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RESEARCH PAPER

Identities of Symmetry for Degenerate Euler Polynomials and Alternating Generalized Falling Factorial Sums

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Abstract Eight basic identities of symmetry in three variables, which are related to degenerate Euler polynomials and alternating generalized falling factorial sums, are derived. These are the degenerate versions of the symmetric identities in three variables obtained in a previous paper. The derivations of identities are based on the p-adic integral expression of the generating function for the degenerate Euler polynomials and the quotient of integrals that can be expressed as the exponential generating function for the alternating generalized falling factorial sums. Those eight basic identities and most of their corollaries are new, since there have been results only about identities of symmetry in two variables.

Keywords Degenerate Euler polynomial · Alternating generalized falling factorial sum · Fermionic p-adic integral · Identities of symmetry

Mathematics Subject Classification 11B83 · 11S80 · 05A19

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1 Introduction and Preliminaries

Let p be a fixed odd prime. Throughout this paper, $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}_p$ will denote respectively the ring of p-adic integers, the field of p-adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$. For a continuous function $f : \mathbb{Z}_p \to \mathbb{C}_p$, the p-adic fermionic integral of f is defined by Kim as

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(z)\mathrm{d}\mu_{-1}(z) = \lim_{N\to\infty} \sum_{j=0}^{p^N-1} f(j)(-1)^j.$$

Then, it is easy to see that

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0), \tag{1}$$

where $f_1(z) = f(z+1)$.

Let $|\cdot|_p$ be the normalized absolute value of $\mathbb{C}_p$ with $|p|_p = \frac{1}{p}$. Throughout this paper, we assume that $\lambda, t \in \mathbb{C}_p$ satisfy

$$0 < |\lambda|_p \le 1, \quad |t|_p \le p^{-\frac{1}{p-1}}.$$

Then, as $|\lambda t|_p < p^{-\frac{1}{p-1}}$, $|\log(1+\lambda t)|_p = |\lambda t|_p$ and hence $\left|\frac{1}{\lambda}\log(1+\lambda t)\right|_p = |t|_p < p^{-\frac{1}{p-1}}$. Thus, $f(t) = (1+\lambda t)^{\frac{z}{\lambda}} = e^{\frac{z}{\lambda}\log(1+\lambda t)}$ is a well-defined analytic function on $\mathbb{Z}_p$. Applying (1) to this f, we obtain the p-adic integral representation of the generating function for the degenerate Euler numbers $\mathcal{E}_n(\lambda)$:

$$\int_{\mathbb{Z}_p} (1+\lambda t)^{\frac{z}{\lambda}}\mathrm{d}\mu_{-1}(z) = \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1} = \sum_{n=0}^{\infty} \mathcal{E}_n(\lambda)\frac{t^n}{n!}. \tag{2}$$

Thus, we have the p-adic integral representation of the generating function for the degenerate Euler polynomials $\mathcal{E}_n(\lambda, x)$:

$$\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{x+z}{\lambda}}d\mu_{-1}(z) = \frac{2}{(1+\lambda t)^{\frac{1}{\lambda}}+1}(1+\lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty}\mathcal{E}_n(\lambda,x)\frac{t^n}{n!}. \tag{3}$$

Note here that $\mathcal{E}_n(\lambda)=\mathcal{E}_n(\lambda,0)$,

$$\lim_{\lambda\to 0}\mathcal{E}_n(\lambda,x)=E_n(x), \tag{4}$$

where $E_n(x)$ is the ordinary Euler polynomial.

The generalized falling factorial $(x\mid\lambda)_n$ is defined as $(x\mid\lambda)_n=x(x-\lambda)\cdots(x-(n-1)\lambda)$, for $n>0$ and $(x\mid\lambda)_0=1$. Let $\tau_k(\lambda,n)$ be the alternating generalized falling factorial sum given by

$$\tau_k(\lambda,n)=\sum_{i=0}^{n}(-1)^i(i\mid\lambda)_k,\quad (n\geq 0). \tag{5}$$

As special cases, we have

$$\tau_0(\lambda,n)=\begin{cases}1, & \text{if } n\equiv 0(\text{mod } 2),\\ 0, & \text{if } n\equiv 1(\text{mod } 2),\end{cases} \qquad \tau_k(\lambda,n)=\begin{cases}1, & \text{for } k=0,\\ 0, & \text{for } k>0.\end{cases} \tag{6}$$

In addition

$$\lim_{\lambda\to 0}\tau_k(\lambda,n)=T_k(n), \tag{7}$$

where $T_k(n)$ denotes the alternating kth power sum of the first $n+1$ nonnegative integers, namely

$$T_k(n)=\sum_{i=0}^{n}(-1)^i i^k=(-1)^0 0^k + (-1)^1\cdot 1^k+\cdots+(-1)^n n^k. \tag{8}$$

In addition, from (1) and (2), we get: for any odd positive integer w

$$\frac{\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{x}{\lambda}}d\mu_{-1}(x)}{\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{wy}{\lambda}}d\mu_{-1}(y)}=\sum_{k=0}^{\infty}\tau_k(\lambda,w-1)\frac{t^k}{k!} \tag{9}$$

$$=\sum_{i=0}^{w-1}(-1)^i(1+\lambda t)^{\frac{i}{\lambda}}. \tag{10}$$

Many authors have done much work on identities of symmetry in two variables involving Bernoulli polynomials or Euler polynomials or q-Bernoulli polynomials or q-Euler polynomials. The reader may refer to the papers (Bayad et al. 2011; Deeba and Rodriguez 1991; Howard 1995; Kim 2008, 2009b; Ozden et al. 2008a, b; Ozden and Simsek 2008; Simsek 2010; Tuenter 2001; Yang 2008). Especially, Kim (2008) is the first paper, where a p-adic approach is introduced. In connection with Bernoulli polynomials and power sums, and also with Euler polynomials and alternating power sums, these results were generalized in Kim (2011) and Kim (2009a) to obtain identities of symmetry involving three variables in contrast to the previous works involving just two variables. These have been done also for the q-Bernoulli and q-Euler polynomials (cf. Kim and Kim 2014; Kim et al. 2015) and for the higher order Bernoulli and higher order Euler polynomials (cf. Kim et al. 2013a, b; Kim 2011, 2012). It turns out that this extension gives not only new identities for three variables, but also those for two variables by specializing one of the variables as 1.

In this paper, we will produce eight basic identities of symmetry in three variables w_1,w_2,w_3 related to degenerate Euler polynomials and alternating generalized falling factorial sums (cf. (48), (49), (52), (55), (59), (61), (63), (64)). These and most of their corollaries seem to be new, since there have been results only about identities of symmetry in two variables in the literature (cf. Kim and Kim 2015, 2016, 2017). These abundance of symmetries shed new light even on the existing identities in two variables. For instance, it has been known that (12) and (13) are equal and (14) and (15) are so (cf. [Kim and Kim 2015, Theorems 1, 2]). In fact, (12)–(15) are all equal, as they can be derived from one and the same p-adic integral. This was neglected to mention in Kim and Kim (2015). All of these were obtained as corollaries (cf. Cor. 4.9, 4.12, 4.15) to some of the basic identities by specializing the variable w_3 as 1. Those would not be unearthed if more symmetries had not been available. The degenerate Euler polynomials were introduced in Carlitz (1979) and Wu and Pan (2014). In view of (4) and (7), all the symmetric identities in this paper approach to the corresponding ones in Kim (2009a), as λ tends to zero, and hence, our symmetric identities are the degenerate versions of the identities in Kim (2009a). Similar results about identities of symmetry in three variables for degenerate Bernoulli polynomials and generalized falling factorial sums were obtained in Kim et al. (2016).

