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RESEARCH PAPER

Hermite–Hadamard–Fejér Type Inequalities for p-Convex Functions via Fractional Integrals

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Abstract
In this paper, firstly, Hermite–Hadamard–Fejér type inequalities for p-convex functions in fractional integral forms are built. Secondly, an integral identity and some Hermite–Hadamard–Fejér type integral inequalities for p-convex functions in fractional integral forms are obtained. Finally, some Hermite–Hadamard and Hermite–Hadamard–Fejér inequalities for convex, harmonically convex and p-convex functions are given. Many results presented here for p-convex functions provide extensions of others given in earlier works for convex, harmonically convex and p-convex functions.

Keywords Hermite–Hadamard inequalities · Hermite–Hadamard–Fejér inequalities · Fractional integrals · Convex functions · Harmonically convex functions · p-Convex functions

Mathematics Subject Classification 26A51 · 26A33 · 26D10

1 Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x \leq \frac{f(a)+f(b)}{2}, \tag{1}$$

is well known in the literature as Hermite–Hadamard's inequality (Hadamard 1893; Hermite 1883).

The most well-known inequalities related to the integral mean of a convex function f are the Hermite–Hadamard inequalities or its weighted versions, the so-called Hermite–Hadamard–Fejér inequalities.

Fejér (1906) established the following Fejér inequality which is the weighted generalization of Hermite–Hadamard inequality (1):

Theorem 1 *Let $f : [a, b] \to \mathbb{R}$ be convex function. Then the inequality*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b w(x)\mathrm{d}x &\leq \int_a^b f(x)w(x)\mathrm{d}x \\ &\leq \frac{f(a)+f(b)}{2}\int_a^b w(x)\mathrm{d}x, \end{aligned} \tag{2}$$

holds, where $w : [a, b] \to \mathbb{R}$ is nonnegative, integrable and symmetric to $(a+b)/2$.

For some results which generalize, improve, and extend the inequalities (1) and (2), see Bombardelli and Varošanec (2009), Chen and Wu (2014), Dragomir and Agarwal (1998), Fang and Shi (2014), İşcan (2013, 2014c, d, 2016b, c), Mihai et al. (2015), Noor et al. (2016), Pearce and Pecaric (2000), Sarıkaya (2012) and Tseng et al. (2011).

We will now give definitions of the right-hand side and left-hand side Riemann–Liouville fractional integrals which are used throughout this paper.

Definition 1 (Kilbas et al. 2006). Let $f \in L[a, b]$. The right-hand side and left-hand side Riemann–Liouville fractional integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $b > a \geq 0$ are defined by

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$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)\mathrm{d}t, \ \ x > a \text{ and}$$
$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}f(t)\mathrm{d}t, \ \ x < b,$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}\mathrm{d}t$.

Because of the wide application of Hermite–Hadamard type inequalities and fractional integrals, many researchers extend their studies to Hermite–Hadamard type inequalities involving fractional integrals not limited to integer integrals. Recently, more and more Hermite–Hadamard inequalities involving fractional integrals have been obtained for different classes of functions; see Dahmani (2010), İşcan (2014a, b, 2015), İşcan and Wu (2014), İşcan et al. (2016d), Sarıkaya et al. (2013) and Wang et al. (2012, 2013).

İşcan (2014d) gave the definition of harmonically convex function and established the following Hermite–Hadamard type inequality for harmonically convex functions as follows:

Definition 2 Let $I \subset \mathbb{R}\backslash\{0\}$ be a real interval. A function $f : I \to \mathbb{R}$ is said to be harmonically convex if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \le tf(y) + (1-t)f(x), \tag{3}$$

for all $x, y \in I$ and $t \in [0,1]$. If the inequality in (3) is reversed, then f is said to be harmonically concave.

Theorem 2 (İşcan 2014d). *Let $f : I \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a,b]$, then the following inequalities hold:*

$$f\left(\frac{2ab}{a+b}\right) \le \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}\mathrm{d}x \le \frac{f(a)+f(b)}{2}. \tag{4}$$

Chen and Wu (2014) presented Hermite–Hadamard–Fejér inequality for harmonically convex functions as follows:

Theorem 3 *Let $f : I \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a harmonically convex function and $a, b \in I$ with $a < b$. If $f \in L[a,b]$ and $w : [a,b] \subseteq \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is nonnegative, integrable and harmonically symmetric with respect to $\frac{2ab}{a+b}$, then*

$$f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{w(x)}{x^2}\mathrm{d}x \le \int_a^b \frac{f(x)w(x)}{x^2}\mathrm{d}x \le \frac{f(a)+f(b)}{2}\int_a^b \frac{w(x)}{x^2}\mathrm{d}x. \tag{5}$$

Sarıkaya et al. (2013) presented Hermite–Hadamard inequality for convex functions via fractional integrals as follows:

Theorem 4 *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $f \in L[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a+}f(b) + J^{\alpha}_{b-}f(a)\right] \le \frac{f(a)+f(b)}{2}, \tag{6}$$

with $\alpha > 0$.

İşcan and Wu (2014) presented Hermite–Hadamard inequality for harmonically convex functions via fractional integrals as follows:

Theorem 5 *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a function such that $f \in L[a,b]$, where $a, b \in I$ with $a < b$. If f is a harmonically convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{2ab}{a+b}\right) \le \frac{\Gamma(\alpha+1)}{2}\left(\frac{ab}{b-a}\right)^{\alpha}\left[J^{\alpha}_{1/a-}(f\circ g)(1/b) + J^{\alpha}_{1/b+}(f\circ g)(1/a)\right] \le \frac{f(a)+f(b)}{2}, \tag{7}$$

with $\alpha > 0$ and $g(x) = \frac{1}{x}$, $x \in \left[\frac{1}{b}, \frac{1}{a}\right]$.

İşcan (2015) presented Hermite–Hadamard–Fejér inequality for convex functions via fractional integrals as follows:

Theorem 6 *Let $f : [a,b] \to \mathbb{R}$ be a convex function with $a < b$ and $f \in L[a,b]$. If w is nonnegative, integrable and symmetric to $(a+b)/2$, then the following inequalities for fractional integrals hold:*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)&\left[J^{\alpha}_{a+}w(b) + J^{\alpha}_{b-}w(a)\right] \\ &\le \left[J^{\alpha}_{a+}(fw)(b) + J^{\alpha}_{b-}(fw)(a)\right] \\ &\le \frac{f(a)+f(b)}{2}\left[J^{\alpha}_{a+}w(b) + J^{\alpha}_{b-}w(a)\right], \end{aligned} \tag{8}$$

with $\alpha > 0$.

İşcan et al. (2016d) presented Hermite–Hadamard–Fejér inequality for harmonically convex functions via fractional integrals as follows:

Theorem 7 *Let $f : [a,b] \to \mathbb{R}$ be a harmonically convex function with $a < b$ and $f \in L[a,b]$. If $w : [a,b] \to \mathbb{R}$ is nonnegative, integrable and harmonically symmetric with respect to $2ab/a+b$, then the following inequalities for fractional integrals holds:*

$$f\left(\frac{2ab}{a+b}\right)\left[J^{\alpha}_{1/b+}(w\circ g)(1/a)+J^{\alpha}_{1/a-}(w\circ g)(1/b)\right]$$
$$\le\left[J^{\alpha}_{1/b+}(fw\circ g)(1/a)+J^{\alpha}_{1/a-}(fw\circ g)(1/b)\right]$$
$$\le\frac{f(a)+f(b)}{2}\left[J^{\alpha}_{1/b+}(w\circ g)(1/a)+J^{\alpha}_{1/a-}(w\circ g)(1/b)\right], \tag{9}$$

with $\alpha>0$ *and* $g(x)=\frac{1}{x}$, $x\in\left[\frac{1}{b},\frac{1}{a}\right]$.

