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RESEARCH PAPER

Finite Spectrum of Sturm–Liouville Problems with n Transmission Conditions

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Abstract For any positive integer n and a set of positive integers m_i, $i = 1, 2, \ldots, n+1$, we construct a class of regular Sturm–Liouville problems with n transmission conditions, which have exactly $\sum_{i=1}^{n+1} m_i + n + 1$ eigenvalues. And further we show that these $\sum_{i=1}^{n+1} m_i + n + 1$ eigenvalues can be distributed arbitrarily throughout the complex plane in the non-self-adjoint case and anywhere along the real line in the self-adjoint case.

Keywords Sturm–Liouville problems · Finite spectrum · Eigenvalues · Transmission conditions

1 Introduction

The Sturm–Liouville problems (SLPs) with transmission conditions at an interior point have always been an important research topic in mathematical physics. Such a problem connected with many assortment of physical problems, such as heat and mass transfer, vibrating string problems and diffraction problems. In recent years the studies of these problems often appear not only in one interior point, but also in two or infinite many interior points. The discussions of these problems include their self-adjointness, eigenvalues and the completeness of eigenfunctions and inverse eigenvalue problems, and so on (Gesztesy et al. 1985; Mukhtarov et al. 2002a, b, 2004; Chanane 2007; Sun and Wang 2008; Titeux and Yakubov 1997).

Also recent years the Sturm–Liouville problems with finite spectrum have been investigated by many authors (Kong et al. 2001, 2009; Ao et al. 2011, 2012, 2013). These problems can be seen as coming from Atkinson's statement in his well-known book (Atkinson 1964). Among these studies there are finite spectrum results of SLPs (Kong et al. 2001, 2009), SLPs with transmission conditions (Ao et al. 2011, 2012), and even SLPs with transmission conditions and eigenparameter-dependent boundary conditions (Ao et al. 2013). However, there is no such results for SLPs with finite transmission conditions. For this reason, in this paper, we shall consider the SLPs with n transmission conditions and prove that for any positive integer n the SLPs with n transmission conditions still have finite spectrum. Similar with the proof in Kong et al. (2001) and Ao et al. (2011), we construct a class of these problems with exactly $\sum_{i=1}^{n+1} m_i + n + 1$ eigenvalues, where m_i are connected with the partition of the interval J. As in Kong et al. (2001) and Ao et al. (2011) our construction based on the characteristic function whose zeros are the eigenvalues. The key to this analysis is still an iterative construction of the characteristic function. Although similar methods are used to get our main results, the specific process of calculations and proofs is not completely the same as in Kong et al. 2001 and Ao et al.

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2011, which is more complicated and include some new items.

2 Notation and Preliminaries

Consider the SLP consisting of the differential equation

$$\begin{aligned} &-(py')' + qy = \lambda wy, \\ &\text{on } J = (a, c_1) \cup (c_1, c_2) \cup \cdots \cup (c_n, b), \end{aligned} \tag{2.1}$$

where $-\infty < a < b < +\infty, c_i \in (a, b), i = 1, 2, \ldots, n$, together with the regular two point boundary conditions (BCs) of the form

$$AY(a) + BY(b) = 0, \;\; Y = \begin{pmatrix} y \\ py' \end{pmatrix}, \;\; A, B \in M_2(\mathbb{C}), \tag{2.2}$$

and the transmission conditions at these n interior points c_i of the form

$$C_iY(c_i-) + D_iY(c_i+) = 0, \;\; i = 1, 2, \ldots, n, \tag{2.3}$$

where $A = (a_{st})_{2\times2}, B = (b_{st})_{2\times2}$ are complex-valued 2×2 matrices, and C_i, D_i are real valued 2×2 matrices satisfying $\det(C_i) = \rho_i > 0$, $\det(D_i) = \theta_i > 0$. $M_2(\mathbb{C})$ denotes the set of square matrices of order 2 over $\mathbb{C}$. Here λ is the complex-valued spectral parameter, and the coefficients satisfy the minimal conditions

$$r = 1/p, q, w \in L(J, \mathbb{C}), \tag{2.4}$$

where $L(J, \mathbb{C})$ denotes the complex-valued functions which are Lebesgue integrable on J. Condition (2.4) is minimal in the sense that it is necessary and sufficient for all initial value problems of Eq. (2.1) to have unique solutions on $[a, b]$; see Everitt and Race (1976), Zettl (2005).