Let w_1,w_2 be any odd positive integers. Then, we have

$$\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_k\left(\frac{\lambda}{w_2},w_1y_1\right)\tau_{n-k}\left(\frac{\lambda}{w_1},w_2-1\right)w_1^{n-k}w_2^k \tag{11}$$

$$=\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_k\left(\frac{\lambda}{w_1},w_2y_1\right)\tau_{n-k}\left(\frac{\lambda}{w_2},w_1-1\right)w_2^{n-k}w_1^k \tag{12}$$

$$=w_1^n\sum_{i=0}^{w_1-1}(-1)^i\mathcal{E}_n\left(\frac{\lambda}{w_1},w_2y_1+\frac{w_2}{w_1}i\right) \tag{13}$$

$$=w_2^n\sum_{i=0}^{w_2-1}(-1)^i\mathcal{E}_n\left(\frac{\lambda}{w_2},w_1y_1+\frac{w_1}{w_2}i\right) \tag{14}$$

$$=\sum_{k+l+m=n}\binom{n}{k,l,m}\mathcal{E}_k\left(\frac{\lambda}{w_1w_2},y_1\right)\tau_l\left(\frac{\lambda}{w_2},w_1-1\right)\tau_m\left(\frac{\lambda}{w_1},w_2-1\right)w_1^{k+m}w_2^{k+l} \tag{15}$$

$$=w_1^n\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{w_1-1}(-1)^i\mathcal{E}_k\left(\frac{\lambda}{w_1w_2},y_1+\frac{i}{w_1}\right)\tau_{n-k}\left(\frac{\lambda}{w_1},w_2-1\right)w_2^k \tag{16}$$

$$=w_2^n\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{w_2-1}(-1)^i\mathcal{E}_k\left(\frac{\lambda}{w_1w_2},y_1+\frac{i}{w_2}\right)\tau_{n-k}\left(\frac{\lambda}{w_2},w_1-1\right)w_1^k \tag{17}$$

$$=(w_1w_2)^n\sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1}(-1)^{i+j}\mathcal{E}_n\left(\frac{\lambda}{w_1w_2},y_1+\frac{i}{w_1}+\frac{j}{w_2}\right). \tag{18}$$

The derivations of identities are based on the p-adic integral expression of the generating function for the degenerate Euler polynomials in (3) and the quotient of integrals in (10) that can be expressed as the exponential generating function for the alternating generalized falling factorial sums.

2 Several Types of Quotients of Fermionic Integrals

Here, we will introduce several types of quotients of p-adic fermionic integrals on $\mathbb{Z}_p$ or $\mathbb{Z}_p^3$ from which some interesting identities follow owing to the built-in symmetries in w_1, w_2, w_3. In the following, w_1, w_2, w_3 are all positive integers and all of the explicit expressions of integrals in (21), (23), (25), and (27) are obtained from the identity in (2). In below, $d\mu_{-1}(x_1,x_2,x_3)$ denotes $d\mu_{-1}(x_1)d\mu_{-1}(x_2)d\mu_{-1}(x_3)$.

(a) Type Λ_{23}^i (for $i=0,1,2,3$)

$$I(\Lambda_{23}^i)=\frac{\int_{\mathbb{Z}_p^3}(1+\lambda t)^{\frac{1}{\lambda}(w_2w_3x_1+w_1w_3x_2+w_1w_2x_3+w_1w_2w_3(\sum_{j=1}^{3-i}y_j))}d\mu_{-1}(x_1,x_2,x_3)}{(\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_1w_2w_3x_4}d\mu_{-1}(x_4))^i} \tag{19}$$

$$=\frac{2^{3-i}(1+\lambda t)^{\frac{w_1w_2w_3}{\lambda}(\sum_{j=1}^{3-i}y_j)}((1+\lambda t)^{\frac{w_1w_2w_3}{\lambda}}+1)^i}{((1+\lambda t)^{\frac{w_2w_3}{\lambda}}+1)((1+\lambda t)^{\frac{w_1w_3}{\lambda}}+1)((1+\lambda t)^{\frac{w_1w_2}{\lambda}}+1)}; \tag{20}$$

(b) Type Λ_{13}^i (for $i=0,1,2,3$)

$$I(\Lambda_{13}^i)=\frac{\int_{\mathbb{Z}_p^3}(1+\lambda t)^{\frac{1}{\lambda}(w_1x_1+w_2x_2+w_3x_3+w_1w_2w_3(\sum_{j=1}^{3-i}y_j))}d\mu_{-1}(x_1)d\mu_{-1}(x_2)d\mu_{-1}(x_3)}{(\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_1w_2w_3x_4}d\mu_{-1}(x_4))^i} \tag{21}$$

$$=\frac{2^{3-i}(1+\lambda t)^{\frac{w_1w_2w_3}{\lambda}(\sum_{j=1}^{3-i}y_j)}((1+\lambda t)^{\frac{w_1w_2w_3}{\lambda}}+1)^i}{((1+\lambda t)^{\frac{w_1}{\lambda}}+1)((1+\lambda t)^{\frac{w_2}{\lambda}}+1)((1+\lambda t)^{\frac{w_3}{\lambda}}+1)}; \tag{22}$$

(c-0) Type Λ_{12}^0

$$I(\Lambda_{12}^0)=\int_{\mathbb{Z}_p^3}(1+\lambda t)^{\frac{1}{\lambda}(w_1x_1+w_2x_2+w_3x_3+w_2w_3y+w_1w_3y+w_1w_2y)}d\mu_{-1}(x_1,x_2,x_3) \tag{23}$$

$$=\frac{8(1+\lambda t)^{\frac{y}{\lambda}(w_2w_3+w_1w_3+w_1w_2)}}{((1+\lambda t)^{\frac{w_1}{\lambda}}+1)((1+\lambda t)^{\frac{w_2}{\lambda}}+1)((1+\lambda t)^{\frac{w_3}{\lambda}}+1)}; \tag{24}$$

(c-1) Type Λ_{12}^1

$$I(\Lambda_{12}^1)=\frac{\int_{\mathbb{Z}_p^3}(1+\lambda t)^{\frac{1}{\lambda}(w_1x_1+w_2x_2+w_3x_3)}d\mu_{-1}(x_1)d\mu_{-1}(x_2)d\mu_{-1}(x_3)}{\int_{\mathbb{Z}_p^3}(1+\lambda t)^{\frac{1}{\lambda}(w_2w_3z_1+w_1w_3z_2+w_1w_2z_3)}d\mu_{-1}(z_1)d\mu_{-1}(z_2)d\mu_{-1}(z_3)} \tag{25}$$

$$=\frac{((1+\lambda t)^{\frac{w_2w_3}{\lambda}}+1)((1+\lambda t)^{\frac{w_1w_3}{\lambda}}+1)((1+\lambda t)^{\frac{w_1w_2}{\lambda}}+1)}{((1+\lambda t)^{\frac{w_1}{\lambda}}+1)((1+\lambda t)^{\frac{w_2}{\lambda}}+1)((1+\lambda t)^{\frac{w_3}{\lambda}}+1)}. \tag{26}$$

All of the above p-adic integrals of various types are invariant under all permutations of w_1, w_2, w_3 as one can see either from p-adic integral representations in (20), (22), (24), and (26) or from their explicit evaluations in (21), (23), (25), and (27).

3 Identities for Degenerate Euler Polynomials

In the following, w_1, w_2, w_3 are all odd positive integers except for (a-0) and (c-0), where they are any positive integers.

First, let us consider Type Λ_{23}^i, for each $i=0,1,2,3$. The following results can be easily obtained from (3), (10), and (11).

