Iran J Sci Technol Trans Sci (2019) 43:249–254
https://doi.org/10.1007/s40995-017-0434-3

RESEARCH PAPER

Approximation Properties of King Type (p, q)-Bernstein Operators

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Received: 28 March 2017 / Accepted: 27 November 2017 / Published online: 12 December 2017

Abstract

The present paper deals mainly with a King type modification of (p, q)-Bernstein operators. By improving the conditions given in Mursaleen et al. (On (p, q)-analogue of Bernstein operators. Appl Math Comput 266:874–882, 2015a), we investigate the Korovkin type approximation of both (p, q)-Bernstein and King type (p, q)-Bernstein operators. We also prove that the error estimation of King type of the operator is better than that of the classical one whenever $0 \le x \le \frac{1}{3}$.

Keywords (p, q)-integers · (p, q)-Bernstein Operators · King type operators · Rate of convergence

1 Introduction

In the last two decades quantum calculus has gained a lot of interest in the field of approximation theory. The q-generalizations of several operators have been constructed and studied by many authors. We can refer the book by Kac and Cheung (2002) for the details of q-calculus and the book by Aral et al. (2013) in which one can find the related studies on this area. Recently, a new generalization of q-calculus has been appeared in the approximation theory, namely (p, q) calculus. Before mentioning the studies on this topic, we recall some notations on (p, q)-integers. Let $0 < q < p \le 1$. For each nonnegative integer n, the (p, q)-numbers is denoted by $[n]_{p,q}$ and given by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}.$$

For each $k, n \in \mathbb{N}$, $n \ge k \ge 0$, the (p, q)-factorial $[n]_{p,q}!$ and (p, q)-binomial are defined by

$$[n]_{p,q}! = \prod_{k=1}^{n} [k]_{p,q}, \quad n \ge 1, \quad [0]_{p,q}! = 1,$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[n-k]_{p,q}![k]_{p,q}!}.$$

The (p, q)-power basis and (p, q)-binomial expansion are defined as

$$(x+y)^n_{p,q} = (x+y)(px+qy)(p^2x+q^2y), \ldots, (p^{n-1}x+q^{n-1}y),$$
$$= \sum_{k=0}^{n} p^{\binom{n-k}{2}} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k y^{n-k},$$

and

$$(1-x)^n_{p,q} = (1-x)(p - qx)(p^2 - q^2x), \ldots, (p^{n-1} - q^{n-1}x).$$

The relation between q-analogues and (p, q)-analogues can be described as:

$$[n]_{p,q} = p^{n-1}[n]_{q/p},$$
$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = p^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{q/p}, \tag{1.1}$$
$$(x-a)^n_{p,q} = p^{n(n-1)/2}(x-a)^n_{q/p}.$$

Note that as taking $p = 1$, (p, q) integers turns out to be q-integers. So q-integers can be regarded as a special case of (p, q)-integers.

Lupas (1987) was the first to study q-calculus in approximation theory by defining a q-generalization of Bernstein operators. Later in 1997, Phillips (1996)

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constructed another modification of Bernstein operators for $n \in \mathbb{N}$ and $0<q<1$ as

$$B_n(f, q; x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q x^k (1-x)_q^{n-k} f\left(\frac{[k]_q}{[n]_q}\right), \quad x \in [0, 1], \tag{1.2}$$

which is known as q-Bernstein operators in the literature.

