



Generalized Statistical Convergence Based on Fractional Order Difference Operator and Its Applications to Approximation Theorems

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Abstract

In this paper, following very recent results of Baliarsingh (Alex Eng J 55(2):1811–1816, 2016), we first introduce the concepts of statistically Ω^Δ -summability and Ω^Δ -statistical convergence by means of fractional-order difference operator $\Delta_n^{\alpha, \beta, \gamma}$. We also present some important inclusion relations between newly proposed methods. Our present investigation deals essentially with various summability techniques and reveals how these methods lead to a number of approximation by positive linear operators. As an application, we prove a Korovkin type approximation theorem and also present an illustrative example using the generating function type Meyer-König and Zeller operator. Furthermore, we estimate the rate of convergence of approximating linear operators by means of the modulus of continuity and some Voronovskaja type results are derived. Finally, we present some computational and geometrical interpretations to illustrate some of our approximation results.

Keywords Statistical convergence and statistical summability · Fractional order difference operator · Korovkin and Voronovskaja type approximation theorems · Modulus of continuity and rate of convergence · Meyer-König and Zeller polynomials

Mathematics Subject Classification Primary 40C05 · 40G15 · 41A36 · Secondary 46A35 · 46A45 · 46B45

1 Introduction and Preliminaries

This work is a combination of two mathematical research areas namely statistical summability and approximation process based on positive linear operators. To reveal the novelties presented by this article, we present some new developments and historical comments in summability theory and its applications to approximation theorems.

The theory of summability arises from the process of summation of series and consists fruitful applications in various contexts, for example, approximation theory, probability theory, quantum mechanics, analytic continuation, Fourier analysis, dynamical systems, the theory of orthogonal series, and fixed point theory. Due to the rapid development of sequence spaces some researchers have focused on

the notion of statistical convergence which brings a new approach to the concept of ordinary convergence. In 1935, statistical convergence was introduced by Zygmund (1959) under the name of almost convergence. In 1951, Steinhaus (1951) and Fast (1951) independently introduced the notion of statistical convergence. At the last quarter of the twentieth century, statistical convergence and statistical summability have been played the significant role in the development of functional analysis. There are many other generalizations of these concepts which have been investigated by many researchers (Kadak et al. 2017; Edely and Mursaleen 2009; Fridy 1993; Kadak 2016; Mohiuddine 2016; Mursaleen et al. 2012; Connor 1989; Başarır and Konca 2017; Yeşilkayağil and Başar 2016; Nuray et al. 2016; Duman and Orhan 2008).

Let A be a subset of the set \mathbb{N} of natural numbers and $A_n = \{j \leq n : j \in A\}$. The *natural density* of A is defined by

$$\delta(A) = \lim_{n \rightarrow \infty} \frac{1}{n} |A_n|$$

provided that the limit exists, where $|A_n|$ denotes the cardinality of set A_n . A sequence $x = (x_j)$ is said to be

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statistically convergent (*st*-convergent) to the number L , denoted by $st - \lim x = L$, if, for each $\varepsilon > 0$, the set:

$$A_\varepsilon = \{j \in \mathbb{N} : |x_j - L| \geq \varepsilon\},$$

has natural density zero, or equivalently:

$$\delta(A_\varepsilon) = \lim_{n \rightarrow \infty} \frac{1}{n} |\{j \leq n : |x_j - L| \geq \varepsilon\}| = 0.$$

By ω , we denote the family of all real valued sequences and any subspace of ω is called a sequence space. We write ℓ_∞, c and c_0 for the classical sequence spaces of all bounded, convergent and null sequences, respectively. With respect to the supremum norm $\|x\|_\infty = \sup_k |x_k|$, it is not hard to show that these are Banach spaces. The theory of difference sequence spaces was initially introduced by Kizmaz (1981). As a generalization of difference sequence spaces, the idea of difference operators with natural order m was introduced by Et and Çolak (1995) by defining:

$$\lambda(\Delta^m) = \{x = (x_k) : \Delta^m(x) \in \lambda, m \in \mathbb{N}\} \quad (\lambda \in \{\ell_\infty, c, c_0\}),$$

where

$$\Delta^0 x = (x_k), \quad \Delta^m x = (\Delta^{m-1} x_k - \Delta^{m-1} x_{k+1}),$$

and

$$\Delta^m x_k = \sum_{i=0}^m (-1)^i \binom{m}{i} x_{k+i}.$$

Moreover, these well-known difference operators were extended and used year after year in many directions (see Aydin and Başar 2004; Kadak 2017a, b). In 2013, Baliarsingh (2013), Baliarsingh and Dutta (2015) (also see Kadak and Baliarsingh 2015) introduced a new kind of difference sequence spaces with respect to the fractional-order difference operator involving Gamma function, as:

$$\Delta^{(\alpha)}(x_k) = \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(\alpha + 1)}{i! \Gamma(\alpha - i + 1)} x_{k-i}.$$

In the year 2016, some new classes of difference sequence spaces of fractional order have been introduced by Baliarsingh (2016) (see Baliarsingh and Nayak 2017).

Given a positive constant h , for each real numbers α, β and $\gamma (\gamma \notin \mathbb{N})$, we define generalized difference sequence associating with the fractional order difference operator $\Delta_h^{\alpha, \beta, \gamma}$ as:

$$(\Delta_h^{\alpha, \beta, \gamma} x)_k = \sum_{i=0}^{\infty} \frac{(-\alpha)_i (-\beta)_i}{i! (-\gamma)_i h^{\alpha + \beta - \gamma}} x_{k-i}, \tag{1}$$

where

$$(u)_k := \begin{cases} 1, & (u = 0 \text{ or } k = 0), \\ \frac{\Gamma(u+k)}{\Gamma(u)} = u(u+1)(u+2)\dots(u+k-1), & (k \in \mathbb{N}). \end{cases}$$

We assume without loss of generality, the summation given in (1) is convergent for all $\gamma \notin \mathbb{N}$ and $\alpha + \beta > \gamma$.

Our main focus of the present study is to generalize the concept of statistical summability using a fractional order linear difference operator $\Delta_h^{\alpha, \beta, \gamma}$. In fact, our present investigation shows how newly proposed summability methods lead to a number of approximation processes. We also establish some important approximation results associating with statistically Ω^Δ -summability and investigate Korovkin and Voronovskaja type approximation results by the help of generalized Meyer-König and Zeller operator. We also present computational and geometrical approaches to illustrate some of our results in this paper.

2 Some New Definitions and Inclusion Relations

In this section, we first give the definitions concerning statistically Ω^Δ -summability and Ω^Δ -statistical convergence by means of the fractional order difference operator $\Delta_h^{\alpha, \beta, \gamma}$. Secondly, we state and prove two theorems and an illustrative example to determine some inclusion relations between proposed methods.

Let $(\lambda_n)_{n=0}^\infty$ be a strictly increasing sequence of positive numbers i.e.

$$0 < \lambda_0 < \lambda_1 < \dots < \lambda_n < \dots \text{ and } \lim_{n \rightarrow \infty} \lambda_n = \infty.$$

Also, let $x = (x_n)$ be a sequence of real or complex number. We then define the following sum involving difference operator $\Delta_h^{\alpha, \beta, \gamma}(\lambda x)$ as follows:

$$\begin{aligned} \Omega_n^\Delta(\lambda x) &= \frac{1}{\Delta \lambda_n} \sum_{k=\lambda_{n-1}}^{\lambda_n} (\Delta_h^{\alpha, \beta, \gamma}(\lambda x))_k \\ &= \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=\lambda_{n-1}}^{\lambda_n} \sum_{i=0}^k \frac{(-\alpha)_i (-\beta)_i}{i! (-\gamma)_i h^{\alpha + \beta - \gamma}} \lambda_{k-i} x_{k-i}, \end{aligned}$$

where α, β and $\gamma \notin \mathbb{N}$ are real numbers, h is a positive constant such that $|\Delta_h^{\alpha, \beta, \gamma}(\lambda_n)| \geq 0$ and $\lambda_{-n} = 0$ for all $n \in \mathbb{N}$. That is,

$$\begin{aligned} \Omega_n^\Delta(\lambda x) &= \frac{1}{\Delta \lambda_n h^{\alpha + \beta - \gamma}} \sum_{k=\lambda_{n-1}}^{\lambda_n} \left\{ \lambda_k x_k - \frac{\alpha \beta}{\gamma} \lambda_{k-1} x_{k-1} + \frac{\alpha(\alpha-1)\beta(\beta-1)}{2\gamma(\gamma-1)} \lambda_{k-2} x_{k-2} \right. \\ &\quad \left. - \frac{\alpha(\alpha-1)(\alpha-2)\beta(\beta-1)(\beta-2)}{3!\gamma(\gamma-1)(\gamma-2)} \lambda_{k-3} x_{k-3} + \dots + \frac{(-\alpha)_k (-\beta)_k}{k! (-\gamma)_k} \lambda_0 x_0 \right\}. \end{aligned}$$