Since the boundary conditions are invariant under left multiplication by a nonsingular matrix, we use the notation $\mathcal{A} = [A : B]$ to denote the equivalence class of BC (2.2). These equivalence classes, endowed with the topology induced by any matrix norm, form the boundary condition quotient space denoted by $\mathbb{B} := M_{2\times4}(\mathbb{C})/GL(2, \mathbb{C})$, where $M_{2\times4}(\mathbb{C})$ is the class of 2×4 matrices over $\mathbb{C}$, and $GL(2, \mathbb{C})$ is the set of nonsingular matrices of order 2 over $\mathbb{C}$. Denote by $\mathbb{B}_s$ the subset of $\mathbb{B}$ consisting of the self-adjoint BCs. $\mathbb{D}_i := \{[C_i : D_i] | \det(C_i) > 0, \; \det(D_i) > 0\}$ denotes the equivalence class of n transmission condition (2.3). Denote by $\mathbb{D}_s$ the subset of $\mathbb{D}$ consisting of the self-adjoint transmission condition. Let

$$\Omega = \{\omega = (a, b, p, q, w) : (2.4) \quad \text{holds}\},$$

and endow Ω with the topology as in Kong et al. (1999). Then the space of SLPs in which we study the dependence of eigenvalues on the problem is given by $\Omega \times \mathbb{B} \times \mathbb{D}(\mathbb{D} = [C_n : D_n] \times [C_{n-1} : D_{n-1}] \times \cdots \times [C_1 : D_1])$ and is called the SLP space with n transmission conditions, and $\Omega \times \mathbb{B}_s \times \mathbb{D}_s$ is called the self-adjoint SLP space with n transmission conditions where p, q, w are real-valued.

Remark 2.1 As usual, the self-adjoint extension of SLPs with n transmission conditions need additional restrictions on C_i, D_i and a new weighted Hilbert space defined as in Mukhtarov et al. (2002b, 2004), Sun and Wang (2008). With this weighted Hilbert space the operator associated with Sturm–Liouville problems with n transmission conditions is self-adjoint if and only if the associated new operator is self-adjoint, and they consist of the same eigenvalues, and satisfying the condition

$$\theta_1 \cdots \theta_n AE^{-1}A^* = \rho_1 \cdots \rho_n BE^{-1}B^*, \text{with } E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

For further details of the self-adjointness of SLPs with n transmission conditions please see Sun and Wang (2008). The results in this paper are not restricted to self-adjoint problems, but include non-self-adjoint problems.

Let $u = y, v = py'$. Then Eq. (2.1) can be transferred into the following first order system:

$$u' = rv, \quad v' = (q - \lambda w)u, \quad \text{on } J. \tag{2.5}$$

Definition 2.1 By trivial solution of Eq. (2.1) on some intervals we mean a solution y which is identically zero and whose quasi-derivative $v = py'$ is also identically zero on this interval.

Lemma 2.1 *Let (2.4) hold and let $\Phi(x, \lambda) = [\phi_{st}(x, \lambda)]$ denote the fundamental matrix of the system (2.5) determined by the initial condition $\Phi(a, \lambda) = I$. Then a complex number λ is an eigenvalue of the Sturm–Liouville problem with n transmission conditions (2.1)–(2.3)* if and only if

$$\Delta(\lambda) = \det[A + B\Phi(b, \lambda)] = 0. \tag{2.6}$$

And in further $\Delta(\lambda)$ can be written as

$$\begin{aligned} \Delta(\lambda) = &\det(A) + \det(B) + h_{11}\phi_{11}(b, \lambda) + h_{12}\phi_{12}(b, \lambda) \\ &+ h_{21}\phi_{21}(b, \lambda) + h_{22}\phi_{22}(b, \lambda), \end{aligned} \tag{2.7}$$

where

$$H = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} := \begin{bmatrix} a_{22}b_{11} - a_{12}b_{21} & a_{11}b_{21} - a_{21}b_{11} \\ a_{22}b_{12} - a_{12}b_{22} & a_{11}b_{22} - a_{21}b_{12} \end{bmatrix}.$$

Proof The proof of the first part of this lemma is similar to the one in Mukhtarov et al. (2004), hence is omitted here. And the second part comes from a straightforward computation. □

Definition 2.2 The SLP with n transmission conditions (2.1)–(2.3), or equivalently (2.2), (2.3), (2.5) is said to be degenerate if in (2.7) either $\Delta(\lambda) \equiv 0$ for all $\lambda \in \mathbb{C}$ or $\Delta(\lambda) \neq 0$ for any $\lambda \in \mathbb{C}$.