(a-0)

$$I(\Lambda_{23}^0)=\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_2w_3(x_1+w_1y_1)}d\mu_{-1}(x_1) \tag{27}$$

$$\times\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_1w_3(x_2+w_2y_2)}d\mu_{-1}(x_2)$$
$$\times\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_1w_2(x_3+w_3y_3)}d\mu_{-1}(x_3)$$
$$=\left(\sum_{k=0}^{\infty}\frac{\mathcal{E}_k\left(\frac{\lambda}{w_2w_3},w_1y_1\right)}{k!}(w_2w_3t)^k\right)$$
$$\times\left(\sum_{l=0}^{\infty}\frac{\mathcal{E}_l\left(\frac{\lambda}{w_1w_3},w_2y_2\right)}{l!}(w_1w_3t)^l\right)$$
$$\times\left(\sum_{m=0}^{\infty}\frac{\mathcal{E}_m\left(\frac{\lambda}{w_1w_2},w_3y_3\right)}{m!}(w_1w_2t)^m\right)$$
$$=\sum_{n=0}^{\infty}\left(\sum_{k+l+m=n}\binom{n}{k,l,m}\mathcal{E}_k\left(\frac{\lambda}{w_2w_3},w_1y_1\right)\right.$$
$$\mathcal{E}_l\left(\frac{\lambda}{w_1w_3},w_2y_2\right)$$
$$\left.\times\,\mathcal{E}_m\left(\frac{\lambda}{w_1w_2},w_3y_3\right)w_1^{l+m}w_2^{k+m}w_3^{k+l}\right)\frac{t^n}{n!},\tag{28}$$

where the inner sum is over all nonnegative integers k, l, m, with $k+l+m=n$, and

$$\binom{n}{k,l,m}=\frac{n!}{k!l!m!}.$$

(a-1) Here, we write $I(\Lambda_{23}^1)$ in two different ways:

(1)

$$I(\Lambda_{23}^1)=\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_2w_3(x_1+w_1y_1)}d\mu_{-1}(x_1)\tag{29}$$

$$\times\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_1w_3(x_2+w_2y_2)}d\mu_{-1}(x_2)$$
$$\times\frac{\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_1w_2x_3}d\mu_{-1}(x_3)}{\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_1w_2w_3x_4}d\mu_{-1}(x_4)}$$
$$=\left(\sum_{k=0}^{\infty}\mathcal{E}_k\left(\frac{\lambda}{w_2w_3},w_1y_1\right)\frac{(w_2w_3t)^k}{k!}\right)$$
$$\times\left(\sum_{l=0}^{\infty}\mathcal{E}_l\left(\frac{\lambda}{w_1w_3},w_2y_2\right)\frac{(w_1w_3t)^l}{l!}\right)$$
$$\times\left(\sum_{m=0}^{\infty}\tau_m\left(\frac{\lambda}{w_1w_2},w_3-1\right)\frac{(w_1w_2t)^m}{m!}\right)$$
$$=\sum_{n=0}^{\infty}\left(\sum_{k+l+m=n}\binom{n}{k,l,m}\mathcal{E}_k\left(\frac{\lambda}{w_2w_3},w_1y_1\right)\right.$$
$$\mathcal{E}_l\left(\frac{\lambda}{w_1w_3},w_2y_2\right)$$
$$\left.\times\,\tau_m\left(\frac{\lambda}{w_1w_2},w_3-1\right)w_1^{l+m}w_2^{k+m}w_3^{k+l}\right)\frac{t^n}{n!}.\tag{30}$$

(2) Invoking (10), (30) can also be written as

$$I(\Lambda_{23}^1)=\sum_{i=0}^{w_3-1}(-1)^i\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_2w_3(x_1+w_1y_1)}d\mu_{-1}(x_1)$$
$$\times\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_1w_3(x_2+w_2y_2+\frac{w_2}{w_3}i)}d\mu_{-1}(x_2)$$
$$=\sum_{i=0}^{w_3-1}(-1)^i\left(\sum_{k=0}^{\infty}\mathcal{E}_k\left(\frac{\lambda}{w_2w_3},w_1y_1\right)\frac{(w_2w_3t)^k}{k!}\right)$$
$$\times\left(\sum_{l=0}^{\infty}\mathcal{E}_l\left(\frac{\lambda}{w_1w_3},w_2y_2+\frac{w_2}{w_3}i\right)\frac{(w_1w_3t)^l}{l!}\right)$$
$$=\sum_{n=0}^{\infty}\left(w_3^n\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_k\left(\frac{\lambda}{w_2w_3},w_1y_1\right)\right.$$
$$\left.\sum_{i=0}^{w_3-1}(-1)^i\mathcal{E}_{n-k}\left(\frac{\lambda}{w_1w_3},w_2y_2+\frac{w_2}{w_3}i\right)w_1^{n-k}w_2^k\right)\frac{t^n}{n!}.\tag{31}$$

(a-2) Here, we write $I(\Lambda_{23}^2)$ in three different ways:

(1)

$$I(\Lambda_{23}^2)=\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_2w_3(x_1+w_1y_1)}d\mu_{-1}(x_1)\tag{32}$$

$$\times\frac{\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_1w_3x_2}d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_1w_2w_3x_4}d\mu_{-1}(x_4)}$$
$$\times\frac{\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_1w_2x_3}d\mu_{-1}(x_3)}{\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_1w_2w_3x_4}d\mu_{-1}(x_4)}$$
$$=\left(\sum_{k=0}^{\infty}\mathcal{E}_k\left(\frac{\lambda}{w_2w_3},w_1y_1\right)\frac{(w_2w_3t)^k}{k!}\right)$$
$$\times\left(\sum_{l=0}^{\infty}\tau_l\left(\frac{\lambda}{w_1w_3},w_2-1\right)\frac{(w_1w_3t)^l}{l!}\right)$$
$$\times\left(\sum_{m=0}^{\infty}\tau_m\left(\frac{\lambda}{w_1w_2},w_3-1\right)\frac{(w_1w_2t)^m}{m!}\right)$$
$$=\sum_{n=0}^{\infty}\left(\sum_{k+l+m=n}\binom{n}{k,l,m}\mathcal{E}_k\left(\frac{\lambda}{w_2w_3},w_1y_1\right)\tau_l\right.$$
$$\times\left(\frac{\lambda}{w_1w_3},w_2-1\right)$$
$$\left.\times\,\tau_m\left(\frac{\lambda}{w_1w_2},w_3-1\right)w_1^{l+m}w_2^{k+m}w_3^{k+l}\right)\frac{t^n}{n!}.\tag{33}$$

(2) Invoking (10), (33) can also be written as

$$I(\Lambda_{23}^2)=\sum_{i=0}^{w_2-1}(-1)^i\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_2w_3(x_1+w_1y_1+\frac{w_1}{w_2}i)}d\mu_{-1}(x_1)\tag{34}$$

$$\times \frac{\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_1w_2x_3}\mathrm{d}\mu_{-1}(x_3)}{\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_1w_2w_3x_4}\mathrm{d}\mu_{-1}(x_4)}$$
$$= \sum_{i=0}^{w_2-1}(-1)^i\left(\sum_{k=0}^{\infty}\mathcal{E}_k\left(\frac{\lambda}{w_2w_3}, w_1y_1+\frac{w_1}{w_2}i\right)\frac{(w_2w_3t)^k}{k!}\right)$$
$$\times\left(\sum_{l=0}^{\infty}\tau_l\left(\frac{\lambda}{w_1w_2}, w_3-1\right)\frac{(w_1w_2t)^l}{l!}\right)$$
$$= \sum_{n=0}^{\infty}\left(w_2^n\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{w_2-1}(-1)^i\mathcal{E}_k\left(\frac{\lambda}{w_2w_3}, w_1y_1+\frac{w_1}{w_2}i\right)\right.$$
$$\left.\times\tau_{n-k}\left(\frac{\lambda}{w_1w_2}, w_3-1\right)w_1^{n-k}w_3^k\right)\frac{t^n}{n!}. \tag{35}$$

