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RESEARCH PAPER

On Dual Sets and Neighborhood of New Subclasses of Analytic Functions Involving q-Derivative

Khalida Inayat Noor[1] · Humayoun Shahid[1]

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Abstract

Quantum theory has wide applications in special functions and quantum physics. In this paper, we discuss the geometric properties of analytic functions using q-differential operator. We introduce some new subclasses of analytic functions which are obtained from the q-derivative and conic domains. We investigate interesting results involving dual sets and convolution properties of these new subclasses. We also study the inclusion properties of neighborhood of analytic functions. Our results continue to hold for the known and new subclasses of analytic functions which can be obtained as special case.

Keywords Convex · Starlike · Quantum calculus · Analytic functions · Dual sets · Neighborhood · Univalent functions · Convolution · Inclusion results

Mathematics Subject Classification Primary 30C45 · 30C10; Secondary 47B38

1 Introduction

Quantum calculus is ordinary calculus without limit. It is also referred as h-calculus, where h stands for Plank's constant. Recently, quantum calculus attracted attention of many researcher due to its vast applications in many branches of mathematics and physics. Jackson (1909, 1910) introduced and studied q-derivative and q-integral in a systematic way. Ismail et al. (1990) generalized the class of starlike functions using quantum calculus. Mohammed and Darus (2013) studied geometric properties of q-operators in some classes of analytic functions. Sahoo and Sharma (2015) introduced studied q- close-to-convex functions. A comprehensive study of geometric properties of q-hypergeometric series can be found in Agarwal and Sahoo (2014). Recent work on q-calculus can be found in Gairola et al. (2017, 2016); Mishra et al. (2013, 2012).

We recall some basic concepts from quantum calculus.

✉ Humayoun Shahid
shahid_humayoun@yahoo.com

Khalida Inayat Noor
khalidainayat@comsats.edu.pk

[1] Department of Mathematics, COMSATS Institute of Information Technology, Park Road, Islamabad, Pakistan

Let A be the class of analytic functions defined on the open unit disc $E = \{z \in \mathbb{C} : |z| < 1\}$ and is of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

The q-derivative of a function $f \in A$ is defined by [see Jackson (1909)]

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}, \quad (z \neq 0) \tag{1.2}$$

and $D_q f(0) = f'(0)$, where $q \in (0, 1)$.

For a function $g(z) = z^n$, the q-derivative is

$$\begin{aligned} D_q g(z) &= \frac{1-q^n}{1-q} z^{n-1} \\ &= [n]_q z^{n-1}, \end{aligned} \tag{1.3}$$

where

$$[n]_q = \frac{1-q^n}{1-q}.$$

We note that as $q \to 1^-$, $D_q f(z) \to f'(z)$, where $f'(z)$ is ordinary derivative and $[n]_q \to n$ as $q \to 1^-$.

For $f \in A$ defined in (1.1) and using (1.3), we conclude that

$$D_qf(z) = 1 + \sum_{n=2}^{\infty}[n]_q a_n z^{n-1}.$$

As an inverse of q-derivative, Jackson (1910), introduced the q-integral of a function f given by

$$\int_0^z f(t)d_qt = z(1-q)\sum_{n=0}^{\infty} q^n f(q^n z), \tag{1.4}$$

provided the series converges.

Let A be the class of analytic functions. Let C and S^* be the subclasses of univalent functions in E which respectively consists of convex and starlike functions. The q-analogous of these classes was introduced in Seoudy and Aouf (2016) which is defined as follows:

$$C_q = \left\{ f \in A : \Re\left(\frac{D_q(zD_qf(z))}{D_qf(z)}\right) > 0,\ z \in E \right\},$$

$$S_q^* = \left\{ f \in A : \Re\left(\frac{zD_qf(z)}{f(z)}\right) > 0,\ z \in E \right\}.$$

We note that when $q \to 1^-$, the above classes reduce to the class of convex and starlike functions.

For a function $f(z)$ defined in (1.1) and $g(z)$ be given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

the convolution (Hadamard product) is defined by

$$(f*g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g*f)(z), z \in \mathrm{E}.$$

Let f and g be analytic in E, then f is subordinate to g, written as $f \prec g$ or $f(z) \prec g(z)$, $z \in E$, if there exist a Schwarz function ω analytic in E with $\omega(0) = 0$ and $|\omega(z)| < 1$ for $z \in E$, such that

$$f(z) = g(\omega(z)).$$

If g is univalent in E, then $f \prec g$ if and only $f(0) = g(0)$ and $f(E) \subset g(E)$.

