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RESEARCH

The modularity of Siegel's zeta functions

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Abstract

Siegel defined zeta functions associated with indefinite quadratic forms, and proved their analytic properties such as analytic continuations and functional equations. Coefficients of these zeta functions are called measures of representations, and play an important role in the arithmetic theory of quadratic forms. In a 1938 paper, Siegel made a comment to the effect that the modularity of his zeta functions would be proved with the help of a suitable converse theorem. In the present paper, we accomplish Siegel's original plan by using a Weil-type converse theorem for Maass forms, which has appeared recently. It is also shown that "half" of Siegel's zeta functions correspond to holomorphic modular forms.

Keywords: Siegel's zeta functions, Maass forms, Modular forms, Weil-type converse theorems, Theory of prehomogeneous vector spaces

Introduction

In 1903, Epstein [3] defined the zeta function $\zeta_0(s)$ associated with a positive definite symmetric matrix S of degree m by

$$\zeta_0(s) = \sum_{a \in \mathbb{Z}^m \setminus \{0\}} \frac{1}{S[a]^s} \qquad (S[a] = {}^t aSa),$$

and studied their analytic properties such as analytic continuations and functional equations. (For a modern treatment of Epstein's zeta functions, we refer to Terras [31, §1.4.2].) In a 1938 paper [23], Siegel defined and investigated the zeta functions associated with quadratic forms of signature $(1, m-1)$, and in a 1939 paper [24], those for general indefinite quadratic forms. Although Siegel's calculations were rather involved, Siegel's results are now well understood in the framework of the theory of prehomogeneous vector spaces. Let Y be a non-degenerate half-integral symmetric matrix of degree m with p positive eigenvalues and $m-p$ negative eigenvalues ($0 < p < m$). Let $SO(Y)$ be the special orthogonal group of Y and denote by $SO(Y)_{\mathbb{Z}}$ its arithmetic subgroup. We put $V_{\pm} = \{v \in \mathbb{R}^m\,;\, \mathrm{sgn}\, Y[v] = \pm\}$. Then Siegel's zeta functions are Dirichlet series associated with the prehomogeneous vector space $(GL_1(\mathbb{C}) \times SO(Y),\ \mathbb{C}^m)$, and are defined by

$$\zeta_{\pm}(s) = \sum_{v \in SO(Y)_{\mathbb{Z}} \setminus (\mathbb{Z}^m \cap V_{\pm})} \frac{\mu(v)}{|Y[v]|^s},$$

where the sum runs over a complete set of representatives of $SO(Y)_{\mathbb{Z}} \setminus (\mathbb{Z}^m \cap V_{\pm})$, and $\mu(v)$ is a certain volume of the fundamental domain related to the isotropy subgroup $SO(Y)_v$

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of $SO(Y)$ at v. In the positive definite case, the *modularity* of Epstein's zeta function $\zeta_0(s)$ is almost obvious; $\zeta_0(s)$ is obtained by taking the Mellin transform of (the restriction to the imaginary axis of) the theta series

$$\theta(S,z)=\sum_{a\in\mathbb{Z}^m} e^{\pi i S[a]z} \qquad (z=x+iy\in\mathbb{C},\ y>0),$$

which is a modular form for a subgroup of $SL_2(\mathbb{Z})$. (cf. Miyake [15, §4.9], Terras [31, §3.4.4].) On the contrary, it is not clear from the definition whether or not Siegel's zeta function arises as an integral transform of some infinite series with modular properties. Rather, in the preface to a 1938 paper [23], Siegel wrote that such theta series would be constucted from his zeta functions, citing the work of Hecke [7], in which Hecke derived the transformation formula for the theta series associated with indefinite binary quadratic forms from the functional equation of zeta functions of real quadratic fields. Furthermore, Siegel made the following remark in the last section of [23]:

> Will man die Transformationstheorie von $f(\mathfrak{S},x)$ für beliebige Modulsubstitutionen entwickeln, so hat man außer $\zeta_1(\mathfrak{S},s)$ auch analog gebildete Zetafunktionen mit Restklassen-Chrakteren zu untersuchen. Die zum Beweise der Sätze 1, 2,3 führenden Überlegungen lassen sich ohne wesentiche Schwierigkeit auf den allgemeinen Fall übertragen. Vermöge der Mellinschen Transformation erhält man dann das wichtige Resultat, daß die durch (53) definierte Funktion $f(\mathfrak{S},x)$ eine Modulform der Dimension $\frac{n}{2}$ und der Stufe $2D$ ist; dabei wird vorausgesetzt, daß n ungerade und $\mathfrak{x}'\mathfrak{S}\mathfrak{x}$ keine ternäre Nullform ist.
>
> (A translation) If one wants to develop the transformation theory of $f(\mathfrak{S},x)$ for arbitrary modular substitutions, then in addition to $\zeta_1(\mathfrak{S},s)$ one also has to investigate zeta functions formed analogously with residual class characters. The considerations leading to the proof of Theorems 1, 2, 3 can be transferred to the general case without any major difficulty. By virtue of the (inverse) Mellin transformation, one then obtains an important result that the function $f(\mathfrak{S},x)$ defined by (53) is a modular form of weight $\frac{n}{2}$ and level $2D$, provided that n is odd and $\mathfrak{x}'\mathfrak{S}\mathfrak{x}$ is not a ternary zero form.

As of 1938, Siegel seemed to have noticed the possibility that by considering the twists of zeta functions by Dirichlet characters, one can prove modularity for congruence subgroups. In the holomorphic case, this fact is known as Weil's converse theorem [34]. It was 1967 when Weil's paper [34] appeared! Revisiting Siegel's prediction in the light of recent developments is one motivation for the present study.

We should note that in the quotation above, Siegel mentioned the parity of n, the number of variables of quadratic forms. This is related to the fact that the concept of non-holomorphic modular forms was not yet in place at that time. In a celebrated paper [12], Maaß introduced the notion of the so-called Maass forms and established a Hecke correspondence for Maass forms. Further, in [13], as its application of his theorem, Maaß proved that in a very special case (when Y is diagonal of even degree with $\det Y=1$), Siegel's zeta functions can be expressed as the product of two standard Dirichlet series such as the Riemann zeta function $\zeta(s)$ and the Dirichlet L-function $L(s,\chi)$. On the other hand, it is only recently that papers on Weil-type converse theorems for Maass forms have emerged (cf. [16,17]). It would be a very natural idea for us to accomplish Siegel's original

plan to prove the modularity via converse theorem including the case of non-holomorphic forms.

Siegel's zeta functions are closely related to the so-called Siegel's main theorem (*Siegelsche HauptSatz*). In a 1951 paper [26], Siegel proved the transformation formula for some theta series arising from indefinite quadratic forms, and the equality between an integral of the indefinite theta series over fundamental domains and some Eisenstein series (cf. [26, Satz 1]). It was shown in [26, Hilfssatz 4] that the coefficients

$$M(Y;\pm n)=\sum_{\substack{v\in SO(Y)_{\mathbb{Z}}\backslash(\mathbb{Z}^m\cap V_{\pm})\\ Y[v]=\pm n}}\mu(v)\qquad(n=1,2,3,\dots)$$

of $\zeta_{\pm}(s)$ coincide with the Fourier coefficients of the non-holomorphic modular forms appearing in Siegel's formula. Here we ignore the differences in the definitions of $\mu(v)$; the definitions of measures are different for each of the papers [23,24,26]. Siegel called $M(Y;n)$ the measures of representations (*Darstellungsmaß*). The measure $M(Y;n)$ of representations is an analogue of the representation number

$$r(S;n)=\sharp\{a\in\mathbb{Z}^m\ ;\ S[a]=n\}$$

for a positive symmetric matrix S, and Siegel's formula can be reformulated as an arithmetic identity that $M(Y,n)$ is equal to the product of local representation densities over all primes. Weil [33] generalized Siegel's result by using the language of adeles, and it is the Siegel-Weil formula—a cornerstone in the modern number theory.

Now we explain the main results of the present paper. First, along the Sato-Shintani theory [21] of prehomogeneous vector spaces, we define Siegel's zeta functions and prove their analytic properties. Here, to treat twisted zeta functions as well as the original Siegel's zeta functions, we first consider Siegel's zeta functions *with congruence conditions*, which are defined using Schwartz-Bruhat functions on $\mathbb{Q}^m$. This idea is due to Sato [20]. Then the converse theorem in [16] is applied to the zeta functions, and the following result is obtained:

Main result 1 (Theorem 2) *Let $m\geq 5$. Assume that at least one of m or p is an odd integer. Take an integer ℓ with $\ell\equiv 2p-m \pmod 4$, and put $D=\det(2Y)$. Let N be the level of $2Y$. Define C^{∞}-function $F(z)=F(x+iy)$ on the Poincaré upper half-plane $\mathcal{H}$ by*

$$F(z)=y^{(m-\ell)/4}\cdot\int_{SO(Y)_{\mathbb{R}}/SO(Y)_{\mathbb{Z}}}d^1g+\alpha(0)\cdot y^{1-(m+\ell)/4}$$
$$+\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}(-1)^{(2p-m-\ell)/4}\cdot\frac{M(Y;n)}{|D|^{\frac{1}{2}}}\frac{\pi^{\frac{m}{4}}\cdot|n|^{-\frac{m}{4}}}{\Gamma\left(\frac{m+\mathrm{sgn}(n)\ell}{4}\right)}\cdot y^{-\frac{\ell}{4}}W_{\frac{\mathrm{sgn}(n)\ell}{4},\frac{m}{4}-\frac{1}{2}}(4\pi|n|y)\mathbf{e}[nx],$$

where d^1g is a suitably normalized Haar measure on $SO(Y)_{\mathbb{R}}$, $\alpha(0)$ is some constant determined by the residues of $\zeta_{\pm}(s)$, and $W_{\mu,\nu}(y)$ denotes the Whittaker function. Then, $F(z)$ is a Maass form of weight $\ell/2$ with respect to the congruence subgroup $\Gamma_0(N)$.

The above formula can be compared with Siegel's calculation [26, Hilfssatz 4] of the Fourier expansions of non-holomorphic modular forms. Our $F(z)$ is essentially the same as the modular form given by Siegel [26]. See Remark 4. The theorem above excludes the case where both m and p are even. Our second result states that if one of $m-p$ and p are even, we can construct holomorphic modular forms from $M(Y;\pm n)$.

Main result 2 (Theorem 3) *Let $m \geq 5$. Assume that $m-p$ is even. We define a holomorphic function $F(z)$ on $\mathcal{H}$ by*

$$F(z) = (-1)^{\frac{m-p}{2}}(2\pi)^{-\frac{m}{2}} \cdot \Gamma\left(\frac{m}{2}\right) \int_{SO(Y)_{\mathbb{R}}/SO(Y)_{\mathbb{Z}}} d^1 g + |D|^{-1/2} \cdot \sum_{n=1}^{\infty} M(Y;n)\mathbf{e}[nz].$$

Then, $F(z)$ is a holomorphic modular form of weight $m/2$ with respect to $\Gamma_0(N)$. (In the case that p is even, we can construct holomorphic modular forms from $M(Y;-n)$ $(n = 1, 2, 3, \ldots)$.)

The theorem above is consistent with a result of Siegel that was published in a 1948 paper [25]. In this paper, Siegel calculated the action of certain differential operators on indefinite theta series, and proved that in the case of $\det Y > 0$, we can construct holomorphic modular forms from indefinite theta series associated with Y.

Before closing Introduction, we give some remarks on related researches, future problems, and possible applications. Special values of Siegel's zeta functions associated with Y of signature $(1, m-1)$ appear in the dimension formula for automorphic forms on orthogonal groups of signature $(2, m)$ (cf. Ibukiyama [8]), and it is important to investigate their arithmetic aspects. Ibukiyama [9] proved an explicit formula expressing Siegel zeta functions (with m even) as linear combinations of products of two shifted Dirichlet L-functions and certain elementary factors. His proof is given by direct calculations using Siegel's main theorem in [26] and not by converse theorems. Ibukiyama's explicit formula is quite general and includes the above-mentioned result of Maaß [13]. We also mention the work [6] of Hafner-Walling, in which they carried out extensive calculations to make Siegel's formula more explicit in terms of standard Eisenstein series. This work is also restricted to the case where m is even. It is worthwhile to investigate the case where m is odd. Finally, in a good situation, the method of converse theorems can be used to prove lifting theorems. In [29], a Shintani-Katok-Sarnak type correspondence is derived from analytic properties of a certain prehomogeneous zeta function whose coefficients involve periods of Maass cusp forms. In [14], Maaß studied a generalization of Siegel's zeta functions, which can be regarded as prehomogeneous zeta functions whose coefficients involve periods of automorphic forms on orthogonal groups. It is quite probable that our method can be applied to these zeta functions, and some lifting theorems will be obtained. We hope to discuss this topic elsewhere.

The present paper is organized as follows. In Sect. 1, we recall a Weil-type converse theorem for Maass forms, and in Sect. 2, we define our prehomogeneous vector spaces and give the local functional equations. Section 3 is devoted to define Siegel's zeta functions with congruence conditions, and analytic properties of Siegel's zeta functions are proved in Sect. 4. We prove our main theorems in Sects. 5 and 6.