For the study of (p, q)-calculus in approximation theory, pioneer works are due to Mursaleen et al. (2015a, b). They defined the (p, q)-Bernstein operators by

$$B_n(f, p, q; x) = \sum_{k=0}^{n} p^{[k(k-1)-n(n-1)]/2} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} x^k (1-x)_{p,q}^{n-k} f\left(\frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}}\right), \tag{1.3}$$

for $n \in \mathbb{N}$ and $0<q<p\leq 1$. Very recently, same authors defined the (p, q)-analogue of Bernstein–Stancu operators in Mursaleen et al. (2015c). Mursaleen et al. (2015d) also studied the approximation properties of (p, q)-Bernstein–Shurer operators. Since the use of (p, q)-calculus in approximation theory is a very new issue and open for development, lately, the (p, q)-generalizations of several operators are being studied by many authors. We refer to Acar et al. (2016), Cai and Zhou (2016), Cai (2017), Gupta (2016a, b) and Karaisa (2016) for some studies about the (p, q)-analogues of some generalizations of Bernstein operators. See also Acar (2016), Gupta (2016c), Mishra and Pandey (2016), Mursaleen et al. (2016) and references therein for some other recent works on the (p, q) generalizations of some other operators.

In the present paper our aim is to construct the King type modification of (p, q)-Bernstein operators and investigate its approximation properties. Before constructing the mentioned operator, we first give the Korovkin type approximation of the (p, q)-Bernstein operators under the weaker conditions than that of the ones given by Mursaleen et al. (2015a). In Sect. 2, we construct King type (p, q)-Bernstein operator and give some auxiliary lemmas and results. Section 3 is devoted to our main results. We give the uniform convergence of the operators via Korovkin's theorem and obtain rate of convergence by means of modulus of continuity. We also compare the error estimates of the classical and King type (p, q)-Bernstein operators.

2 Construction of the Operators and Auxiliary Lemmas

Before constructing our new operators recall that q-Bernstein operators defined by (1.2) and (p, q)-Bernstein operators defined by (1.3) satisfy the following identities (see Mursaleen et al. 2015b; Phillips 1996 for details):

$$\begin{aligned} B_n(e_0, q; x) &= 1, \\ B_n(e_1, q; x) &= x, \\ B_n(e_2, q; x) &= x^2 + \frac{x(1-x)}{[n]_q}, \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} B_n(e_0, p, q; x) &= 1, \\ B_n(e_1, p, q; x) &= x, \\ B_n(e_2, p, q; x) &= x^2 + \frac{p^{n-1}x(1-x)}{[n]_{p,q}}. \end{aligned} \tag{2.2}$$

Remark 1 Since $\frac{1}{[n]_{q/p}} = \frac{p^{n-1}}{[n]_{p,q}}$ and $x \in [0, 1]$, if we take p and q as sequences p_n and q_n satisfying the conditions

$$0<q_n<p_n\leq 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{q_n}{p_n} = 1, \tag{2.3}$$

then the (p, q)-Bernstein operator (1.3) converges uniformly to the function f on $[0, 1]$. Note that with these conditions $[n]_{q/p}$ tends to infinity as $n \to \infty$, and hence uniform convergence holds from Korovkin's theorem.

Example 1 We can always find such sequences: for example, letting $(q_n) = \left(1-\frac{1}{n}\right)$ and $(p_n) = \left(1-\frac{1}{n+1}\right)$ we get $\lim_{n\to\infty}\frac{(q_n)}{(p_n)} = \lim_{n\to\infty}\frac{\left(1-\frac{1}{n}\right)}{\left(1-\frac{1}{n+1}\right)} = 1$ and $\lim_{n\to\infty}[n]_{q/p} = \lim_{n\to\infty}\frac{[n]_{p,q}}{p^{n-1}} = \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^{1-n}\frac{\left(1-\frac{1}{n+1}\right)^n-\left(1-\frac{1}{n}\right)^n}{\left(1-\frac{1}{n+1}\right)-\left(1-\frac{1}{n}\right)} = \infty.$ Similarly, if we take $(q_n) = \frac{1}{2}-\frac{1}{2n+0.7}$ and $(p_n) = \frac{1}{2}-\frac{1}{e^n}$ we get, $\lim_{n\to\infty}\frac{(q_n)}{(p_n)} = \lim_{n\to\infty}\frac{\left(\frac{1}{2}-\frac{1}{2n+0.7}\right)}{\left(\frac{1}{2}-\frac{1}{e^n}\right)} = 1$ and $\lim_{n\to\infty}[n]_{q/p} = \lim_{n\to\infty}\frac{[n]_{p,q}}{p^{n-1}} = \lim_{n\to\infty}\left(\frac{1}{2}-\frac{1}{e^n}\right)^{1-n}\frac{\left(\frac{1}{2}-\frac{1}{e^n}\right)^n-\left(\frac{1}{2}-\frac{1}{2n+0.7}\right)^n}{\left(\frac{1}{2}-\frac{1}{e^n}\right)-\left(\frac{1}{2}-\frac{1}{2n+0.7}\right)} = \infty.$ Note that in the second example, (q_n) and (p_n) tends to 1/2 separately, as n tends to infinity.