Definition 1 Let α, β and $\gamma \notin \mathbb{N}$ be real numbers and h be any positive constant. A sequence $x = (x_n)$ is said to be Ω^Δ -summable to the number L , if

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=\lambda_{n-1}}^{\lambda_n} (\Delta_h^{\alpha,\beta,\gamma}(\lambda x))_k = L.$$

We also say that the sequence $x = (x_n)$ is strongly Ω_q^Δ -summable to L , if

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=\lambda_{n-1}}^{\lambda_n} |(\Delta_h^{\alpha,\beta,\gamma}(\lambda x))_k - L|^q = 0 \quad (0 < q < \infty).$$

Definition 2 A sequence $x = (x_n)$ is said to be Ω^Δ -statistical convergent to a number L , if, for every $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n - \lambda_{n-1}} \left| \left\{ k \leq \lambda_n - \lambda_{n-1} : |(\Delta_h^{\alpha,\beta,\gamma}(\lambda x))_k - L| \geq \epsilon \right\} \right| = 0.$$

We denote it by $S_{\Omega^\Delta} - \lim x_n = L$. We also say that the sequence $x = (x_n)$ is statistically Ω^Δ -summable to L , if

$$st - \lim \Omega_n^\Delta(\lambda x) = L.$$

Equivalently, we may write

$$\lim_{j \rightarrow \infty} \frac{1}{j} \left| \left\{ n \leq j : \left| \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=\lambda_{n-1}}^{\lambda_n} (\Delta_h^{\alpha,\beta,\gamma}(\lambda x))_k - L \right| \geq \epsilon \right\} \right| = 0.$$

We write it as $\bar{N}_{\Omega^\Delta} - \lim x_n = L$.

Now, we shall give the following special cases to show the effectiveness of above definitions.

(1) Let us take $\alpha = 2, \beta = \gamma, h = 1$, i.e.

$$(\Delta_1^{2,\beta,\beta}(\lambda x))_k = \lambda_k x_k - 2\lambda_{k-1} x_{k-1} + \lambda_{k-2} x_{k-2}.$$

Then statistically Ω^Δ -summability given in Definition 2 is reduced to the Λ^2 -statistically summability introduced in Braha et al. (2015) (see also Kadak 2016; Alotaibi and Mursaleen 2013).

(2) Let $\alpha = 2, \beta = \gamma, h = 1$, then Ω^Δ -statistical convergence given in Definition 2 reduces to weighted Λ^2 -statistically convergence. In addition, the notion of strongly Ω_q^Δ -summability can be interpreted as strongly Λ^2 -summability (see Kadak 2016; Braha et al. 2015; Mursaleen 2000).

We now present the following theorem which gives the relation between Ω^Δ -statistical convergence and statistically Ω^Δ -summability.

Theorem 1 Let $|(\Delta_h^{\alpha,\beta,\gamma}(\lambda x))_k - L| \leq M$ for all $k \in \mathbb{N}$. If a sequence $x = (x_n)$ is Ω^Δ -statistical convergent to the number L , then it is statistically Ω^Δ -summable to the same limit, but not conversely.

Proof Let $h > 0$ be any constant, α, β and $\gamma \notin \mathbb{N}$ be real numbers. Let $|(\Delta_h^{\alpha,\beta,\gamma}(\lambda x))_k - L| \leq M$ for all $k \in \mathbb{N}$. Since $S_{\Omega^\Delta} - \lim x_n = L$, we have

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n - \lambda_{n-1}} \left| \left\{ k \leq \lambda_n - \lambda_{n-1} : |(\Delta_h^{\alpha,\beta,\gamma}(\lambda x))_k - L| \geq \epsilon \right\} \right| = 0.$$

We thus find that

$$\begin{aligned} |\Omega_n^\Delta(\lambda x) - L| &= \left| \left(\frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=\lambda_{n-1}}^{\lambda_n} (\Delta_h^{\alpha,\beta,\gamma}(\lambda x))_k \right) - L \right| \\ &= \left| \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=\lambda_{n-1}}^{\lambda_n} [(\Delta_h^{\alpha,\beta,\gamma}(\lambda x))_k - L] + \left(\frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=\lambda_{n-1}}^{\lambda_n} L - L \right) \right| \\ &\leq \frac{1}{\lambda_n - \lambda_{n-1}} \left\{ \left| \sum_{k=\lambda_{n-1}}^{\lambda_n} [(\Delta_h^{\alpha,\beta,\gamma}(\lambda x))_k - L] \right| + \left| \sum_{k=\lambda_{n-1}}^{\lambda_n} [(\Delta_h^{\alpha,\beta,\gamma}(\lambda x))_k - L] \right| \right\} \\ &\quad (k \in K_{\lambda_n}(\epsilon)) \qquad \qquad \qquad (k \in K_{\lambda_n}^C(\epsilon)) \\ &\quad + \left| \frac{1}{\lambda_n - \lambda_{n-1}} \sum_{k=\lambda_{n-1}}^{\lambda_n} L - L \right| \\ &\leq \frac{1}{\lambda_n - \lambda_{n-1}} \left\{ \sum_{k=\lambda_{n-1}}^{\lambda_n} |(\Delta_h^{\alpha,\beta,\gamma}(\lambda x))_k - L| + \sum_{k=\lambda_{n-1}}^{\lambda_n} |(\Delta_h^{\alpha,\beta,\gamma}(\lambda x))_k - L| \right\} \\ &\quad (k \in K_{\lambda_n}(\epsilon)) \qquad \qquad \qquad (k \in K_{\lambda_n}^C(\epsilon)) \\ &\leq \frac{M|K_{\lambda_n}(\epsilon)| + \epsilon|K_{\lambda_n}^C(\epsilon)|}{\lambda_n - \lambda_{n-1}} \rightarrow \epsilon(n \rightarrow \infty), \end{aligned}$$

where

$$K_{\lambda_n}(\epsilon) := \{k \leq \lambda_n - \lambda_{n-1} : |(\Delta_h^{\alpha,\beta,\gamma}(\lambda x))_k - L| \geq \epsilon\}$$

and

$$K_{\lambda_n}^C(\epsilon) := \{k \leq \lambda_n - \lambda_{n-1} : |(\Delta_h^{\alpha,\beta,\gamma}(\lambda x))_k - L| < \epsilon\}.$$

That is to say that $x = (x_n)$ is Ω^Δ -summable to L and hence statistically Ω^Δ -summable to the same limit. On the other hand the converse is not true, as can be seen by considering the following example.

Example 1 Define the sequence $x = (x_n)$ by

$$x_n := \begin{cases} 1/m^2, & n = m^2 - m, m^2 - m + 1, \dots, m^2 - 1, m > 1; \\ -1/m^3, & n = m^2, m > 1; \\ 0, & \text{otherwise} \end{cases}$$

for all $m \in \mathbb{N}$. Let $\alpha \in (0, 1)$, $\beta = \gamma$, $h = 1$ and $\lambda_n = n^2$ for all $n \in \mathbb{N}$. We thus find that

$$\begin{aligned} \Omega_n^\Delta(\lambda x) &= \frac{1}{2n-1} \left\{ n^4 x_{n^2} + (1-\alpha)(n^2-1)^2 x_{n^2-1} \right. \\ &\quad + \left[1-\alpha + \frac{\alpha(\alpha-1)}{2} \right] (n^2-2)^2 x_{n^2-2} + \dots \\ &\quad \left. + \left[1-\alpha + \frac{\alpha(\alpha-1)}{2} - \frac{\alpha(\alpha-1)(\alpha-2)}{3!} + \dots \right] (n-1)^4 x_{(n-1)^2} \right\}. \end{aligned} \tag{2}$$

Letting $n \rightarrow \infty$ in (2), we find that $\Omega_n^\Delta(\lambda x) \rightarrow 0$ which yields $st - \lim \Omega_n^\Delta(\lambda x) = 0$. On the other hand, for a fixed $\alpha \in (0, 1)$, since

$$S_{\Omega^\Delta} - \liminf x_n = \liminf_n \frac{1}{2n-1} \left| \left\{ k \leq 2n-1 : |\Delta_h^{\alpha,\beta,\gamma}(x_k)| \geq \epsilon \right\} \right| = 0$$

and

$$S_{\Omega^\Delta} - \limsup x_n = \limsup_n \frac{1}{2n-1} \left| \left\{ k \leq 2n-1 : |\Delta_h^{\alpha,\beta,\gamma}(x_k)| \geq \epsilon \right\} \right| = 1,$$

then the sequence $x = (x_n)$ is not Ω^Δ -statistical convergent.