Zhang and Wan (2007) gave the definition of p-convex function on $I\subset\mathbb{R}$, İşcan (2016c) gave a different definition of p-convex function on $I\subset(0,\infty)$ as follows:

Definition 3 Let $I\subset(0,\infty)$ be a real interval and $p\in\mathbb{R}\backslash\{0\}$. A function $f:I\to\mathbb{R}$ is said to be p-convex, if

$$f\left(\left[tx^p+(1-t)y^p\right]^{1/p}\right)\le tf(x)+(1-t)f(y) \tag{10}$$

for all $x,y\in I$ and $t\in[0,1]$.

It can be easily seen that for $p=1$ and $p=-1$, p-convexity reduces to ordinary convexity and harmonically convexity of functions defined on $I\subset(0,\infty)$, respectively.

In Fang and Shi (2014), Theorem 5, if we take $I\subset(0,\infty)$, $p\in\mathbb{R}\backslash\{0\}$ and $h(t)=t$, then we have the following theorem.

Theorem 8 *Let* $f:I\subset(0,\infty)\to\mathbb{R}$ *be a p-convex function,* $p\in\mathbb{R}\backslash\{0\}$, *and* $a,b\in I$ *with* $a<b$. *If* $f\in L[a,b]$ *then the following inequalities hold*:

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\le\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}\mathrm{d}x\le\frac{f(a)+f(b)}{2}. \tag{11}$$

For some results related to p-convex functions and its generalizations, we refer the reader to see Fang and Shi (2014), İşcan (2016a, b, c), Mihai et al. (2015), Noor et al. (2016) and Zhang and Wan (2007).

In this paper, we built Hermite–Hadamard–Fejér type inequalities for p-convex functions in fractional integral forms. We obtain an integral identity and some Hermite–Hadamard–Fejér type integral inequalities for p-convex functions in fractional integral forms. We give some Hermite–Hadamard and Hermite–Hadamard–Fejér inequalities for convex, harmonically convex and p-convex functions.

2 Main Results

Throughout this section, $\|w\|_\infty=\sup_{t\in[a,b]}|w(t)|$, for the continuous function $w:[a,b]\to\mathbb{R}$.

Definition 4 Let $p\in\mathbb{R}\backslash\{0\}$. A function $w:[a,b]\subset(0,\infty)\to\mathbb{R}$ is said to be p-symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{1/p}$ if

$$w(x)=w\left(\left[a^p+b^p-x^p\right]^{1/p}\right),$$

holds for all $x\in[a,b]$.

Lemma 1 *Let* $p\in\mathbb{R}\backslash\{0\}$, $\alpha>0$ *and* $w:[a,b]\subset(0,\infty)\to\mathbb{R}$ *is integrable, p-symmetric with respect to* $\left[\frac{a^p+b^p}{2}\right]^{1/p}$, *then*

(i) If $p>0$,

$$J^{\alpha}_{a^p+}(w\circ g)(b^p)=J^{\alpha}_{b^p-}(w\circ g)(a^p)=\frac{1}{2}\left[J^{\alpha}_{a^p+}(w\circ g)(b^p)+J^{\alpha}_{b^p-}(w\circ g)(a^p)\right],$$

with $g(x)=x^{\frac{1}{p}}$, $x\in[a^p,b^p]$.

(ii) If $p<0$,

$$J^{\alpha}_{b^p+}(w\circ g)(a^p)=J^{\alpha}_{a^p-}(w\circ g)(b^p)=\frac{1}{2}\left[J^{\alpha}_{b^p+}(w\circ g)(a^p)+J^{\alpha}_{a^p-}(w\circ g)(b^p)\right],$$

with $g(x)=x^{\frac{1}{p}}$, $x\in[b^p,a^p]$.

Proof

(i) Let $p>0$. Since w is p-symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{1/p}$, using Definition 4 we have $w\left(x^{1/p}\right)=w\left(\left[a^p+b^p-x\right]^{1/p}\right)$ for all $x\in[a^p,b^p]$. Hence in the following integral setting $t=a^p+b^p-x$ and $\mathrm{d}t=-\mathrm{d}x$ gives

$$\begin{aligned}J^{\alpha}_{a^p+}(w\circ g)(b^p)&=\frac{1}{\Gamma(\alpha)}\int_{a^p}^{b^p}(b^p-x)^{\alpha-1}w\left(x^{1/p}\right)\mathrm{d}x\\&=\frac{1}{\Gamma(\alpha)}\int_{a^p}^{b^p}(b^p-x)^{\alpha-1}w\left(\left[a^p+b^p-x\right]^{1/p}\right)\mathrm{d}x\\&=\frac{1}{\Gamma(\alpha)}\int_{a^p}^{b^p}(x-a^p)^{\alpha-1}w\left(x^{1/p}\right)\mathrm{d}x=J^{\alpha}_{b^p-}(w\circ g)(a^p).\end{aligned}$$

This completes the proof of i.

(ii) The proof is similar to i.

Theorem 9 *Let* $f:I\subset(0,\infty)\to\mathbb{R}$ *be a p-convex function,* $p\in\mathbb{R}\backslash\{0\}$, $\alpha>0$ *and* $a,b\in I$ *with* $a<b$. *If* $f\in L[a,b]$ *and* $w:[a,b]\to\mathbb{R}$ *is nonnegative, integrable and p-symmetric with respect to* $\left[\frac{a^p+b^p}{2}\right]^{1/p}$, *then the following inequalities for fractional integrals hold*:

(i) *If* $p > 0$,

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\left[J^{\alpha}_{a^p+}(w\circ g)(b^p)+J^{\alpha}_{b^p-}(w\circ g)(a^p)\right] \le \left[J^{\alpha}_{a^p+}(fw\circ g)(b^p)+J^{\alpha}_{b^p-}(fw\circ g)(a^p)\right] \le \frac{f(a)+f(b)}{2}\left[J^{\alpha}_{a^p+}(w\circ g)(b^p)+J^{\alpha}_{b^p-}(w\circ g)(a^p)\right], \tag{12}$$

with $g(x) = x^{1/p}$, $x \in [a^p, b^p]$.

(ii) *If* $p<0$,

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\left[J^{\alpha}_{b^p+}(w\circ g)(a^p)+J^{\alpha}_{a^p-}(w\circ g)(b^p)\right] \le \left[J^{\alpha}_{b^p+}(fw\circ g)(a^p)+J^{\alpha}_{a^p-}(fw\circ g)(b^p)\right] \le \frac{f(a)+f(b)}{2}\left[J^{\alpha}_{b^p+}(w\circ g)(a^p)+J^{\alpha}_{a^p-}(w\circ g)(b^p)\right], \tag{13}$$

with $g(x) = x^{1/p}$, $x \in [b^p, a^p]$.