In the derivation of our main results an important role is played by the "Continuity Principle" established in Kong et al. (1999), which reads.

Lemma 2.2 *Let $\mathcal{N} \subset \mathbb{C}$ be a bounded open set in the complex plane $\mathbb{C}$ and let $m \in \mathbb{N}$. If an SLP $(\omega, \mathcal{A}) \in \Omega \times \mathbb{B}$ has exactly m eigenvalues, counting multiplicity, in $\mathcal{N}$, and none on the boundary of $\mathcal{N}$, then every SLP $(\sigma, \mathcal{B})$ sufficiently close to $(\omega, \mathcal{A})$ also has exactly m eigenvalues, counting multiplicity, in $\mathcal{N}$.*

Proof See Kong et al. (1999), Kong and Zettl (1996). □

This lemma can be easily generated to SLP space with n transmission conditions.

3 The Finite Spectrum of Sturm–Liouville Problems with *n* Transmission Conditions

In this section, we assume (2.4) holds and there exists a partition of interval J

$$\begin{aligned} a = a_0^1 < a_1^1 < a_2^1 < \cdots < a_{2m_1}^1 < a_{2m_1+1}^1 = c_1, \text{on } [a, c_1),\\ c_1 = a_0^2 < a_1^2 < a_2^2 < \cdots < a_{2m_2}^2 < a_{2m_2+1}^2 = c_2, \text{on } (c_1, c_2),\\ \cdots\cdots\cdots\cdots\cdots\cdots \\ c_n = a_0^{n+1} < a_1^{n+1} < a_2^{n+1} < \cdots < a_{2m_{n+1}}^{n+1} < a_{2m_{n+1}+1}^{n+1} \\ = b, \text{on } (c_n, b], \end{aligned} \tag{3.1}$$

for n and some integers $m_i, i = 1, 2, \ldots, n+1$, such that

$$r = \frac{1}{p} = 0 \text{ on } (a_{2k}^i, a_{2k+1}^i), \int_{a_{2k}^i}^{a_{2k+1}^i} w \neq 0, \quad k = 0, 1, \ldots, m_i, i = 1, 2, \ldots, n+1, \tag{3.2}$$

and

$$q = w = 0 \text{ on } (a_{2k+1}^i, a_{2k+2}^i), \int_{a_{2k+1}^i}^{a_{2k+2}^i} r \neq 0, \quad k = 0, 1, 2, \ldots, m_i - 1, i = 1, 2, \ldots, n+1. \tag{3.3}$$

Next we let

$$\begin{aligned} r_k^i &= \int_{a_{2k+1}^i}^{a_{2k+2}^i} r(x)\mathrm{d}x, k = 0, 1, \ldots, m_i - 1, i = 1, 2, \ldots, n+1;\\ q_k^i &= \int_{a_{2k}^i}^{a_{2k+1}^i} q(x)\mathrm{d}x, k = 0, 1, \ldots, m_i, i = 1, 2, \ldots, n+1;\\ w_k^i &= \int_{a_{2k}^i}^{a_{2k+1}^i} w(x)\mathrm{d}x, k = 0, 1, \ldots, m_i, i = 1, 2, \ldots, n+1. \end{aligned} \tag{3.4}$$

These notations will be used in our iterative construction process.

Following Kong et al. (2001), Ao et al. (2011), we first determine the structure of the principal fundamental matrix of system (2.5) which, together with the "Continuity Principle", is basic to our results.

Lemma 3.1 *Let (2.4) and (3.1)–(3.3) hold. Let $\Phi(x, \lambda) = [\phi_{st}(x, \lambda)]$ be the fundamental matrix solution of the system (2.5) determined by the initial condition $\Phi(a, \lambda) = I$ for each $\lambda \in \mathbb{C}, x \in [a, c_1)$. Then we have that*

$$\Phi(a_1^1, \lambda) = \begin{bmatrix} 1 & 0 \\ q_0^1 - \lambda w_0^1 & 1 \end{bmatrix} \Phi(a, \lambda), \tag{3.5}$$

$$\Phi(a_3^1, \lambda) = \begin{bmatrix} 1 + (q_0^1 - \lambda w_0^1) r_0^1 & r_0^1 \\ \phi_{21}^1(a_3^1, \lambda) & 1 + (q_1^1 - \lambda w_1^1) r_0^1 \end{bmatrix}, \tag{3.6}$$

where

$$\phi_{21}^1(a_3^1, \lambda) = (q_0^1 - \lambda w_0^1) + (q_1^1 - \lambda w_1^1) + (q_0^1 - \lambda w_0^1)(q_1^1 - \lambda w_1^1) r_0^1 .$$