Theorem 2

(a) Let us suppose that $x = (x_n)$ is strongly Ω_q^Δ -summable ($0 < q < \infty$) to the number L . If the following conditions hold, then the sequence x is Ω^Δ -statistical convergent to L :

- (i) $q \in (0, 1)$ and $0 \leq |(\Delta_h^{\alpha,\beta,\gamma}(\lambda x))_k - L| < 1$ or
- (ii) $q \in [1, \infty)$ and $1 \leq |(\Delta_h^{\alpha,\beta,\gamma}(\lambda x))_k - L| < \infty$.

(b) Assume that $x = (x_n)$ is Ω^Δ -statistical convergent to L and $|(\Delta_h^{\alpha,\beta,\gamma}(\lambda x))_k - L| \leq M$ for all $k \in \mathbb{N}$. If the

following conditions hold, then the sequence x is strongly Ω_q^Δ -summable to L :

- (i) $q \in (0, 1]$ and $M \in [1, \infty)$ or
- (ii) $q \in [1, \infty)$ and $M \in [0, 1)$.

Proof

(a) Let $x = (x_n)$ be strongly Ω_q^Δ -summable ($0 < q < \infty$) to L . Under the above conditions, we get

$$\begin{aligned} \frac{1}{\epsilon \Delta \lambda_n} \sum_{k=\lambda_{n-1}}^{\lambda_n} |(\Delta_h^{\alpha,\beta,\gamma}(\lambda x))_k - L|^q &\geq \frac{1}{\epsilon \Delta \lambda_n} \sum_{k=\lambda_{n-1}}^{\lambda_n} |(\Delta_h^{\alpha,\beta,\gamma}(\lambda x))_k - L| \\ &\geq \frac{1}{\epsilon \Delta \lambda_n} \sum_{k=\lambda_{n-1}}^{\lambda_n} |(\Delta_h^{\alpha,\beta,\gamma}(\lambda x))_k - L| \\ &\quad (k \in K_{\lambda_n}(\epsilon)) \\ &\geq \frac{1}{\epsilon \Delta \lambda_n} \sum_{k=\lambda_{n-1}}^{\lambda_n} \epsilon \\ &\quad (k \in K_{\lambda_n}^C(\epsilon)) \\ &= \frac{1}{\lambda_n - \lambda_{n-1}} |K_{\lambda_n}^C(\epsilon)| \quad (\epsilon > 0) \end{aligned}$$

which leads us by passing to limit as $n \rightarrow \infty$ that

$$\frac{1}{\lambda_n - \lambda_{n-1}} \left| \left\{ k \leq \lambda_n - \lambda_{n-1} : |(\Delta_h^{\alpha,\beta,\gamma}(\lambda x))_k - L| \geq \epsilon \right\} \right| \rightarrow 0.$$

Hence, $x = (x_n)$ is Ω^Δ -statistical convergent to L .

(b) Let $x = (x_n)$ be Ω^Δ -statistical convergent to L and

$$|(\Delta_h^{\alpha,\beta,\gamma}(\lambda x))_k - L| \leq M \quad \text{for all } k \in \mathbb{N}.$$

Thus, clearly, we have

$$\begin{aligned} \frac{1}{\Delta \lambda_n} \sum_{k=\lambda_{n-1}}^{\lambda_n} |(\Delta_h^{\alpha,\beta,\gamma}(\lambda x))_k - L|^q &= \frac{1}{\Delta \lambda_n} \sum_{k=\lambda_{n-1}}^{\lambda_n} |(\Delta_h^{\alpha,\beta,\gamma}(\lambda x))_k - L|^q + \frac{1}{\Delta \lambda_n} \\ &\quad (k \in K_{\lambda_n}^C(\epsilon)) \\ &\times \sum_{k=\lambda_{n-1}}^{\lambda_n} |(\Delta_h^{\alpha,\beta,\gamma}(\lambda x))_k - L|^q \\ &\quad (k \in K_{\lambda_n}(\epsilon)) \\ &\leq \frac{1}{\Delta \lambda_n} \sum_{k=\lambda_{n-1}}^{\lambda_n} |(\Delta_h^{\alpha,\beta,\gamma}(\lambda x))_k - L| + \frac{1}{\Delta \lambda_n} \\ &\quad (k \in K_{\lambda_n}^C(\epsilon)) \\ &\times \sum_{k=\lambda_{n-1}}^{\lambda_n} |(\Delta_h^{\alpha,\beta,\gamma}(\lambda x))_k - L| \\ &\quad (k \in K_{\lambda_n}(\epsilon)) \\ &\leq \frac{\epsilon |K_{\lambda_n}^C(\epsilon)|}{\lambda_n - \lambda_{n-1}} + \frac{M |K_{\lambda_n}(\epsilon)|}{\lambda_n - \lambda_{n-1}} \rightarrow \epsilon \quad (n \rightarrow \infty). \end{aligned}$$

Hence, for $0 < q < \infty$, our sequence $x = (x_n)$ is strongly Ω_q^Δ -summable to L . \square

3 Applications to Korovkin Type Approximation Theorem

Let $C[a, b]$ be the space of all continuous real valued functions on $[a, b]$. It is well known that $C[a, b]$ is a Banach space with the norm defined by

$$\|f\|_\infty = \sup \{ |f(x)| : x \in [a, b] \}.$$

Let $L : C[a, b] \rightarrow C[a, b]$ be a linear operator. Then, L is said to be positive provided by $f \geq 0$ implies $Lf \geq 0$. We also use the notation $Lf(x)$ for the value of Lf at a point x .

The classical Korovkin type approximation theorem (see Korovkin 1953; Bohman 1952) states as follows:

Let $T_n : C[a, b] \rightarrow C[a, b]$ be a sequence of positive linear operators. Then

$$\lim_{n \rightarrow \infty} \|T_n(f; x) - f(x)\|_\infty = 0, \quad \text{for all } f \in C[a, b]$$

if and only if

$$\lim_{n \rightarrow \infty} \|T_n(f_i; x) - f_i(x)\|_\infty = 0, \quad \text{for each } i = 0, 1, 2,$$

where the test function $f_i(x) = x^i$.

The statistical version of Korovkin theorem was given by Gadjiev and Orhan (2002). With the development of summability methods, this type approximation has been widely used and extended by many authors the reader may refer to (Kadak 2016, 2017; Orhan and Demirci 2014; Srivastava et al. 2012; Edely et al. 2010).

In this section using the test function $f_i(x) = (\frac{x}{1-x})^i, i = 0, 1, 2$ for $x \in [0, A]$ where $A \leq \frac{1}{2}$, we try to obtain a Korovkin type approximation theorem which is stronger than that both classical and statistical cases of Korovkin theorem.

Theorem 3 Let α, β and $\gamma \notin \mathbb{N}$ be real numbers and h be a positive constant. Also let $(T_k)_{k \geq 1}$ be a sequence of positive linear operators from $C[0, A]$ into itself. Then, for all $f \in C[0, A]$,

$$\overline{N}_{\Omega^\Delta} - \lim_{n \rightarrow \infty} \|T_k(f(s); x) - f(x)\|_\infty = 0 \tag{3}$$

if and only if

$$\overline{N}_{\Omega^\Delta} - \lim_{n \rightarrow \infty} \|T_k(1; x) - 1\|_\infty = 0 \tag{4}$$

$$\overline{N}_{\Omega^\Delta} - \lim_{n \rightarrow \infty} \left\| T_k \left(\frac{s}{1-s}; x \right) - \frac{x}{1-x} \right\|_\infty = 0 \tag{5}$$

$$\overline{N}_{\Omega^\Delta} - \lim_{n \rightarrow \infty} \left\| T_k \left(\left(\frac{s}{1-s} \right)^2; x \right) - \left(\frac{x}{1-x} \right)^2 \right\|_\infty = 0. \tag{6}$$