Proof

(i) Let $p > 0$. Since $f : I \subset (0,\infty) \to \mathbb{R}$ is a p-convex function, we have, for all $x, y \in I$ (with $t = \frac{1}{2}$ in the inequality (10))

$$f\left(\left[\frac{x^p+y^p}{2}\right]^{1/p}\right) \le \frac{f(x)+f(y)}{2}.$$

Choosing $x = [ta^p + (1-t)b^p]^{1/p}$ and $y = [tb^p + (1-t)a^p]^{1/p}$, we get

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right) \le \frac{f\left([ta^p+(1-t)b^p]^{1/p}\right)+f\left([tb^p+(1-t)a^p]^{1/p}\right)}{2}. \tag{14}$$

Multiplying both sides of (14) by $2t^{\alpha-1}w\left([ta^p+(1-t)b^p]^{1/p}\right)$ and integrating with respect to t over $[0,1]$, using Lemma 1-i, we get

$$f\left(\left[\frac{a^p+b^p}{2}\right]^{1/p}\right)\left[J^{\alpha}_{a^p+}(w\circ g)(b^p)+J^{\alpha}_{b^p-}(w\circ g)(a^p)\right] \le \left[J^{\alpha}_{a^p+}(fw\circ g)(b^p)+J^{\alpha}_{b^p-}(fw\circ g)(a^p)\right],$$

the left hand side of (12). For the proof of the second inequality in (12), we first note that if f is a p-convex function, then, for all $t \in [0,1]$, it yields

$$\frac{f\left([ta^p+(1-t)b^p]^{1/p}\right)+f\left([tb^p+(1-t)a^p]^{1/p}\right)}{2} \le \frac{f(a)+f(b)}{2}. \tag{15}$$

Multiplying both sides of (15) by $2t^{\alpha-1}w\left([ta^p+(1-t)b^p]^{1/p}\right)$ and integrating with respect to t over $[0,1]$, using Lemma 1-i, we get

$$\left[J^{\alpha}_{a^p+}(fw\circ g)(b^p)+J^{\alpha}_{b^p-}(fw\circ g)(a^p)\right] \le \frac{f(a)+f(b)}{2}\left[J^{\alpha}_{a^p+}(w\circ g)(b^p)+J^{\alpha}_{b^p-}(w\circ g)(a^p)\right],$$

the right hand side of (12). This completes the proof of i.

(ii) The proof is similar to i.

□

Remark 1 In Theorem 9, one can see the following.

(1) If one takes $p = 1$, one has (8).
(2) If one takes $p = 1$ and $w(x) = 1$, one has (6).
(3) If one takes $p = 1$ and $\alpha = 1$, one has (2).
(4) If one takes $p = 1$, $\alpha = 1$ and $w(x) = 1$, one has (1).
(5) If one takes $p = -1$, one has (9).
(6) If one takes $p = -1$ and $w(x) = 1$, one has (7),
(7) If one takes $p = -1$ and $\alpha = 1$, one has (5).
(8) If one takes $p = -1$, $\alpha = 1$ and $w(x) = 1$, one has (4).
(9) If one takes $\alpha = 1$ and $w(x) = 1$, one has (11).

Lemma 2 *Let* $f : I \subset (0,\infty) \to \mathbb{R}$ *be a differentiable function on* I° *and* $a, b \in I^\circ$ *with* $a<b$, $p \in \mathbb{R}\backslash\{0\}$ *and* $\alpha > 0$. *If* $f' \in L[a,b]$ *and* $w : [a,b] \to \mathbb{R}$ *is integrable and p-symmetric with respect to* $\left[\frac{a^p+b^p}{2}\right]^{1/p}$, *then the following equalities for fractional integrals hold*:

(i) *If* $p > 0$,

$$\frac{f(a)+f(b)}{2}\left[J^{\alpha}_{a^p+}(w\circ g)(b^p)+J^{\alpha}_{b^p-}(w\circ g)(a^p)\right] - \left[J^{\alpha}_{a^p+}(fw\circ g)(b^p)+J^{\alpha}_{b^p-}(fw\circ g)(a^p)\right] = \frac{1}{\Gamma(\alpha)}\int_{a^p}^{b^p}\left[\begin{array}{l}\int_{a^p}^{t}(b^p-s)^{\alpha-1}(w\circ g)(s)\mathrm{d}s\\ -\int_{t}^{b^p}(s-a^p)^{\alpha-1}(w\circ g)(s)\mathrm{d}s\end{array}\right](f\circ g)'(t)\mathrm{d}t, \tag{16}$$

with $g(x) = x^{1/p}$, $x \in [a^p, b^p]$.

(ii) *If* $p<0$,

$$\frac{f(a)+f(b)}{2}\left[J^{\alpha}_{b^p+}(w\circ g)(a^p)+J^{\alpha}_{a^p-}(w\circ g)(b^p)\right] - \left[J^{\alpha}_{b^p+}(fw\circ g)(a^p)+J^{\alpha}_{a^p-}(fw\circ g)(b^p)\right] = \frac{1}{\Gamma(\alpha)}\int_{b^p}^{a^p}\left[\begin{array}{l}\int_{b^p}^{t}(a^p-s)^{\alpha-1}(w\circ g)(s)\mathrm{d}s\\ -\int_{t}^{a^p}(s-b^p)^{\alpha-1}(w\circ g)(s)\mathrm{d}s\end{array}\right](f\circ g)'(t)\mathrm{d}t, \tag{17}$$

with $g(x)=x^{1/p}$, $x\in[b^p,a^p]$.

Proof

(i) Let $p>0$. It suffices to note that

$$\begin{aligned} I &= \frac{1}{\Gamma(\alpha)}\int_{a^p}^{b^p}\left[\begin{array}{l}\int_{a^p}^{t}(b^p-s)^{\alpha-1}(w\circ g)(s)\mathrm{d}s\\ -\int_{t}^{b^p}(s-a^p)^{\alpha-1}(w\circ g)(s)\mathrm{d}s\end{array}\right](f\circ g)'(t)\mathrm{d}t\\ &= \frac{1}{\Gamma(\alpha)}\int_{a^p}^{b^p}\left[\int_{a^p}^{t}(b^p-s)^{\alpha-1}(w\circ g)(s)\mathrm{d}s\right](f\circ g)'(t)\mathrm{d}t\\ &\quad -\frac{1}{\Gamma(\alpha)}\int_{a^p}^{b^p}\left[\int_{t}^{b^p}(s-a^p)^{\alpha-1}(w\circ g)(s)\mathrm{d}s\right](f\circ g)'(t)\mathrm{d}t\\ &= I_1-I_2. \end{aligned} \tag{18}$$

By integration by parts and using Lemma 1-i, we have

$$\begin{aligned} I_1 &= \frac{1}{\Gamma(\alpha)}(f\circ g)(t)\left(\int_{a^p}^{t}(b^p-s)^{\alpha-1}(w\circ g)(s)\mathrm{d}s\right)\Bigg|_{a^p}^{b^p}\\ &\quad -\frac{1}{\Gamma(\alpha)}\int_{a^p}^{b^p}(b^p-t)^{\alpha-1}(fw\circ g)(t)\mathrm{d}t\\ &= f(b)\frac{1}{\Gamma(\alpha)}\int_{a^p}^{b^p}(b^p-s)^{\alpha-1}(w\circ g)(s)\mathrm{d}s\\ &\quad -\frac{1}{\Gamma(\alpha)}\int_{a^p}^{b^p}(b^p-t)^{\alpha-1}(fw\circ g)(t)\mathrm{d}t\\ &= \frac{f(b)}{2}\left[J^{\alpha}_{a^p+}(w\circ g)(b^p)+J^{\alpha}_{b^p-}(w\circ g)(a^p)\right]\\ &\quad -J^{\alpha}_{a^p+}(fw\circ g)(b^p) \end{aligned} \tag{19}$$

and similarly

$$\begin{aligned} I_2 &= \frac{1}{\Gamma(\alpha)}(f\circ g)(t)\left(\int_{t}^{b^p}(s-a^p)^{\alpha-1}(w\circ g)(s)\mathrm{d}s\right)\Bigg|_{a^p}^{b^p}\\ &\quad +\frac{1}{\Gamma(\alpha)}\int_{a^p}^{b^p}(t-a^p)^{\alpha-1}(fw\circ g)(t)\mathrm{d}t\\ &= -f(a)\frac{1}{\Gamma(\alpha)}\int_{a^p}^{b^p}(s-a^p)^{\alpha-1}(w\circ g)(s)\mathrm{d}s\\ &\quad +\frac{1}{\Gamma(\alpha)}\int_{a^p}^{b^p}(t-a^p)^{\alpha-1}(fw\circ g)(t)\mathrm{d}t\\ &= -\frac{f(a)}{2}\left[J^{\alpha}_{a^p+}(w\circ g)(b^p)+J^{\alpha}_{b^p-}(w\circ g)(a^p)\right]\\ &\quad +J^{\alpha}_{b^p-}(fw\circ g)(a^p). \end{aligned} \tag{20}$$

A combination of (18), (19) and (20) gives (16). This completes the proof of i.