For $k \in [0, \infty)$, the conic domain Ω_k is defined in Kanas and Wisniaskawa (1999) as follows:

$$\Omega_k = \left\{ u + iv : u > k\sqrt{(u-1)^2 + v^2} \right\}. \tag{1.5}$$

For fixed k, Ω_k represents the conic region bounded successively by the imaginary axis ($k = 0$), the right branch of a hyperbola ($0 < k < 1$), a parabola ($k = 1$) and an ellipse ($k > 1$). In addition, we note that, for no choice of k ($k > 1$), Ω_k reduces to a disc, see Kanas (2003).

We shall choose $k \in [0, 1]$, for these values of k, the following functions $p_k(z)$ are univalent in E, continuous as regards to k, have real coefficients and map E onto Ω_k, such that $p_k(0) = 1$, $p_k'(0) > 0$:

$$p_k(z) = \begin{cases} \dfrac{1+z}{1-z}, & (k=0) \\ 1 + \dfrac{2}{\pi^2}\left(\log\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & (k=1) \\ 1 + \dfrac{2}{1-k^2}\sinh^2\left\{\dfrac{2}{\pi}\arccos(k)\arctan(\sqrt{z})\right\}, \\ (0 <; k <; 1). \end{cases} \tag{1.6}$$

Utilizing the q-derivative and the conic domain given in (1.5), we now define the following subclasses of analytic functions.

Definition 1.1 Let $f \in A$, $k \in [0, 1]$ and $0 < q < 1$. Then, $f \in k-$qST, if

$$\frac{zD_qf(z)}{f(z)} \prec p_k(z),$$

where $p_k(z)$ is given by (1.6).

Using the Alexander-type relation, the class $k-$qCV is defined as follows:

$$f \in k - \mathrm{qCV} \Longleftrightarrow \mathrm{zD_qf \in k - qST}.$$

As a special case, when $q \to 1^-$, the above classes reduces to well-known classes of $k-$ST ($k-$uniformly starlike functions)and $k-$UCV ($k-$uniformly convex functions) respectively, see Kanas and Wisniaskawa (1999).

The dual set of a set V is defined as follows:

Definition 1.2 Ruscheweyh (1975). Let $V \subset A$, dual set V^* of V is defined as

$$V^* = \left\{ g \in A : \frac{(f*g)(z)}{z} \neq 0, \text{ for all } f \in V \right\}. \tag{1.7}$$

Our main focus in this paper is to discuss the geometric properties of subclasses of analytic functions using dual sets.

2 Main Results

Theorem 2.1 *Let* $k \in [0, 1]$ *and* $0 < q < 1$. *Then,* $V^* = k-$qST, *where*

$$V = \left\{ g \in A : g(z) = \frac{z(1 - L + Lqz)}{(1-L)(1-z)(1-qz)} \right\},$$

and

$$L = L(\alpha) = k\alpha \pm \sqrt{\alpha^2 + (\alpha k - 1)^2}, \alpha^2 + (\alpha k - 1)^2 \geq 0. \tag{2.1}$$

Proof Let $f \in A$ and is of the form (1.1) and let $f \in k-\mathrm{qST}$.

Then

$$\frac{zD_q f(z)}{f(z)} \in \Omega_k, \tag{2.2}$$

where Ω_k is given in (1.5) and $k \in [0,1]$.

Equivalently, (2.2) can be written as

$$\frac{zD_q f(z)}{f(z)} \notin \partial\Omega_k.$$

Using parametric form of $\partial\Omega_k$, we obtain

$$\frac{zD_q f(z)}{f(z)} \neq L(\alpha), \tag{2.3}$$

where $L(\alpha)$ is given by (2.1).