Proof Since each $1, \frac{x}{1-x}, (\frac{x}{1-x})^2$ functions belongs to $C[0, A]$ then the assertions (4), (5) and (6) follow immediately from the first assertion (3). Let us take $f \in C[0, A]$ and $x \in [0, A]$ be fixed. Then there exists a constant $M > 0$ such that $|f(x)| \leq M$ for all $x \in [0, A]$. Thus, we have

$$|f(s) - f(x)| \leq 2M \quad (s, x \in [0, A]). \tag{7}$$

By continuity of f at x , for given $\varepsilon > 0$, there exists a number $\delta = \delta(\varepsilon) > 0$ such that for all $s, x \in [0, A]$ satisfying

$$\left| \frac{s}{1-s} - \frac{x}{1-x} \right| < \delta$$

we have

$$|f(s) - f(x)| < \varepsilon. \tag{8}$$

Setting $\varphi(s, x) = (\frac{s}{1-s} - \frac{x}{1-x})$. For $|\frac{s}{1-s} - \frac{x}{1-x}| \geq \delta$, then $\varphi^2(s, x) \geq \delta^2$. Hence, we derive the consequence from (7) and (8) that

$$|f(s) - f(x)| < \varepsilon + \frac{2M}{\delta^2} \varphi^2(s, x).$$

It follows from the linearity and positivity of T_k that

$$\begin{aligned} |T_k(f(s); x) - f(x)| &= |T_k(f(s) - f(x); x) + f(x)(T_k(f_0; x) - f_0(x))| \\ &\leq T_k(|f(s) - f(x); x) + M|T_k(1; x) - 1| \\ &\leq \left| T_k \left(\varepsilon + \frac{2M}{\delta^2} \varphi^2; x \right) \right| + M|T_k(1; x) - 1| \\ &\leq \varepsilon + \left(\varepsilon + M + \frac{4M}{\delta^2} \right) |T_k(1; x) - 1| + \frac{4M}{\delta^2} \\ &\quad \times \left| T_k \left(\frac{s}{1-s}; x \right) - \frac{x}{1-x} \right| \\ &\quad + \frac{2M}{\delta^2} \left| T_k \left(\left(\frac{s}{1-s} \right)^2; x \right) - \left(\frac{x}{1-x} \right)^2 \right|. \end{aligned}$$

Taking the supremum over $x \in [0, A]$, we obtain

$$\begin{aligned} \|T_k(f(s); x) - f(x)\|_\infty &\leq \varepsilon + N \left\{ \left\| T_k(1; x) - 1 \right\|_\infty \right. \\ &\quad + \left\| T_k \left(\frac{s}{1-s}; x \right) - \frac{x}{1-x} \right\|_\infty \\ &\quad \left. + \left\| T_k \left(\left(\frac{s}{1-s} \right)^2; x \right) - \left(\frac{x}{1-x} \right)^2 \right\|_\infty \right\} \tag{9} \end{aligned}$$

where

$$N := \left\{ \varepsilon + M + \frac{4M}{\delta^2} \right\}.$$

We now replace $T_k(\cdot; x)$ by

$$\Omega_k^\Delta(T(\cdot; x)) = \frac{1}{\lambda_k - \lambda_{k-1}} \sum_{j=\lambda_{k-1}}^{\lambda_k} \Delta_h^{\alpha, \beta, \gamma}(\lambda T(\cdot; x))_j$$

in (9). For a given $\varepsilon' > 0$, we choose a number $\varepsilon > 0$ such that $\varepsilon < \varepsilon'$. Then, upon setting

$$\begin{aligned} \mathcal{B} &:= \{k \leq n : \|\Omega_k^\Delta(T(f(s); x)) - f(x)\|_\infty \geq \varepsilon'\}, \\ \mathcal{B}_0 &:= \left\{ k \leq n : \|\Omega_k^\Delta(T(1; x)) - 1\|_\infty \geq \frac{\varepsilon' - \varepsilon}{3N} \right\}, \\ \mathcal{B}_1 &:= \left\{ k \leq n : \|\Omega_k^\Delta(T\left(\frac{s}{1-s}; x\right)) - \frac{x}{1-x}\|_\infty \geq \frac{\varepsilon' - \varepsilon}{3N} \right\}, \\ \mathcal{B}_2 &:= \left\{ k \leq n : \|\Omega_k^\Delta(T\left(\left(\frac{s}{1-s}\right)^2; x\right)) - \left(\frac{x}{1-x}\right)^2\|_\infty \geq \frac{\varepsilon' - \varepsilon}{3N} \right\}. \end{aligned}$$

Then, it is clear that $\mathcal{B} \subset \mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_2$ and hence using the conditions (4–6) we obtain

$$\overline{N}_{\Omega^\Delta} - \lim_{n \rightarrow \infty} \|T_k(f(s); x) - f(x)\|_\infty = 0, \quad \text{for all } f \in C[0, A],$$

which completes the proof.

Now we may give an example for Theorem 3. Before giving this example, we present a short introduction related with the generating function type Meyer-König and Zeller operators (see Altın et al. 2005).

For a function f on $[0, 1)$, the operators

$$M_n(f; x) = \sum_{k=0}^{\infty} f\left(\frac{k}{k+n+1}\right) m_{nk}(x)$$

are known as Meyer-König and Zeller operators Meyer-König and Zeller (1960) where

$$m_{nk}(x) = \binom{n+k}{k} x^k (1-x)^{n+1}.$$

This operator were also generalized in Altın et al. (2005) using linear generating functions

$$L_n(f(s); x) = \frac{1}{h_n(x, s)} \sum_{k=0}^{\infty} f\left(\frac{a_{k,n}}{a_{k,n} + b_n}\right) \Gamma_{k,n}(s) x^k \quad (10)$$

where $0 < \frac{a_{k,n}}{a_{k,n} + b_n} \leq \tilde{A}$, $\tilde{A} \in (0, 1)$, and $h_n(x, s)$ is the generating function for the sequence of $\{\Gamma_{k,n}(s)\} (s \in I)$ with the form

$$h_n(x, s) = \sum_{k=0}^{\infty} \Gamma_{k,n}(s) x^k \quad (s \in I \subset \mathbb{R}).$$

We also suppose that the following conditions hold true:

$$(i) \quad h_n(x, s) = (1-x)h_{n+1}(x, s);$$

- (ii) $b_n \Gamma_{k,n+1}(s) = a_{k+1,n} \Gamma_{k+1,n}(s)$ and $\Gamma_{k,n}(s) \geq 0$ for all $s \in I \subset \mathbb{R}$;
- (iii) $b_n \rightarrow \infty, \frac{b_{n+1}}{b_n} \rightarrow 1$ and $b_n \neq 0$ for all $n \in \mathbb{N}$;
- (iv) $a_{k+1,n} - a_{k,n+1} = \varphi_n$ where $|\varphi_n| \leq m < \infty$, and $a_{0n} = 0$.

It is easy to see that L_n defined by (10) is positive and linear. We also observe that

$$\begin{aligned} L_n(1; x) &= 1, \quad L_n\left(\frac{s}{1-s}; x\right) = \frac{x}{1-x} \quad \text{and} \\ L_n\left(\left(\frac{s}{1-s}\right)^2; x\right) &= \frac{x^2}{(1-x)^2} \frac{b_{n+1}}{b_n} + \frac{\varphi_n}{b_n} \frac{x}{1-x}. \end{aligned}$$

Example 2 Let $\{T_k\}$ be a sequence of positive linear operators from $C[0, \tilde{A}]$ into itself defined by

$$T_k(f(s); x) = (1 + x_k)L_k(f(s); x) \quad (11)$$

where $x = (x_k)$ is defined as in Example 1. Since $\overline{N}_{\Omega^\Delta} - \lim x_n = 0$, it is easy to see that

$$\overline{N}_{\Omega^\Delta} - \lim_n \|T_k(1; x) - 1\|_\infty = 0 \text{ and}$$

$$\overline{N}_{\Omega^\Delta} - \lim_n \left\| T_k\left(\frac{s}{1-s}; x\right) - \frac{x}{1-x} \right\|_\infty = 0.$$

In view of (iii), we have

$$\begin{aligned} T_k\left(\left(\frac{s}{1-s}\right)^2; x\right) - \left(\frac{x}{1-x}\right)^2 \\ = (1 + x_k) \left[\left(\frac{x}{1-x}\right)^2 \frac{b_{k+1}}{b_k} + \left(\frac{\varphi_k}{b_k} \frac{x}{1-x}\right) \right] - \left(\frac{x}{1-x}\right)^2 \\ \leq (1 + x_k) \left\{ \left[\frac{b_{k+1}}{b_k} - 1 \right] + \frac{m}{b_k} \right\} \end{aligned}$$

which yields that

$$\overline{N}_{\Omega^\Delta} - \lim_n \left\| T_k\left(\left(\frac{s}{1-s}\right)^2; x\right) - \left(\frac{x}{1-x}\right)^2 \right\|_\infty = 0.$$

Therefore, we obtain

$$\overline{N}_{\Omega^\Delta} - \lim_n \|T_k(f_i(s); x) - f_i(x)\|_\infty = 0, \quad i = 0, 1, 2.$$

By taking Theorem 3 into account, and hence by letting $n \rightarrow \infty$, we are led to the fact that

$$\overline{N}_{\Omega^\Delta} - \lim_n \|T_k(f(s); x) - f(x)\|_\infty = 0.$$

In view of above example, we say that our proposed method works successfully but classical and statistical forms of Korovkin theorem do not work for this sequence $\{T_k\}$ of positive linear operators.