(ii) The proof is similar to i.

Remark 2 In Lemma 2, one can see the following.

(1) If one takes $p=1$, one has İşcan (2015), Lemma 2.4.
(2) If one takes $p=1$ and $w(x)=1$, one has Sarıkaya et al. (2013), Lemma 2.
(3) If one takes $p=1$ and $\alpha=1$, one has Sarıkaya (2012), Lemma 2.6.
(4) If one takes $p=1$, $\alpha=1$ and $w(x)=1$, one has Dragomir and Agarwal (1998), Lemma 2.1.
(5) If one takes $p=-1$, one has İşcan et al. (2016d), Lemma 3.
(6) If one takes $p=-1$ and $w(x)=1$, one has İşcan and Wu (2014), Lemma 3.
(7) If one takes $p=-1$, $\alpha=1$ and $w(x)=1$, one has İşcan (2014d), 2.5. Lemma.
(8) If one takes $\alpha=1$ and $w(x)=1$, one has Noor et al. (2016), Lemma 2.4.

Theorem 10 *Let* $f:I\subset(0,\infty)\to\mathbb{R}$ *be a differentiable function on* I° *such that* $f'\in L[a,b]$, *where* $a,b\in I$ *and* $a<b$. *If* $|f'|$ *is p-convex function on* $[a,b]$ *for* $p\in\mathbb{R}\backslash\{0\}$ *and* $\alpha>0$, $w:[a,b]\to\mathbb{R}$ *is continuous and p-symmetric with respect to* $\left[\frac{a^p+b^p}{2}\right]^{1/p}$, *then the following inequality for fractional integrals hold*:

(i) *If* $p>0$,

$$\left|\frac{f(a)+f(b)}{2}\left[J^{\alpha}_{a^p+}(w\circ g)(b^p)+J^{\alpha}_{b^p-}(w\circ g)(a^p)\right] -\left[J^{\alpha}_{a^p+}(fw\circ g)(b^p)+J^{\alpha}_{b^p-}(fw\circ g)(a^p)\right]\right| \le \frac{\|w\|_\infty(b^p-a^p)^{\alpha+1}}{\Gamma(\alpha+1)}\left[C_1(\alpha,p)|f'(a)|+C_2(\alpha,p)|f'(b)|\right],$$

where

$$C_1(\alpha,p)=\int_0^1\frac{|(1-u)^\alpha-u^\alpha|}{p[ua^p+(1-u)b^p]^{1-(1/p)}}u\mathrm{d}u \quad\text{and}\quad C_2(\alpha,p)=\int_0^1\frac{|(1-u)^\alpha-u^\alpha|}{p[ua^p+(1-u)b^p]^{1-(1/p)}}(1-u)\mathrm{d}u$$

with $g(x)=x^{1/p}$, $x\in[a^p,b^p]$.

(ii) *If* $p<0$,

$$\left|\frac{f(a)+f(b)}{2}\left[J^{\alpha}_{b^p+}(w\circ g)(a^p)+J^{\alpha}_{a^p-}(w\circ g)(b^p)\right] -\left[J^{\alpha}_{b^p+}(fw\circ g)(a^p)+J^{\alpha}_{a^p-}(fw\circ g)(b^p)\right]\right| \le \frac{\|w\|_\infty(a^p-b^p)^{\alpha+1}}{\Gamma(\alpha+1)}\left[C_3(\alpha,p)|f'(a)|+C_4(\alpha,p)|f'(b)|\right],$$

where

$$C_3(\alpha,p)=\int_0^1\frac{-|(1-u)^\alpha-u^\alpha|}{p[ua^p+(1-u)b^p]^{1-(1/p)}}u\mathrm{d}u \quad \text{and} \quad C_4(\alpha,p)=\int_0^1\frac{-|(1-u)^\alpha-u^\alpha|}{p[ua^p+(1-u)b^p]^{1-(1/p)}}(1-u)\mathrm{d}u$$

with $g(x)=x^{1/p}$, $x\in[b^p,a^p]$.

Proof

(i) Let $p>0$. Using Lemma 2-i, it follows that

$$\left|\frac{f(a)+f(b)}{2}\left[J^\alpha_{a^p+}(w\circ g)(b^p)+J^\alpha_{b^p-}(w\circ g)(a^p)\right]-\left[J^\alpha_{a^p+}(fw\circ g)(b^p)+J^\alpha_{b^p-}(fw\circ g)(a^p)\right]\right| \leq\frac{1}{\Gamma(\alpha)}\int_{a^p}^{b^p}\left|\begin{array}{l}\int_{a^p}^{t}(b^p-s)^{\alpha-1}(w\circ g)(s)\mathrm{d}s\\-\int_t^{b^p}(s-a^p)^{\alpha-1}(w\circ g)(s)\mathrm{d}s\end{array}\right||(f\circ g)'(t)|\mathrm{d}t. \tag{21}$$

Since w is p-symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{1/p}$, using Definition 4 we have $w(x^{1/p})=w\left([a^p+b^p-x]^{1/p}\right)$ for all $x\in[a^p,b^p]$:

$$\begin{aligned}&\left|\int_{a^p}^{t}(b^p-s)^{\alpha-1}(w\circ g)(s)\mathrm{d}s-\int_t^{b^p}(s-a^p)^{\alpha-1}(w\circ g)(s)\mathrm{d}s\right|\\&=\left|\int_{a^p+b^p-t}^{b^p}(s-a^p)^{\alpha-1}(w\circ g)(s)\mathrm{d}s+\int_{b^p}^{t}(s-a^p)^{\alpha-1}(w\circ g)(s)\mathrm{d}s\right|\\&=\left|\int_{a^p+b^p-t}^{t}(s-a^p)^{\alpha-1}(w\circ g)(s)\mathrm{d}s\right|\\&\leq\begin{cases}\int_t^{a^p+b^p-t}\left|(s-a^p)^{\alpha-1}(w\circ g)(s)\right|\mathrm{d}s, & t\in\left[a^p,\frac{a^p+b^p}{2}\right]\\ \int_{a^p+b^p-t}^{t}\left|(s-a^p)^{\alpha-1}(w\circ g)(s)\right|\mathrm{d}s, & t\in\left[\frac{a^p+b^p}{2},b^p\right]\end{cases}.\end{aligned} \tag{22}$$