It is known that when $0<q<1$,

$$zD_q f(z) = f(z) * \frac{z}{(1-z)(1-qz)}, \tag{2.4}$$

and

$$f(z) * \frac{z}{1-z} = f(z). \tag{2.5}$$

Using (2.4) and (2.5) in (2.3), we obtain

$$\frac{1}{z}\left[f * \left\{\frac{z(1-L+Lqz)}{(1-L)(1-z)(1-qz)}\right\}\right] \neq 0.$$

Using dual set defined by (1.7), we obtain the required result. □

Theorem 2.2 *Let* $k \in [0,1]$ *and* $0<q<1$. *Then,* $W^* = k-\mathrm{qCV}$, *where*

$$W = \left\{g \in A : g(z) = \frac{z}{(1-L)(1-z)(1-qz)}\left(\frac{1+qz}{1-q^2z} - L\right)\right\},$$

where L is given by (2.1).

Proof Let $f \in A$ and is of the form (1.1) and let $f \in k - q\mathrm{CV}$.

Then, by definition

$$\frac{D_q(zD_q f(z))}{D_q f(z)} \in \Omega_k, \tag{2.6}$$

where Ω_k is given in (1.5) and $k \in [0,1]$. we can write (2.6) as

$$\frac{D_q(zD_q f(z))}{D_q f(z)} \notin \partial\Omega_k.$$

Using parametric form of $\partial\Omega_k$, we get

$$\frac{D_q(zD_q f(z))}{D_q f(z)} \neq L(\alpha), \tag{2.7}$$

where $L(\alpha)$ is given by (2.1). We know that for $0<q<1$,

$$zD_q f(z) = f(z) * \frac{z}{(1-z)(1-qz)}, \tag{2.8}$$

and

$$zD_q\left(f(z) * \frac{z}{(1-z)(1-qz)}\right) = f(z) * \frac{z(1+qz)}{(1-z)(1-qz)(1-q^2z)}. \tag{2.9}$$

Using (2.8) and (2.9) in (2.7), we obtain

$$\frac{1}{z}\left[f * \left\{\frac{z}{(1-L)(1-z)(1-qz)}\left(\frac{1+qz}{1-q^2z} - L\right)\right\}\right] \neq 0.$$

Using the Definition 1.3, we obtain the required result. □

When $q \to 1^-$, The above theorems reduce to following:

Corollary 2.3 [Kanas and Wisniaskawa (1999)]. *Let* $k \in [0,1]$ *and L is given by* (2.1). *Then,* $V^* = k-\mathrm{ST}$ *and* $W^* = k-\mathrm{UCV}$, *where*

$$V = \left\{g \in A : g(z) = \frac{z(1-L+Lz)}{(1-L)(1-z)^2}\right\},$$

and

$$W = \left\{g \in A : g(z) = \frac{z}{(1-L)(1-z)^2}\left(\frac{1+z}{1-z} - L\right)\right\}.$$

We now discuss coefficient bounds of functions in set V and W.

Theorem 2.4 *Let* $k \in [0,1]$ *and* $0<q<1$. *Then, for all* $h(z) = z + \sum_{n=2}^{\infty} c_n z^n \in V$,

$$|c_n| \leq [n]_q + k\left([n]_q - 1\right)$$

and

$$|c_n| \geq \sqrt{1 - k^2\left(\frac{[n]_q - 1}{[n]_q + 1}\right)}.$$

Proof Let $h \in V$, using series representation of $h(z)$ and after some simplifications, we obtain

$$c_n = \frac{[n]_q - L(\alpha)}{1 - L(\alpha)}, \tag{2.10}$$

where $L(\alpha)$ is given in (2.1). Using (2.1) in (2.10), we obtain

$$|c_n|^2 = 1 + \frac{\left([n]_q\right)^2 - 1}{\alpha^2} + \frac{2k\left(1 - [n]_q\right)}{\alpha} = g(\alpha).$$

Since $k \in [0,1]$, we note that $\alpha \geq \frac{1}{k+1}$. In addition, $g(\alpha)$ attains its minimum at $\alpha_0 = \frac{[n]_q+1}{k}$. Note that $\alpha_0 \geq \frac{1}{k+1}$ and $g(\alpha) \leq g\left(\frac{1}{k+1}\right) = \left([n]_q + k\left([n]_q - 1\right)\right)^2$ for all $\alpha \geq 0$. Thus, we have

$$|c_n| \leq [n]_q + k\left([n]_q - 1\right). \tag{2.11}$$

Since α_0 is the minimum, therefore, $g(\alpha) \geq g(\alpha_0) = 1 - k^2\left(\frac{[n]_q-1}{[n]_q+1}\right)$.

