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RESEARCH PAPER

Boundary Determination of the Inverse Heat Conduction Problem in One and Two Dimensions via the Collocation Method Based on the Satisfier Functions

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Abstract In this paper, we are concerned with the numerical solutions of the inverse heat conduction problems (IHCP) in one and two dimensions with free boundary conditions. For the one-dimensional problem, we first apply the Landau's transformation to replace the physical domain with a rectangular one. Reciprocally, some nonlinear terms appear thus an iterative scheme based on the application of the satisfier function is proposed for solving the problem. Second, we treat with the nonlinear two-dimensional problem by providing a collocation technique which takes advantage of the satisfier functions. Throughout this work, the presented schemes make the reader free of solving any nonlinear system of algebraic equations. Moreover, an admissible regularization strategy, namely, the Landweber's iterations method is used to overcome the numerical instability and achieve the acceptable approximations. Illustrative examples are included to show the efficiency of the presented algorithms.

Keywords Parabolic equation · Inverse heat conduction problem · Collocation method · Satisfier function · Landweber's iterations

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1 Introduction

Heat conduction problems are of vital importance in many areas of applied sciences (Beck et al. 1985; Cannon 1984; Kirsch 2011), e.g., heat exchangers, mathematical finance, and various chemical and biological systems (Cannon 1984; Johansson et al. 2011c, d). The multidisciplinary features of these problems have drawn the attention of many researchers and the practical importance is to obtain the applicable methods for solving them from both analytical and numerical aspects (Dehghan 2001a, b, 2005; Ebel and Davitashvili 2007; Farcas and Lesnic 2006; Fatullayev 2002, 2004; Fatullayev and Cula 2009; Johansson and Lesnic 2007, 2008; Lakestani and Dehghan 2010; Rashedi et al. 2014; Shamsi and Dehghan 2012, 2007). Although the task of developing the solutions for these problems has been well studied, but for multi-dimensional cases and especially problems with irregular domains, the literature has been less examined (Johansson et al. 2014; Rashedi et al. 2014). The organization of these problems is roughly twofold. Briefly stated,

- For the direct problem, the heat flux or temperature histories are known Beck et al. (1985) as functions of time and finding the temperature distribution is requested.
- For the inverse problem, insufficient detail is given in the description of a surface condition, particularly, the surface heat flux and temperature histories and then approximating them from transient temperature measurements at some interior locations are aimed.

As a general description of various aspects of the IHCP, it is pointed out that many experimental impediments may appear in measuring or providing the boundary conditions. For example, some data are expensive to carry out or relatively inaccurate measurements are extracted by placing the sensors in the remote locations. Thus, it makes more sense to use the data coming from interior sensors which are much more reliable and cost effective. An important conclusion that has been reported is that the solution might depend discontinuously on the data. There is increased difficulty in approximating the heat flux Beck et al. (1985).

As is said in Beck et al. (1985), Guo and Murio (1991), and Qian and Feng (2013), real applications of IHCP include retrieving the thermal constants in some freezing and quenching processes, estimation of surface heat transfer measurements taken within the skin of a re-entry space vehicle, the motion of a projectile over a gun barrel surface, determination of aerodynamic heating in wind tunnels and rocket nozzles, and infrared computerized tomography.

A number of studies dealing with analytical (Slota 2006, 2007, 2011) and numerical aspects have appeared in the literature including the Lie-group shooting method (Liu 2011), mesh-free methods based on the radial basis functions (Rashedi et al. 2014; Vrankar et al. 2006, 2007, 2010), and method of fundamental solutions (Johansson et al. 2011a, b, c, 2013).

The application of the satisfier functions has been verified in solving the linear and nonlinear partial differential equations (Dehghan et al. 2013; Lesnic et al. 2013; Rashedi et al. 2013, 2014). The main task of these functions is fulfilling all the initial and boundary conditions. Taking this fact into account, it is known that the smaller system of algebraic equations can be obtained. Nevertheless, for problems with complex domains, and on top of that, for improperly posed problems which involve noisy input data, care enough must be taken in introducing the appropriate satisfier functions. Since the satisfier functions take part directly in the numerical computations. This paper presents a framework which helps to overcome these obstacles.

The layout of this paper is as follows.

We give the presentation of the inverse problems in Sect. 2. In Sect. 3, we describe the numerical techniques to solve the problems. Section 4 gives the numerical examples to observe the performance of the suggested procedures. At last, we give a concluding remark in Sect. 5.

2 Problem statements

Case 1: The one-dimensional inverse heat conduction problem

Inverse problem 1 (IP1): In a 1D phase change problem Liu and Guerrier (1997), let us consider the moving boundary domain (Fig. 1a) corresponding to the solid phase $\Omega_1 = \{(y,t) \in \mathbb{R}^2 | 0 < y < \xi(t), 0 < t < t_f\}$ with the solid/liquid interface position $\xi(t)$ and the heat conduction equation:

$$\frac{\partial A(y,t)}{\partial t} = \frac{\partial^2 A(y,t)}{\partial y^2}, \qquad \text{in} \qquad \Omega_1, \tag{1}$$

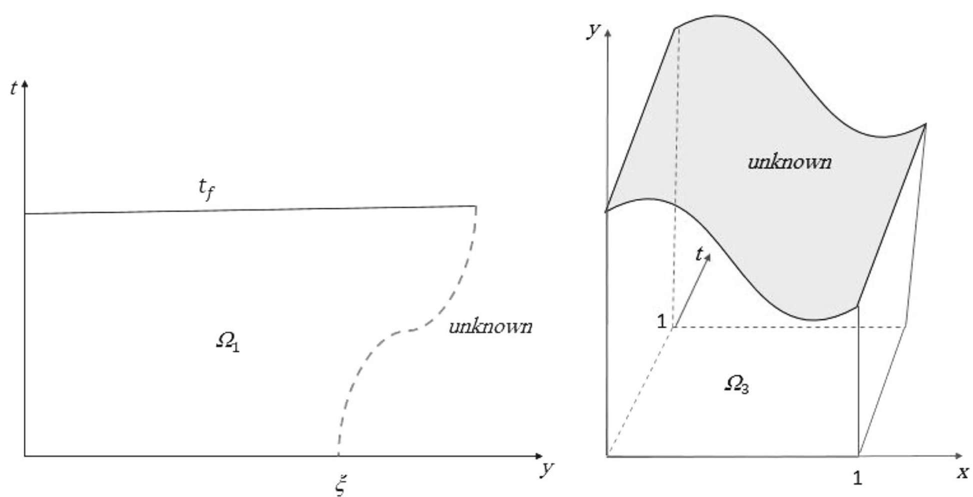
with the boundary conditions:

$$\frac{\partial A(y,t)}{\partial y}\Big|_{y=0} = z(t), \quad A(\xi(t),t) = 0, \qquad \text{in} \qquad (0,t_f), \tag{2}$$

and the an initial condition:

$$A(y,0) = f(y), \qquad 0 < y < \xi(0). \tag{3}$$

The inverse problem studied here deals with retrieving $\xi(t)$ from the Dirichlet boundary data:



(a) (b)

Fig. 1 **a** Physical domain of IP1. **b** Physical domain of IP2

$$A(0,t) = g(t) + \alpha(t). \tag{4}$$

Since the observation obtained from a real experiment is always noisy, a particular assumption $\alpha(t)$ as contaminations with the internal data is made. Moreover, we suppose that $\xi(0) = \xi > 0$ is either known or available as a priori information. This problem in the physical applications is referred as the inverse boundary Stefan problem (Johansson et al. 2011a, b); however, the initial temperature (3) may also be unknown in some cases (Wang et al. 2010; Wei and Yamamoto 2009).