Notation. We denote by $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ the ring of integers, the field of rational numbers, the field of real numbers, and the field of complex numbers, respectively. The set of non-zero real numbers and the set of positive real numbers are denoted by $\mathbb{R}^{\times}$ and $\mathbb{R}_{+}$, respectively. The set of positive integers and the set of non-negative integers are denoted by $\mathbb{Z}_{>0}$ and $\mathbb{Z}_{\geq 0}$, respectively. The real part and the imaginary part of a complex number s are denoted by $\Re(s)$ and $\Im(s)$, respectively. For complex numbers α, z with $\alpha \neq 0$, α^z always

stands for the principal value, namely, $\alpha^z = \exp((\log|\alpha| + i\arg\alpha)z)$ with $-\pi < \arg\alpha \le \pi$. We use $\mathbf{e}[x]$ to denote $\exp(2\pi ix)$. The quadratic residue symbol $\left(\frac{*}{*}\right)$ has the same meaning as in Shimura [22, p. 442]. For a meromorphic function $f(s)$ with a pole at $s = \alpha$, we denote its residue at $s = \alpha$ by $\operatorname*{Res}_{s=\alpha} f(s)$.

1 A Weil-type converse theorem for Maass forms

In this section, we define Maass forms on the Poincaré upper half-plane $\mathcal{H} = \{z \in \mathbb{C} \mid \Im(z) > 0\}$ of integral and half-integral weight, and recall a Weil-type converse theorem for Maass forms that is proved in [16]. We refer to Cohen-Strömberg [2] for an overview of the theory of Maass forms. Let $\Gamma = SL_2(\mathbb{Z})$ be the modular group, and for a positive integer N, we denote by $\Gamma_0(N)$ the congruence subgroup defined by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \;\middle|\; c \equiv 0 \pmod{N} \right\}.$$

As usual, Γ acts on $\mathcal{H}$ by the linear fractional transformation

$$gz = \frac{az+b}{cz+d} \quad \text{for} \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

We put $j(\gamma, z) = cz + d$, and define $\theta(z)$ and $J(\gamma, z)$ by

$$\theta(z) = \sum_{n=-\infty}^{\infty} \exp(2\pi i n^2 z), \qquad J(\gamma, z) = \frac{\theta(\gamma z)}{\theta(z)}.$$

Then it is well-known that

$$J(\gamma, z) = \varepsilon_d^{-1} \cdot \left(\frac{c}{d}\right) \cdot (cz+d)^{1/2} \quad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4),$$

where

$$\varepsilon_d = \begin{cases} 1 & (d \equiv 1 \pmod 4), \\ i & (d \equiv 3 \pmod 4). \end{cases} \tag{1}$$

For an integer ℓ, the hyperbolic Laplacian $\Delta_{\ell/2}$ of weight $\ell/2$ on $\mathcal{H}$ is defined by

$$\Delta_{\ell/2} = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \frac{i\ell y}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \quad (z = x + iy \in \mathcal{H}). \tag{2}$$

Let χ be a Dirichlet character mod N. Then we use the same symbol χ to denote the character of $\Gamma_0(N)$ defined by

$$\chi(\gamma) = \chi(d) \qquad \text{for} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N). \tag{3}$$

Definition 1 (Maass forms) Let $\ell \in \mathbb{Z}$, and N be a positive integer, with $4|N$ when ℓ is odd. A complex-valued C^∞-function $F(z)$ on $\mathcal{H}$ is called *a Maass form for* $\Gamma_0(N)$ of weight $\ell/2$ with character χ, if the following three conditions are satisfied;

(i) for every $\gamma \in \Gamma_0(N)$,

$$F(\gamma z) = \begin{cases} \chi(\gamma) j(\gamma, z)^{\ell/2} \cdot F(z) & (\ell \text{ is even}) \\ \chi(\gamma) J(\gamma, z)^{\ell} \cdot F(z) & (\ell \text{ is odd}) \end{cases},$$

(ii) $\Delta_{\ell/2}F = \Lambda \cdot F$ with some $\Lambda \in \mathbb{C}$,

(iii) F is of moderate growth at every cusp, namely, for every $A \in SL_2(\mathbb{Z})$, there exist positive constants C, K and ν depending on F and A such that

$$|F(Az)| \cdot |j(A,z)|^{-\ell/2} < Cy^{\nu} \quad \text{if } y = \Im(z) > K.$$

We call Λ the *eigenvalue* of F.