Which gives us

$$|c_n| \geq \sqrt{1 - k^2\left(\frac{[n]_q - 1}{[n]_q + 1}\right)}. \tag{2.12}$$

Note that (2.11) and (2.12) are valid for $k \in [0,1]$ and $0 < q < 1$. □

Using Alexander-type relation between k-qST and k-qCV, we have the following.

Corollary 2.5 *Let $k \in [0,1]$ and $0 < q < 1$. Then, for all $h(z) = z + \sum_{n=2}^{\infty} c_n z^n \in W$*

$$|c_n| \leq [n]_q\left([n]_q + k\left([n]_q - 1\right)\right)$$

and

$$|c_n| \geq [n]_q\sqrt{1 - k^2\left(\frac{[n]_q - 1}{[n]_q + 1}\right)}.$$

Corollary 2.6 *Let $k \in [0,1]$, $0 < q < 1$ and let $f(z) = z + \lambda z^n$, $n \geq 2$. Then, $f \in k - q\mathrm{ST}$, if and only if*

$$|\lambda| \leq \frac{1}{[n]_q + k\left([n]_q - 1\right)}, \tag{2.13}$$

and $f \in k - qCV$, if and only if

$$|\lambda| \leq \frac{1}{[n]_q\{[n]_q + k\left([n]_q - 1\right)\}}. \tag{2.14}$$

Proof Let $f(z) = z + \lambda z^n$, with λ satisfies inequality (2.13).

Let $g \in V$ and

$$\left|\frac{(f * g)(z)}{z}\right| \geq 1 - |A||c_n||z|^{n-1} > 1 - |z| > 0.$$

Applying Theorem 2.1, we obtain $f \in k-$qST. Conversely, let $f \in k-$qST and let $g(z) = z + \sum_{n=2}^{\infty}\left([n]_q + k\left([n]_q - 1\right)\right)z^n$.

Then

$$\frac{(f * g)(z)}{z} = 1 + \lambda\left([n]_q + k\left([n]_q - 1\right)\right)z^{n-1} \neq 0.$$

Let $|\lambda| > \frac{1}{[n]_q + k\left([n]_q - 1\right)}$, then there exists $u \in E$, such that

$$\frac{(f * g)(u)}{u} = 0, \mathrm{u} \in \mathrm{E}.$$

which is a contradiction. Hence, $|\lambda| \leq \frac{1}{[n]_q + k\left([n]_q - 1\right)}$.

Using the Alexander-type relation between k-qST and k-qCV, one can obtain condition given in (2.14). □

Using Theorem 2.1, we now obtain the following result which is a special case of theorem given in Dziok (2011).

Corollary 2.7 *Let $k \in [0,1]$, $0 < q < 1$ and let $f \in A$ and is of the form (1.1). If*

$$\sum_{n=2}^{\infty}\left([n]_q + k\left([n]_q - 1\right)\right)|a_n| \leq 1, \tag{2.15}$$

then $f \in k - qST$. In addition, if

$$\sum_{n=2}^{\infty}[n]_q\left([n]_q + k\left([n]_q - 1\right)\right)|a_n| \leq 1, \tag{2.16}$$

then $f \in k - qCV$.

Proof Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = z + \sum_{n=2}^{\infty} c_n z^n \in V$. The convolution

$$\frac{(f * g)(z)}{z} = 1 + \sum_{n=2}^{\infty} c_n a_n z^{n-1}, \quad z \in E. \tag{2.17}$$

It is known from Theorem 2.1 that $f \in k - qST$ if and only if $\frac{(f*g)(z)}{z} \neq 0$, for all $g \in V$. Using Theorem 2.4 in (2.17), we have

$$\left|\frac{(f*g)(z)}{z}\right| \geq 1 - \sum_{n=2}^{\infty}\left([n]_q + k\left([n]_q - 1\right)\right)|a_n||z|^{n-1} > 0, \; z \in E. \tag{2.18}$$

Which can be written as

$$\sum_{n=2}^{\infty}\left([n]_q + k\left([n]_q - 1\right)\right)|a_n| \leq 1.$$

For (2.16), we apply the same method with $g \in W$. □

As a special case when $q \to 1^-$, we have the following special case.