Inverse problem 2 (IP2): Consider the following mathematical formulation satisfying the heat equation (Liu and Wei 2011):

$$\frac{\partial A(x,y,t)}{\partial t} = \frac{\partial^2 A(x,y,t)}{\partial x^2} + \frac{\partial^2 A(x,y,t)}{\partial y^2}, \quad in \quad \Omega_3, \tag{5}$$

$$\frac{\partial A(x,y,t)}{\partial x}\Big|_{x=0} = g_1(y,t), \quad 0 \le y \le p(0,t), 0 \le t \le 1, \tag{6}$$

$$\frac{\partial A(x,y,t)}{\partial x}\Big|_{x=1} = g_2(y,t), \quad 0 \le y \le p(1,t), 0 \le t \le 1, \tag{7}$$

$$A(x,0,t) = f(x,t), \quad 0 \le x \le 1, 0 \le t \le 1, \tag{8}$$

$$\frac{\partial A(x,y,t)}{\partial y}\Big|_{y=0} = q(x,t), \quad 0 \le x \le 1, 0 \le t \le 1, \tag{9}$$

which is set up on the physical domain (Fig. 2a)

$$\Omega_3 = \{(x,y,t) \in \mathbb{R}^3 | 0 < x < 1, 0 < y < p(x,t), 0 < t < 1\},$$

and the goal is mainly devoted to approximate the free boundary $p(x, t)$ from the measured boundary data:

$$A(x, p(x,t), t) = A_0 + \alpha(x,t), \tag{10}$$

where $\alpha(x,t)$ denotes an error function that naturally exists with the measurement. Both problems IP1 and IP2 are ill-posed in the sense that the solution (if it exists) does not depend continuously on the data (Liu and Wei 2011; Liu 2011; Liu and Guerrier 1997; Wang et al. 2012). Here, assuming the sufficient constrains for the existence and uniqueness of the solution for each problem, our main struggle is to present an appropriate numerical scheme for approximating their solution. The sufficient condition for obtaining the unique solution for IP1 is investigated in (Liu and Guerrier 1997).

3 The Numerical Approximations

3.1 The Solution of IP1

First, we assume that the initial and boundary conditions are sufficiently smooth to guarantee a unique solution (Johansson et al. 2011a; Liu 2011; Liu and Guerrier 1997). Besides, by applying the Landau's coordinate transformation

$$x = \frac{y}{\xi(t)}, \qquad W(x,t) = A(y,t), \tag{11}$$

Liu (2011); Liu and Guerrier (1997), the moving boundary Ω_1 and Eqs. (1)–(4) are transformed into the rectangular domain $\Omega_1^{'} = \{(x,t) \in \mathbb{R}^2 | 0 < x < 1, 0 < t < t_f\}$.

Thus, we have

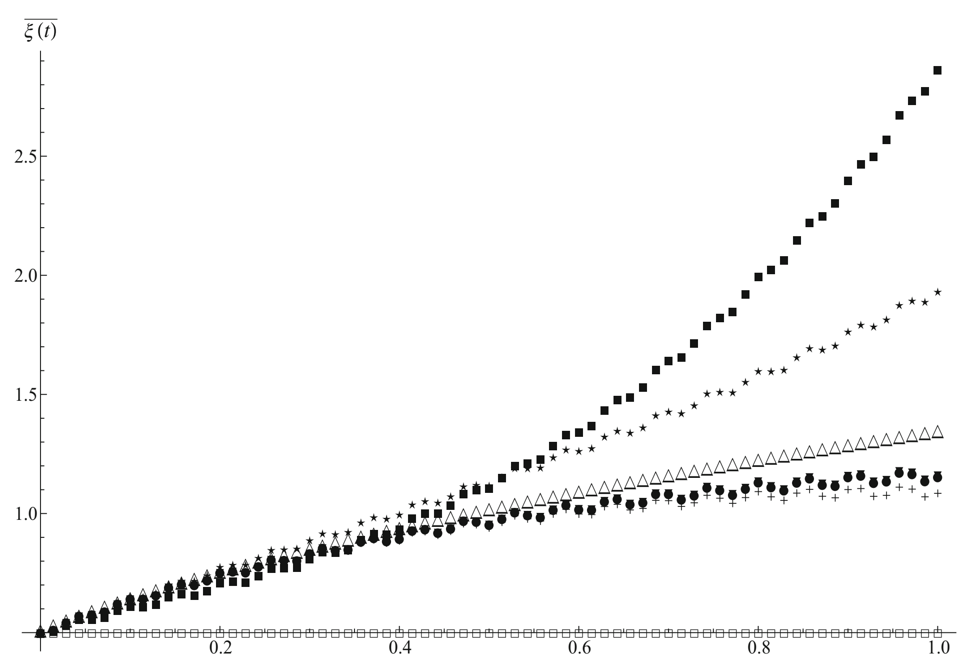


Fig. 2 Graph of the approximate solutions when (*triple open square* $k = 0$), (*triple filled square* $k = 1$), (*triple star* k=2), ($+ + + : k = 6$), (*triple filled circle* $k = 7$), (*triple filled inverted triangle* $k = 9$) for $\xi(t)$ with the exact solution, i.e., *triple open triangle*, and the noise level, i.e., $\lambda = 1\%$ discussed in Example 4.1

$$\frac{\partial^2 W(x,t)}{\partial x^2} = \xi(t)^2 \frac{\partial W(x,t)}{\partial t} - x\xi(t)\frac{d\xi(t)}{dt}\frac{\partial W(x,t)}{\partial x}, \tag{12}$$

$$\frac{\partial W(x,t)}{\partial x}|_{x=0} = \xi(t)z(t), \qquad 0<t<t_f, \tag{13}$$

$$W(1,t) = 0, \qquad 0<t<t_f, \tag{14}$$

$$W(x,0) = f(\xi x), \qquad 0<x<1, \tag{15}$$

$$W(0,t) = \alpha(t) + g(t), \qquad 0<t<t_f. \tag{16}$$

Noting (13)–(16), the following compatibility conditions hold:

$$f(\xi) = 0, \qquad f'(0) = z(0), \qquad f(0) = \alpha(0) + g(0). \tag{17}$$

Here, both functions $(W(x,t), \xi(t))$ are unknown and the advection–diffusion equation (12) contains the product of terms $\xi^2(t)$, $\xi(t)\frac{d\xi(t)}{dt}$, and $W(x, t)$. Hence, we have to deal with a nonlinear problem. We try to iteratively calculate $\xi(t)$ and $W(x, t)$ upon using the satisfier function and solving a linear system of algebraic equations for each iteration. At the first step, we set

$$\xi^{(0)}(t) = \xi, \qquad B^{(0)}(x,t) = (1-x)(g(t)+\alpha(t)),$$
$$W^{(0)}(x,t) = f(\xi x) + B^{(0)}(x,t) - B^{(0)}(x,0). \tag{18}$$

Now, by integrating (12) over [0, 1] and [0, t], respectively, utilizing (14)–(16) and after a simple calculation, we get

$$\int_{[0,t]} \frac{\partial W(x,\tau)}{\partial x}|_{x=1} d\tau = \int_{[0,t]} z(\tau)\xi(\tau)d\tau - \xi^2 \int_{[0,1]} f(\xi x)dx + \xi^2(t)\int_{[0,1]} W(x,t)dx - \int_{[0,t]}\int_{[0,1]} \xi(\tau)\frac{d\xi(\tau)}{d\tau} W(x,\tau)dxd\tau, \quad 0<t<t_f. \tag{19}$$

Defining

$$\overline{W_x(1,t)} = \sum_{i=1}^{N} c_i t\psi_i(t) + \xi f'(\xi), \tag{20}$$

as an approximation for $\frac{\partial W(x,\tau)}{\partial x}|_{x=1}$ and substituting both (20) and (18) into (19), we get

$$\int_{[0,t]} \overbrace{W_x^{(1)}(1,\tau)}^{=\sum_{i=1}^{N} c_i^{(1)}\tau\psi_i(\tau)+\xi f'(\xi)} d\tau = \int_{[0,t]} z(\tau)\xi^{(0)}(\tau)d\tau - \xi^2 \int_{[0,1]} f(\xi x)dx + (\xi^{(0)}(t))^2 \int_{[0,1]} W^{(0)}(x,t)dx - \int_{[0,t]}\int_{[0,1]} \xi^{(0)}(\tau)\frac{d\xi^{(0)}(\tau)}{d\tau} W^{(0)}(x,\tau)dxd\tau, \quad 0<t<t_f. \tag{21}$$

Solving the Volterra integral Eq. (21) of the first kind by taking N collocation points over the interval $(0, t_f)$, we find the unknown coefficients $\{c_i^{(1)}\}_{i=\overline{1,N}}$. To obtain the stable solution, we use the Landweber's iterations as an admissible regularization strategy (Kirsch 2011). For the next iterations, i.e., $k \geq 1$, we introduce

$$B^{(k)}(x,t) = x(x-1)W_x^{(k)}(1,t) + (x-1)^2(g(t)+\alpha(t)),$$
$$W^{(k)}(x,t) = f(\xi x) + B^{(k)}(x,t) - B^{(k)}(x,0),$$
$$\overline{W_x^{(k)}(0,t)} = z(t)\xi^{(k)}(t), \qquad \xi^{(k)}(t) \simeq \sum_{i=1}^{N} d_i^{(k)} t\psi_i(t) + \xi. \tag{22}$$