(3) Invoking (10) once again, (35) can be written as

$$I(\Lambda_{23}^2)$$
$$= \sum_{i=0}^{w_2-1}\sum_{j=0}^{w_3-1}(-1)^{i+j}$$
$$\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_2w_3(x_1+w_1y_1+\frac{w_1}{w_2}i+\frac{w_1}{w_3}j)}\mathrm{d}\mu_{-1}(x_1)$$
$$= \sum_{i=0}^{w_2-1}\sum_{j=0}^{w_3-1}(-1)^{i+j}$$
$$\sum_{n=0}^{\infty}\mathcal{E}_n\left(\frac{\lambda}{w_2w_3}, w_1y_1+\frac{w_1}{w_2}i+\frac{w_1}{w_3}j\right)\frac{(w_2w_3t)^n}{n!}$$
$$= \sum_{n=0}^{\infty}\left((w_2w_3)^n\sum_{i=0}^{w_2-1}\sum_{j=0}^{w_3-1}(-1)^{i+j}\right.$$
$$\left.\mathcal{E}_n\left(\frac{\lambda}{w_2w_3}, w_1y_1+\frac{w_1}{w_2}i+\frac{w_1}{w_3}j\right)\right)\frac{t^n}{n!}. \tag{36}$$

(a-3)

$$I(\Lambda_{23}^3)=\frac{\int_{\mathbb{Z}_p}(1+\lambda t)^{w_2w_3x_1t}\mathrm{d}\mu_{-1}(x_1)}{\int_{\mathbb{Z}_p}(1+\lambda t)^{w_1w_2w_3x_4t}\mathrm{d}\mu_{-1}(x_4)}\frac{\int_{\mathbb{Z}_p}(1+\lambda t)^{w_1w_3x_2t}\mathrm{d}\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p}(1+\lambda t)^{w_1w_2w_3x_4t}\mathrm{d}\mu_{-1}(x_4)}$$
$$\times\frac{\int_{\mathbb{Z}_p}(1+\lambda t)^{w_1w_2x_3t}\mathrm{d}\mu_{-1}(x_3)}{\int_{\mathbb{Z}_p}(1+\lambda t)^{w_1w_2w_3x_4t}\mathrm{d}\mu_{-1}(x_4)}$$
$$=\left(\sum_{k=0}^{\infty}\tau_k\left(\frac{\lambda}{w_2w_3}, w_1-1\right)\frac{(w_2w_3t)^k}{k!}\right)$$
$$\left(\sum_{l=0}^{\infty}\tau_l\left(\frac{\lambda}{w_1w_3}, w_2-1\right)\frac{(w_1w_3t)^l}{l!}\right)$$
$$\times\left(\sum_{m=0}^{\infty}\tau_m\left(\frac{\lambda}{w_1w_2}, w_3-1\right)\frac{(w_1w_2t)^m}{m!}\right)$$
$$=\sum_{n=0}^{\infty}\sum_{k+l+m=n}\left(\binom{n}{k,l,m}\tau_k\left(\frac{\lambda}{w_2w_3}, w_1-1\right)\right.$$
$$\tau_l\left(\frac{\lambda}{w_1w_3}, w_2-1\right)$$
$$\left.\times\tau_m\left(\frac{\lambda}{w_1w_2}, w_3-1\right)w_1^{l+m}w_2^{k+m}w_3^{k+l}\right)\frac{t^n}{n!}. \tag{37}$$

(b) For Type $\Lambda_{13}^i (i = 0, 1, 2, 3)$, we may consider the analogous things to the ones in (a-0), (a-1), (a-2), and (a-3). However, these do not lead us to new identities. Indeed, if we substitute w_2w_3, w_1w_3, w_1w_2, respectively, for w_1, w_2, w_3 in (20), this amounts to replacing λ by $\frac{\lambda}{w_1w_2w_3}$ and t by $w_1w_2w_3t$ in (22). Therefore, upon replacing w_1, w_2, w_3, respectively ,by w_2w_3, w_1w_3, w_1w_2, and then replacing λ by $w_1w_2w_3\lambda$ and dividing by $(w_1w_2w_3)^n$, in each of the expressions of Theorem 4.1 through Corollary 4.15, we will get the corresponding symmetric identities for Type $\Lambda_{13}^i (i = 0, 1, 2, 3)$.

(c-0)

$$I(\Lambda_{12}^0) = \int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_1(x_1+w_2y)}\mathrm{d}\mu_{-1}(x_1)$$
$$\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_2(x_2+w_3y)}\mathrm{d}\mu_{-1}(x_2)$$
$$\times\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_3(x_3+w_1y)}\mathrm{d}\mu_{-1}(x_3)$$
$$=\left(\sum_{k=0}^{\infty}\frac{\mathcal{E}_k(\frac{\lambda}{w_1}, w_2y)}{k!}(w_1t)^k\right)\left(\sum_{l=0}^{\infty}\frac{\mathcal{E}_l(\frac{\lambda}{w_2}, w_3y)}{l!}(w_2t)^l\right)$$
$$\times\left(\sum_{m=0}^{\infty}\frac{\mathcal{E}_m(\frac{\lambda}{w_3}, w_1y)}{m!}(w_3t)^m\right)$$
$$=\sum_{n=0}^{\infty}\left(\sum_{k+l+m=n}\binom{n}{k,l,m}\mathcal{E}_k\left(\frac{\lambda}{w_1}, w_2y\right)\mathcal{E}_l\left(\frac{\lambda}{w_2}, w_3y\right)\right.$$
$$\left.\mathcal{E}_m\left(\frac{\lambda}{w_3}, w_1y\right)w_1^kw_2^lw_3^m\right)\frac{t^n}{n!}. \tag{38}$$

(c-1)

$$I(\Lambda_{12}^1) = \frac{\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_1x_1}\mathrm{d}\mu_{-1}(x_1)}{\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_1w_2z_3}\mathrm{d}\mu_{-1}(z_3)}$$
$$\times\frac{\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_2x_2}\mathrm{d}\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_2w_3z_1}\mathrm{d}\mu_{-1}(z_1)}$$
$$\times\frac{\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_3x_3}\mathrm{d}\mu_{-1}(x_3)}{\int_{\mathbb{Z}_p}(1+\lambda t)^{\frac{1}{\lambda}w_3w_1z_2}\mathrm{d}\mu_{-1}(z_2)}$$
$$=\left(\sum_{k=0}^{\infty}\tau_k\left(\frac{\lambda}{w_1}, w_2-1\right)\frac{(w_1t)^k}{k!}\right)$$
$$\times\left(\sum_{l=0}^{\infty}\tau_l\left(\frac{\lambda}{w_2}, w_3-1\right)\frac{(w_2t)^l}{l!}\right)$$
$$\times\left(\sum_{m=0}^{\infty}\tau_m\left(\frac{\lambda}{w_3}, w_1-1\right)\frac{(w_3t)^m}{m!}\right)$$
$$=\sum_{n=0}^{\infty}\left(\sum_{k+l+m=n}\binom{n}{k,l,m}\tau_k\left(\frac{\lambda}{w_1}, w_2-1\right)\right.$$
$$\tau_l\left(\frac{\lambda}{w_2}, w_3-1\right)$$
$$\left.\times\tau_m\left(\frac{\lambda}{w_3}, w_1-1\right)w_1^kw_2^lw_3^m\right)\frac{t^n}{n!}. \tag{39}$$

4 Main Theorems

As we noted earlier in the last paragraph of Sect. 2, the various types of quotients of p-adic fermionic integrals are invariant under any permutation of w_1, w_2, w_3. Therefore, the corresponding expressions in Sect. 3 are also invariant under any permutation of w_1, w_2, w_3. Thus, our results about identities of symmetry will be immediate consequences of this observation.