Motivated by the method of King (2003), we now modify (p, q)-Bernstein operators in a way that the new operator preserves x^2. For this purpose, for each $n \in \mathbb{N}$ and $0<q<p\leq 1$, we define the King type (p, q)-Bernstein operators $B_n^* : C[0, 1] \to C[0, 1]$ as,

$$B_n^*(f, p, q; x) = \sum_{k=0}^{n} p^{[k(k-1)-n(n-1)]/2} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \left(v_{n,p,q}(x)\right)^k \times \left(1-v_{n,p,q}(x)\right)_{p,q}^{n-k} f\left(\frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}}\right), \tag{2.4}$$

for $0\leq x\leq 1$, where

$$v_{n,p,q}(x) = -\frac{p^{n-1}}{2([n]_{p,q} - p^{n-1})} + \sqrt{\frac{[n]_{p,q}}{([n]_{p,q} - p^{n-1})}x^2 + \frac{p^{2n-2}}{4([n]_{p,q} - p^{n-1})^2}}. \tag{2.5}$$

Remark 2 $[n]_{p,q} - p^{n-1} = q[n-1]_{p,q} \geq 0$, $\sqrt{\frac{[n]_{p,q}}{([n]_{p,q}-p^{n-1})}x^2 + \frac{p^{2n-2}}{4([n]_{p,q}-p^{n-1})^2}} \geq \frac{p^{n-1}}{2([n]_{p,q}-p^{n-1})}$ from which we have $v_{n,p,q}(x) \geq 0$. On the other hand, since $x \in [0, 1]$, we get $\sqrt{\frac{[n]_{p,q}}{([n]_{p,q}-p^{n-1})}x^2 + \frac{p^{2n-2}}{4([n]_{p,q}-p^{n-1})^2}} = \frac{\sqrt{4([n]_{p,q}-p^{n-1})[n]x^2+p^{2n-2}}}{2([n]_{p,q}-p^{n-1})} \leq \frac{2[n]_{p,q}-p^{n-1}}{2([n]_{p,q}-p^{n-1})}$. Using this inequality in (2.5), we obtain $v_{n,p,q}(x) \leq 1$. Hence, $0 \leq v_{n,p,q}(x) \leq 1$ and B_n^* is a linear and positive operator.

Lemma 1 *For the operators B_n^* defined by (2.4), the following identities hold*:

$$\begin{aligned} &B_n^*(e_0, p, q; x) = 1, \\ &B_n^*(e_1, p, q; x) = v_{n,p,q}(x), \\ &B_n^*(e_2, p, q; x) = x^2. \end{aligned} \tag{2.6}$$