Corollary 2.8 Kanas and Wisniaskawa (1999). *Let $k \in [0,1]$ and let $f \in A$.*

$$\textit{If} \sum_{n=2}^{\infty}(n + k(n-1))|a_n| \leq 1, \textit{then } f \in k-\mathrm{ST},$$

and

$$\text{If} \sum_{n=2}^{\infty} n(n+k(n-1))|a_n| \leq 1, \text{then} f \in k\text{-}UCV.$$

Let $k - S_q$ be the class of satisfying the condition:

$$\sum_{n=2}^{\infty} \left([n]_q + k\left([n]_q - 1\right)\right)|a_n| < 1. \tag{2.19}$$

From Corollary 2.7, we note $k - S_q \subset k - qST$. We prove the following theorem.

Theorem 2.9 *Let $k \in [0,1]$, $0<q<1$. If $f \in k - S_q^*$ and $g \in C$, then $(f * g) \in k - qST$.*

Proof Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in k - S_q$ and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in C$ and $h(z) = z + \sum_{n=2}^{\infty} c_n z^n \in V$.

Since $f \in k - S_q^*$, therefore, by definition

$$1 - \sum_{n=2}^{\infty} \left([n]_q + k\left([n]_q - 1\right)\right)|a_n| > 0. \tag{2.20}$$

To prove that $(f * g) \in k - qST$, it is enough to show that

$$\frac{(f * g * h)(z)}{z} \neq 0.$$

Consider

$$\left|\frac{(f * g * h)(z)}{z}\right| \geq 1 - \sum_{n=2}^{\infty} |a_n||b_n||c_n||z|^{n-1}.$$

As $z \in E$ and $g \in C$, therefore, $|b_n| \leq 1$. Using coefficient bounds of $g(z)$ from Theorem 2.4 and (2.20), we obtain

$$\left|\frac{(f * g * h)(z)}{z}\right| > 1 - \sum_{n=2}^{\infty} \left([n]_q + k\left([n]_q - 1\right)\right)|a_n| > 0, \; z \in E.$$

Thus, $f * g \in k - qST$.

Similarly, we can construct k-C_q using (2.16) and prove the following.

Corollary 2.10 *Let $k \in [0,1]$ and $0<q<1$. Then*

$$\left(k - C_q\right) * C \subset k - qCV.$$

We now prove the generalization of Theorem 2.9.

Theorem 2.11 *Let $V \subset A$ and V^* is a dual set of V. Let $V_1 \subset V^*$ consist of functions satisfying the condition:*

$$\sum_{n=2}^{\infty} |a_n||c_n| < 1.$$

Then

$$C * V_1 \subset V^*.$$

Proof Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in V^*$, $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in C$ and $h(z) = z + \sum_{n=2}^{\infty} c_n z^n \in V$.

Then, by definition of V^*

$$\frac{(f * h)(z)}{z} \neq 0, \quad z \in E.$$

Let $f \in V_1$. Then

$$1 - \sum_{n=2}^{\infty} |a_n||c_n| > 0. \tag{2.21}$$

Now, consider

$$\left|\frac{(f * g * h)(z)}{z}\right| \geq 1 - \sum_{n=2}^{\infty} |a_n||b_n||c_n||z|^{n-1}, \quad z \in E. \tag{2.22}$$

Since $g \in C$, it is known that $|b_n| \leq 1$. Using coefficient bounds of $g(z)$ and (2.21) in (2.22), we obtain

$$\left|\frac{(f * g * h)(z)}{z}\right| > 0, \quad z \in E.$$

Thus, by definition of dual set $f * g \in V^*$. This completes the proof. □

3 Applications of Theorem 2.9

Consider the following operators:

(1) $f_1(z) = \int_0^z \frac{f(\xi)}{\xi} d\xi = (f_1 * \phi_1)(z)$.

(2) $f_2(z) = \frac{2}{z} \int_0^z f(\xi) d\xi = (f_2 * \phi_2)(z)$.

(3) $f_2(z) = \int_0^z \frac{f(\xi) - f(x\xi)}{\xi - x\xi} d\xi = (f_3 * \phi_3)(z), |x| \leq 1, x \neq 1$.