However, not all permutations of an expression in the above yield distinct ones. In fact, as these expressions are obtained by permuting w_1, w_2, w_3 in a single equation labeled by them, there is a natural transitive action of the group S_3 on those set of expressions, and hence, they are in bijective correspondence with a quotient of S_3. In particular, the number of possible distinct expressions are 1, 2, 3, or 6. Indeed, (a-0), (a-1(1)), (a-1(2)), and (a-2(2)) give the full six identities of symmetry, (a-2(1)) and (a-2(3)) yield three identities of symmetry, and (c-0) and (c-1) give two identities of symmetry, while the expression in (a-3) yields no identities of symmetry.

Here, we will just consider the cases of Theorems 4.8 and 4.17, leaving the others as easy exercises for the reader. As for the case of Theorem 4.8, in addition to (54)–(56), we get the following three ones:

$$\sum_{k+l+m=n}\binom{n}{k,l,m}\mathcal{E}_k\left(\frac{\lambda}{w_2w_3},w_1y_1\right)\tau_l\left(\frac{\lambda}{w_1w_2},w_3-1\right) \times\tau_m\left(\frac{\lambda}{w_1w_3},w_2-1\right)w_1^{l+m}w_3^{k+m}w_2^{k+l}, \tag{40}$$

$$\sum_{k+l+m=n}\binom{n}{k,l,m}\mathcal{E}_k\left(\frac{\lambda}{w_1w_3},w_2y_1\right)\tau_l\left(\frac{\lambda}{w_2w_3},w_1-1\right) \times\tau_m\left(\frac{\lambda}{w_1w_2},w_3-1\right)w_2^{l+m}w_1^{k+m}w_3^{k+l}, \tag{41}$$

$$\sum_{k+l+m=n}\binom{n}{k,l,m}\mathcal{E}_k\left(\frac{\lambda}{w_1w_2},w_3y_1\right)\tau_l\left(\frac{\lambda}{w_1w_3},w_2-1\right) \times\tau_m\left(\frac{\lambda}{w_2w_3},w_1-1\right)w_3^{l+m}w_2^{k+m}w_1^{k+l}. \tag{42}$$

However, by interchanging l and m, we see that (41), (42), and (43) are, respectively, equal to (54), (55), and (56).

As to Theorem 4.17, in addition to (64) and (65), we have

$$\sum_{k+l+m=n}\binom{n}{k,l,m}\tau_k\left(\frac{\lambda}{w_1},w_2-1\right)\tau_l\left(\frac{\lambda}{w_2},w_3-1\right) \tau_m\left(\frac{\lambda}{w_3},w_1-1\right)w_1^kw_2^lw_3^m, \tag{43}$$

$$\sum_{k+l+m=n}\binom{n}{k,l,m}\tau_k\left(\frac{\lambda}{w_2},w_3-1\right)\tau_l\left(\frac{\lambda}{w_3},w_1-1\right) \tau_m\left(\frac{\lambda}{w_1},w_2-1\right)w_2^kw_3^lw_1^m, \tag{44}$$

$$\sum_{k+l+m=n}\binom{n}{k,l,m}\tau_k\left(\frac{\lambda}{w_1},w_3-1\right)\tau_l\left(\frac{\lambda}{w_3},w_2-1\right) \tau_m\left(\frac{\lambda}{w_2},w_1-1\right)w_1^kw_3^lw_2^m, \tag{45}$$

$$\sum_{k+l+m=n}\binom{n}{k,l,m}\tau_k\left(\frac{\lambda}{w_3},w_2-1\right)\tau_l\left(\frac{\lambda}{w_2},w_1-1\right) \tau_m\left(\frac{\lambda}{w_1},w_3-1\right)w_3^kw_2^lw_1^m. \tag{46}$$

However, (44) and (45) are equal to (64), as we can see by applying the permutations $k\rightarrow l, l\rightarrow m, m\rightarrow k$ for (44) and $k\rightarrow m, l\rightarrow k, m\rightarrow l$ for (45). Similarly, we see that (46) and (47) are equal to (65), by applying permutations $k\rightarrow l, l\rightarrow m, m\rightarrow k$ for (46) and $k\rightarrow m, l\rightarrow k, m\rightarrow l$ for (47).

Theorems 4.1, 4.2, 4.5, 4.8, 4.11, 4.14, 4.16, and 4.17 follow respectively from the considerations in (a-0), (a-1(1)), (a-1(2)), (a-2(1)), (a-2(2)), (a-2(3)), (c-0), and (c-1) in Sect. 3.

Theorem 4.1 *Let w_1, w_2, w_3 be any positive integers. Then, the following expression is invariant under any permutation of w_1, w_2, w_3, so that it gives us six symmetries:*

$$\begin{aligned}
&\sum_{k+l+m=n}\binom{n}{k,l,m}\mathcal{E}_k\left(\frac{\lambda}{w_2w_3},w_1y_1\right)\mathcal{E}_l\left(\frac{\lambda}{w_1w_3},w_2y_2\right)\mathcal{E}_m\left(\frac{\lambda}{w_1w_2},w_3y_3\right)w_1^{l+m}w_2^{k+m}w_3^{k+l}\\
&=\sum_{k+l+m=n}\binom{n}{k,l,m}\mathcal{E}_k\left(\frac{\lambda}{w_2w_3},w_1y_1\right)\mathcal{E}_l\left(\frac{\lambda}{w_1w_2},w_3y_2\right)\mathcal{E}_m\left(\frac{\lambda}{w_1w_3},w_2y_3\right)w_1^{l+m}w_3^{k+m}w_2^{k+l}\\
&=\sum_{k+l+m=n}\binom{n}{k,l,m}\mathcal{E}_k\left(\frac{\lambda}{w_1w_3},w_2y_1\right)\mathcal{E}_l\left(\frac{\lambda}{w_2w_3},w_1y_2\right)\mathcal{E}_m\left(\frac{\lambda}{w_1w_2},w_3y_3\right)w_2^{l+m}w_1^{k+m}w_3^{k+l}\\
&=\sum_{k+l+m=n}\binom{n}{k,l,m}\mathcal{E}_k\left(\frac{\lambda}{w_1w_3},w_2y_1\right)\mathcal{E}_l\left(\frac{\lambda}{w_1w_2},w_3y_2\right)\mathcal{E}_m\left(\frac{\lambda}{w_2w_3},w_1y_3\right)w_2^{l+m}w_3^{k+m}w_1^{k+l}\\
&=\sum_{k+l+m=n}\binom{n}{k,l,m}\mathcal{E}_k\left(\frac{\lambda}{w_1w_2},w_3y_1\right)\mathcal{E}_l\left(\frac{\lambda}{w_2w_3},w_1y_2\right)\mathcal{E}_m\left(\frac{\lambda}{w_1w_3},w_2y_3\right)w_3^{l+m}w_1^{k+m}w_2^{k+l}\\
&=\sum_{k+l+m=n}\binom{n}{k,l,m}\mathcal{E}_k\left(\frac{\lambda}{w_1w_2},w_3y_1\right)\mathcal{E}_l\left(\frac{\lambda}{w_1w_3},w_2y_2\right)\mathcal{E}_m\left(\frac{\lambda}{w_2w_3},w_1y_3\right)w_3^{l+m}w_2^{k+m}w_1^{k+l}.
\end{aligned}\tag{47}$$