Again, by taking N collocation points over the interval $(0, t_f)$ and employing the Landweber's iterations, we find $\xi^{(k)}(t)$ from (22). By inserting $W^{(k)}(x,t)$ and $\xi^{(k)}(t)$ in (19) and resolving the following Volterra integral equation for the elements $c_i^{(k+1)}$, we find $\overline{W_x^{(k+1)}(1,t)}$.

$$\int_{[0,t]} \overbrace{W_x^{(k+1)}(1,\tau)}^{=\sum_{i=1}^{N} c_i^{(k+1)}\tau\psi_i(\tau)+\xi f'(\xi)} d\tau = \int_{[0,t]} z(\tau)\xi^{(k)}(\tau)d\tau - \xi^2 \int_{[0,1]} f(\xi x)dx + (\xi^{(k)}(t))^2 \int_{[0,1]} W^{(k)}(x,t)dx - \int_{[0,t]}\int_{[0,1]} \xi^{(k)}(\tau)\frac{d\xi^{(k)}(\tau)}{d\tau} W^{(k)}(x,\tau)dxd\tau, \quad 0<t<t_f. \tag{23}$$

Similarly, using

$$\overline{W_x^{(k+1)}(0,t)} = z(t)\xi^{(k+1)}(t), \qquad \xi^{(k+1)}(t) \simeq \sum_{i=1}^{N} d_i^{(k+1)} t\psi_i(t) + \xi, \tag{24}$$

we can find $\xi^{(k+1)}(t)$. The presented scheme will continue until for some values of $r \geq 1$ and ε_f, we have

$$\|\xi^{(r)}(t) - \xi^{(r-1)}(t)\|_\infty \leq \varepsilon_f + \|\alpha(t)\|_\infty. \tag{25}$$

Assuming the sufficient conditions for the existence and uniqueness of the solution for IP1 and defining

$$LW^{(i)} = \int_{[0,t]} z(\tau)\xi^{(i)}d\tau - \xi^2 \int_{[0,1]} f(\xi x)dx + (\xi^{(i)}(t))^2 \int_{[0,1]} W^{(i)}dx - \int_{[0,t]}\int_{[0,1]} \xi^{(i)}\frac{d\xi^{(i)}}{d\tau} W^{(i)}dxd\tau, \tag{26}$$

the following theorem guarantees the convergence of the solution presented by (18)–(24).

Theorem 3.1 Define $L_k = \limsup_{i\longrightarrow\infty} \sqrt[i]{\| \frac{\partial^k}{\partial t^k} LW^{(i)} \|}$, $k = \overline{1,2}$ and suppose that both following conditions:

(i) $\max\{L_1, L_2\} < 1$,

(ii) $z(t) \in M = \{s(t) : C^1[0,t_f] \longrightarrow \mathbb{R}, \| \frac{1}{s(t)} \|_\infty < 1, \| \frac{d\frac{1}{s(t)}}{dt} \|_\infty < \infty\}$,

hold, then sequences $\{w^{(n)}(x,t)\}_n, \{\xi^{(n)}(t)\}_n, \frac{d\xi^{(n)}(t)}{dt}$ converge to the solution of the problem in the complete Hilbert spaces $L^2([0,1]\times[0,t_f]), L^2([0,1])$.

Proof Suppose that the arbitrary $\epsilon > 0$ is given, now for $m > n > N^*$

$$\begin{aligned}\|W^{(m)}(x,t) - W^{(n)}(x,t)\| &= \|B^{(m)}(x,t) - B^{(n)}(x,t) \\ &\quad + B^{(n)}(x,0) - B^{(m)}(x,0)\| \\ &\le \|B^{(m)}(x,t) - B^{(n)}(x,t)\| + \|B^{(n)}(x,0) - B^{(m)}(x,0)\| \le U^* \\ &= \|W_x^{(m)}(1,t) - W_x^{(n)}(1,t)\| \\ &\quad + \|W_x^{(m)}(1,0) - W_x^{(n)}(1,0)\|.\end{aligned}$$

On the other hand, if (i) holds, then there exists

$$N^* \in \mathbb{N}, \delta < 1, \quad s.t \quad \forall n > N^* \| \frac{\partial}{\partial t} LW^{(n)} \| < \delta^n.$$

Thus, setting $M^* = \|x(x-1)\|$

$$\begin{aligned}U^* &\le 2M^*\left(\delta_1^{(n-1)} + \delta_2^{(m-1)}\right) \le 4M^* \sup\{\delta_1,\delta_2\}^{\inf\{m-1,n-1\}} \\ &\le 4M^* \sup\{\delta_1,\delta_2\}^{N^*-1} = \frac{4M^*}{\inf\{\frac{1}{\delta_1},\frac{1}{\delta_2}\}^{N^*-1}},\end{aligned}$$

now it is sufficient to take $N^* \ge [\frac{\log\left(\frac{4M^*}{\epsilon}\right)}{\log\left(\inf\{\frac{1}{\delta_1},\frac{1}{\delta_2}\}\right)}] + 2$ to find

$$\|W^{(m)}(x,t) - W^{(n)}(x,t)\| < \epsilon. \tag{27}$$

This implies that $\{W^{(n)}(x,t)\}_n$ is uniformly Cauchy. Similarly, using (i) and (22), we can see that $\{W_x^{(n)}(0,t)\}_n$ defined by

$$\begin{aligned}W_x^{(k)}(0,t) &= W_x^{(k)}(1,0) - W_x^{(k)}(1,t) + \xi f'(0) \\ &\quad + 2\big(g(0) + \alpha(0) - g(t) - \alpha(t)\big),\end{aligned}$$

is uniformly Cauchy. Thus, for the given $\epsilon > 0$

$$\exists N_1 \in \mathbb{N} s.t \forall m > n > \max\{N_1, N^*\}, \|W_x^{(m)}(0,t) - W_x^{(n)}(0,t)\| < \epsilon,$$

ultimately, using (ii), we have

$$\begin{aligned}\|\xi^{(m)}(t) - \xi^{(n)}(t)\| &= \left\| \frac{W_x^{(m)}(0,t) - W_x^{(n)}(0,t)}{z(t)} \right\| \\ &< \|W_x^{(m)}(0,t) - W_x^{(n)}(0,t)\| < \epsilon.\end{aligned}$$

Paying attention to the definition of $\{\xi^{(k)}(t)\}_k$ and using $L_2 < 1$, it is easy to see that the uniform Cauchy property of it concludes that $\{\frac{d\xi^{(k)}(t)}{dt}\}_k$ is uniformly Cauchy and finally since every uniformly Cauchy sequences converges in the complete Hilbert space L^2, so are the three sequences $\{\xi^k(t), \frac{d\xi^k(t)}{dt}, w^{(k)}(x,t)\}_k$ in $L^2([0,1]), L^2([0,1]\times[0,t_f])$. In addition, since the operator $LW^{(i)}$ is continuous, then it tends to LW, as the solution of the problem. □

3.2 The Solution of IP2

We calculate both unknown functions $\big(A(x,y,t), p(x,t)\big)$ from the given boundary conditions (6)–(10). The measurement (10) produces the nonlinear property of the problem. On top of that the overspecification (10) contains some kind of errors (e.g., $\alpha(x,t)$) that if it contributes directly to the solution scheme including the differential operators, a strong propagation of errors can be expected, and ultimately, the produced solution is unacceptable. Taking these considerations into account, we introduce a procedure to pass these difficulties and conclude the effective solution:

(i) If the known functions $g_1(y,t), g_2(y,t)$ are at most of degree one with respect to the variable y:

Take the satisfier function for (6)–(9) as

$$\begin{aligned}B_1(x,y,t) &= f(x,t) + yq(x,t), \quad SF(x,y,t) \\ &= x(x-1)^2 g_1(y,t) + x^2(x-1) g_2(y,t) \\ &\quad + B_1(x,y,t) - \big(x(x-1)^2 B_1(0,y,t) \\ &\quad + x^2(x-1) B_1(1,y,t)\big),\end{aligned} \tag{28}$$

and the approximations for $p(x, t)$ and $A(x, y, t)$ as

$$\begin{aligned}\overline{p(x,t)} &:= \sum_{i_1,i_2=1}^{N_1,N_2} c_{i_1,i_2} \psi_{i_1,i_2}(x,t), \\ \overline{A(x,y,t)} &:= \sum_{i_1,i_2,i_3}^{N_1,N_2,N_3} x^2(x-1)^2 y^2 \big(y - \overline{p(x,t)}\big) \\ &\quad c_{i_1,i_2,i_3} \psi_{i_1,i_2,i_3}(x,y,t) + SF(x,y,t),\end{aligned} \tag{29}$$

respectively. By applying (29) in (10), we find an approximation for $p(x, t)$ by solving a linear system of equations with respect to the elements c_{i_1,i_2}.