And in general, for $1 \leq k \leq m_1$

$$\Phi(a_{2k+1}^1, \lambda) = \begin{bmatrix} 1 & r_{k-1}^1 \\ q_k^1 - \lambda w_k^1 & 1 + (q_k^1 - \lambda w_k^1) r_{k-1}^1 \end{bmatrix} \Phi(a_{2k-1}^1, \lambda). \tag{3.7}$$

Proof Observe from (2.5) that u is constant on each subinterval where r is identically zero and v is constant on each subinterval where both q and w are identically zero. The result follows from repeated applications of (2.5). □

Lemma 3.2 *Let (2.4) and (3.1)–(3.3) hold. Let*

$$\Phi_i(x, \lambda) = [\phi_{st}^i(x, \lambda)](x \in (c_i, c_{i+1}), c_{n+1} = b = a_{2m_{n+1}+1}^{n+1})$$

be the fundamental matrix solution of the system (2.5) determined by the initial condition $\Phi_i(c_i+, \lambda) = I$ (here $\Phi_i(c_i+, \lambda) = \Phi_i(a_0^{i+1}, \lambda) = \Phi(c_i+, \lambda), i = 1, 2, \ldots, n)$ denote the right limit at point c_i for each $\lambda \in \mathbb{C}$. Then we have that

$$\Phi_i(a_1^{i+1},\lambda)=\begin{bmatrix}1 & 0\\ q_0^{i+1}-\lambda w_0^{i+1} & 1\end{bmatrix}\Phi_i(a_0^{i+1},\lambda), \quad (3.8)$$

$$\Phi_i(a_3^{i+1},\lambda)=\begin{bmatrix}1+(q_0^{i+1}-\lambda w_0^{i+1})r_0^{i+1} & r_0^{i+1}\\ \phi_{21}^{i+1}(a_3^{i+1},\lambda) & 1+(q_1^{i+1}-\lambda w_1^{i+1})r_0^{i+1}\end{bmatrix} \quad (3.9)$$

where

$$\phi_{21}^{i+1}(a_3^{i+1},\lambda)=(q_0^{i+1}-\lambda w_0^{i+1})+(q_1^{i+1}-\lambda w_1^{i+1})+(q_0^{i+1}-\lambda w_0^{i+1})(q_1^{i+1}-\lambda w_1^{i+1})r_0^{i+1}.$$

And in general, for $1\le k\le m_{i+1}$,

$$\Phi_i(a_{2k+1}^{i+1},\lambda)=\begin{bmatrix}1 & r_{k-1}^j\\ q_k^{i+1}-\lambda w_k^{i+1} & 1+(q_k^{i+1}-\lambda w_k^{i+1})r_{k-1}^{i+1}\end{bmatrix}\Phi_i(a_{2k-1}^{i+1},\lambda). \quad (3.10)$$

Proof The proof is similar to the one as in Lemma 3.1. □

Lemma 3.3 *Let (2.4) and (3.1)–(3.3) hold. Let* $\Phi(x,\lambda)=[\phi_{st}(x,\lambda)]$ *be the fundamental matrix solution of the system (2.5) determined by the initial condition* $\Phi(a,\lambda)=I$ *for each* $\lambda\in\mathbb{C}$, *and* $\Phi_i(x,\lambda)=[\phi_{st}^i(x,\lambda)]$ *be given as in Lemma 3.2. Then we have that*

$$\Phi(b,\lambda)=\Phi_n(b,\lambda)G_n\Phi_{n-1}(c_n,\lambda)G_{n-1}\Phi_{n-2}(c_{n-1},\lambda)\cdots G_1\Phi(c_1,\lambda), \quad (3.11)$$

where $G_i=(g_{st})_{2\times2}=-D_i^{-1}C_i$, *and*

$$\Phi(c_1,\lambda)=\Phi(c_1-,\lambda)=\Phi(a_{2m_1+1}^1,\lambda),$$

$$\Phi_i(c_{i+1},\lambda)=\Phi_i(c_{i+1}-,\lambda)=\Phi_i(a_{2m_{i+1}+1}^{i+1},\lambda)=\Phi(c_{i+1},\lambda),(i=1,2,\ldots,n-1),$$

$$\Phi_n(b,\lambda)=\Phi_n(a_{2m_{n+1}+1}^{n+1},\lambda)$$

denote the left limit at point $c_i(i=1,2,\ldots,n)$.