Theorem 4.2 *Let w_1, w_2, w_3 be any odd positive integers. Then, the following expression is invariant under any permutation of w_1, w_2, w_3, so that it gives us six symmetries*:

$$\begin{aligned}
&\sum_{k+l+m=n}\binom{n}{k,l,m}\mathcal{E}_k\left(\frac{\lambda}{w_2w_3},w_1y_1\right)\mathcal{E}_l\left(\frac{\lambda}{w_1w_3},w_2y_2\right)\tau_m\left(\frac{\lambda}{w_1w_2},w_3y_3\right)w_1^{l+m}w_2^{k+m}w_3^{k+l}\\
&=\sum_{k+l+m=n}\binom{n}{k,l,m}\mathcal{E}_k\left(\frac{\lambda}{w_2w_3},w_1y_1\right)\mathcal{E}_l\left(\frac{\lambda}{w_1w_2},w_3y_2\right)\tau_m\left(\frac{\lambda}{w_1w_3},w_2y_3\right)w_1^{l+m}w_3^{k+m}w_2^{k+l}\\
&=\sum_{k+l+m=n}\binom{n}{k,l,m}\mathcal{E}_k\left(\frac{\lambda}{w_1w_3},w_2y_1\right)\mathcal{E}_l\left(\frac{\lambda}{w_2w_3},w_1y_2\right)\tau_m\left(\frac{\lambda}{w_1w_2},w_3y_3\right)w_2^{l+m}w_1^{k+m}w_3^{k+l}\\
&=\sum_{k+l+m=n}\binom{n}{k,l,m}\mathcal{E}_k\left(\frac{\lambda}{w_1w_3},w_2y_1\right)\mathcal{E}_l\left(\frac{\lambda}{w_1w_2},w_3y_2\right)\tau_m\left(\frac{\lambda}{w_2w_3},w_1y_3\right)w_2^{l+m}w_3^{k+m}w_1^{k+l}\\
&=\sum_{k+l+m=n}\binom{n}{k,l,m}\mathcal{E}_k\left(\frac{\lambda}{w_1w_2},w_3y_1\right)\mathcal{E}_l\left(\frac{\lambda}{w_1w_3},w_2y_2\right)\tau_m\left(\frac{\lambda}{w_2w_3},w_1y_3\right)w_3^{l+m}w_2^{k+m}w_1^{k+l}\\
&=\sum_{k+l+m=n}\binom{n}{k,l,m}\mathcal{E}_k\left(\frac{\lambda}{w_1w_2},w_3y_1\right)\mathcal{E}_l\left(\frac{\lambda}{w_2w_3},w_1y_2\right)\tau_m\left(\frac{\lambda}{w_1w_3},w_2y_3\right)w_3^{l+m}w_1^{k+m}w_2^{k+l}.
\end{aligned}\tag{48}$$

Putting $w_3 = 1$ in (49), we get the following corollary.

Corollary 4.3 *Let w_1, w_2 be any odd positive integers*:

$$
\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_k\left(\frac{\lambda}{w_2},w_1y_1\right)\mathcal{E}_{n-k}\left(\frac{\lambda}{w_1},w_2y_2\right)w_1^{n-k}w_2^{k}\\
&\quad=\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_k\left(\frac{\lambda}{w_1},w_2y_1\right)\mathcal{E}_{n-k}\left(\frac{\lambda}{w_2},w_1y_2\right)w_2^{n-k}w_1^{k}\\
&\quad=\sum_{k+l+m=n}\binom{n}{k,l,m}\mathcal{E}_k\left(\frac{\lambda}{w_1w_2},y_1\right)\mathcal{E}_l\left(\frac{\lambda}{w_1},w_2y_2\right)\tau_m\left(\frac{\lambda}{w_2},w_1-1\right)w_2^{k+m}w_1^{k+l}\\
&\quad=\sum_{k+l+m=n}\binom{n}{k,l,m}\mathcal{E}_k\left(\frac{\lambda}{w_1},w_2y_1\right)\mathcal{E}_l\left(\frac{\lambda}{w_1w_2},y_2\right)\tau_m\left(\frac{\lambda}{w_2},w_1-1\right)w_2^{l+m}w_1^{k+l}\\
&\quad=\sum_{k+l+m=n}\binom{n}{k,l,m}\mathcal{E}_k\left(\frac{\lambda}{w_1w_2},y_1\right)\mathcal{E}_l\left(\frac{\lambda}{w_2},w_1y_2\right)\tau_m\left(\frac{\lambda}{w_1},w_2-1\right)w_1^{k+m}w_2^{k+l}\\
&\quad=\sum_{k+l+m=n}\binom{n}{k,l,m}\mathcal{E}_k\left(\frac{\lambda}{w_2},w_1y_1\right)\mathcal{E}_l\left(\frac{\lambda}{w_1w_2},y_2\right)\tau_m\left(\frac{\lambda}{w_1},w_2-1\right)w_1^{l+m}w_2^{k+l}.
\end{aligned}
\tag{49}
$$

Letting further $w_2 = 1$ in (50), we have the following corollary.

Corollary 4.4 *Let w_1 be any odd positive integer*:

$$
\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_k(\lambda,w_1y_1)\mathcal{E}_{n-k}\left(\frac{\lambda}{w_1},y_2\right)w_1^{n-k}\\
&\quad=\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_k\left(\frac{\lambda}{w_1},y_1\right)\mathcal{E}_{n-k}(\lambda,w_1y_2)w_1^{k}\\
&\quad=\sum_{k+l+m=n}\binom{n}{k,l,m}\mathcal{E}_k\left(\frac{\lambda}{w_1},y_1\right)\mathcal{E}_l\left(\frac{\lambda}{w_1},y_2\right)\\
&\quad\tau_m(\lambda,w_1-1)w_1^{k+l}.
\end{aligned}
\tag{50}
$$

Theorem 4.5 *Let w_1, w_2, w_3 be any odd positive integers. Then, the following expression is invariant under any permutation of w_1, w_2, w_3, so that it gives us six symmetries*:

$$
\begin{aligned}
&w_1^n\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_k\left(\frac{\lambda}{w_1w_2},w_3y_1\right)\sum_{i=0}^{w_1-1}(-1)^i\mathcal{E}_{n-k}\left(\frac{\lambda}{w_1w_3},w_2y_2+\frac{w_2}{w_1}i\right)w_3^{n-k}w_2^{k}\\
&\quad=w_1^n\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_k\left(\frac{\lambda}{w_1w_3},w_2y_1\right)\sum_{i=0}^{w_1-1}(-1)^i\mathcal{E}_{n-k}\left(\frac{\lambda}{w_1w_2},w_3y_2+\frac{w_3}{w_1}i\right)w_2^{n-k}w_3^{k}\\
&\quad=w_2^n\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_k\left(\frac{\lambda}{w_1w_2},w_3y_1\right)\sum_{i=0}^{w_2-1}(-1)^i\mathcal{E}_{n-k}\left(\frac{\lambda}{w_2w_3},w_1y_2+\frac{w_1}{w_2}i\right)w_3^{n-k}w_1^{k}\\
&\quad=w_2^n\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_k\left(\frac{\lambda}{w_2w_3},w_1y_1\right)\sum_{i=0}^{w_2-1}(-1)^i\mathcal{E}_{n-k}\left(\frac{\lambda}{w_1w_2},w_3y_2+\frac{w_3}{w_2}i\right)w_1^{n-k}w_3^{k}\\
&\quad=w_3^n\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_k\left(\frac{\lambda}{w_1w_3},w_2y_1\right)\sum_{i=0}^{w_3-1}(-1)^i\mathcal{E}_{n-k}\left(\frac{\lambda}{w_2w_3},w_1y_2+\frac{w_1}{w_3}i\right)w_2^{n-k}w_1^{k}\\
&\quad=w_3^n\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_k\left(\frac{\lambda}{w_2w_3},w_1y_1\right)\sum_{i=0}^{w_3-1}(-1)^i\mathcal{E}_{n-k}\left(\frac{\lambda}{w_1w_3},w_2y_2+\frac{w_2}{w_3}i\right)w_1^{n-k}w_2^{k}.
\end{aligned}
\tag{51}
$$

Letting $w_3 = 1$ in (52), we obtain alternative expressions for the identities in (50).