(ii) For the general form of $g_1(y,t)$ and $g_2(y,t)$, we use the piecewise linear basis functions:

$$Y_i^*(s) = \begin{cases} s, & \frac{i-1}{Q} < s < \frac{i}{Q}, i = \overline{1,Q}, \\ 0, & Otherwise \end{cases}$$

and suppose the approximations for $g_1(y,t), g_2(y,t)$ as

$$g_1(y,t) \simeq G(t)\left(\sum_{i=1}^{Q} g_{i1} Y_i^*(y) + \beta_{i1}\right), g_2(y,t) \simeq G(t)\left(\sum_{i=1}^{Q} g_{i2} Y_i^*(y) + \beta_{i2}\right). \tag{30}$$

Now, by applying the approximations (30) in Eqs. (28), (29), we find the linear system of equations as

$$\sum_{i_1,i_2=1}^{N_1,N_2} c_{i_1,i_2}\psi_{i_1,i_2}(x,t)H_i(x,t) = R_i(x,t). \tag{31}$$

The unknown parameters $c_{i_1,i_2}, i_{j\in\{1,2\}} = \overline{1,N_j}$ can be found by using an interpolation technique on $(x,t) \in (0,1)\times(0,1)$ for $\frac{R_i(x,t)}{H_i(x,t)}, i = \overline{1,Q}$. The investigated functions given by Eq. (31) are

$$H_i(x,t) = x(x-1)^2\left\{\sum_{i=1}^{Q} g_{i1}G(t) - q(0,t)\right\} + q(x,t) + x^2(x-1)\left\{\sum_{i=1}^{Q} g_{i2}G(t) - q(1,t)\right\}, i = \overline{1,Q},$$

$$R_i(x,t) = \alpha(x,t) + A_0 + x(x-1)^2\left\{f(0,t) - G(t)\sum_{i=1}^{Q}\beta_{i1}\right\}, + x^2(x-1)\left\{f(1,t) - G(t)\sum_{i=1}^{Q}\beta_{i2}\right\} - f(x,t).$$

After employing the Landweber's iterations to find the stable solution for $p(x, t)$ and inserting this approximation in (29), we apply the Galerkin equations Rashedi et al. (2013); Yousefi et al. (2013):

$$\int_{(-\infty,\infty)} \big(Residual(x,y,t)\big)\delta(x-x_i)\mathrm{d}x = 0, \quad x_i \in \Omega_3, i = \overline{1, N_1N_2N_3}, \tag{32}$$

where $\mathbf{x} = (\mathbf{x},\mathbf{y},\mathbf{t})$ and

$$residual(x,y,t) := A_0 + \alpha(x,t) - f(x,t) - \overline{p(x,t)}q(x,t) - \int_{[0,\overline{p(x,t)}]}\int_{[0,y]}\Big(\frac{\partial\overline{A(x,s,t)}}{\partial t} - \frac{\partial^2\overline{A(x,s,t)}}{\partial x^2}\Big)\mathrm{d}s\mathrm{d}y. \tag{33}$$

In a similar way, to derive the stable solution for $A(x, y, t)$, we use the Landweber's iterations to solve the final linear system of equations resulted from (32). Considering the linear system of equations as $Kx = y^\delta$, the regularized solution $x^{m,\delta}$ is computed by the iterations:

$$x^{0,\delta} = 0, \quad x^{m,\delta} = \left(I - aK^tK\right)x^{m-1,\delta} + aK^ty^\delta, \tag{34}$$

where K^t is the transpose of the matrix K, I denotes the identity matrix, y^δ is the right hand side vector of the equations that supposed to contain errors basically generated from input perturbed data and $0 < a < \frac{1}{\|K\|^2}$. Indeed, these iterations define a regularization strategy in which for $m = 1, 2, \ldots$ subject to $m(\delta) \longrightarrow \infty(\delta \longrightarrow 0)$ with $\delta^2 m(\delta) \longrightarrow 0(\delta \longrightarrow 0)$ is admissible. Equivalently, the linear and bounded operator:

$$R_m = a\sum_{r=0}^{m-1}(I - aK^tK)^rK^t, \quad m = 1,2,\ldots$$

as the approximation of K^{-1} is used to find

$$x^{m,\delta} = R_m y^\delta, \tag{35}$$

where $\|R_m\| \le \sqrt{am}$. For more details, see Kaltenbacher et al. (2008), Kirsch (2011), Landweber (1951), and Wang and Yagola (2010).

Remark 3.2 It should be clarified that the notations

$$\psi_i(t), \psi_{i_1,i_2}(x,t) = \psi_{i_1}(x)\psi_{i_2}(t), \psi_{i_1,i_2,i_3}(x,y,t) = \psi_{i_1}(x)\psi_{i_2}(y)\psi_{i_3}(t)$$

used in this section imply the general form of the basis functions. Here, we employ the Bernstein basis functions which can be seen all over the approximation theory (Dehghan et al. 2013; Idrees Bhatti and Bracken 2007; Rivlin 1969). Other basis functions, such as the orthogonal ones, e.g., Legendre or Chebyshev polynomials can be applied as an alternative by the interested reader. In addition, the function $G(t)$ which appeared in Eq. (30) can be considered as the combination of the basis functions, i.e., $G(t) = \sum_{s=0}^{W} z_s\psi_s(t)$.

4 Numerical Experiments

To test the effectiveness of the proposed techniques, we solve four benchmark test examples. They are chosen for reporting the results of implementing the proposed methods for IP1–IP2, respectively. For all computational experiments, the exact solution is available. We investigated the stability of the numerical solution by performing the mentioned methods in the presence of various amount of noise levels $\lambda\% = \lambda \times 10^{-3}$. The numerical implementation is carried out in MATHEMATICA 7, with hardware configuration: desktop 32-bit Intel Core 2 Duo CPU, 4 GB of RAM, 32-bit Operating System (Windows 7).

4.1 Example 1

Consider **IP1** given by Eqs. (1)–(4) with the following properties Johansson et al. (2011b):

Table 1 Maximum absolute errors between exact solution and $\overline{\xi^{(k)}(t)}$ discussed in Example 4.1 for exact data

Iterations	$k=0$	$k=3$	$k=5$	$k=7$	$k=8$	$k=9$
$Max_{0<t<1}\lvert\xi(t)-\overline{\xi^{(k)}(t)}\rvert$	0.8	0.14	0.06	0.01	0.0007	0.0002

Table 2 Approximate solutions of $W(\frac{i}{10},\frac{i}{10})$ s.t $i=\overline{0,10}$, discussed in Example 4.1 for exact data

Function r	$W(r,r)$ *Exact*	$\overline{W^{(k)}(r,r)}$ $k=1$	$\overline{W^{(k)}(r,r)}$ $k=3$	$\overline{W^{(k)}(r,r)}$ $k=5$	$\overline{W^{(k)}(r,r)}$ $k=7$	$\overline{W^{(k)}(r,r)}$ $k=8$	$\overline{W^{(k)}(r,r)}$ $k=9$
0.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
0.1	0.8871	0.8923	0.8860	0.8871	0.8872	0.8872	0.8872
0.2	0.7752	0.7868	0.7712	0.7753	0.7754	0.7753	0.7752
0.3	0.6650	0.6746	0.6570	0.6655	0.6654	0.6651	0.6650
0.4	0.5573	0.5520	0.5451	0.5587	0.5579	0.5574	0.5572
0.5	0.4528	0.4200	0.4370	0.4555	0.4537	0.4530	0.4527
0.6	0.3523	0.2845	0.3341	0.3564	0.3534	0.3525	0.3522
0.7	0.2563	0.1566	0.2378	0.2614	0.2575	0.2564	0.2561
0.8	0.1653	0.0519	0.1492	0.1706	0.1664	0.1654	0.1652
0.9	0.0798	−0.0088	0.0693	0.0837	0.0805	0.0798	0.0797
1.0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