Proof From the transmission conditions (2.3) we know that

$$C_i\Phi(c_i-,\lambda)+D_i\Phi(c_i+,\lambda)=0,$$

thus

$$\Phi(c_i+,\lambda)=-D_i^{-1}C_i\Phi(c_i-,\lambda).$$

By using the Lemma 3.3 in Ao et al. (2011), when $i=1$, in $(a,c_1)\cup(c_1,c_2)$, we have that

$$\Phi(c_2,\lambda)=\Phi_1(c_2,\lambda)G_1\Phi(c_1,\lambda),$$

when $i=2$, we have that in $(a,c_1)\cup(c_1,c_2)\cup(c_2,c_3)$,

$$\Phi(c_3,\lambda)=\Phi_2(c_3,\lambda)G_2\Phi_1(c_2,\lambda)G_1\Phi(c_1,\lambda).$$

By repeated application of Lemmas 3.1, 3.2 and the Lemma 3.3 in Ao et al. (2011), it can be concluded that (3.11) follows. □

Note that $c_i=a_{2m_i+1}^i=a_0^{i+1}$, $b=a_{2m_{n+1}+1}^{n+1}$ and (3.11). Then the structure of fundamental matrix solution $\Phi(b,\lambda)$ given in Lemmas 3.1, 3.2 and mathematical induction yield the following.

Corollary 3.1 *If* $g_{12}^i\neq0, i=1,2,\ldots,n$, *then for the fundamental matrix* $\Phi(b,\lambda)$ *we have that*

$$\phi_{11}(b,\lambda)=G_{0n}\cdot\prod_{i=1}^{n-1}g_{12}^i\cdot\prod_{i=1}^{n+1}R_i\cdot\prod_{i=1}^{m_{n+1}-1}\cdot\prod_{i=0}^{m_n-1}\cdot\prod_{i=0}^{m_{n-1}}\cdots\prod_{i=0}^{m_2}\cdot\prod_{i=0}^{m_1}+\tilde{\phi}_{11}(\lambda), \quad (3.12)$$

$$\phi_{12}(b,\lambda)=G_{0n}\cdot\prod_{i=1}^{n-1}g_{12}^i\cdot\prod_{i=1}^{n+1}R_i\cdot\prod_{i=1}^{m_{n+1}-1}\cdot\prod_{i=0}^{m_n-1}\cdot\prod_{i=0}^{m_{n-1}}\cdots\prod_{i=0}^{m_2}\cdot\prod_{i=1}^{m_1}+\tilde{\phi}_{12}(\lambda), \quad (3.13)$$

$$\phi_{21}(b,\lambda)=G_{0n}\cdot\prod_{i=1}^{n-1}g_{12}^i\cdot\prod_{i=1}^{n+1}R_i\cdot\prod_{i=1}^{m_{n+1}}\cdot\prod_{i=0}^{m_n-1}\cdot\prod_{i=0}^{m_{n-1}}\cdots\prod_{i=0}^{m_2}\cdot\prod_{i=0}^{m_1}+\tilde{\phi}_{21}(\lambda), \quad (3.14)$$

$$\phi_{22}(b,\lambda)=G_{0n}\cdot\prod_{i=1}^{n-1}g_{12}^i\cdot\prod_{i=1}^{n+1}R_i\cdot\prod_{i=1}^{m_{n+1}}\cdot\prod_{i=0}^{m_n-1}\cdot\prod_{i=0}^{m_{n-1}}\cdots\prod_{i=0}^{m_2}\cdot\prod_{i=1}^{m_1}+\tilde{\phi}_{22}(\lambda), \quad (3.15)$$

where

$$G_{0n}=[g_{12}^n(q_{m_n}^n-\lambda\omega_{m_n}^n)(q_0^{n+1}-\lambda\omega_0^{n+1})+g_{11}^n(q_0^{n+1}-\lambda\omega_0^{n+1})+g_{22}^n(q_{m_n}^n-\lambda\omega_{m_n}^n)+g_{21}^n],$$

$$R_i=\prod_{k=0}^{m_i-1}r_k^i, i=1,2,\ldots,n+1,$$

$$\prod_{k=1}^{m_{n+1}-1}=\prod_{k=1}^{m_{n+1}-1}(q_k^{n+1}-\lambda w_k^{n+1}),$$

$$\prod_{k=0}^{m_n-1}=\prod_{k=0}^{m_n-1}(q_k^n-\lambda w_k^n),$$

$$\prod_k^{m_i}=\prod_k^{m_i}(q_k^i-\lambda w_k^i),\ i=1,2,\ldots,n-1,$$

$\tilde{\phi}_{st}(\lambda) = o(\prod_{i=1}^{n+1} R_i), s, t = 1, 2$ *as* $\min\{r_k^i\} \to \infty$ *for fixed* q, w *and* λ.