Proof With the help of the identities in (2.2) and the test functions of q-Bernstein operators given by (2.1), we can write

$$\begin{aligned} B_n^*(e_0, p, q; x) &= \sum_{k=0}^{n} p^{[k(k-1)-n(n-1)]/2} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \left(v_{n,p,q}(x)\right)^k \\ &\quad \times \left(1 - v_{n,p,q}(x)\right)_{p,q}^{n-k} \\ &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{q/p} \left(v_{n,p,q}(x)\right)^k \left(1 - v_{n,p,q}(x)\right)_{q/p}^{n-k} = 1, \end{aligned}$$

$$\begin{aligned} B_n^*(e_1, p, q; x) &= \sum_{k=1}^{n} \frac{p^{n-k}[n-1]!_{p,q}}{[k-1]!_{p,q}[n-k]!_{p,q}} p^{[k(k-1)-n(n-1)]/2} \\ &\quad \times \left(v_{n,p,q}(x)\right)^k \left(1 - v_{n,p,q}(x)\right)_{p,q}^{n-k} \\ &= \sum_{k=0}^{n-1} p^{[k(k-1)-(n-2)(n-1)]/2} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \left(v_{n,p,q}(x)\right)^{k+1} \\ &\quad \times \left(1 - v_{n,p,q}(x)\right)_{p,q}^{n-k-1} \\ &= v_{n,p,q}(x) \end{aligned}$$

and

$$\begin{aligned} B_n^*(e_2, p, q; x) &= \sum_{k=0}^{n} p^{[k(k-1)-n(n-1)]/2} \frac{p^{2n-2k}[k]_{p,q}^2}{[n]_{p,q}^2} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \left(v_{n,p,q}(x)\right)^k \\ &\quad \times \left(1 - v_{n,p,q}(x)\right)_{p,q}^{n-k} \\ &= \frac{p^{n-1}}{[n]_{p,q}} v_{n,p,q}(x) + q\left(v_{n,p,q}(x)\right)^2 \frac{[n-1]_{p,q}}{[n]_{p,q}} \\ &= \frac{p^{n-1}}{[n]_{p,q}} \frac{1}{2([n]_{p,q} - p^{n-1})} \\ &\quad \times \left(-p^{n-1} + \sqrt{4[n]_{p,q}\left([n]_{p,q} - p^{n-1}\right)x^2 + p^{2n-2}}\right) \\ &\quad + q\frac{[n-1]}{[n]} \frac{1}{4([n]_{p,q} - p^{n-1})^2} \\ &\quad \times \left(p^{2n-2} + 4[n]_{p,q}\left([n]_{p,q} - p^{n-1}\right)x^2 + p^{2n-2}\right. \\ &\quad \left. -2p^{n-1}\sqrt{4[n]_{p,q}\left([n]_{p,q} - p^{n-1}\right)x^2 + p^{2n-2}}\right). \end{aligned}$$

Using the identity $q[n-1]_{p,q} = [n]_{p,q} - p^{n-1}$ and arranging the terms one can easily obtain that

$$B_n^*(e_2, p, q; x) = x^2.$$

□

3 Main Results

3.1 Korovkin Type Approximation

Before giving the Korovkin type approximation theorem for the operator (2.4), let us remind the following remark:

Remark 3 For fixed p, q with $0 < q < p \leq 1$, one can see that $\lim_{n\to\infty} [n]_{p,q} = \lim_{n\to\infty} \frac{p^n - q^n}{p-q} = 0$ and $\lim_{n\to\infty} p^{1-n} [n]_{p,q} = \lim_{n\to\infty} \frac{p}{p-q} \frac{p^n - q^n}{p^n} = \lim_{n\to\infty} \frac{p}{p-q} \left(1 - \left(\frac{q}{p}\right)^n\right) = \frac{p}{p-q}$ which is different from infinity. So, $v_{n,p,q}(x)$ cannot converge to x as $n \to \infty$ and this means that we are not able to investigate approximation properties of the operators $B_n^*(f, p, q; \cdot)$ given by (2.4) via Korovkin's theorem. To obtain convergency results, we assume p and q as sequences p_n and q_n satisfying the conditions given in (2.3).