A combination of (21) and (22) gives

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}\left[J^\alpha_{a^p+}(w\circ g)(b^p)+J^\alpha_{b^p-}(w\circ g)(a^p)\right]-\left[J^\alpha_{a^p+}(fw\circ g)(b^p)+J^\alpha_{b^p-}(fw\circ g)(a^p)\right]\right|\\
&\leq\frac{1}{\Gamma(\alpha)}\left[\begin{array}{l}\int_{a^p}^{\frac{a^p+b^p}{2}}\left(\int_t^{a^p+b^p-t}\left|(s-a^p)^{\alpha-1}(w\circ g)(s)\right|\mathrm{d}s\right)|(f\circ g)'(t)|\mathrm{d}t\\+\int_{\frac{a^p+b^p}{2}}^{b^p}\left(\int_{a^p+b^p-t}^{t}\left|(s-a^p)^{\alpha-1}(w\circ g)(s)\right|\mathrm{d}s\right)|(f\circ g)'(t)|\mathrm{d}t\end{array}\right]\\
&\leq\frac{\|w\|_\infty}{\Gamma(\alpha)}\left[\begin{array}{l}\int_{a^p}^{\frac{a^p+b^p}{2}}\left(\int_t^{a^p+b^p-t}(s-a^p)^{\alpha-1}\mathrm{d}s\right)|(f\circ g)'(t)|\mathrm{d}t\\+\int_{\frac{a^p+b^p}{2}}^{b^p}\left(\int_{a^p+b^p-t}^{t}(s-a^p)^{\alpha-1}\mathrm{d}s\right)|(f\circ g)'(t)|\mathrm{d}t\end{array}\right]\\
&\leq\frac{\|w\|_\infty}{\Gamma(\alpha)}\left[\begin{array}{l}\int_{a^p}^{\frac{a^p+b^p}{2}}\left(\int_t^{a^p+b^p-t}(s-a^p)^{\alpha-1}\mathrm{d}s\right)\frac{1}{pt^{1-(1/p)}}\left|f'\left(t^{1/p}\right)\right|\mathrm{d}t\\+\int_{\frac{a^p+b^p}{2}}^{b^p}\left(\int_{a^p+b^p-t}^{t}(s-a^p)^{\alpha-1}\mathrm{d}s\right)\frac{1}{pt^{1-(1/p)}}\left|f'\left(t^{1/p}\right)\right|\mathrm{d}t\end{array}\right]\\
&\leq\frac{\|w\|_\infty}{\Gamma(\alpha+1)}\left[\begin{array}{l}\int_{a^p}^{\frac{a^p+b^p}{2}}\frac{(b^p-t)^\alpha-(t-a^p)^\alpha}{pt^{1-(1/p)}}\left|f'\left(t^{1/p}\right)\right|\mathrm{d}t\\+\int_{\frac{a^p+b^p}{2}}^{b^p}\frac{(t-a^p)^\alpha-(b^p-t)^\alpha}{pt^{1-(1/p)}}\left|f'\left(t^{1/p}\right)\right|\mathrm{d}t\end{array}\right].\end{aligned}$$

Setting $t=ua^p+(1-u)b^p$ and $\mathrm{d}t=(a^p-b^p)\mathrm{d}u$ gives

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}\left[J^\alpha_{a^p+}(w\circ g)(b^p)+J^\alpha_{b^p-}(w\circ g)(a^p)\right]-\left[J^\alpha_{a^p+}(fw\circ g)(b^p)+J^\alpha_{b^p-}(fw\circ g)(a^p)\right]\right|\\
&\leq\frac{\|w\|_\infty(b^p-a^p)^{\alpha+1}}{\Gamma(\alpha+1)}\left[\begin{array}{l}\int_0^{\frac12}\frac{(1-u)^\alpha-u^\alpha}{p[ua^p+(1-u)b^p]^{1-(1/p)}}\left|f'\left([ua^p+(1-u)b^p]^{1/p}\right)\right|\mathrm{d}u\\+\int_{\frac12}^1\frac{u^\alpha-(1-u)^\alpha}{p[ua^p+(1-u)b^p]^{1-(1/p)}}\left|f'\left([ua^p+(1-u)b^p]^{1/p}\right)\right|\mathrm{d}u\end{array}\right]\\
&=\frac{\|w\|_\infty(b^p-a^p)^{\alpha+1}}{\Gamma(\alpha+1)}\int_0^1\frac{|(1-u)^\alpha-u^\alpha|}{p[ua^p+(1-u)b^p]^{1-(1/p)}}\left|f'\left([ua^p+(1-u)b^p]^{1/p}\right)\right|\mathrm{d}u.\end{aligned} \tag{23}$$

Since $|f'|$ is p-convex function on $[a,b]$, we have

$$\left|f'\left(\left[ua^p+(1-u)b^p\right]^{1/p}\right)\right| \le u|f'(a)|+(1-u)|f'(b)|. \tag{24}$$

A combination of (23) and (24) gives

$$\begin{aligned}&\Big|\frac{f(a)+f(b)}{2}\left[J^{\alpha}_{a^p+}(w\circ g)(b^p)+J^{\alpha}_{b^p-}(w\circ g)(a^p)\right]\\ &\quad-\left[J^{\alpha}_{a^p+}(fw\circ g)(b^p)+J^{\alpha}_{b^p-}(fw\circ g)(a^p)\right]\Big|\\ &\le\frac{\|w\|_{\infty}(b^p-a^p)^{\alpha+1}}{\Gamma(\alpha+1)}\\ &\left[\begin{array}{l}\int_0^1\frac{|(1-u)^{\alpha}-u^{\alpha}|}{p[ua^p+(1-u)b^p]^{1-(1/p)}}u\,\mathrm{d}u|f'(a)|\\ +\int_0^1\frac{|(1-u)^{\alpha}-u^{\alpha}|}{p[ua^p+(1-u)b^p]^{1-(1/p)}}(1-u)\,\mathrm{d}u|f'(b)|\end{array}\right]\\ &=\frac{\|w\|_{\infty}(b^p-a^p)^{\alpha+1}}{\Gamma(\alpha+1)}\left[C_1(\alpha,p)|f'(a)|+C_2(\alpha,p)|f'(b)|\right].\end{aligned}$$

This completes the proof of i.

(ii) The proof is similar to i.

Remark 3 In Theorem 10, one can see the following.

(1) If one takes $p=1$, one has İşcan (2015), Theorem 2.6.
(2) If one takes $p=1$ and $w(x)=1$, one has Sarıkaya et al. (2013), Theorem 3.
(3) If one takes $p=1$, $\alpha=1$ and $w(x)=1$, one has Dragomir and Agarwal (1998), Theorem 2.2.
(4) If one takes $\alpha=1$ and $w(x)=1$, one has Noor et al. (2016), Theorem 3.1.

Corollary 1 *In Theorem 10, one can see the following.*

(1) *If one takes $p=1$ and $\alpha=1$, one has the following Hermite–Hadamard–Fejér inequality for convex functions:*

$$\begin{aligned}&\left|\frac{1}{b-a}\frac{f(a)+f(b)}{2}\int_a^b w(x)\mathrm{d}x-\frac{1}{b-a}\int_a^b f(x)w(x)\mathrm{d}x\right|\\ &\quad\le\frac{\|w\|_{\infty}(b-a)}{2}\left[C_1(1,1)|f'(a)|+C_2(1,1)|f'(b)|\right].\end{aligned}$$

(2) *If one takes $p=-1$, one has the following Hermite–Hadamard–Fejér inequality for harmonically convex functions via fractional integrals:*

$$\begin{aligned}&\Big|\frac{f(a)+f(b)}{2}\left[J^{\alpha}_{1/b+}(w\circ g)(1/a)+J^{\alpha}_{1/a-}(w\circ g)(1/b)\right]\\ &-\left[J^{\alpha}_{1/b+}(fw\circ g)(1/a)+J^{\alpha}_{1/a-}(fw\circ g)(1/b)\right]\Big|\\ &\le\frac{\|w\|_{\infty}ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha}\left[C_3(\alpha,-1)|f'(a)|+C_4(\alpha,-1)|f'(b)|\right].\end{aligned}$$

(3) *If one takes $p=-1$, $\alpha=1$ and $w(x)=1$, one has the following Hermite–Hadamard inequality for harmonically convex functions:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\mathrm{d}x\right|\\ &\quad\le\left(\frac{b-a}{ab}\right)\left[C_3(1,-1)|f'(a)|+C_4(1,-1)|f'(b)|\right].\end{aligned}$$

(4) *If one takes $p=-1$ and $\alpha=1$, one has the following Hermite–Hadamard–Fejér inequality for harmonically convex functions:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}\int_a^b\frac{w(x)}{x^2}\mathrm{d}x-\int_a^b\frac{f(x)w(x)}{x^2}\mathrm{d}x\right|\\ &\quad\le\frac{\|w\|_{\infty}(b-a)^2}{2}\left[C_3(1,-1)|f'(a)|+C_4(1,-1)|f'(b)|\right].\end{aligned}$$