From Corollary 3.1 we can see that each of the entries in Φ (i.e. $\phi_{st}, s, t = 1, 2$) is a polynomial of λ.

Now we construct regular SLPs with n transmission conditions (2.3) with general self-adjoint and non-self-adjoint BCs (2.2) which have exactly m eigenvalues for each $m \in \mathbb{N}$.

Theorem 3.1 *Let* $m_i \in \mathbb{N} (i = 1, 2, \ldots, n+1)$, $g_{12}^i \neq 0, i = 1, 2, \ldots, n$, *and let* (2.4) *and* (3.1)–(3.3) *hold. Let* $H = (h_{st})_{2\times 2}$ *be defined as in Lemma* 2.1. *Then:*

(1) *If* $h_{21} \neq 0$, *then the SLP with* n *transmission conditions* (2.1)–(2.3) *has exactly* $\sum_{i=1}^{n+1} m_i + n + 1$ *eigenvalues* $\lambda_j, j = 0, 1, \ldots, \sum_{i=1}^{n+1} m_i + n$.

(2) *If* $h_{21} = 0$, *and* $h_{11}\omega_0^1 + h_{22}\omega_{m_{n+1}}^{n+1} \neq 0$, *then the SLP with* n *transmission conditions* (2.1)–(2.3) *has exactly* $\sum_{i=1}^{n+1} m_i + n$ *eigenvalues* $\lambda_j, j = 0, 1, \ldots, \sum_{i=1}^{n+1} m_i + n - 1$.

(3) *If* $h_{21} = h_{11} = h_{22} = 0$, *but* $h_{12} \neq 0$, *then the SLP with* n *transmission conditions* (2.1)–(2.3) *has exactly* $\sum_{i=1}^{n+1} m_i + n - 1$ *eigenvalues* $\lambda_j, j = 0, 1, \ldots, \sum_{i=1}^{n+1} m_i + n - 2$.

(4) *If none of the above conditions holds, then the SLP with n transmission conditions* (2.1)–(2.3) *either has* l *eigenvalues for* $l \in \{1, 2, \ldots, \sum_{i=1}^{n+1} m_i + n - 2\}$ *or is degenerate.*

Proof We prove the case (1), and the other cases can be proved in the same way. From Lemma 2.1 we know that

$$\Delta(\lambda) = \det(A) + \det(B) + h_{11}\phi_{11}(b, \lambda) + h_{12}\phi_{12}(b, \lambda) + h_{21}\phi_{21}(b, \lambda) + h_{22}\phi_{22}(b, \lambda),$$

note that from (3.2) and Corollary 3.1 that the degree of $\phi_{11}(b, \lambda), \phi_{12}(b, \lambda), \phi_{21}(b, \lambda), \phi_{22}(b, \lambda)$ in λ are $\sum_{i=1}^{n+1} m_i + n, \sum_{i=1}^{n+1} m_i + n - 1, \sum_{i=1}^{n+1} m_i + n + 1, \sum_{i=1}^{n+1} m_i + n$, respectively. Thus when $h_{21} \neq 0$, we can conclude from (2.7) that the characteristic function $\Delta(\lambda)$ is also a polynomial function of λ and with the degree of $\sum_{i=1}^{n+1} m_i + n + 1$, hence from Fundamental Theorem of Algebra we know that $\Delta(\lambda)$ has exactly $\sum_{i=1}^{n+1} m_i + n + 1$ roots, i.e. SLP (2.1)–(2.3) has exactly $\sum_{i=1}^{n+1} m_i + n + 1$ eigenvalues. Then we complete the proof of case (1). □

Theorem 3.2 *Let* $m_i \in \mathbb{N} (i = 1, 2, \ldots, n+1)$, $g_{12}^n = 0$, *but* $g_{11}^n\omega_0^{n+1} + g_{22}^n\omega_{m_n}^n \neq 0$, $g_{12}^i \neq 0, i = 1, 2, \ldots, n-1$, *and let* (2.4) *and* (3.1)–(3.3) *hold. Let* $H = (h_{st})_{2\times 2}$ *be defined as in Lemma* 2.1. *Then:*

(1) *If* $h_{21} \neq 0$, *then the SLP with* n *transmission conditions* (2.1)–(2.3) *has exactly* $\sum_{i=1}^{n+1} m_i + n$ *eigenvalues* $\lambda_j, j = 0, 1, \ldots, \sum_{i=1}^{n+1} m_i + n - 1$.