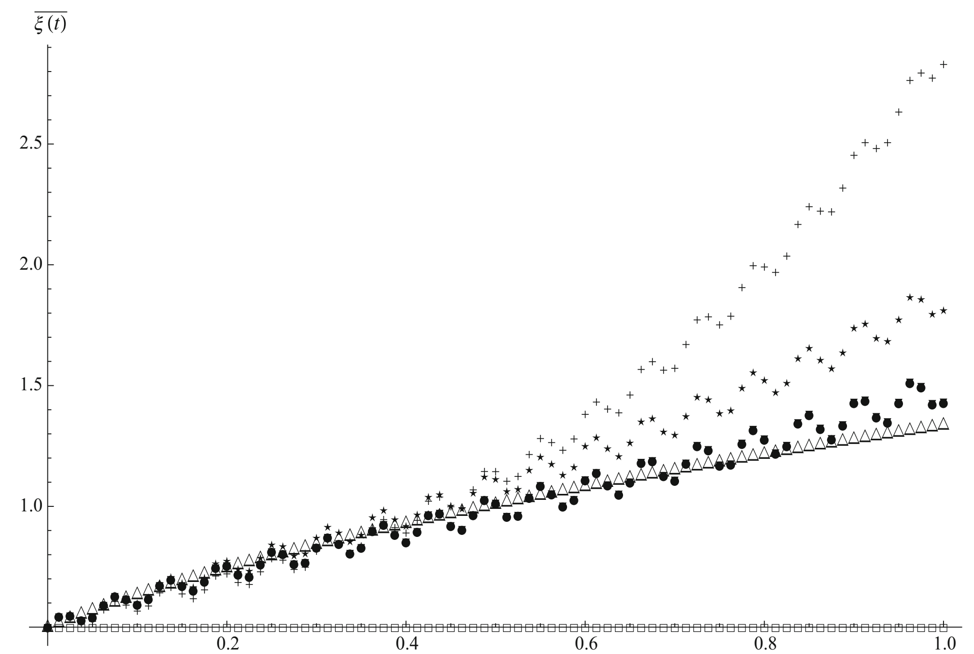


Fig. 3 Graph of the approximate solutions when (*triple open square* $k=0$), (*triple plus* $k=2$), (*triple star* $k=5$), (*triple filled circle* $k=7$), (*triple filled inverted square* $k=8$) for $\xi(t)$ with the exact solution, i.e., *triple open triangle*, and the noise level, i.e., $\lambda=3\%$ discussed in Example 4.1

$$z(t)=-\frac{1}{\sqrt{\pi(t+t_0)}erf(a^*)},f(y)=1-\frac{erf\left(\frac{y}{2\sqrt{t_0}}\right)}{erf(a^*)},$$
$$g(t)=1,\alpha(t)=\lambda\sin\left(\frac{t}{\lambda^2}\right),\lambda\in\{1,3\}\%,$$

where

$$erf(y)=\frac{2}{\sqrt{\pi}}\int_{[0,y]}\exp(-s^2)ds,\quad a^*=0.620063,$$
$$t_0=0.162558,t_f=1.$$

We aim to approximate the solutions for both continuous functions:

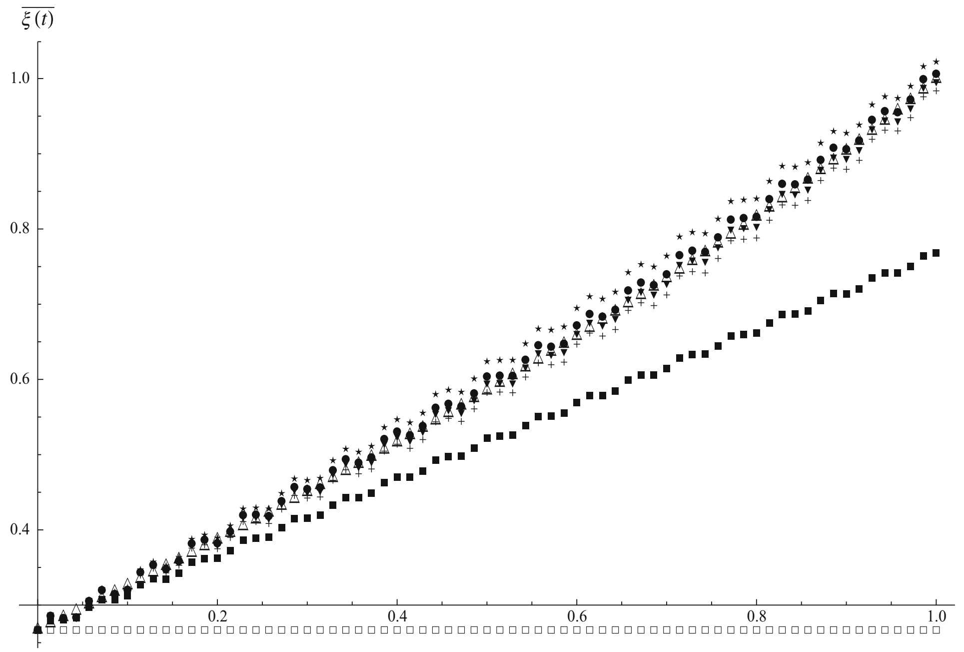


Fig. 4 Graph of the approximate solutions when (*triple open square* $k = 0$), (*triple filled square* $k = 1$), (*triple star* $k = 2$), (*triple plus* $k = 3$), (*triple filled circle* $k = 4$), (*triple filled inverted triangle* $k = 5$) for $\xi(t)$ with the exact solution, i.e., *triple open triangle*, and the noise level, i.e., $\lambda = 1\%$ discussed in Example 4.2

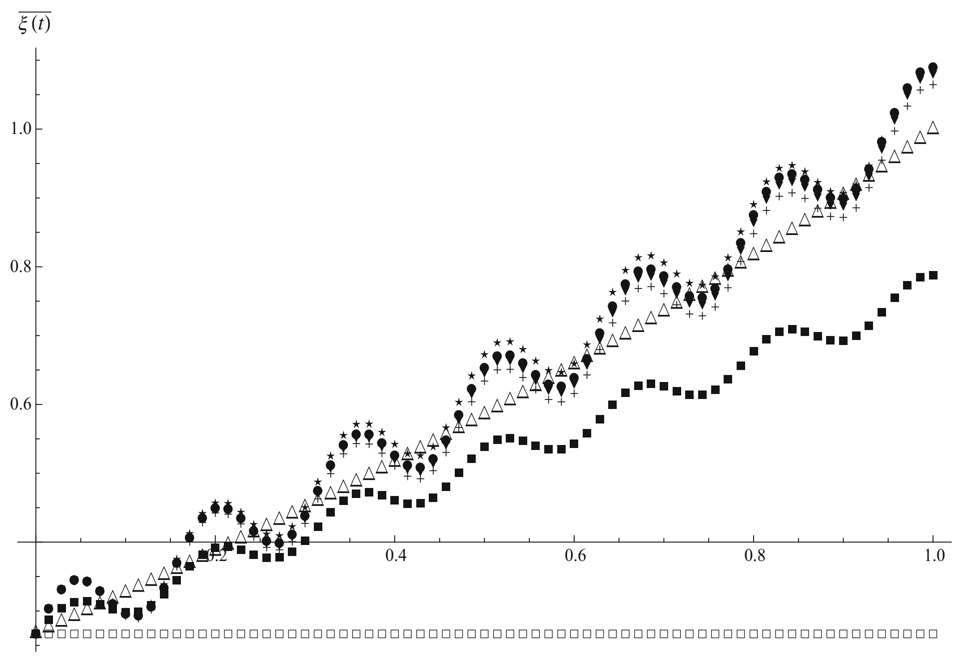
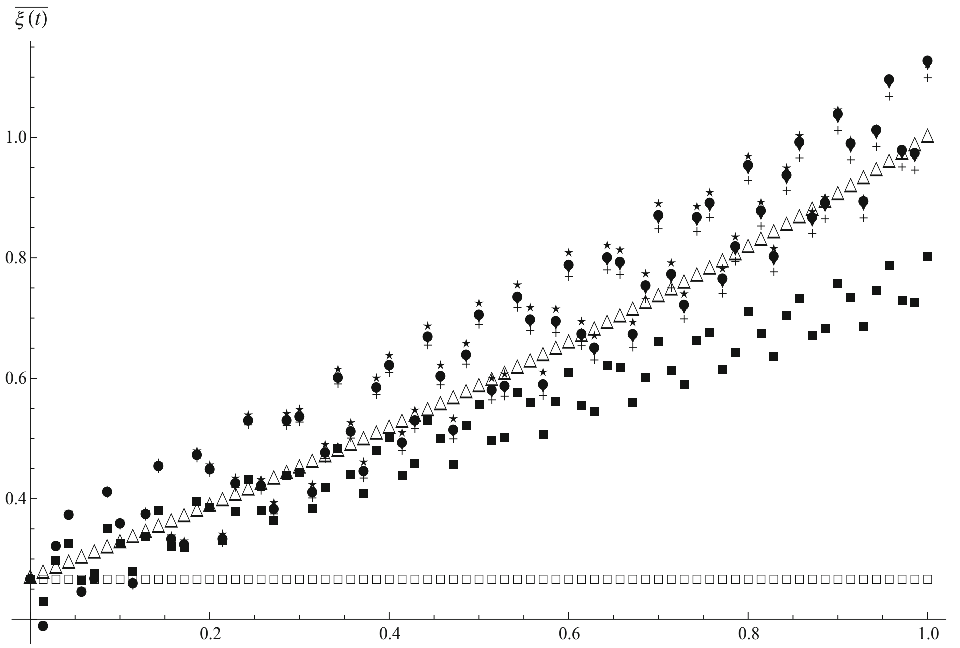