(5) *If one takes $p=-1$ and $w(x)=1$, one has the following Hermite–Hadamard inequality for harmonically convex functions via fractional integrals:*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2\left(\frac{1}{b}-\frac{1}{a}\right)^{\alpha}}\left[J^{\alpha}_{1/b+}(f\circ g)(1/a)+J^{\alpha}_{1/a-}(f\circ g)(1/b)\right]\right|\\ &\le\left(\frac{b-a}{ab}\right)\left[C_3(\alpha,-1)|f'(a)|+C_4(\alpha,-1)|f'(b)|\right].\end{aligned}$$

Theorem 11 *Let $f: I\subset(0,\infty)\to\mathbb{R}$ be a differentiable function on I° such that $f'\in L[a,b]$, where $a,b\in I$ and $a<b$. If $|f'|^q$, $q\ge1$, is p-convex function on $[a,b]$ for $p\in\mathbb{R}\backslash\{0\}$, $\alpha>0$, $w:[a,b]\to\mathbb{R}$ is continuous and p-symmetric with respect to $\left[\frac{a^p+b^p}{2}\right]^{1/p}$, then the following inequality for fractional integrals holds:*

(i) *If $p>0$,*

$$\begin{aligned}&\Big|\frac{f(a)+f(b)}{2}\left[J^{\alpha}_{a^p+}(w\circ g)(b^p)+J^{\alpha}_{b^p-}(w\circ g)(a^p)\right]\\ &\quad-\left[J^{\alpha}_{a^p+}(fw\circ g)(b^p)+J^{\alpha}_{b^p-}(fw\circ g)(a^p)\right]\Big|\\ &\le\frac{\|w\|_{\infty}(b^p-a^p)^{\alpha+1}}{\Gamma(\alpha+1)}C_5^{1-\frac{1}{q}}(\alpha,p)\\ &\quad\times\left[C_1(\alpha,p)|f'(a)|^q+C_2(\alpha,p)|f'(b)|^q\right]^{\frac{1}{q}},\end{aligned}$$

where $C_1(\alpha,p)$, $C_2(\alpha,p)$ are the same in Theorem 10,

$$C_5(\alpha,p)=\int_0^1\frac{|(1-u)^{\alpha}-u^{\alpha}|}{p[ua^p+(1-u)b^p]^{1-(1/p)}}\mathrm{d}u$$

with $g(x)=x^{1/p}$, $x\in[a^p,b^p]$.

(ii) *If $p<0$,*

$$\left|\frac{f(a)+f(b)}{2}\left[J^{\alpha}_{b^p+}(w\circ g)(a^p)+J^{\alpha}_{a^p-}(w\circ g)(b^p)\right]\right.$$
$$\left.-\left[J^{\alpha}_{b^p+}(fw\circ g)(a^p)+J^{\alpha}_{a^p-}(fw\circ g)(b^p)\right]\right|$$
$$\leq\frac{\|w\|_{\infty}(a^p-b^p)^{\alpha+1}}{\Gamma(\alpha+1)}C_6^{1-\frac{1}{q}}(\alpha,p)$$
$$\left[C_3(\alpha,p)|f'(a)|^q+C_4(\alpha,p)|f'(b)|^q\right]^{\frac{1}{q}},$$

where $C_3(\alpha,p)$, $C_4(\alpha,p)$ *are the same in Theorem 10,*

$$C_6(\alpha,p)=\int_0^1\frac{-|(1-u)^{\alpha}-u^{\alpha}|}{p[ua^p+(1-u)b^p]^{1-(1/p)}}du$$

with $g(x)=x^{1/p}$, $x\in[b^p,a^p]$.

Proof

(i) Let $p>0$. Using (23), power mean inequality and the p-convexity of $|f'|^q$, it follows that

$$\left|\frac{f(a)+f(b)}{2}\left[J^{\alpha}_{a^p+}(w\circ g)(b^p)+J^{\alpha}_{b^p-}(w\circ g)(a^p)\right]\right.$$
$$\left.-\left[J^{\alpha}_{a^p+}(fw\circ g)(b^p)+J^{\alpha}_{b^p-}(fw\circ g)(a^p)\right]\right|$$
$$\leq\frac{\|w\|_{\infty}(b^p-a^p)^{\alpha+1}}{\Gamma(\alpha+1)}\int_0^1\frac{|(1-u)^{\alpha}-u^{\alpha}|}{p[ua^p+(1-u)b^p]^{1-(1/p)}}$$
$$\left|f'\left(\left[ua^p+(1-u)b^p\right]^{1/p}\right)\right|du$$
$$\leq\frac{\|w\|_{\infty}(b^p-a^p)^{\alpha+1}}{\Gamma(\alpha+1)}\left(\int_0^1\frac{|(1-u)^{\alpha}-u^{\alpha}|}{p[ua^p+(1-u)b^p]^{1-(1/p)}}du\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\frac{|(1-u)^{\alpha}-u^{\alpha}|}{p[ua^p+(1-u)b^p]^{1-(1/p)}}\left|f'\left(\left[ua^p+(1-u)b^p\right]^{1/p}\right)\right|^q du\right)^{\frac{1}{q}}$$
$$\leq\frac{\|w\|_{\infty}(b^p-a^p)^{\alpha+1}}{\Gamma(\alpha+1)}\left(\int_0^1\frac{|(1-u)^{\alpha}-u^{\alpha}|}{p[ua^p+(1-u)b^p]^{1-(1/p)}}du\right)^{1-\frac{1}{q}}$$
$$\times\left[\left(\int_0^1\frac{|(1-u)^{\alpha}-u^{\alpha}|}{p[ua^p+(1-u)b^p]^{1-(1/p)}}u\,du\right)|f'(a)|^q\right.$$
$$\left.+\left(\int_0^1\frac{|(1-u)^{\alpha}-u^{\alpha}|}{p[ua^p+(1-u)b^p]^{1-(1/p)}}(1-u)du\right)|f'(b)|^q\right]^{\frac{1}{q}}$$
$$=\frac{\|w\|_{\infty}(b^p-a^p)^{\alpha+1}}{\Gamma(\alpha+1)}C_5^{1-\frac{1}{q}}(\alpha,p)\left[C_1(\alpha,p)|f'(a)|^q+C_2(\alpha,p)|f'(b)|^q\right]^{\frac{1}{q}}.$$

This completes the proof of i.

(ii) The proof is similar to i.

Remark 4 In Theorem 11, one can see the following.

(1) If one takes $p=1$, one has İşcan (2015), Theorem 2.8.
(2) If one takes $p=1$, $\alpha=1$ and $w(x)=1$, one has Pearce and Pecaric (2000), Theorem 1.
(3) If one takes $p=-1$ and $w(x)=1$, one has İşcan and Wu (2014), Theorem 5.
(4) If one takes $p=-1$, $\alpha=1$ and $w(x)=1$, one has İşcan (2014d), 2.6. Theorem.
(5) If one takes $\alpha=1$ and $w(x)=1$, one has Noor et al. (2016), Theorem 3.2.