(4) $f_4(z) = \frac{1+c}{z^c} \int_0^z \xi^{c-1} f(\xi) d\xi = (f_4 * \phi_4)(z), \Re(c) > 0$.

Where

$$\phi_1 = -\log(1-z),$$

$$\phi_2 = -2[z + \log(1-z)],$$

$$\phi_3 = \frac{1}{1-x} \log\left[\frac{1-xz}{1-z}\right], \quad |x| = 1, x \neq 1,$$

$$\phi_4 = \sum_{n=1}^{\infty} \frac{1+c}{n+c} z^n, \Re(\mathrm{c}) > 0,$$

ϕ_i, $1 \leq i \leq 3$, can easily be verified to be convex in E and for $\phi_4 \in C$, we refer to Ruscheweyh (1975). If $f \in k - S_q$ or $k - C_q$, then $f_i \in k - qST$ or $k - qCV$. For more applications, see Ruscheweyh and Sheil-Small (1973).

4 Neighborhood of Analytic Functions

For $f \in A$ and is of form (1.1) and $\delta \geq 0$, the T_δ neighborhood of function f is defined as follows:

$$T_\delta(f) = \left\{ g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in A : \sum_{n=2}^{\infty} t_n |b_n - a_n| \leq \delta \right\}. \tag{2.23}$$

Ruscheweyh Ruscheweyh (1981) proved many inclusion properties including when $t_n = n$, especially $T_{\frac{1}{4}}(f) \subset S^*$ for all $f \in C$.

We prove the following.

Theorem 2.12 *Let $f \in T_1(e)$ and is of the form (1.1) with $t_n = [n]_q$ and $e(z) = z$. Then*

$$\left| \frac{zD_q f(z)}{f(z)} - 1 \right| < 1, \tag{2.24}$$

where $z \in E$ and $0 < q < 1$.

Proof Let $f \in A$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Consider

$$\begin{aligned} |zD_q f(z) - f(z)| &= \left| \sum_{n=2}^{\infty} \left([n]_q - 1 \right) a_n z^{n-1} \right| \\ &\leq \sum_{n=2}^{\infty} [n]_q |a_n| - \sum_{n=2}^{\infty} [n]_q |a_n| |z|^{n-1} \\ &\leq |z| - \sum_{n=2}^{\infty} [n]_q |a_n| |z|^{n-1}. \\ &\leq |f(z)|, \quad z \in E. \end{aligned}$$

This gives us the required result.

The procedure already described in Theorem 2.12 leads to the following new result. □

Theorem 2.13 *If $f \in T_1(e)$ with $t_n = \left([n]_q\right)^2$ and $e(z) = z$, then*

$$\left| \frac{D_q\left(zD_q f(z)\right)}{D_q f(z)} - 1 \right| < 1. \tag{2.25}$$

The inequalities (2.24) and (2.25) describe subclasses of q- starlike and q-convex functions with $\phi(z) = 1 + z$, for more details, see Seoudy and Aouf (2016).

We now discuss the geometric properties of $T_\delta(f)$ defined in (2.23). Consider a class of functions

$$F_\alpha(z) = \frac{f(z) + \alpha z}{1 + \alpha}, \tag{2.26}$$

where α is a non-zero complex number. We note that if $f \in A$, then $F_\alpha \in A$. We discuss the relationship between $f(z)$ and $F_\alpha(z)$ in the following Lemma.

Lemma 2.14 *Let $k \in [0, 1]$, $f \in A$ and $\delta > 0$. If $F_\alpha \in k - qST$ for all $\alpha \in \mathbb{C} \backslash \{0\}$, then $f \in k - qST$. Furthermore, for all $g \in V$*

$$\left| \frac{(f * g)(z)}{z} \right| > \delta,$$

where $|\alpha| < \delta$ and $z \in E$.

Proof Since $F_\alpha \in k - qST$, Therefore, by Theorem 2.1, we have

$$\frac{(F_\alpha * g)(z)}{z} \neq 0, \text{ for all } g \in V \text{ and } z \in E.$$

Using (2.26) and after some simplifications, we obtain

$$\frac{(f * g)(z)}{z} \neq -\alpha, \text{ for all } \alpha \in \mathbb{C} \backslash \{0\}.$$

For $|\alpha| < \delta$

$$\left| \frac{(f * g)(z)}{z} \right| > \delta. \tag{2.27}$$

Applying Theorem 2.1, we obtain that $f \in k-$qST. □

Applying the similar method, we have the following Lemma.