4 Rates of Ω^Δ -Statistical Convergence

In this section, we estimate the rate of Ω^Δ -statistical convergence of positive linear operators defined from $C[0, A]$ into itself.

We first present the following definition.

Definition 3 Let α, β and $\gamma \notin \mathbb{N}$ be real numbers and h be a positive constant. Also let (θ_n) be any non-increasing sequence of positive real numbers. We say that a sequence $x = (x_n)$ is Ω^Δ -statistical convergent to the number L with the rate $o(\theta_n)$ if, for every $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{\theta_n \Delta \lambda_n} \left\{ k \leq \lambda_n - \lambda_{n-1} : |(\Delta_h^{\alpha, \beta, \gamma}(\lambda x))_k - L| \geq \epsilon \right\} = 0. \tag{12}$$

In this case, we can write $x_k - L = \Omega^\Delta - o(\theta_n)$.

Lemma 1 Let (a_n) and (b_n) be positive non-increasing sequences. Assume that $x = (x_k)$ and $y = (y_k)$ are two sequences such that

$$x_k - L_1 = \Omega^\Delta - o(a_n) \text{ and } y_k - L_2 = \Omega^\Delta - o(b_n).$$

Then

- (1) $(x_k - L_1) \pm (y_k - L_2) = \Omega^\Delta - o(c_n)$
- (2) $(x_k - L_1)(y_k - L_2) = \Omega^\Delta - o(a_n b_n)$
- (3) $\mu(x_k - L_1) = \Omega^\Delta - o(a_n)$, for any scalar $\mu \in \mathbb{R}$,

where $c_n = \max_n \{a_n, b_n\}$.

Proof Assume that $x_k - L_1 = \Omega^\Delta - o(a_n)$ and $y_k - L_2 = \Omega^\Delta - o(b_n)$. In addition, for $\epsilon > 0$, let us set

$$\mathcal{D} := \left\{ k \leq \lambda_n - \lambda_{n-1} : |(\Delta_h^{\alpha, \beta, \gamma}(\lambda x))_k + (\Delta_h^{\alpha, \beta, \gamma}(\lambda y))_k - (L_1 + L_2)| \geq \epsilon \right\},$$

$$\mathcal{D}_0 := \left\{ k \leq \lambda_n - \lambda_{n-1} : |(\Delta_h^{\alpha, \beta, \gamma}(\lambda x))_k - L_1| \geq \frac{\epsilon}{2} \right\}$$

and

$$\mathcal{D}_1 := \left\{ k \leq \lambda_n - \lambda_{n-1} : |(\Delta_h^{\alpha, \beta, \gamma}(\lambda y))_k - L_2| \geq \frac{\epsilon}{2} \right\}.$$

We then observe that $\mathcal{D} \subset \mathcal{D}_0 \cup \mathcal{D}_1$. Thus, we obtain

$$\frac{|\mathcal{D}|}{c_n(\lambda_n - \lambda_{n-1})} \leq \frac{|\mathcal{D}_0|}{a_n(\lambda_n - \lambda_{n-1})} + \frac{|\mathcal{D}_1|}{b_n(\lambda_n - \lambda_{n-1})} \tag{13}$$

where $c_n = \max\{a_n, b_n\}$. Now letting $n \rightarrow \infty$ in (13) and using by hypothesis, we have

$$\lim_{n \rightarrow \infty} \frac{1}{c_n \Delta \lambda_n} \left\{ k \leq \lambda_n - \lambda_{n-1} : |(\Delta_h^{\alpha, \beta, \gamma}(\lambda x))_k + (\Delta_h^{\alpha, \beta, \gamma}(\lambda y))_k - (L_1 + L_2)| \geq \epsilon \right\} = 0,$$

as asserted by Lemma 1 (1). Since the other assertions can be proved similarly, we choose to omit the details involved.

We now recall the following basic definition and notation on the modulus of continuity to get the rates of Ω^Δ -statistical convergence using Definition 3.

The modulus of continuity for a function $f \in C[0, A]$ is defined as follows:

$$\omega(f, \delta) = \sup_{|h| < \delta} |f(x+h) - f(x)|.$$

It is well-known that for any $\delta > 0$ and each $s \in [0, A]$,

$$|f(s) - f(x)| \leq \omega(f, \delta) \left(\frac{|s-x|}{\delta} + 1 \right), \quad (f \in C[0, A]). \tag{14}$$

Theorem 4 Let α, β and $\gamma (\notin \mathbb{N})$ be real numbers and h be a positive constant. Let $\{T_k\}$ be a sequence of positive linear operators from $C[0, A]$ into itself. Assume further that (a_n) and (b_n) be positive non-increasing sequences. Suppose that the following conditions hold true:

- (i) $\|T_k(1; x) - 1\|_\infty = \Omega^\Delta - o(a_n)$ on $[0, A]$,
- (ii) $\omega(f, \xi_k) = \Omega^\Delta - o(b_n)$ on $[0, A]$, where $\xi_k := \sqrt{\|T_k(\varphi_x(s); x)\|_\infty}$ with

$$\varphi_x(s) = \left(\frac{s}{1-s} - \frac{x}{1-x} \right)^2.$$

Then we have, for all $f \in C[0, A]$,

$$\|T_k(f(s); x) - f(x)\|_\infty = \Omega^\Delta - o(c_n)$$

where $c_n = \max\{a_n, b_n\}$.

Proof Let $f \in C[0, A]$ and $x \in [0, A]$ be fixed. Since $\{T_k\}$ is linear and monotone, we see (for any $\delta > 0$) that

$$\begin{aligned} |T_k(f(s); x) - f(x)| &\leq T_k(|f(s) - f(x)|; x) + |f(x)| |T_k(1; x) - 1| \\ &\leq \omega(f, \delta) T_k \left(\frac{\left| \frac{s}{1-s} - \frac{x}{1-x} \right|}{\delta} + 1; x \right) + N |T_k(1; x) - 1| \\ &\leq \omega(f, \delta) T_k \left(\frac{\left(\frac{s}{1-s} - \frac{x}{1-x} \right)^2}{\delta^2} + 1; x \right) + N |T_k(1; x) - 1| \\ &\leq \omega(f, \delta) \left\{ \frac{1}{\delta^2} T_k(\varphi_x; x) + T_k(1; x) \right\} + N |T_k(1; x) - 1| \end{aligned}$$

where $N = \|f\|_\infty$. Taking the supremum over $x \in [0, A]$ on both sides, we get

$$\begin{aligned} \|T_k(f(s); x) - f(x)\|_\infty &\leq \omega(f, \delta) \left\{ \frac{1}{\delta^2} \|T_k(\varphi_x; x)\|_\infty + \|T_k(1; x) - 1\|_\infty + 1 \right\} \\ &\quad + N \|T_k(1; x) - 1\|_\infty. \end{aligned}$$

Now, if we take

$$\delta = \xi_k = \sqrt{\|T_k(\varphi_x; x)\|_\infty}$$

in the last relation, we deduce that

$$\begin{aligned} \|T_k(f(s); x) - f(x)\|_\infty &\leq \omega(f, \xi_k) \{ \|T_k(1; x) - 1\|_\infty + 2 \} \\ &\quad + N \|T_k(1; x) - 1\|_\infty \\ &= \omega(f, \xi_k) \|T_k(1; x) - 1\|_\infty \\ &\quad + 2\omega(f, \xi_k) + N \|T_k(1; x) - 1\|_\infty. \end{aligned}$$

Now, we replace $T_k(\cdot; x)$ by

$$H_k(\cdot; x) := \Delta_h^{\alpha, \beta, \gamma} (\lambda T(\cdot; x))_k.$$

Using Lemma 1, for a given $\epsilon > 0$, we obtain that

$$\begin{aligned} &\frac{1}{c_n(\lambda_n - \lambda_{n-1})} |\{k \leq \lambda_n - \lambda_{n-1} : \|H_k(f(s); x) - f(x)\|_\infty \geq \epsilon\}| \\ &\leq \frac{1}{a_n(\lambda_n - \lambda_{n-1})} |\{k \leq \lambda_n - \lambda_{n-1} : \|H_k(1; x) - 1\|_\infty \geq \epsilon/3N\}| \\ &\quad + \frac{1}{a_n b_n(\lambda_n - \lambda_{n-1})} |\{k \leq \lambda_n - \lambda_{n-1} : \omega(f, \xi_k) \|H_k(1; x) - 1\|_\infty \geq \epsilon/3\}| \\ &\quad + \frac{1}{b_n(\lambda_n - \lambda_{n-1})} |\{k \leq \lambda_n - \lambda_{n-1} : \omega(f, \xi_k) \geq \epsilon/6\}|. \end{aligned}$$

Letting $n \rightarrow \infty$ which leads us to the fact that

$$\|T_k(f(s); x) - f(x)\|_\infty = \Omega^\Delta - o(c_n) \quad (f \in C[0, A]),$$

as desired.