Theorem 1 *Let p_n and q_n be sequences satisfying the conditions given in (2.3). Then for every $f \in C[0, 1]$, $B_n^*(f, p_n, q_n; \cdot)$ converges uniformly to f.*

Proof We know that B_n^* is a linear and positive operator. Let us examine the second moment of the operator.

$$\begin{aligned}B_n^*(e_1,\,p,\,q;\,x) =& v_{n,p,q}(x)\\ =& -\frac{p^{n-1}}{2([n]_{p,q}-p^{n-1})}\\ &+\sqrt{\frac{[n]_{p,q}}{([n]_{p,q}-p^{n-1})}x^2+\frac{p^{2n-2}}{4([n]_{p,q}-p^{n-1})^2}}\\ =& -\frac{1}{2\left(\frac{[n]_{p,q}}{p^{n-1}}-1\right)}+\sqrt{\frac{[n]_{p,q}}{p^{n-1}}\frac{1}{\left(\frac{[n]_{p,q}}{p^{n-1}}-1\right)}x^2+\frac{1}{4\left(\frac{[n]_{p,q}}{p^{n-1}}-1\right)^2}}.\end{aligned}$$

Using the identity in (1.1), we can rewrite the above equality as

$$\begin{aligned}B_n^*(e_1,\,p,\,q;\,x) =& -\frac{1}{2}\frac{1}{([n]_{q/p}-1)}\\ &+\sqrt{[n]_{q/p}\frac{1}{([n]_{q/p}-1)}x^2+\frac{1}{4([n]_{q/p}-1)^2}}.\end{aligned}$$

Hence, from the conditions given in (2.3), we get

$$\lim_{n\to\infty}\left\|B_n^*(e_1,\,p,\,q;\,\cdot)-x\right\|=0.$$

Finally the result follows from Korovkin's theorem. □

3.2 Rate of Convergence

Let $f\in C[0,\,1]$. The modulus of continuity of f, denoted by $\omega_f(\delta)$, is defined as

$$\omega_f(\delta)=\sup_{\substack{x,y\in[0,1]\\|x-y|\le\delta}}|f(x)-f(y)|. \tag{3.1}$$

Note that for $f\in C[0,\,1]$, $\lim_{\delta\to 0}\omega_f(\delta)=0$ and for each $t,\,x\in[0,\,1]$

$$|f(t)-f(x)|\le\omega_f(\delta)\left(\frac{|t-x|}{\delta}+1\right). \tag{3.2}$$

We now give the rate of convergence of the operators $B_n^*(f,\,p,\,q,\,\cdot)$ by means of modulus of continuity.

Theorem 2 *Let f be a function in $C[0,\,1]$ and $x\in[0,\,1]$. The operators B_n^*, $n\in\mathbb{N}$, defined by (2.4) satisfy*

$$\left|B_n^*(f,\,p,\,q;\,x)-f(x)\right|\le 2\omega_f\left(\delta_{n,p,q,x}\right),$$

where $\delta_{n,p,q,x}=\sqrt{2x(x-v_{n,p,q}(x))}$ and $v_{n,p,q}$ is given by (2.5).

Proof Note that based on Cauchy's inequality we can write $(B_n^*(e_1,\,p,\,q;\,x))^2\le B_n^*(e_0,\,p,\,q;\,x)\;\; B_n^*(e_2,\,p,\,q;\,x)$. So, by Lemma 1, we have $B_n^*(e_1,\,p,\,q;\,x)\le x$ for each $x\in[0,\,1]$, from which we get $2x(x-v_{n,p,q}(x))\ge 0$. From (3.1) and (3.2) we have

$$\begin{aligned}\left|B_n^*(f,\,p,\,q;\,x)-f(x)\right|\le&\Bigg\{\sum_{k=0}^{n}p^{[k(k-1)-n(n-1)]/2}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\left(v_{n,p,q}(x)\right)^k\\ &\times\left(1-v_{n,p,q}(x)\right)_{p,q}^{n-k}\\ &\times\left(\frac{\left|\frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}}-x\right|}{\delta}+1\right)w(f;\,\delta)\Bigg\}.\end{aligned}$$