Lemma 2.15 *Let $k \in [0, 1]$, $f \in A$ and $\delta > 0$ and let α be a non-zero complex number. if $F_\alpha \in k - qCV$, then $f \in k - qCV$, furthermore for all $g \in W$:*

$$\left| \frac{(f * g)(z)}{z} \right| > \delta,$$

where $|\alpha| < \delta$ and $z \in E$.

We now prove the following.

Theorem 2.16 *Let $k \in [0, 1]$ and $\delta > 0$. If $F_\alpha \in k - qST$ for all $\alpha \in \mathbb{C} \backslash \{0\}$, then $T_{\delta_1}(f) \subset k - qST$ with $t_n = [n]_q$ and*

$$\delta_1 = \frac{\delta}{k+1},$$

where $|\alpha| < \delta$.

Proof Let $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in T_{\delta_1}(f)$. Using Theorem 2.1, $g \in k - qST$, if and only if

$$\frac{(g * h)(z)}{z} \neq 0,$$

for all $h \in V$.

Consider

$$\begin{aligned} \left| \frac{(g * h)(z)}{z} \right| &= \left| \frac{(f * h)(z)}{z} + \frac{((g - f) * h)(z)}{z} \right| \\ &\geq \left| \frac{(f * h)(z)}{z} \right| - \left| \frac{((g - f) * h)(z)}{z} \right|. \end{aligned} \tag{2.28}$$

Using Lemma 2.14 and series representations of $f(z)$ and $g(z)$ with $h(z)=\sum_{n=2}^{\infty}\left([n]_q+k\left([n]_q-1\right)\right)z^n$, we obtain

$$\left|\frac{(g*h)(z)}{z}\right|>\delta-\sum_{n=2}^{\infty}\left([n]_q+k\left([n]_q-1\right)\right)|b_n-a_n|. \tag{2.29}$$

Now

$$\sum_{n=2}^{\infty}\left([n]_q+k\left([n]_q-1\right)\right)|b_n-a_n|\le(k+1)$$
$$\sum_{n=2}^{\infty}[n]_q|b_n-a_n|\le(k+1)\delta_1. \tag{2.30}$$

Using (2.29) and (2.30) in (2.28), we obtain

$$\left|\frac{(g*h)(z)}{z}\right|>\delta-(k+1)\delta_1>0.$$

Where

$$\delta_1=\frac{\delta}{k+1}.$$

Remark 2.17 In Theorem 2.16, we can replace t_n by n and obtain same result as $[n]_q<n$ when $0<q<1$.

Theorem 2.18 *Let $f\in k-qST$. Then, $T_{\delta_1}(f)\subset k-qST$ with*

$$\delta_1<\frac{c}{k+1}, \tag{2.31}$$

where c is a non-zero real number with $c\le\left|\frac{(f*h)(z)}{z}\right|$, $z\in E$.

Proof Let $g\in T_{\delta_1}(f)$, with $t_n=[n]_q$ and let $h\in V$.

Consider

$$\left|\frac{(g*h)(z)}{z}\right|\ge\left|\frac{(f*h)(z)}{z}\right|-\left|\frac{((g-f)*h)(z)}{z}\right|. \tag{2.32}$$

Since $f\in k-qST$, therefore applying Theorem 2.1, we obtain

$$\left|\frac{(f*g)(z)}{z}\right|\ge c, \tag{2.33}$$

where c is a non-zero real number and $z\in E$.

Now

$$\begin{aligned}\left|\frac{((g-f)*h)(z)}{z}\right|&=\sum_{n=2}^{\infty}|c_n||b_n-a_n||z|^{n-1}\\&\le\sum_{n=2}^{\infty}\left([n]_q+k\left([n]_q-1\right)\right)|b_n-a_n|\\&\le(k+1)\delta_1,\end{aligned} \tag{2.34}$$

where we have used Theorem 2.4. Using (2.33) and (2.34) in (2.32), we obtain

$$\left|\frac{(g*h)(z)}{z}\right|>c-(k+1)\delta_1>0.$$

where δ_1 is given in (2.31). This completes the proof. □

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