5 A Voronovskaja-Type Theorem

In this section, using the notion of statistically Ω^Δ -summability we obtain a Voronovskaja-type approximation theorem by the help of T_k family of linear operators defined

by (11) for $h_n(x, s) = (1 - x)^{-n-1}$, $a_{k,n} = k$, $\Gamma_{k,n}(s) =$

$$\binom{n+k}{k} \text{ and } b_n = n + 1 \text{ for all } n \in \mathbb{N}.$$

Lemma 2 Let α, β and $\gamma (\notin \mathbb{N})$ be real numbers and h be a positive constant. Suppose also that $\eta_x(s) = (\frac{s}{1-s} - \frac{x}{1-x})$ where $x, s \in [0, A]$. Then, we get

$$\bar{N}_{\Omega^\Delta} - \lim_n \{ (k+1) T_k(\eta_x^2; x) \} = \frac{x}{(1-x)^2}. \tag{15}$$

Proof Let h be a positive constant, $x \in [0, A]$ and, α, β and $\gamma (\notin \mathbb{N})$ be real numbers. Since

$$L_k\left(\frac{s}{1-s}; x\right) = \frac{x}{1-x},$$

we deduce that

$$T_k(\eta_x; x) = (1+x_k) \left[L_k\left(\frac{s}{1-s}; x\right) - \frac{x}{1-x} L_k(1; x) \right] = 0.$$

In addition, since

$$L_k\left(\frac{s^2}{(1-s)^2}; x\right) = \frac{k+2}{k+1} \left(\frac{x}{1-x}\right)^2 + \frac{1}{k+1} \left(\frac{x}{1-x}\right),$$

we find that

$$\begin{aligned} T_k(\eta_x^2; x) &= (1+x_k) \left[L_k\left(\frac{s^2}{(1-s)^2}; x\right) - 2\frac{x}{1-x} L_k\left(\frac{s}{1-s}; x\right) \right. \\ &\quad \left. + \left(\frac{x}{1-x}\right)^2 L_k(1; x) \right] \\ &= (1+x_k) \left[\frac{1}{k+1} \left(\frac{x}{1-x}\right)^2 + \frac{1}{k+1} \left(\frac{x}{1-x}\right) \right] \end{aligned}$$

which yields, for all $x \in [0, A] (A \leq 1/2)$, that

$$\begin{aligned} (k+1) T_k(\eta_x^2; x) &= \left[\left(\frac{x}{1-x}\right)^2 + \frac{x}{1-x} \right] \\ &= x_k \left[\left(\frac{x}{1-x}\right)^2 + \frac{x}{1-x} \right] \leq 2x_k. \end{aligned}$$

Since

$$\bar{N}_{\Omega^\Delta} - \lim_{n \rightarrow \infty} 2x_k = 0,$$

the desired result follows, that is,

$$\bar{N}_{\Omega^\Delta} - \lim_{n \rightarrow \infty} (k+1) T_k(\eta_x^2; x) = \frac{x}{(1-x)^2}.$$

Corollary 1 Assume that h is a positive constant, $x \in [0, A]$. Let $\eta_x(s)$ be given as in Lemma 2. Then there is a positive constant $M_0(x)$ depending only on x , such that

$$\bar{N}_{\Omega^\Delta} - \lim_{n \rightarrow \infty} \{ (k+1)^2 T_k(\eta_x^4; x) \} = M_0(x).$$

Theorem 5 Let α, β and $\gamma (\notin \mathbb{N})$ be real numbers and h be a positive constant. Then, for every $f \in C[0, A]$ such that $f', f'' \in C[0, A]$,

$$\begin{aligned} \bar{N}_{\Omega^\Delta} - \lim_n \left\{ (k+1) \left[T_k\left(f\left(\frac{s}{1-s}\right); x\right) - f\left(\frac{x}{1-x}\right) \right] \right\} \\ = \frac{x}{2(1-x)^2} f''\left(\frac{x}{1-x}\right). \end{aligned}$$

Proof Let $x \in [0, A]$ and $f, f', f'' \in C[0, A]$. Now we consider the following function defined by

where $\theta_x(\frac{x}{1-x}) = 0$ and $\theta_x \in C[0, A]$. Using the Taylor formula for $f \in C[0, A]$, we can write

$$\theta_x\left(\frac{s}{1-s}\right) = \begin{cases} \frac{f\left(\frac{s}{1-s}\right) - f\left(\frac{x}{1-x}\right) - \left(\frac{s}{1-s} - \frac{x}{1-x}\right)f'\left(\frac{x}{1-x}\right) - \frac{1}{2}\left(\frac{s}{1-s} - \frac{x}{1-x}\right)^2 f''\left(\frac{x}{1-x}\right)}{\left(\frac{s}{1-s} - \frac{x}{1-x}\right)^2}, & s \neq x, \\ 0, & s = x, \end{cases}$$

$$f\left(\frac{s}{1-s}\right) = f\left(\frac{x}{1-x}\right) + \left(\frac{s}{1-s} - \frac{x}{1-x}\right)f'\left(\frac{x}{1-x}\right) + \frac{1}{2}\left(\frac{s}{1-s} - \frac{x}{1-x}\right)^2 f''\left(\frac{x}{1-x}\right) + \theta_x\left(\frac{s}{1-s}\right)\left(\frac{s}{1-s} - \frac{x}{1-x}\right)^2.$$

We then observe that the operator T_k is linear and that

$$T_k\left(f\left(\frac{s}{1-s}\right); x\right) = T_k\left(f\left(\frac{x}{1-x}\right); x\right) + f'\left(\frac{x}{1-x}\right)T_k\left(\left(\frac{s}{1-s} - \frac{x}{1-x}\right); x\right) + \frac{1}{2}f''\left(\frac{x}{1-x}\right)T_k\left(\left(\frac{s}{1-s} - \frac{x}{1-x}\right)^2; x\right) + T_k\left(\theta_x\left(\frac{s}{1-s}\right)\left(\frac{s}{1-s} - \frac{x}{1-x}\right)^2; x\right).$$

In view of Lemma 2, one obtains

$$T_k\left(f\left(\frac{s}{1-s}\right); x\right) = f\left(\frac{x}{1-x}\right)T_k(1; x) + f'\left(\frac{x}{1-x}\right)T_k(\eta_x; x) + \frac{1}{2}f''\left(\frac{x}{1-x}\right)T_k(\eta_x^2; x) + T_k\left(\theta_x\left(\frac{s}{1-s}\right) \cdot \eta_x^2; x\right) = f\left(\frac{x}{1-x}\right)(1+x_k) + \frac{1+x_k}{2}f''\left(\frac{x}{1-x}\right)\left[\frac{1}{k+1}\left(\frac{x}{1-x}\right)^2 + \frac{1}{k+1}\left(\frac{x}{1-x}\right)\right] + T_k\left(\theta_x\left(\frac{s}{1-s}\right) \cdot \eta_x^2; x\right).$$