Corollary 2 *In Theorem 11, one can see the following.*

(1) *If one takes* $p=1$ *and* $w(x)=1$, *one has the following Hermite–Hadamard inequality for convex functions via fractional integrals*:

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a+}f(b)-J^{\alpha}_{b-}f(a)\right]\right|$$
$$\leq\frac{(b-a)}{2}C_5^{1-\frac{1}{q}}(\alpha,1)\left[C_1(\alpha,1)|f'(a)|^q+C_2(\alpha,1)|f'(b)|^q\right]^{\frac{1}{q}}.$$

(2) *If one takes* $p=1$ *and* $\alpha=1$, *one has the following Hermite–Hadamard–Fejér inequality for convex functions*:

$$\left|\frac{1}{b-a}\frac{f(a)+f(b)}{2}\int_a^b w(x)dx-\frac{1}{b-a}\int_a^b f(x)w(x)dx\right|$$
$$\leq\frac{\|w\|_{\infty}(b-a)}{2}C_5^{1-\frac{1}{q}}(1,1)$$
$$\left[C_1(1,1)|f'(a)|^q+C_2(1,1)|f'(b)|^q\right]^{\frac{1}{q}}.$$

(3) *If one takes* $p=-1$, *one has the following Hermite–Hadamard–Fejér inequality for harmonically convex functions via fractional integrals*:

$$\left|\frac{f(a)+f(b)}{2}\left[J^{\alpha}_{1/b+}(w\circ g)(1/a)+J^{\alpha}_{1/a-}(w\circ g)(1/b)\right]\right.$$
$$\left.-\left[J^{\alpha}_{1/b+}(fw\circ g)(1/a)+J^{\alpha}_{1/a-}(fw\circ g)(1/b)\right]\right|$$
$$\leq\frac{\|w\|_{\infty}ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha}$$
$$C_6^{1-\frac{1}{q}}(\alpha,-1)\left[C_3(\alpha,-1)|f'(a)|^q+C_4(\alpha,-1)|f'(b)|^q\right]^{\frac{1}{q}}.$$

(4) *If one takes* $p=-1$ *and* $\alpha=1$, *one has the following Hermite–Hadamard–Fejér inequality for harmonically convex functions*:

$$\left|\frac{f(a)+f(b)}{2}\int_a^b\frac{w(x)}{x^2}dx-\int_a^b\frac{f(x)w(x)}{x^2}dx\right|$$
$$\leq\frac{\|w\|_{\infty}(b-a)^2}{2}C_6^{1-\frac{1}{q}}(1,-1)$$
$$\left[C_3(1,-1)|f'(a)|^q+C_4(1,-1)|f'(b)|^q\right]^{\frac{1}{q}}.$$

Theorem 12 *Let* $f:I\subset(0,\infty)\to\mathbb{R}$ *be a differentiable function on* I° *such that* $f'\in L[a,b]$, *where* $a,b\in I$ *and* $a<b$. *If* $|f'|^q$, $q>1$, *is p-convex function on* $[a,b]$ *for* $p\in\mathbb{R}\backslash\{0\}$, $\alpha>0$, $\frac{1}{q}+\frac{1}{r}=1$, $w:[a,b]\to\mathbb{R}$ *is continuous and p-symmetric with respect to* $\left[\frac{a^p+b^p}{2}\right]^{1/p}$, *then the following inequality for fractional integrals holds*:

(i) *If* $p > 0$,

$$\left|\frac{f(a)+f(b)}{2}\left[J^{\alpha}_{a^p+}(w\circ g)(b^p)+J^{\alpha}_{b^p-}(w\circ g)(a^p)\right] - \left[J^{\alpha}_{a^p+}(fw\circ g)(b^p)+J^{\alpha}_{b^p-}(fw\circ g)(a^p)\right]\right| \le \frac{\|w\|_\infty (b^p-a^p)^{\alpha+1}}{\Gamma(\alpha+1)} C_7^{\frac{1}{r}}(\alpha,p,r)\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}},$$

where

$$C_7(\alpha,p,r)=\int_0^1\left(\frac{|(1-u)^\alpha-u^\alpha|}{p[ua^p+(1-u)b^p]^{1-(1/p)}}\right)^r \mathrm{d}u$$

with $g(x)=x^{1/p}$, $x\in[a^p,b^p]$.

(ii) *If* $p<0$,

$$\left|\frac{f(a)+f(b)}{2}\left[J^{\alpha}_{b^p+}(w\circ g)(a^p)+J^{\alpha}_{a^p-}(w\circ g)(b^p)\right] - \left[J^{\alpha}_{b^p+}(fw\circ g)(a^p)+J^{\alpha}_{a^p-}(fw\circ g)(b^p)\right]\right| \le \frac{\|w\|_\infty (a^p-b^p)^{\alpha+1}}{\Gamma(\alpha+1)} C_8^{\frac{1}{r}}(\alpha,p,r)\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}},$$

where

$$C_8(\alpha,p,r)=\int_0^1\left(\frac{-|(1-u)^\alpha-u^\alpha|}{p[ua^p+(1-u)b^p]^{1-(1/p)}}\right)^r \mathrm{d}u$$

with $g(x)=x^{1/p}$, $x\in[b^p,a^p]$.

Proof

(i) Let $p > 0$. Using (23), Hölder's inequality and the p-convexity of $|f'|^q$, it follows that

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}\left[J^{\alpha}_{a^p+}(w\circ g)(b^p)+J^{\alpha}_{b^p-}(w\circ g)(a^p)\right] - \left[J^{\alpha}_{a^p+}(fw\circ g)(b^p)+J^{\alpha}_{b^p-}(fw\circ g)(a^p)\right]\right|\\
&\le \frac{\|w\|_\infty (b^p-a^p)^{\alpha+1}}{\Gamma(\alpha+1)}\int_0^1 \frac{|(1-u)^\alpha-u^\alpha|}{p[ua^p+(1-u)b^p]^{1-(1/p)}}\times\left|f'\left([ua^p+(1-u)b^p]^{1/p}\right)\right|\mathrm{d}u \le \frac{\|w\|_\infty (b^p-a^p)^{\alpha+1}}{\Gamma(\alpha+1)}\\
&\times\left(\int_0^1\left(\frac{|(1-u)^\alpha-u^\alpha|}{p[ua^p+(1-u)b^p]^{1-(1/p)}}\right)^r\mathrm{d}u\right)^{\frac{1}{r}}\times\left(\int_0^1\left|f'\left([ua^p+(1-u)b^p]^{1/p}\right)\right|^q\mathrm{d}u\right)^{\frac{1}{q}}\\
&\le \frac{\|w\|_\infty (b^p-a^p)^{\alpha+1}}{\Gamma(\alpha+1)}\left(\int_0^1\left(\frac{|(1-u)^\alpha-u^\alpha|}{p[ua^p+(1-u)b^p]^{1-(1/p)}}\right)^r\mathrm{d}u\right)^{\frac{1}{r}}\times\left(\int_0^1 u|f'(a)|^q+(1-u)|f'(b)|^q\mathrm{d}u\right)^{\frac{1}{q}}\\
&= \frac{\|w\|_\infty (b^p-a^p)^{\alpha+1}}{\Gamma(\alpha+1)}\left(\int_0^1\left(\frac{|(1-u)^\alpha-u^\alpha|}{p[ua^p+(1-u)b^p]^{1-(1/p)}}\right)^r\mathrm{d}u\right)^{\frac{1}{r}}\times\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}\\
&= \frac{\|w\|_\infty (b^p-a^p)^{\alpha+1}}{\Gamma(\alpha+1)}C_7^{\frac{1}{r}}(\alpha,p,r)\times\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}.\end{aligned}$$

This completes the proof of i.

(ii) The proof is similar to i.

Remark 5 In Theorem 12, one can see the following.

(1) If one takes $p=1$, one has İşcan (2015), Theorem 2.9-i.

(2) If one takes $p=1$, $\alpha=1$ and $w(x)=1$, one has Dragomir and Agarwal (1998), Theorem 2.3.

(3) If one takes $p=1$ and $\alpha=1$, one has Sarıkaya (2012), Theorem 2.8.

