *be the Hermite–Sheffer polynomial sequences with generating functions*

$$\frac{1}{g_1(f^{-1}(t))}\exp\left(xf^{-1}(t) + y(f^{-1}(t))^2 + z(f^{-1}(t))^3\right) = \sum_{n=0}^{\infty}{}_Hs_n^1(x,y,z)\frac{t^n}{n!} \quad (67)$$

and

$$\frac{1}{g_2(f^{-1}(t))}\exp\left(xf^{-1}(t) + y(f^{-1}(t))^2 + z(f^{-1}(t))^3\right) = \sum_{n=0}^{\infty}{}_Hs_n^2(x,y,z)\frac{t^n}{n!}, \quad (68)$$

respectively. Then

$$ {}_Hs_n^1(x,y,z) = \sum_{k=0}^{n}\binom{n}{k}h^{(n-k)}(0)\,{}_Hs_k^2(x,y,z), \quad (69)$$

where $h(t) = \frac{g_2(f^{-1}(t))}{g_1(f^{-1}(t))}$.

Proof Rewriting the vector form of ${}_Hs_n^1(x,y,z)$ as:

$$\overline{\mathbf{{}_Hs_n^1}}(x,y,z) = \mathcal{W}_n\left[\frac{1}{g_1(f^{-1}(t))}\frac{g_2(f^{-1}(t))}{g_2(f^{-1}(t))}\exp\left(xf^{-1}(t) + y(f^{-1}(t))^2 + z(f^{-1}(t))^3\right)\right]_{t=0}, \quad (70)$$

which on using Eq. (13) gives

$$\overline{\mathbf{{}_Hs_n^1}}(x,y,z) = \mathcal{P}_n\left[\frac{g_2(f^{-1}(t))}{g_1(f^{-1}(t))}\right]_{t=0} \mathcal{W}_n\left[\frac{1}{g_2(f^{-1}(t))}\exp\left(xf^{-1}(t) + y(f^{-1}(t))^2 + z(f^{-1}(t))^3\right)\right]_{t=0}. \quad (71)$$

Again, using vector form of ${}_Hs_n^2(x,y,z)$ in the r.h.s. of Eq. (71), so that we have

$$\overline{\mathbf{{}_Hs_n^1}}(x,y,z) = \mathcal{P}_n\left[\frac{g_2(f^{-1}(t))}{g_1(f^{-1}(t))}\right]_{t=0}\overline{\mathbf{{}_Hs_n^2}}(x,y,z), \quad (72)$$

which on simplification becomes

$$\begin{bmatrix} {}_Hs_0^1(x,y,z) \\ {}_Hs_1^1(x,y,z) \\ {}_Hs_2^1(x,y,z) \\ \vdots \\ {}_Hs_n^1(x,y,z) \end{bmatrix} = \begin{bmatrix} h(0) & 0 & 0 & \cdots & 0 \\ h^{(1)}(0) & h(0) & 0 & \cdots & 0 \\ h^{(2)}(0) & \binom{2}{1}h^{(1)}(0) & h(0) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h^{(n)}(0) & \binom{n}{1}h^{(n-1)}(0) & \binom{n}{2}h^{(n-2)}(0) & \cdots & h(0) \end{bmatrix} \begin{bmatrix} {}_Hs_0^2(x,y,z) \\ {}_Hs_1^2(x,y,z) \\ {}_Hs_2^2(x,y,z) \\ \vdots \\ {}_Hs_n^2(x,y,z) \end{bmatrix}. \quad (73)$$

Equating the last rows of Eq. (73), assertion (69) follows. □

As an illustration of Theorem 5.1, we consider the following example.

Example 5.1 The Poisson–Charlier polynomials $c_n(x;a)$ belong to the Sheffer sequence for

$$\mathsf{g}(t) = e^{a(e^t-1)},$$

$$f(t) = a(e^t - 1)$$

for $a \neq 0$ (Roman 1984). These polynomials are important from the fact that, for $a \neq 0$, they are orthogonal w.r.t. the Poisson distribution:

$$\sum_{k=0}^{\infty} j(k) c_n(k;a) c_m(k;a) = a^{-n} n! \delta_{n,m},$$

where $j(k)$ is the Poisson density

$$j(k) = (a^k/k!)e^{-a}$$

for $k = 0, 1, 2, \ldots$.

With these values of $\mathsf{g}(t)$, $f(t)$, the Hermite–Sheffer polynomials become the Hermite–Poisson–Charlier polynomials ${}_Hc_n(x,y,z;a)$, defined by the following generating function:

$$e^{-t} \exp\left(x \ln\left(1+\frac{t}{a}\right) + y\left(\ln\left(1+\frac{t}{a}\right)\right)^2 + z\left(\ln\left(1+\frac{t}{a}\right)\right)^3\right) = \sum_{n=0}^{\infty} {}_Hc_n(x,y,z;a)\frac{t^n}{n!}. \tag{74}$$

Here we consider the Hermite–Poisson–Charlier polynomials for $a = 1$, which are defined by the following generating function:

$$e^{-t} \exp\left(x \ln(1+t) + y(\ln(1+t))^2 + z(\ln(1+t))^3\right) = \sum_{n=0}^{\infty} {}_Hc_n(x,y,z;1)\frac{t^n}{n!}. \tag{75}$$

Further, for $\mathsf{g}(t) = \frac{1+e^t}{2}$ and $f(t) = e^t - 1$, the Sheffer polynomials become the related polynomials $r_n(x)$ (Jordan 1965). For these values of $\mathsf{g}(t)$ and $f(t)$, the Hermite–Sheffer polynomials become the Hermite-related polynomials ${}_Hr_n(x,y,z)$, defined by the following generating function:

$$\frac{2}{2+t} \exp\left(x \ln(1+t) + y(\ln(1+t))^2 + z(\ln(1+t))^3\right) = \sum_{n=0}^{\infty} {}_Hr_n(x,y,z)\frac{t^n}{n!}. \tag{76}$$

Now applying Theorem 5.1 to generating functions (75) and (76), we obtain the following connection formula between the Hermite–Poisson–Charlier polynomials ${}_Hc_n(x,y,z;1)$ and Hermite-related polynomials ${}_Hr_n(x,y,z)$:

$${}_Hc_n(x,y,z;1) = \sum_{k=0}^{n} \binom{n}{k} h^{(n-k)}(0)\, {}_Hr_n(x,y,z), \tag{77}$$

where $h(t) = \frac{2e^t}{2+t}$.

This article is first attempt in the direction of using matrix approach to a hybrid family of special polynomials. This approach is general and may be used to study the properties of other hybrid polynomial sequences.

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Compliance with ethical standards

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