(2) *If* $h_{21} = 0$, *and* $h_{11}\omega_0^1 + h_{22}\omega_{m_{n+1}}^{n+1} \neq 0$, *then the SLP with* n *transmission conditions* (2.1)–(2.3) *has exactly* $\sum_{i=1}^{n+1} m_i + n - 1$ *eigenvalues* $\lambda_j, j = 0, 1, \ldots, \sum_{i=1}^{n+1} m_i + n - 2$.

(3) *If* $h_{21} = h_{11} = h_{22} = 0$, *but* $h_{12} \neq 0$, *then the SLP with* n *transmission conditions* (2.1)–(2.3) *has exactly* $\sum_{i=1}^{n+1} m_i + n - 2$ *eigenvalues* $\lambda_j, j = 0, 1, \ldots, \sum_{i=1}^{n+1} m_i + n - 3$.

(4) *If none of the above conditions holds, then the SLP with n transmission conditions* (2.1)–(2.3) *either has* l *eigenvalues for* $l \in \{1, 2, \ldots, \sum_{i=1}^{n+1} m_i + n - 3\}$ *or is degenerate.*

Proof The proof is similar with Theorem 3.1 only by noting that $g_{12}^n = 0$, but $g_{11}^n\omega_0^{n+1} + g_{22}^n\omega_{m_n}^n \neq 0$, and the degree of λ will decrease by one, hence is omitted here. □

The next theorem will show that these eigenvalues can be located anywhere in the complex plane in the non-self-adjoint case and anywhere along the real line in the self-adjoint case.

Theorem 3.3 *Given any k disjoint open sets* $\mathcal{N}_i$ *in* $\mathbb{C}$ *and any k integers* n_i, *there exists an SLP with n transmission conditions with exactly* n_i *eigenvalues in* $\mathcal{N}_i$, *for* $i = 1, 2, \ldots, k$. *Given any k disjoint open intervals* J_i *of the real line and any k integers* n_i, *there exists a self-adjoint SLP with n transmission conditions with exactly* n_i *eigenvalues in the intervals* J_i, *for* $i = 1, 2, \ldots, k$.

Proof We prove the former case, and the latter can be proved in the same way. Let $\sum_{i=1}^{n+1} m_i + n = \sum_{i=1}^{k} n_i$. Construct an SLP with n transmission conditions in the form of (2.1), (2.2) and (2.3) with the assumptions (2.4) and (3.1)–(3.4), $g_{12}^i \neq 0, i = 1, 2, \ldots, n$, $a_{11} = a_{21} = a_{22} = b_{22} = 0$, and $a_{12} = b_{21} = 1 (or a_{11} = a_{12} = a_{21} = b_{12} = 0, a_{22} = b_{11} = 1)$. Then by Corollary 3.1 the characteristic function defined by (2.7) becomes

$$\Delta(\lambda) = \phi_{11}(b, \lambda) = G_{0n} \cdot \prod_{i=1}^{n-1} g_{12}^i \cdot \prod_{i=1}^{n+1} R_i \cdot \prod_{i=1}^{m_{n+1}-1} \cdot \prod_{i=0}^{m_n - 1} \cdot \prod_{i=0}^{m_{n-1}} \cdots \prod_{i=0}^{m_2} \cdot \prod_{i=0}^{m_1} + \tilde{\phi}_{11}(\lambda), \tag{3.16}$$

where $\tilde{\phi}_{11}(\lambda) = o(\prod_{i=1}^{n+1} R_i)$ as $\min\{r_k^i\} \to \infty$ for fixed q, w and λ. Since q and w can be chosen arbitrarily, we can choose them such that

$$\tilde{\Delta}(\lambda) := \prod_{i=0}^{m_{n+1}-1} \cdot \prod_{i=0}^{m_n} \cdot \prod_{i=0}^{m_{n-1}} \cdots \prod_{i=0}^{m_2} \cdot \prod_{i=0}^{m_1}$$