Upon multiplying both sides by $k+1$, we have

$$(k+1)\left[T_k\left(f\left(\frac{s}{1-s}\right); x\right) - f\left(\frac{x}{1-x}\right)\right] = (k+1)f\left(\frac{x}{1-x}\right)x_k + \frac{1+x_k}{2}f''\left(\frac{x}{1-x}\right)\frac{x}{(1-x)^2} + (k+1)T_k\left(\theta_x\left(\frac{s}{1-s}\right) \cdot \eta_x^2; x\right)$$

and hence

$$\begin{aligned} & \left| (k+1)\left(T_k\left(f\left(\frac{s}{1-s}\right); x\right) - f\left(\frac{x}{1-x}\right)\right) - \frac{1}{2}\frac{x}{(1-x)^2}f''\left(\frac{x}{1-x}\right) \right| \\ & \leq (k+1)M_1x_k + M_2\frac{x_k}{2} + (k+1)\left|T_k\left(\theta_x\left(\frac{s}{1-s}\right) \cdot \eta_x^2; x\right)\right| \end{aligned} \tag{16}$$

where $M_1 = \|f(\frac{x}{1-x})\|_\infty$ and $M_2 = \|f''(\frac{x}{1-x})\|_\infty$. Applying the Cauchy-Schwarz inequality in (16), we obtain

$$(k+1)\left|T_k\left(\theta_x\left(\frac{s}{1-s}\right) \cdot \eta_x^2; x\right)\right| \leq \sqrt{T_k\left(\theta_x^2\left(\frac{s}{1-s}\right); x\right)}\sqrt{(k+1)^2T_k(\eta_x^4; x)}.$$

From Theorem 3, we observe that

$$\bar{N}_{\Omega^\Delta} - \lim T_k\left(\theta_x^2\left(\frac{s}{1-s}\right); x\right) = 0.$$

Using Lemma 2 and Corollary 1, it is not hard to see that

$$\bar{N}_{\Omega^\Delta} - \lim_n (k+1)\left|T_k\left(\theta_x\left(\frac{s}{1-s}\right) \cdot \eta_x^2; x\right)\right| = 0.$$

Thus, by taking $n \rightarrow \infty$ in (16), we get

$$\bar{N}_{\Omega^\Delta} - \lim_n \left\{ (k+1)M_1x_k + \frac{M_2x_k}{2} + (k+1)\left|T_k\left(\theta_x\left(\frac{s}{1-s}\right) \cdot \eta_x^2; x\right)\right| \right\} = 0,$$

which leads us to the desired assertion of Theorem 5. \square

6 Computational and Geometrical Approaches

In this section, we provide the computational and geometrical approaches of Theorem 3 with respect to the linear operator $L_n(f; x)$ given in (10) under different choices for the parameters. Here, we have found it to be convenient to

Fig. 1 The convergence of $\Omega_m^\Delta(\lambda L(1;x))$ for different values of m

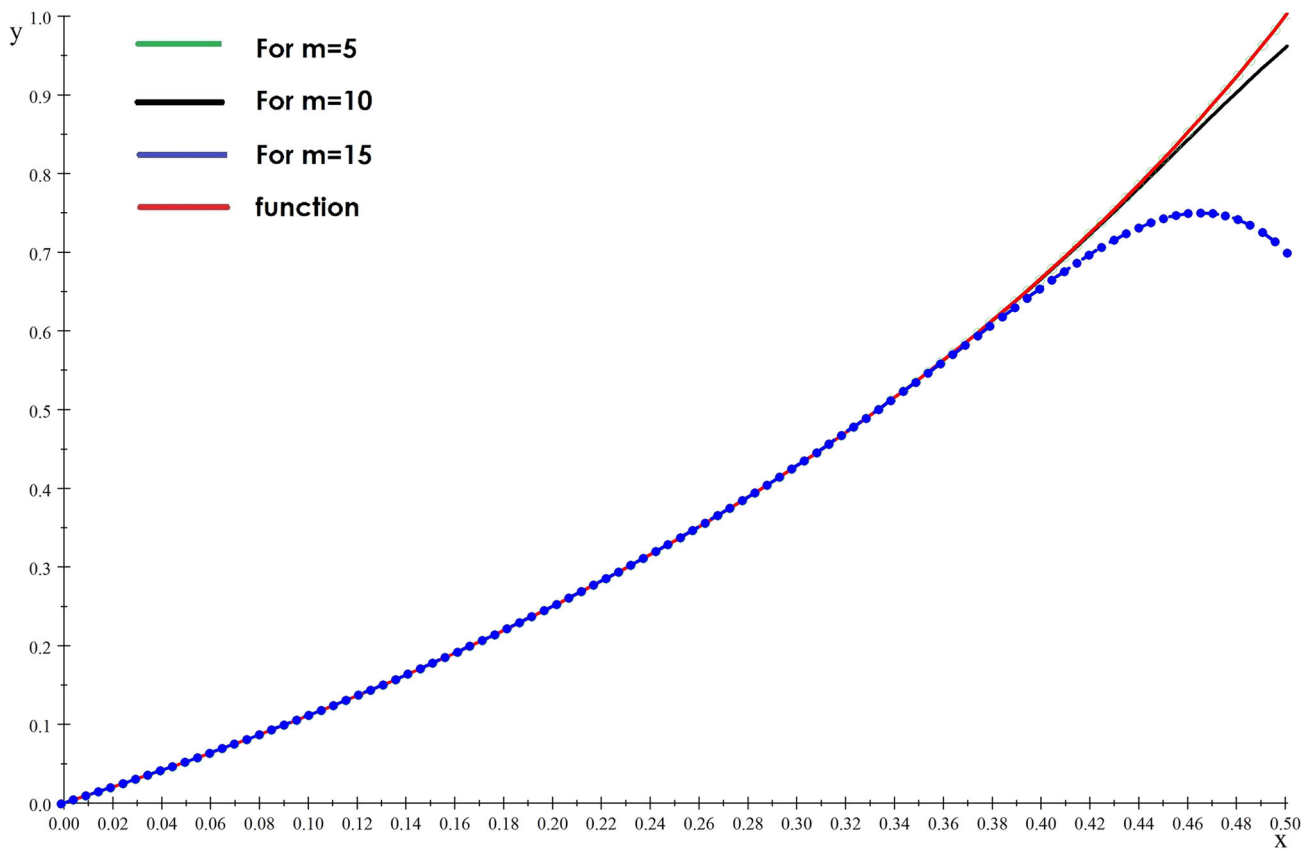
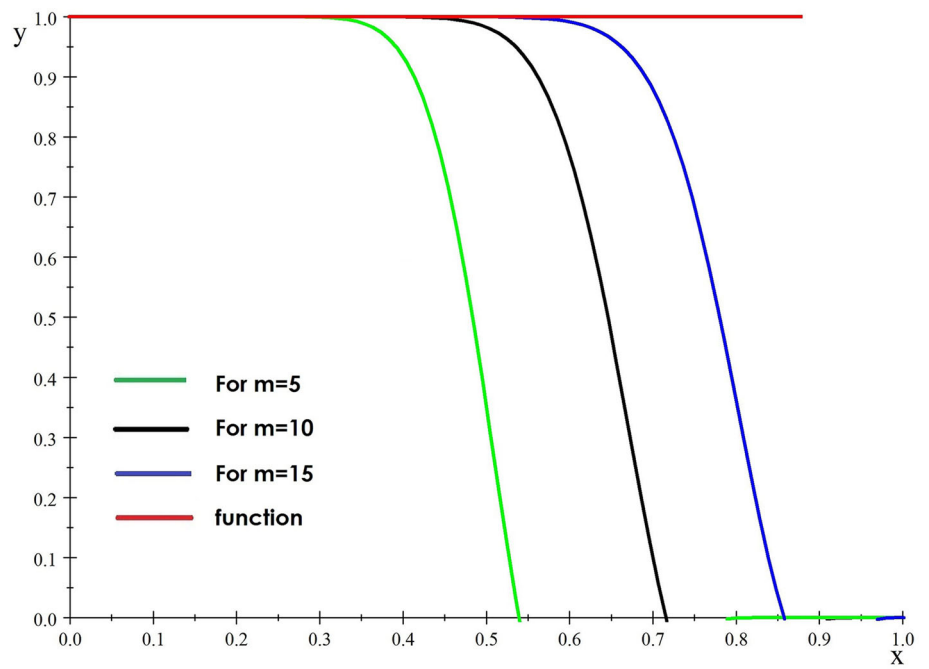


Fig. 2 The convergence of $\Omega_m^\Delta(\lambda L(\frac{x}{1-s}; x))$ for different values of m