Corollary 4.6 *Let w_1, w_2 be any odd positive integers:*

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_k\left(\frac{\lambda}{w_2}, w_1y_1\right)\mathcal{E}_{n-k}\left(\frac{\lambda}{w_1}, w_2y_2\right)w_1^{n-k}w_2^k\\
&= \sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_k\left(\frac{\lambda}{w_1}, w_2y_1\right)\mathcal{E}_{n-k}\left(\frac{\lambda}{w_2}, w_1y_2\right)w_2^{n-k}w_1^k\\
&= w_1^n\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_k\left(\frac{\lambda}{w_1w_2}, y_1\right)\sum_{i=0}^{w_1-1}(-1)^i\mathcal{E}_{n-k}\left(\frac{\lambda}{w_1}, w_2y_2 + \frac{w_2}{w_1}i\right)w_2^k\\
&= w_1^n\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_k\left(\frac{\lambda}{w_1}, w_2y_1\right)\sum_{i=0}^{w_1-1}(-1)^i\mathcal{E}_{n-k}\left(\frac{\lambda}{w_1w_2}, y_2 + \frac{i}{w_1}\right)w_2^{n-k}\\
&= w_2^n\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_k\left(\frac{\lambda}{w_1w_2}, y_1\right)\sum_{i=0}^{w_2-1}(-1)^i\mathcal{E}_{n-k}\left(\frac{\lambda}{w_2}, w_1y_2 + \frac{w_1}{w_2}i\right)w_1^k\\
&= w_2^n\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_k\left(\frac{\lambda}{w_2}, w_1y_1\right)\sum_{i=0}^{w_2-1}(-1)^i\mathcal{E}_{n-k}\left(\frac{\lambda}{w_1w_2}, y_2 + \frac{i}{w_2}\right)w_1^{n-k}
\end{aligned} \tag{52}$$

Putting further $w_2 = 1$ in (53), we have the alternative expressions for the identities for (51).

Corollary 4.7 *Let w_1 be any odd positive integer:*

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_k(\frac{\lambda}{w_1}, y_1)\mathcal{E}_{n-k}(\lambda, w_1y_2)w_1^k\\
&= \sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_k\left(\frac{\lambda}{w_1}, y_2\right)\mathcal{E}_{n-k}(\lambda, w_1y_1)w_1^k\\
&= w_1^n\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_k\left(\frac{\lambda}{w_1}, y_1\right)\sum_{i=0}^{w_1-1}(-1)^i\mathcal{E}_{n-k}\left(\frac{\lambda}{w_1}, y_2 + \frac{i}{w_1}\right).
\end{aligned}$$

Theorem 4.8 *Let w_1, w_2, w_3 be any odd positive integers. Then, we have the following three symmetries in w_1, w_2, w_3:*

$$\sum_{k+l+m=n}\binom{n}{k,l,m}\mathcal{E}_k\left(\frac{\lambda}{w_2w_3}, w_1y_1\right)\tau_l\left(\frac{\lambda}{w_1w_3}, w_2 - 1\right) \tag{53}$$

$$\begin{aligned}
&\times\tau_m\left(\frac{\lambda}{w_1w_2}, w_3 - 1\right)w_1^{l+m}w_2^{k+m}w_3^{k+l}\\
&= \sum_{k+l+m=n}\binom{n}{k,l,m}\mathcal{E}_k\left(\frac{\lambda}{w_1w_3}, w_2y_1\right)\tau_l\left(\frac{\lambda}{w_1w_2}, w_3 - 1\right)
\end{aligned} \tag{54}$$

$$\begin{aligned}
&\times\tau_m\left(\frac{\lambda}{w_2w_3}, w_1 - 1\right)w_2^{l+m}w_3^{k+m}w_1^{k+l}\\
&= \sum_{k+l+m=n}\binom{n}{k,l,m}\mathcal{E}_k\left(\frac{\lambda}{w_1w_2}, w_3y_1\right)\tau_l\left(\frac{\lambda}{w_2w_3}, w_1 - 1\right)\\
&\times\tau_m\left(\frac{\lambda}{w_1w_3}, w_2 - 1\right)w_3^{l+m}w_1^{k+m}w_2^{k+l}.
\end{aligned} \tag{55}$$

Putting $w_3 = 1$ in (54)–(56), we get the following corollary.

Corollary 4.9 *Let w_1, w_2 be any odd positive integers:*

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_k\left(\frac{\lambda}{w_2}, w_1y_1\right)\tau_{n-k}\left(\frac{\lambda}{w_1}, w_2 - 1\right)w_1^{n-k}w_2^k\\
&= \sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_k\left(\frac{\lambda}{w_1}, w_2y_1\right)\tau_{n-k}\left(\frac{\lambda}{w_2}, w_1 - 1\right)w_2^{n-k}w_1^k\\
&= \sum_{k+l+m=n}\binom{n}{k,l,m}\mathcal{E}_k\left(\frac{\lambda}{w_1w_2}, y_1\right)\tau_l\left(\frac{\lambda}{w_2}, w_1 - 1\right)\\
&\tau_m\left(\frac{\lambda}{w_1}, w_2 - 1\right)w_1^{k+m}w_2^{k+l}.
\end{aligned} \tag{56}$$

Letting further $w_2 = 1$ in (57), we get the following corollary.

Corollary 4.10 *Let w_1 be any odd positive integer:*

$$\mathcal{E}_n(\lambda, w_1y_1) = \sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_k\left(\frac{\lambda}{w_1}, y_1\right)\tau_{n-k}(\lambda, w_1 - 1)w_1^k. \tag{57}$$

Theorem 4.11 *Let w_1, w_2, w_3 be any odd positive integers. Then, the following expression is invariant under any permutation of w_1, w_2, w_3, so that it gives us six symmetries:*

$$
\begin{aligned}
&w_1^n \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{w_1-1} (-1)^i \mathcal{E}_k\left(\frac{\lambda}{w_1 w_3}, w_2 y_1 + \frac{w_2}{w_1} i\right) \tau_{n-k}\left(\frac{\lambda}{w_1 w_2}, w_3 - 1\right) w_2^{n-k} w_3^k \\
&= w_1^n \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{w_1-1} (-1)^i \mathcal{E}_k\left(\frac{\lambda}{w_1 w_2}, w_3 y_1 + \frac{w_3}{w_1} i\right) \tau_{n-k}\left(\frac{\lambda}{w_1 w_3}, w_2 - 1\right) w_3^{n-k} w_2^k \\
&= w_2^n \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{w_2-1} (-1)^i \mathcal{E}_k\left(\frac{\lambda}{w_2 w_3}, w_1 y_1 + \frac{w_1}{w_2} i\right) \tau_{n-k}\left(\frac{\lambda}{w_1 w_2}, w_3 - 1\right) w_1^{n-k} w_3^k \\
&= w_2^n \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{w_2-1} (-1)^i \mathcal{E}_k\left(\frac{\lambda}{w_1 w_2}, w_3 y_1 + \frac{w_3}{w_2} i\right) \tau_{n-k}\left(\frac{\lambda}{w_2 w_3}, w_1 - 1\right) w_3^{n-k} w_1^k \\
&= w_3^n \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{w_3-1} (-1)^i \mathcal{E}_k\left(\frac{\lambda}{w_2 w_3}, w_1 y_1 + \frac{w_1}{w_3} i\right) \tau_{n-k}\left(\frac{\lambda}{w_1 w_3}, w_2 - 1\right) w_1^{n-k} w_2^k \\
&= w_3^n \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{w_3-1} (-1)^i \mathcal{E}_k\left(\frac{\lambda}{w_1 w_3}, w_2 y_1 + \frac{w_2}{w_3} i\right) \tau_{n-k}\left(\frac{\lambda}{w_2 w_3}, w_1 - 1\right) w_2^{n-k} w_1^k.
\end{aligned}
\tag{58}
$$