Corollary 3 *In Theorem 12, one can see the following.*

(1) *If one takes* $p=1$ *and* $w(x)=1$, *one has the following Hermite–Hadamard inequality for convex functions via fractional integrals*:

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a+}f(b)-J^\alpha_{b-}f(a)\right]\right| \le \frac{b-a}{2}C_7^{\frac{1}{r}}(\alpha,1,r)\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}.$$

(2) *If one takes* $p=-1$ *and* $w(x)=1$, *one has the following Hermite–Hadamard inequality for harmonically convex functions via fractional integrals*:

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2\left(\frac{1}{b}-\frac{1}{a}\right)^{\alpha}}\left[J^{\alpha}_{1/b+}(f\circ g)(1/a)+J^{\alpha}_{1/a-}(f\circ g)(1/b)\right]\right| \le \left(\frac{b-a}{ab}\right)C_8^{\frac{1}{r}}(\alpha,-1,r)\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}.$$

(3) *If one takes $p=-1$, $\alpha=1$ and $w(x)=1$, one has the following Hermite–Hadamard inequality for harmonically convex functions*:

$$\left|\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\mathrm{d}x\right| \le \left(\frac{b-a}{ab}\right)C_8^{\frac{1}{r}}(1,-1,r)\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}.$$

(4) *If one takes $p=-1$, one has the following Hermite–Hadamard–Fejér inequality for harmonically convex functions via fractional integrals*:

$$\left|\frac{f(a)+f(b)}{2}\left[J^{\alpha}_{1/b+}(w\circ g)(1/a)+J^{\alpha}_{1/a-}(w\circ g)(1/b)\right] - \left[J^{\alpha}_{1/b+}(fw\circ g)(1/a)+J^{\alpha}_{1/a-}(fw\circ g)(1/b)\right]\right| \le \frac{\|w\|_\infty ab(b-a)}{\Gamma(\alpha+1)}\left(\frac{b-a}{ab}\right)^{\alpha}C_8^{\frac{1}{r}}(\alpha,-1,r)\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}.$$

(5) *If one takes $p=-1$ and $\alpha=1$, one has the following Hermite–Hadamard–Fejér inequality for harmonically convex functions*:

$$\left|\frac{f(a)+f(b)}{2}\int_a^b\frac{w(x)}{x^2}\mathrm{d}x-\int_a^b\frac{f(x)w(x)}{x^2}\mathrm{d}x\right| \le \frac{\|w\|_\infty(b-a)^2}{2}C_8^{\frac{1}{r}}(1,-1,r)\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}.$$

(6) *If one takes $\alpha=1$ and $w(x)=1$, one has the following Hermite–Hadamard inequality for p-convex functions*:

$$\left|\frac{f(a)+f(b)}{2}-\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}\mathrm{d}x\right| \le \left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}} \times \begin{cases}\frac{(b^p-a^p)}{2}C_7^{\frac{1}{r}}(1,p,r), & p>0\\ \frac{(a^p-b^p)}{2}C_8^{\frac{1}{r}}(1,p,r), & p<0\end{cases}.$$

3 Conclusion

In Theorem 9, Hermite–Hadamard–Fejér type inequalities for p-convex functions in fractional integral forms are built. In Lemma 2, an integral identity, and in Theorems 10, 11 and 12, some Hermite–Hadamard–Fejér type integral inequalities for p-convex functions in fractional integral forms are obtained. In Corollaries 1, 2 and 3, some Hermite–Hadamard and Hermite–Hadamard–Fejér inequalities for convex, harmonically convex and p-convex functions are given. Some results presented in Remarks 3, 4 and 5 provide extensions of others given in earlier works for convex, harmonically convex and p-convex functions.

Compliance with ethical standards

Conflict of interest The authors declare that they have no competing interests.

References

Bombardelli M, Varošanec S (1869) Properties of h-convex functions related to the Hermite Hadamard Fejér inequalities. Comput Math Appl 58(2009):1877

Chen F, Wu S (2014) Fejér and Hermite–Hadamard type inqequalities for harmonically convex functions. J Appl Math 2014. Article id:386806

Dahmani Z (2010) On Minkowski and Hermite–Hadamard integral inequalities via fractional integration. Ann Funct Anal 1(1):51–58

Dragomir SS, Agarwal RP (1998) Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula. Appl Math Lett 11(5):91–95

Fejér L (1906) Uber die Fourierreihen, II. Math Naturwise Anz Ungar Akad Wiss 24:369–390 (in Hungarian)

Fang ZB, Shi R (2014) On the (p,h)-convex function and some integral inequalities. J Inequal Appl 2014:45

Hadamard J (1893) Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann. J Math Pures Appl 58:171–215

Hermite Ch (1883) Sur deux limites d'une intégrale définie. Mathesis 3:82–83

İşcan İ (2013) New estimates on generalization of some integral inequalities for s-convex functions and their applications. Int J Pure Appl Math 86(4):727–746

İşcan İ (2014a) Generalization of different type integral inequalities for s-convex functions via fractional integrals. Appl Anal 93(9):1846–1862

İşcan İ (2014b) On generalization of different type integral inequalities for s-convex functions via fractional integrals. Math Sci Appl E Notes 2(1):55–67

İşcan İ (2014c) Some new general integral inequalities for h-convex and h-concave functions. Adv Pure Appl Math 5(1):21–29

İşcan İ (2014d) Hermite–Hadamard type inequalities for harmonically convex functions. Hacet J Math Stat 43(6):935–942

İşcan İ, Wu S (2014) Hermite–Hadamard type inequalities for harmonically convex functions via fractional integrals. Appl Math Comput 238:237–244

İşcan İ (2015) Hermite–Hadamard-Fejér type inequalities for convex functions via fractional integrals. Studia Universitatis Babeş Bolyai Mathematica 60(3):355–366

İşcan İ (2016a) Ostrowski type inequalities for p-convex functions. N Trends Math Sci 4(3):140–150

İşcan İ (2016b) Hermite–Hadamard type inequalities for p-convex functions. Int J Anal Appl 11(2):137–145

İşcan İ (2016c) Hermite–Hadamard and Simpson-like type inequalities for differentiable p-quasi-convex functions. https://doi.org/

10.13140/RG.2.1.2589.4801. https://www.researchgate.net/publication/299610889

İşcan İ, Kunt M, Yazıcı N (2016d) Hermite–Hadamard–Fejér type inequalities for harmonically convex functions via fractional integrals. N Trends Math Sci 4(3):239–253

Kilbas AA, Srivastava HM, Trujillo JJ (2006) Theory and applications of fractional differential equations. Elsevier, Amsterdam

Mihai MV, Noor MA, Noor KI, Awan MU (2015) New estimates for trapezoidal like inequalities via differentiable (p, h)-convex functions. https://doi.org/10.13140/RG.2.1.5106.5046. https://www.researchgate.net/publication/282912293

Noor MA, Noor KI, Mihai MV, Awan MU (2016) Hermite-Hadamard inequalities for differentiable p-convex functions using hypergeometric functions. Publications de L'institut Mathematique, Nouvelle série, tome 100(114):251–257

Pearce CEM, Pecaric J (2000) Inequalities for differentiable mappings with application to special means and quadrature formulae. Appl Math Lett 13(2):51–55

Sarıkaya MZ (2012) On new Hermite Hadamard Fejér type integral inequalities. Stud Univ Babeş Bolyai Math 57(3):377–386

Sarıkaya MZ, Set E, Yaldız H, Başak N (2013) Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities. Math Comput Model 57(9):2403–2407

Tseng K-L, Yang G-S, Hsu K-C (2011) Some inequalities for differentiable mappings and applications to Fejér inequality and weighted trapezoidal formula. Taiwan J Math 15(4):1737–1747

Wang J, Li X, Fečkan M, Zhou Y (2012) Hermite–Hadamard-type inequalities for Riemann–Liouville fractional integrals via two kinds of convexity. Appl Anal 92(11):2241–2253

Wang J, Zhu C, Zhou Y (2013) New generalized Hermite–Hadamard type inequalities and applications to special means. J Inequal Appl 2013:325

Zhang KS, Wan JP (2007) p-Convex functions and their properties. Pure Appl Math 23(1):130–133