Fig. 5 Graph of the approximate solutions when (*triple open square* k=0), (*triple filled square* k=1), (*triple star* k=2), (*triple plus* k=3), (*triple filled circle* k=4), (*triple filled inverted triangle* k=5) for $\xi(t)$ with the exact solution, i.e., *triple open triangle*, and the noise level, i.e., $\lambda = 5\%$ discussed in Example 4.2

$$\left(\xi(t), A(y,t)\right) = \left(2a^*\sqrt{t+t_0}, 1 - \frac{erf\left(\frac{y}{2\sqrt{t+t_0}}\right)}{erf(a^*)}\right),$$
$$(y,t) \in \left(0, \xi(t)\right) \times (0,1).$$

Utilizing the method presented in Eqs. (11)–(24) and using the Bernstein basis functions of degree one Rashedi et al. (2013), we obtain the results which are tabulated in Tables 1 and 2. The approximations in the presence of the exact input data, i.e., $\alpha(t) = 0$, are derived using the Landweber's iterations with $a = 1, m = 10^4$. Following them, it is seen that the approximations are improved by increasing the number of iterations k. Moreover, for the perturbed values of the boundary conditions with $\lambda \in \{1,3\}\%$, we find the approximations illustrated in

Fig. 6 Graph of the approximate solutions when (*triple open square* $k = 0$), (*triple filled square* $k = 1$), (*triple star* $k = 2$), (*triple plus* $k = 3$), (*triple filled circle* $k = 4$), (*triple filled inverted triangle* $k = 5$) for $\xi(t)$ with the exact solution, i.e., *triple open triangle*, and the noise level, i.e., $\lambda = 9\%$ discussed in Example 4.2



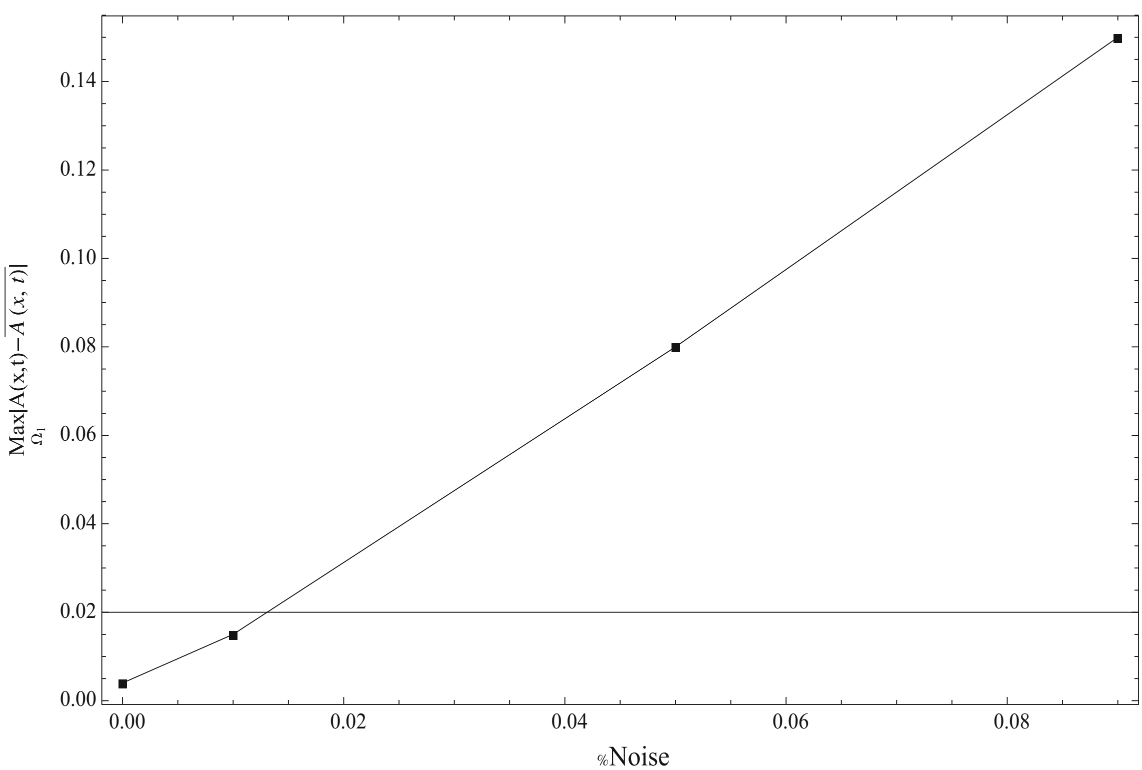


Fig. 7 Graph of the maximum absolute errors between the exact solutions for $\overline{A(x,t)}$ against the amount of noise levels $\lambda \in \{1, 5, 9\}\%$ with $k = 5$ for Example 4.2

Table 3 Maximum absolute errors between the exact and numerical solutions of $p(x, t)$, discussed in Example 4.3 for exact data

$N_{i\in\{1,2,3\}}$	3	4	5	6	7	8
$Max_{[0,1]\times[0,1]}\lvert p(x,t) - \overline{p(x,t)}\rvert$	0.05	0.016	0.0035	5×10^{-4}	7×10^{-5}	6×10^{-6}

Figs. 2 and 3. In fact, starting with the initial guess (□), one can observe that after a few number of iterations, the figures of the numerical solution tend to overlap until we reach the final iteration (▼). They show the fair agreement between the numerical and the exact solutions in both views of accuracy and stability. However, it should be noted that for the large amount of errors with the input data (i.e., $\lambda \geq 5\%$), we faced to drawback and could not get acceptable solutions.

4.2 Example 2

As the second example of **IP1**, the algorithm given by (11)-(24) is tested for approximating the pair Johansson et al. (2011a):

$$\big(\xi(t), A(y,t)\big) = \big(2-\sqrt{3-2t}, -\frac{y^2}{2} - 2y - \frac{1}{2} - t\big), \quad (y,t) \in \big(0, \xi(t)\big) \times (0,1),$$

by taking the boundary conditions

$$z(t) = -2, f(y) = -\frac{y^2}{2} - 2y - \frac{1}{2}, g(t) = -\frac{1}{2} - t, \alpha(t) = \lambda \sin(\frac{t}{\lambda^2}), \lambda \in \{1,5,9\}\%.$$

By applying the Bernstein basis functions of degree two (Rashedi et al. 2013) and solving the final linear system of equations using the Landweber's iterations with $a = 1, m = 10^5$, we arrive at the consequences depicted by Figs. 4, 5, 6, and 7. Similar to the previous example, we observe that the produced results are fair and accurate. It can be seen that as the percentage of the imposed errors decreases gradually, the agreement between exact and approximate solutions becomes uniformly good. In addition, the stability of the solutions encounters less difficulty. The exhibited approximate solutions corresponding to the noise levels greater than 5% verify this fact.