Putting $w_3 = 1$ in (59), we obtain the following corollary. In Sect. 1, the identities in (57), (60), and (62) are combined to give those in (12)–(19):

Corollary 4.12 *Let w_1, w_2 be any odd positive integers*:

$$
\begin{aligned}
&w_1^n \sum_{i=0}^{w_1-1} (-1)^i \mathcal{E}_n\left(\frac{\lambda}{w_1}, w_2 y_1 + \frac{w_2}{w_1} i\right) \\
&= w_2^n \sum_{i=0}^{w_2-1} (-1)^i \mathcal{E}_n\left(\frac{\lambda}{w_2}, w_1 y_1 + \frac{w_1}{w_2} i\right) \\
&= \sum_{k=0}^{n} \binom{n}{k} \mathcal{E}_k\left(\frac{\lambda}{w_1}, w_2 y_1\right) \tau_{n-k}\left(\frac{\lambda}{w_2}, w_1 - 1\right) w_2^{n-k} w_1^k \\
&= \sum_{k=0}^{n} \binom{n}{k} \mathcal{E}_k\left(\frac{\lambda}{w_2}, w_1 y_1\right) \tau_{n-k}\left(\frac{\lambda}{w_1}, w_2 - 1\right) w_1^{n-k} w_2^k \\
&= w_1^n \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{w_1-1} (-1)^i \mathcal{E}_k\left(\frac{\lambda}{w_1 w_2}, y_1 + \frac{i}{w_1}\right) \tau_{n-k}\left(\frac{\lambda}{w_1}, w_2 - 1\right) w_2^k \\
&= w_2^n \sum_{k=0}^{n} \binom{n}{k} \sum_{i=0}^{w_2-1} (-1)^i \mathcal{E}_k\left(\frac{\lambda}{w_1 w_2}, y_1 + \frac{i}{w_2}\right) \tau_{n-k}\left(\frac{\lambda}{w_2}, w_1 - 1\right) w_1^k.
\end{aligned}
\tag{59}
$$

Letting further $w_2 = 1$ in (60), we get the following corollary. This is the multiplication formula for Euler polynomials (cf. [Kim and Kim 2015, Page 6]) together with the new identity mentioned in (58).

Corollary 4.13 *Let w_1 be any odd positive integer*:

$$
\begin{aligned}
\mathcal{E}_n(\lambda, w_1 y_1) &= w_1^n \sum_{i=0}^{w_1-1} (-1)^i \mathcal{E}_n\left(\frac{\lambda}{w_1}, y_1 + \frac{i}{w_1}\right) \\
&= \sum_{k=0}^{n} \binom{n}{k} \mathcal{E}_k\left(\frac{\lambda}{w_1}, y_1\right) \tau_{n-k}(\lambda, w_1 - 1) w_1^k.
\end{aligned}
$$

Theorem 4.14 *Let w_1, w_2, w_3 be any odd positive integers. Then, we have the following three symmetries in w_1, w_2, w_3*:

$$
\begin{aligned}
&(w_1 w_2)^n \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} (-1)^{i+j} \mathcal{E}_n\left(\frac{\lambda}{w_1 w_2}, w_3 y_1 + \frac{w_3}{w_1} i + \frac{w_3}{w_2} j\right) \\
&= (w_2 w_3)^n \sum_{i=0}^{w_2-1} \sum_{j=0}^{w_3-1} (-1)^{i+j} \mathcal{E}_n\left(\frac{\lambda}{w_2 w_3}, w_1 y_1 + \frac{w_1}{w_2} i + \frac{w_1}{w_3} j\right) \\
&= (w_3 w_1)^n \sum_{i=0}^{w_3-1} \sum_{j=0}^{w_1-1} (-1)^{i+j} \mathcal{E}_n\left(\frac{\lambda}{w_3 w_1}, w_2 y_1 + \frac{w_2}{w_3} i + \frac{w_2}{w_1} j\right).
\end{aligned}
\tag{60}
$$

Letting $w_3 = 1$ in (61), we have the following corollary.

Corollary 4.15 *Let w_1, w_2 be any odd positive integers*:

$$
\begin{aligned}
&w_1^n \sum_{j=0}^{w_1-1} (-1)^j \mathcal{E}_n\left(\frac{\lambda}{w_1}, w_2 y_1 + \frac{w_2}{w_1} j\right) \\
&= w_2^n \sum_{i=0}^{w_2-1} (-1)^i \mathcal{E}_n\left(\frac{\lambda}{w_2}, w_1 y_1 + \frac{w_1}{w_2} i\right) \\
&= (w_1 w_2)^n \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_2-1} (-1)^{i+j} \mathcal{E}_n\left(\frac{\lambda}{w_1 w_2}, y_1 + \frac{i}{w_1} + \frac{j}{w_2}\right).
\end{aligned}
\tag{61}
$$

Theorem 4.16 *Let w_1, w_2, w_3 be any positive integers. Then, we have the following two symmetries in w_1, w_2, w_3*:

$$\sum_{k+l+m=n}\binom{n}{k,l,m}\mathcal{E}_k\left(\frac{\lambda}{w_3},w_1y\right)\mathcal{E}_l\left(\frac{\lambda}{w_1},w_2y\right)$$
$$\mathcal{E}_m\left(\frac{\lambda}{w_2},w_3y\right)w_3^kw_1^lw_2^m$$
$$=\sum_{k+l+m=n}\binom{n}{k,l,m}\mathcal{E}_k\left(\frac{\lambda}{w_2},w_1y\right)\mathcal{E}_l\left(\frac{\lambda}{w_1},w_3y\right)$$
$$\mathcal{E}_m\left(\frac{\lambda}{w_3},w_2y\right)w_2^kw_1^lw_3^m. \quad (62)$$

Theorem 4.17 *Let w_1, w_2, w_3 be any odd positive integers. Then, we have the following two symmetries in w_1, w_2, w_3:*

$$\sum_{k+l+m=n}\binom{n}{k,l,m}\tau_k\left(\frac{\lambda}{w_3},w_1-1\right)\tau_l\left(\frac{\lambda}{w_1},w_2-1\right)$$
$$\tau_m\left(\frac{\lambda}{w_2},w_3-1\right)w_3^kw_1^lw_2^m \quad (63)$$
$$=\sum_{k+l+m=n}\binom{n}{k,l,m}\tau_k\left(\frac{\lambda}{w_2},w_1-1\right)$$
$$\tau_l\left(\frac{\lambda}{w_1},w_3-1\right)\tau_m\left(\frac{\lambda}{w_3},w_2-1\right)w_2^kw_1^lw_3^m. \quad (64)$$

Putting $w_3 = 1$ in (64) and (65), we get the following corollary.

Corollary 4.18 *Let w_1, w_2 be any odd positive integers:*

$$\sum_{k=0}^{n}\binom{n}{k}\tau_k\left(\frac{\lambda}{w_1},w_2-1\right)\tau_{n-k}(\lambda,w_1-1)w_1^k$$
$$=\sum_{k=0}^{n}\binom{n}{k}\tau_k\left(\frac{\lambda}{w_2},w_1-1\right)\tau_{n-k}(\lambda,w_2-1)w_2^k.$$

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