Cauchy's inequality and the identities given by (2.6) imply

$$\begin{aligned}\left|B_n^*(f,\,p,\,q;\,x)-f(x)\right|\le&\Bigg\{\frac{1}{\delta^2}\sum_{k=0}^{n}p^{[k(k-1)-n(n-1)]/2}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}\left(v_{n,p,q}(x)\right)^k\\ &\times\left(1-v_{n,p,q}(x)\right)_{p,q}^{n-k}\\ &\times\left(\frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}}-x\right)^2+1\Bigg\}w(f;\,\delta)\\ =&\left\{\frac{1}{\delta^2}\left(B_n^*(e_2,\,p,\,q;\,x)-2xB_n^*(e_1,\,p,\,q;\,x)+x^2\right)\right.\\ &\left.+1\right\}w(f;\,\delta)\\ =&\left\{\frac{1}{\delta^2}\left(2x(x-v_{n,p,q}(x)\right)+1\right\}w(f;\,\delta).\end{aligned}$$

Choosing $\delta=\delta_{n,p,q,x}=\sqrt{2x(x-v_{n,p,q})}$, the proof is completed. □

Theorem 3 *Let f be a function in $C^1[0,\,1]$ and $x\in[0,\,1]$. The operators B_n^*, $n\in\mathbb{N}$, defined by (2.4) verify for $\delta>0$*

$$\begin{aligned}\left|B_n^*(f,\,p,\,q;\,x)-f(x)\right|\le&|f'(x)|\left(x-v_{n,p,q}(x)\right)\\ &+\left(\frac{1}{\delta}\sqrt{2x\left(x-v_{n,p,q}(x)\right)}+1\right)\\ &\sqrt{2x\left(x-v_{n,p,q}(x)\right)}\omega_{f'}(\delta),\end{aligned}$$

where $v_{n,p,q}$ is given in (2.5).

Proof For every $f\in C^1[0,\,1]$ and $x,\,t\in[0,\,1]$, we have

$$f(t)-f(x)=f'(x)(t-x)+\int_x^t(f'(u)-f'(x))\mathrm{d}u. \tag{3.3}$$

On the other hand, we can write

$$\left|\int_x^t f'(u)-f'(x)\right|\le\omega_{f'}(\delta)\left(\frac{(t-x)^2}{\delta}+|t-x|\right).$$

Applying B_n^* to both the sides of (3.3) and using the above inequality we get,

$$\begin{aligned}\left|B_n^*(f,\,p,\,q;\,x)-f(x)\right|\le&|f'(x)|B_n^*(|t-x|,\,p,\,q;\,x)\\ &+\left(\frac{1}{\delta}B_n^*\left((t-x)^2,\,p,\,q;\,x\right)\right.\\ &\left.+B_n^*(|t-x|,\,p,\,q;\,x)\right)\omega_{f'}(\delta).\end{aligned}$$

Using Cauchy's inequality, we have

$$\left|B_n^*(f, p, q; x) - f(x)\right| \le |f'(x)| B_n^*(|t-x|, p, q; x) + \left(\frac{1}{\delta}\sqrt{B_n^*\left((t-x)^2, p, q; x\right)} + 1\right) \sqrt{B_n^*\left((t-x)^2, p, q; x\right)}\,\omega_{f'}(\delta).$$

So, from the second central moment of the operator B_n^*, we obtain

$$\left|B_n^*(f, p, q; x) - f(x)\right| \le |f'(x)|\left(x - v_{n,p,q}(x)\right) + \left(\frac{1}{\delta}\sqrt{2x\left(x - v_{n,p,q}(x)\right)} + 1\right) \sqrt{2x\left(x - v_{n,p,q}(x)\right)}\,\omega_{f'}(\delta).$$

which is the desired result. □

Note that for the (p, q)-Bernstein operators, for $f \in C[0, 1]$, $x \in [0, 1]$, one has

$$|B_n(f, p, q; x) - f(x)| \le \left(1 + \frac{1}{\delta}\sqrt{\frac{p^{n-1}x(1-x)}{[n]_{p,q}}}\right)\omega_f(\delta). \tag{3.4}$$