4.3 Example 3

Take the exact solution for **IP2** given by Eqs. (5)–(10) as Liu and Wei (2011)

$$A(x,y,t) = \exp(-4t)\sin(2x-1) - y + 1, \quad p(x,t) = \exp(-4t)\sin(2x-1) + 1, \quad (x,y,t) \in \Omega_3,$$

along with the following specifications:

$$g_1(y,t) = g_2(y,t) = 2\cos(1)\exp(-4t), f(x,t) = 1 + \exp(-4t)\sin(2x-1), q(x,t) = -1, A_0 = 0. \tag{36}$$

By applying the numerical technique presented by (28)–(33) and using the Bernstein basis functions when $N_{i=\overline{1,3}} \in \{3,4,5,6,7,8\}$, we get the results shown by Table 3. Of special interest is testifying the numerical convergence of the solution of this problem and following the demonstrations, one observes that for the exact input data ($\alpha(x,t) = 0$), the accuracy of the numerical solution grows as the number of basis increases gradually. In

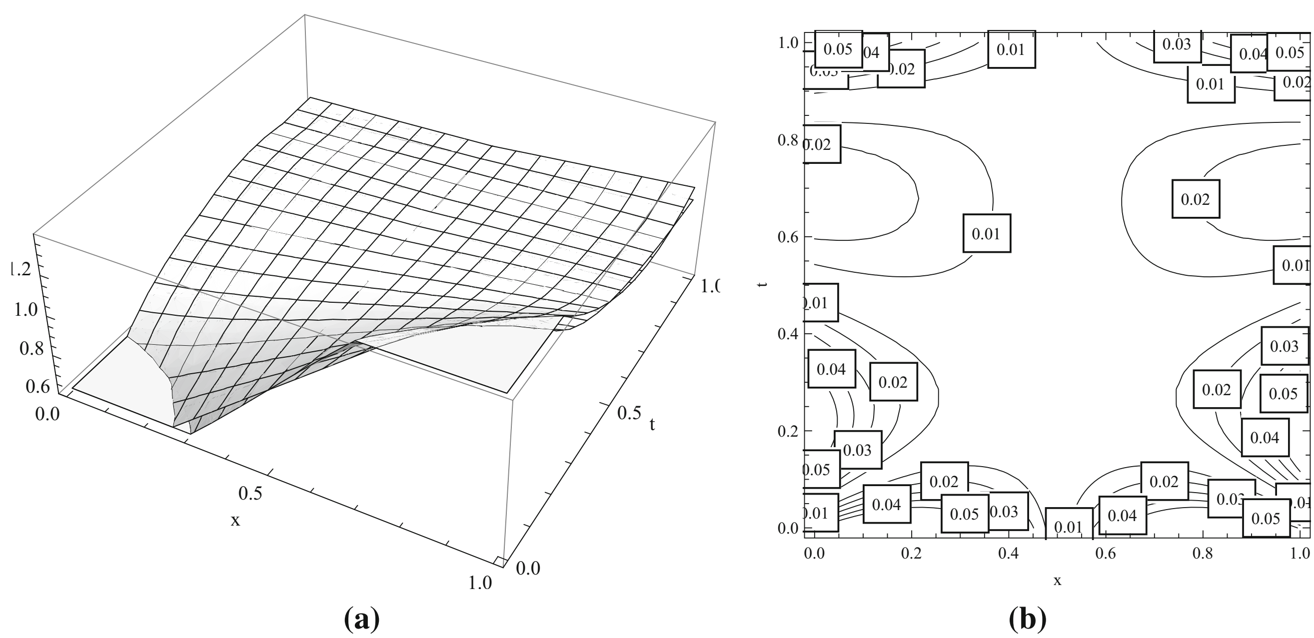


Fig. 8 **a** Exact and approximate solutions. **b** Contour lines corresponding to absolute error of the exact solution and its approximation for $p(x,t), (x,t) \in [0,1] \times [0,1]$. All plots were obtained with $a = 1, m = 10^6, \lambda = 9\%$ for Example 4.3

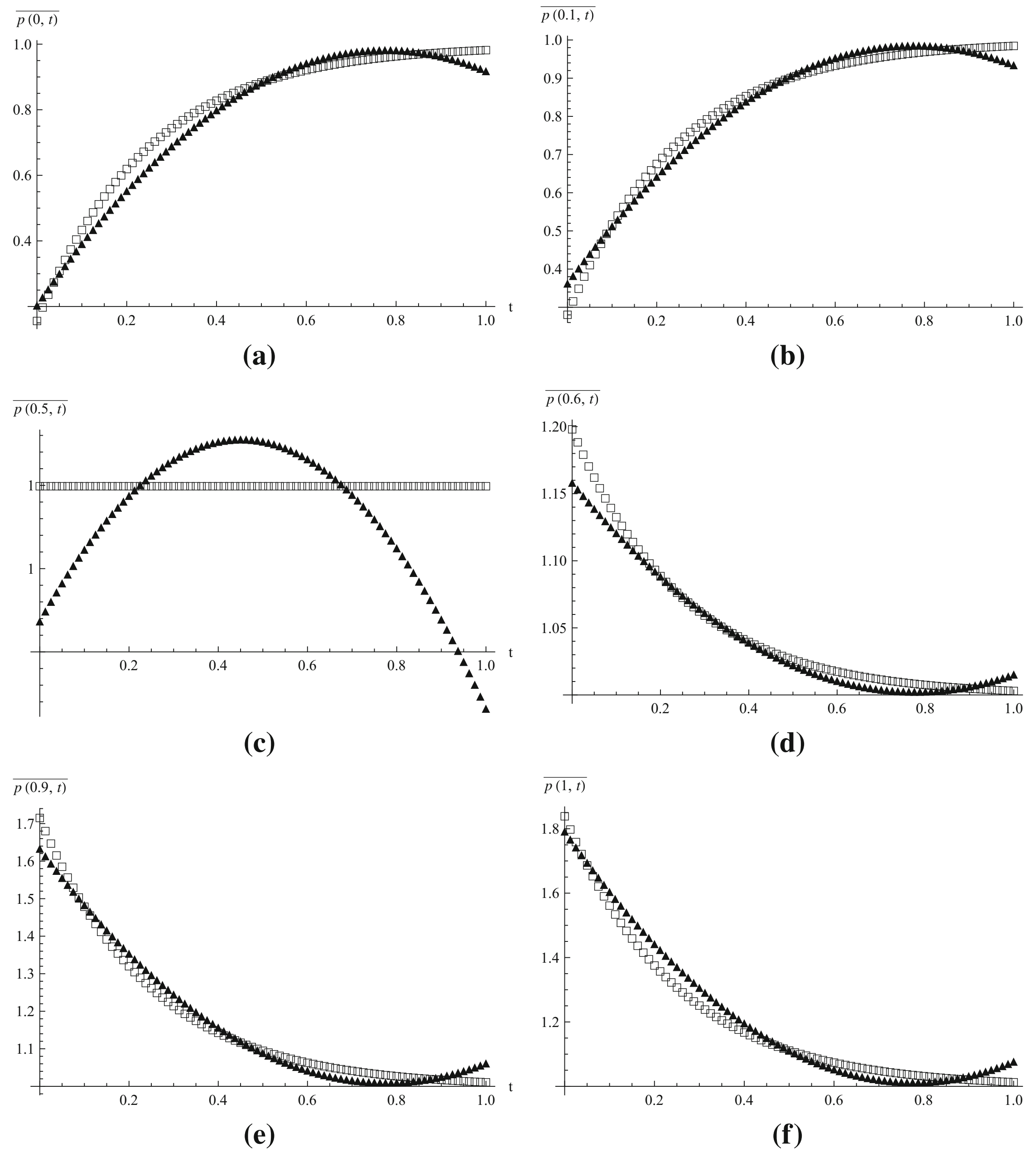


Fig. 9 Graphs of the exact *triple open square* and approximate *triple filled triangle* solutions for $p(\frac{i}{10}, t)$, $i \in \{1, 2, 5, 6, 9, 10\}$. All plots for $\lambda = 9\%, a = 1, m = 10^6$ for Example 4.3

addition, for the contaminated input data, we apply the proposed technique with

$$N_{i=\overline{1,3}} = 3, a = 1, m = 10^6, \alpha(x, t) = \lambda \sin(\frac{x + t}{\lambda^2}), \lambda \in \{1, 3, 9\}\%.$$

For the sake of brevity, only the results with $\lambda = 9\%$ are shown by Figs. 8a, b and 9a, f. They imply that the stability is maintained for the solution with respect to the boundary conditions.