has exactly n_i roots in $\mathcal{N}_i$ and none on the boundary of $\mathcal{N}_i, i = 1, 2, \ldots, k$. Choose $r_k^i, k = 0, 1, 2, \ldots, m_{n+1} - 1$, $i = 1, 2, \ldots, n$ and $|g_{12}^n|$ so large that

$$|\tilde{\phi}_{11}(\lambda)| < |g_{12}^n| \cdot |\prod_{i=1}^{n-1} g_{12}^i| \cdot \prod_{i=1}^{n+1} R_i \cdot \prod_{i=0}^{m_{n+1}-1} \cdot \prod_{i=0}^{m_n} \cdot \prod_{i=0}^{m_{n-1}} \cdots \prod_{i=0}^{m_2} \cdot \prod_{i=0}^{m_1}. \tag{3.17}$$

Then it follows from Rouche's theorem that the $\Delta(\lambda)$ has exactly n_i roots in $\mathcal{N}_i, i = 1, 2, \ldots, k$.

The other case of $\Delta(\lambda)$ can be proved similarly. □

Remark 3.1 If $n = 0$, the results will reduce to the finite spectrum of general SLP in Kong et al. (2001). If $n = 1$, the results will reduce to the results in Ao et al. (2011), where only one transmission condition at an interior point is considered.

Finally, we give an example to illustrate our main result.

Example 1 Let $n = 2$ and consider the SLP with two transmission conditions

$$\begin{cases} -(py')' + qy = \lambda wy, \ on\ J = (-5, -2) \cup (-2, 1) \cup (1, 6), \\ (py')(-5) = 0, \quad y(6) = 0, \\ -2(py')(-2-) + y(-2+) = 0, \quad y(-2-) + (py')(-2+) = 0, \\ 2y(1-) - (py')(1+) = 0, \quad (py')(1-) + y(1+) = 0. \end{cases} \tag{3.18}$$

Choose $m_1 = 1, m_2 = 1, m_3 = 2$ and suppose $p, \ q, \ w$ are piecewise polynomial functions defined as follows:

$$p(x) = \begin{cases} \infty, (-5, -4) \\ 1, (-4, -3) \\ \infty, (-3, -2) \\ \infty, (-2, -1) \\ 1/2, (-1, 0) \\ \infty, (0, 1) \\ \infty, (1, 2) \\ 1, (2, 3) \\ \infty, (3, 4) \\ 1/2, (4, 5) \\ \infty, (5, 6), \end{cases} \quad q(x) = \begin{cases} 0, (-5, -4) \\ 0, (-4, -3) \\ 1, (-3, -2) \\ 2, (-2, -1) \\ 0, (-1, 0) \\ -1, (0, 1) \\ 1, (1, 2) \\ 0, (2, 3) \\ 3, (3, 4) \\ 0, (4, 5) \\ 1, (5, 6), \end{cases} \quad w(x) = \begin{cases} 1, (-5, -4) \\ 0, (-4, -3) \\ 1, (-3, -2) \\ 1, (-2, -1) \\ 0, (-1, 0) \\ 1, (0, 1) \\ 1, (1, 2) \\ 0, (2, 3) \\ 1, (3, 4) \\ 0, (4, 5) \\ 1, (5, 6). \end{cases} \tag{3.19}$$

From (3.18) we have

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, C_1 = \begin{bmatrix} 0 & -2 \\ 1 & 0 \end{bmatrix}, D_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$C_2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \ D_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

and

$$\mathbf{det}(C_1) = 2 > 0, \quad \mathbf{det}(D_1) = 1 > 0,$$
$$\mathbf{det}(C_2) = 2 > 0, \quad \mathbf{det}(D_2) = 1 > 0,$$

$$G_1 = -D_1^{-1}C_1 = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}, G_2 = -D_2^{-1}C_2 = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix},$$
$$g_{12}^1 = 2 \neq 0, g_{12}^2 = 1 \neq 0.$$

By deduction it can be obtained that the characteristic function

$$\Delta(\lambda) = -8\lambda^6 + 92\lambda^5 - 348\lambda^4 + 398\lambda^3 + 357\lambda^2 - 849\lambda + 224.$$

Hence the SLP (3.18), (3.19) has exactly $m_1 + m_2 + m_3 + n = 6$ eigenvalues

$$\lambda_0 = -1.2260, \lambda_1 = 0.3174, \lambda_2 = 1.9598, \lambda_3 = 2.3213,$$
$$\lambda_4 = 3.2305, \lambda_5 = 4.8971.$$

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