We claim that the error estimation in Theorem 2 is better than that of (3.4) provided that $0 \le x \le 1/3$. Indeed, if we can show that the inequality

$$2x\left(x - v_{n,p,q}(x)\right) \le \frac{p^{n-1}x(1-x)}{[n]_{p,q}}$$

is satisfied for all $0 \le x \le 1/3$, then we are done. So we are dealing with finding $x \ge 0$ such that

$$2x\left(x + \frac{p^{n-1}}{2([n]_{p,q} - p^{n-1})} - \sqrt{\frac{[n]_{p,q}}{([n]_{p,q} - p^{n-1})}x^2 + \frac{p^{2n-2}}{4([n]_{p,q} - p^{n-1})^2}}\right) \le \frac{p^{n-1}x(1-x)}{[n]_{p,q}},$$

is satisfied for all $n \in \mathbb{N}$. We may rewrite the above inequality in the form

$$-\frac{1}{2([n]_{q/p} - 1)} + \sqrt{\frac{[n]_{q/p}}{([n]_{q/p} - 1)}x^2 + \frac{1}{4([n]_{q/p} - 1)^2}} \ge x\left(1 + \frac{1}{2[n]_{q/p}}\right) - \frac{1}{2[n]_{q/p}}. \tag{3.5}$$

After some calculations one can see that, for each $n \ge 2$, the equality holds for the values $x_n = \frac{[n]_{q/p}+1}{3[n]_{q/p}+1}$ or $x = 1$. One can also see that

$$x_n > \frac{1}{3} \quad \text{and} \quad x_n \to \frac{1}{3} \tag{3.6}$$

as $n \to \infty$. The inequality given by (3.5) is satisfied by those x values in the interval $\left[0, \frac{[n]_{q/p}+1}{3[n]_{q/p}+1}\right]$. Since (3.6) is satisfied, (3.5) holds for all $0 \le x \le \frac{1}{3}$ (see Mahmudov 2009, for q-analogue). So, the error estimation for the King type (p, q)-Bernstein operators is better than that of the classical (p, q)-Bernstein operators whenever $x \in \left[0, \frac{1}{3}\right]$. The converse inequality in (3.5) is satisfied for those x values in the interval $A_n := \left[\frac{[n]_{q/p}+1}{3[n]_{q/p}+1}, 1\right]$ for all $n \ge 2$. We claim that the error estimation for the classical (p, q)-Bernstein operators is better than that of the King type (p, q)-Bernstein operators in the interval $\left(\frac{1}{3}, 1\right]$. Note that $A_1 \subset A_2 \subset A_3 \cdots$ holds. Let $x \in \bigcup_{n \ge 2} A_n$. Since (3.6) is satisfied, $[x_n, 1] \subset \left(\frac{1}{3}, 1\right]$ for all $n \ge 2$. Hence, the union of these intervals also satisfy

$$\bigcup_{n \ge 2} A_n \subset \left(\frac{1}{3}, 1\right]. \tag{3.7}$$

Now let $x \in \left(\frac{1}{3}, 1\right]$. Since (3.6) holds, there exists N_0 such that for all $n \ge N_0$ $\left|x_n - \frac{1}{3}\right| < \delta = \left|x - \frac{1}{3}\right|$. This implies $x \in [x_n, 1]$ and eventually $x \in \bigcup_{n \ge 2} A_n$. So we obtain

$$\left(\frac{1}{3}, 1\right] \subset \bigcup_{n \ge 2} A_n. \tag{3.8}$$

From (3.7) and (3.8) we get $\left(\frac{1}{3}, 1\right] = \bigcup_{n \ge 2} A_n$ and our claim is true.

Acknowledgements We would like to thank the referees for their suggestions that improved the presentation of the paper.

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