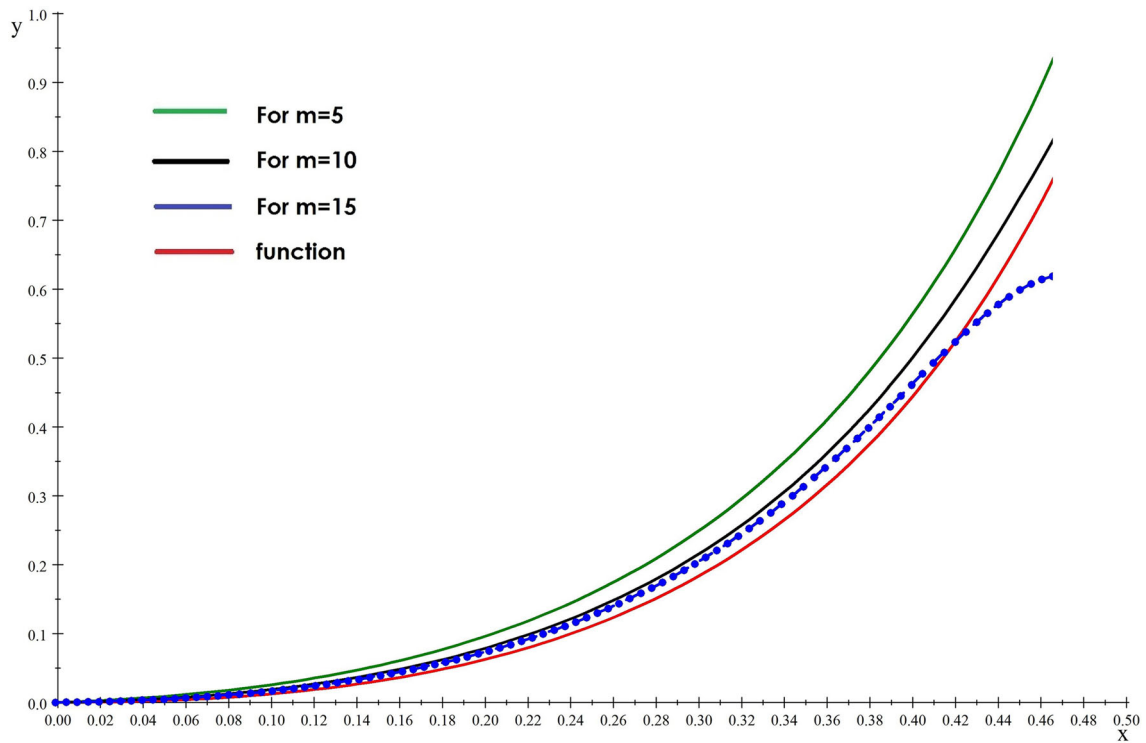


Fig. 3 The convergence of $\Omega_m^\Delta\left(\lambda L\left(\left(\frac{x}{1-x}\right)^2; x\right)\right)$ for different values of m

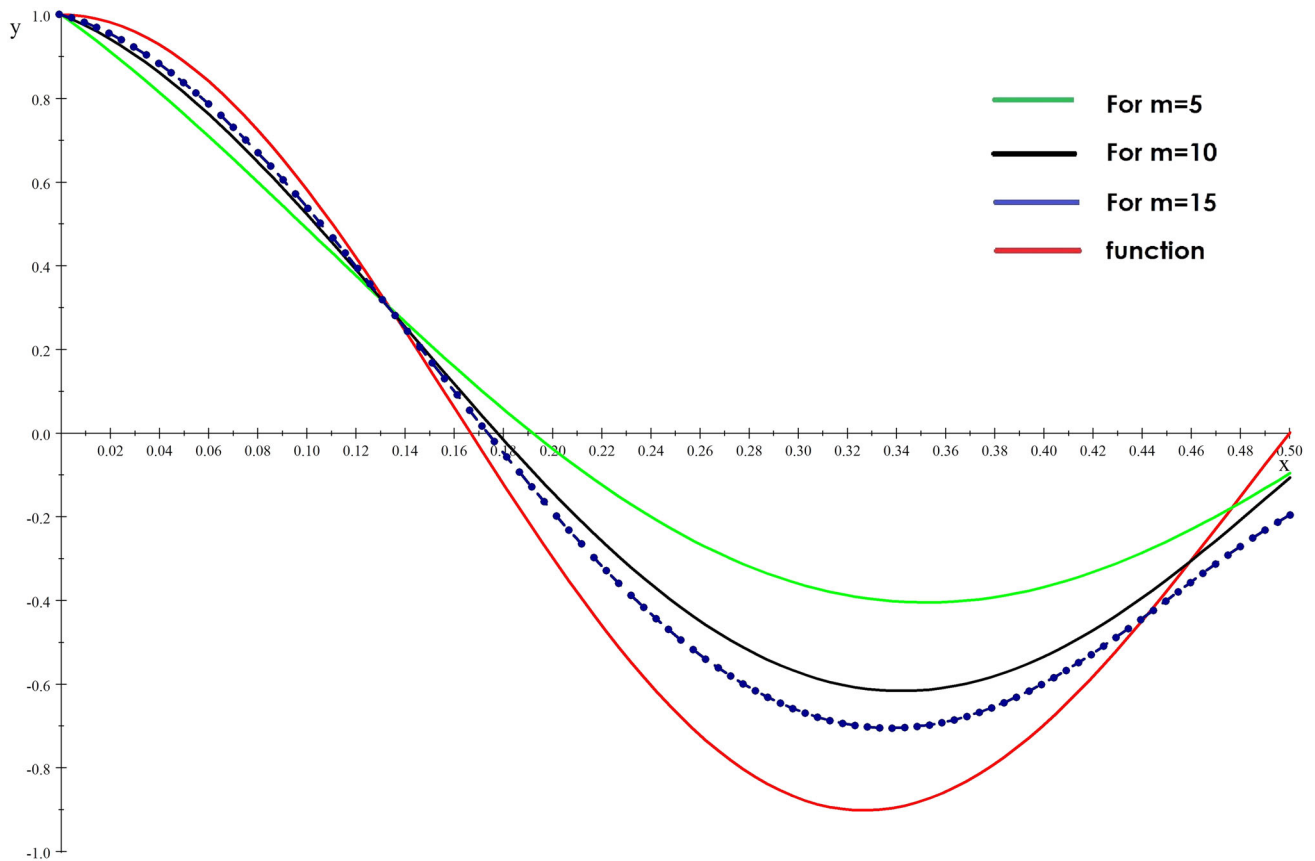


Fig. 4 The convergence of $\Omega_m^\Delta\left(\lambda L\left(\frac{\cos(3\pi s)}{1+s^2}; x\right)\right)$ for different values of m

investigate our series only for finite sums. More powerful equipments with higher speed can easily compute the more complicated infinite series in a similar manner.

Here, in our computations, we take

- $h_n(x, s) = (1-x)^{-n-1}$ and $\Gamma_{k,n}(s) = \binom{n+k}{k}$;
- $a_{k,n} = k$ and $b_n = n+1$;
- $\alpha = 2, \beta = \gamma$ and $h = 1$;
- $\lambda(n) = n^2$ for all $n \in \mathbb{N}$.

Based upon the above choices, we may define the following operator $\Omega_m^\Delta(\lambda Lf)$ by

$$\Omega_m^\Delta(\lambda Lf) = \frac{1}{2m-1} \sum_{n=(m-1)^2}^{m^2} \Delta_1^{2,\beta,\beta}(\lambda_n L_n(f; x)) \quad (17)$$

where

$$L_n(f; x) = (1-x)^{n+1} \sum_{k=0}^{\infty} f\left(\frac{k}{k+n+1}\right) \binom{n+k}{k} x^k. \quad (18)$$

Under above conditions, we obtain

$$\Omega_m^\Delta(\lambda Lf) = \frac{1}{2m-1} \left[m^2 L_m(f; x) - (m-1)^2 L_{m-1}(f; x) \right].$$

In fact, in Fig. 1, the value of k runs from $k=0$ to 25 for $m=5$, $m=10$ and $m=15$, respectively. As the value of m increases, the sequence

$$\Omega_m^\Delta(\lambda L(1; x))$$

converges towards to the function $f_0(x) = 1$.

In addition, from Fig. 2, it can be observed that, as the value of m increases, the sequence

$$\Omega_m^\Delta\left(\lambda L\left(\frac{s}{1-s}; x\right)\right) \quad (s, x \in [0, A], A \leq 1/2)$$

converges to the function $f_1(x) = \frac{x}{1-x}$.

Similarly, from Fig. 3, it can be easily seen that, as the value of m increases, the sequence

$$\Omega_m^\Delta\left(\lambda L\left(\left(\frac{s}{1-s}\right)^2; x\right)\right)$$

converges to the function $f_2(x)$ given by $f_2(x) = \left(\frac{x}{1-x}\right)^2$.

Figures 1, 2 and 3 clearly show that the conditions (4), (5) and (6) of Theorem 3 are satisfied.

We also observe from Fig. 4 that, as the value of m increases, the operators given by (17) converge towards the function. Indeed, Fig. 4 shows that the condition (3) holds true for the function

$$f(s) = \frac{\cos(3\pi s)}{1+s^2}$$

in $C[0, A]$ where $A \leq 1/2$.

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