4.4 Example 4

As the last example Liu and Wei (2011), we consider **IP2** with the following properties:

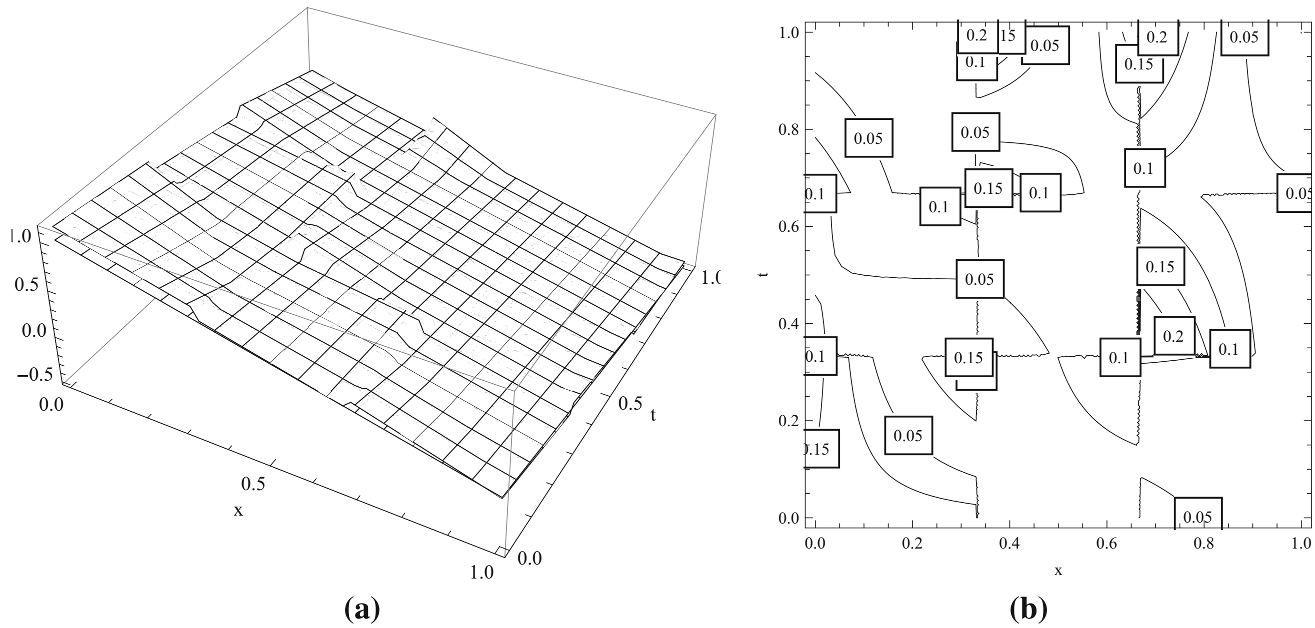


Fig. 10 **a** Exact and approximate solutions. **b** Contour lines corresponding to absolute error between the exact and numerical solutions for $p(x, t), (x, t) \in [0, 1] \times [0, 1]$. All plots were obtained with $a = 1, m = 10^6, \lambda = 1\%$ for Example 4.4

$$g_1(y,t) = \frac{1}{4}\exp\left(\frac{t}{8} + \frac{y-1}{4}\right), \quad g_2(y,t) = \frac{1}{4}\exp\left(\frac{t}{8} + \frac{y-1}{4} + \frac{1}{4}\right), \tag{37}$$

$$f(x,t) = \exp\left(\frac{t}{8} + \frac{x}{4} - \frac{1}{4}\right) - 1, \quad q(x,t) = \frac{1}{4}\exp\left(\frac{t}{8} + \frac{x}{4} - \frac{1}{4}\right), \tag{38}$$

$$A(x,y,t) = -1 + \exp\left(\frac{t}{8} + \frac{x}{4} + \frac{y-1}{4}\right), \quad p(x,t) = 1 - x - \frac{t}{2}. \tag{39}$$

In this example, the known functions $g_1(y, t)$ and $g_2(y, t)$ are nonlinear with respect to y. First, by setting $Q = 3$, we take the linear approximations for $g_1(y, t), g_2(y, t)$. These approximations are valid on $(y, t) \subseteq [0, x_f] \times [0, x_f]$ which, without loss of generality, we assume that a priori information about the unknown bounded function $p(x, t)$ is available subject to

$$\sup_{0 \le x,t \le 1} p(x,t) = x_f.$$

Hereafter, by applying the procedure given by Eqs. (28)–(33) and taking into account the perturbed noisy data with $\lambda = 1\%$ and solving the final linear system of equations using the Landweber's iterations method with $a = 1, m = 10^6$, we obtain the results shown by Figs. 10a, b and 11a–f. Considering the ill-posedness of the problem and the size of the data that should be retrieved, the approximation is good. In addition, results are stable for the noise levels $\lambda \le 1\%$.

5 Concluding Remarks

The paper addresses two numerical techniques for boundary identification of the inverse heat conduction problems in one and two dimensions. Apart from the ill-posedness of the problems, they include some nonlinear terms that make it difficult to find the acceptable solutions. For the one-dimensional problem, we first apply the Landau's transformation to replace the physical domain with a rectangular one. Reciprocally, some nonlinear terms appear thus an iterative scheme based on the application of the satisfier function is proposed for solving the problem. Second, we treat with the nonlinear two-dimensional problem by providing a collocation technique which takes advantage of the satisfier functions. Throughout this work, the presented schemes make the reader free of solving any nonlinear system of algebraic equations. Moreover, an admissible regularization strategy, namely, the Landweber's iterations method is used to overcome the numerical instability and achieve the

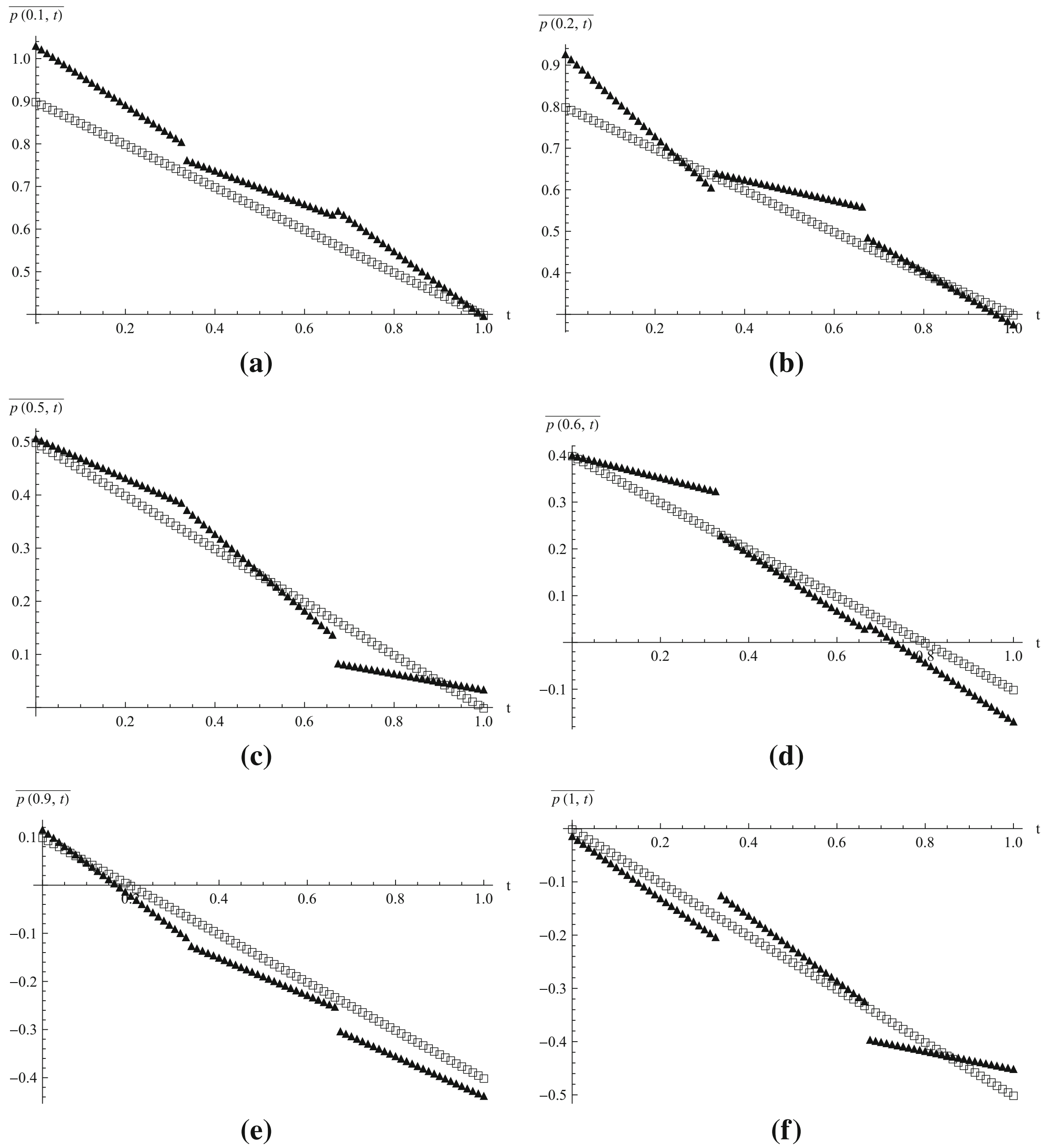


Fig. 11 Graphs of the exact *triple open square* and approximate *triple filled triangle* solution for $p(\frac{i}{10}, t)$, $i \in \{1, 2, 5, 6, 9, 10\}$. All plots for $\lambda = 1\%, a = 1, m = 10^6$ for Example 4.4

acceptable approximations. Applicability of the presented schemes with both views of numerical convergence and numerical stability while solving several test problems is verified in the presence of noisy input data.

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