Let λ be a complex number with $\lambda \notin 1 - \frac{1}{2}\mathbb{Z}_{\geq 0}$. Let $\alpha = \{\alpha(n)\}_{n\in\mathbb{Z}\setminus\{0\}}$ and $\beta = \{\beta(n)\}_{n\in\mathbb{Z}\setminus\{0\}}$ be complex sequences of polynomial growth. For α, β, we can define the L-functions $\xi_{\pm}(\alpha;s)$, $\xi_{\pm}(\beta;s)$ and the completed L-functions $\Xi_{\pm}(\alpha;s)$, $\Xi_{\pm}(\beta;s)$ by

$$\xi_{\pm}(\alpha;s) = \sum_{n=1}^{\infty} \frac{\alpha(\pm n)}{n^s}, \quad \Xi_{\pm}(\alpha;s) = (2\pi)^{-s}\Gamma(s)\xi_{\pm}(\alpha;s),$$
$$\xi_{\pm}(\beta;s) = \sum_{n=1}^{\infty} \frac{\beta(\pm n)}{n^s}, \quad \Xi_{\pm}(\beta;s) = (2\pi)^{-s}\Gamma(s)\xi_{\pm}(\beta;s).$$

In the following, for simplicity, we put

$$\xi_e(\alpha;s) = \xi_+(\alpha;s) + \xi_-(\alpha;s), \quad \xi_o(\alpha;s) = \xi_+(\alpha;s) - \xi_-(\alpha;s).$$

Now we assume the following conditions [A1] – [A4]:

[A1] The L-functions $\xi_{\pm}(\alpha;s)$, $\xi_{\pm}(\beta;s)$ have meromorphic continuations to the whole s-plane, and $(s-1)(s-2+2\lambda)\xi_{\pm}(\alpha;s)$ and $(s-1)(s-2+2\lambda)\xi_{\pm}(\alpha;s)$ are entire functions, which are of finite order in any vertical strip.

Here a function $f(s)$ on a vertical strip $\sigma_1 \leq \Re(s) \leq \sigma_2$ ($\sigma_1, \sigma_2 \in \mathbb{R}, \sigma_1 < \sigma_2$) is said to be *of finite order* on the strip if there exist some positive constants A, B, ρ such that

$$|f(s)| < Ae^{B|\Im(s)|^{\rho}}, \quad \sigma_1 \leq \Re(s) \leq \sigma_2.$$

[A2] The residues of $\xi_{\pm}(\alpha;s)$ and $\xi_{\pm}(\beta;s)$ at $s=1$ satisfy

$$\operatorname*{Res}_{s=1} \xi_+(\alpha;s) = \operatorname*{Res}_{s=1} \xi_-(\alpha;s), \quad \operatorname*{Res}_{s=1} \xi_+(\beta;s) = \operatorname*{Res}_{s=1} \xi_-(\beta;s).$$

[A3] The following functional equation holds:

$$\gamma(s)\begin{pmatrix} \Xi_+(\alpha;s) \\ \Xi_-(\alpha;s) \end{pmatrix} = N^{2-2\lambda-s} \cdot \Sigma(\ell) \cdot \gamma(2-2\lambda-s)\begin{pmatrix} \Xi_+(\beta;2-2\lambda-s) \\ \Xi_-(\beta;2-2\lambda-s) \end{pmatrix},$$

where $\gamma(s)$ and $\Sigma(\ell)$ are defined by

$$\gamma(s) = \begin{pmatrix} e^{\pi si/2} & e^{-\pi si/2} \\ e^{-\pi si/2} & e^{\pi si/2} \end{pmatrix}, \qquad \Sigma(\ell) = \begin{pmatrix} 0 & i^{\ell} \\ 1 & 0 \end{pmatrix}. \tag{4}$$

[A4] If $\lambda = \frac{q}{2}$ ($q \in \mathbb{Z}_{\geq 0},\ q \geq 4$), then

$$\xi_+(\alpha;-k) + (-1)^k\xi_-(\alpha;-k) = 0 \quad (k = 1, 2, \ldots, q-3).$$

Under the assumptions [A1] – [A4], we define $\alpha(0)$, $\beta(0)$, $\alpha(\infty)$, $\beta(\infty)$ by

$$\alpha(0) = -\xi_e(\alpha; 0) \tag{5}$$

$$\alpha(\infty) = \frac{N}{2} \operatorname*{Res}_{s=1} \xi_e(\beta; s), \tag{6}$$

$$\beta(0) = -\xi_e(\beta; 0) \tag{7}$$

$$\beta(\infty) = \frac{i^{-\ell}}{2} \operatorname*{Res}_{s=1} \xi_e(\alpha; s), \tag{8}$$

For an odd prime number r with $(N, r) = 1$ and a Dirichlet character ψ mod r, the twisted L-functions $\xi_\pm(\alpha, \psi; s)$, $\Xi_\pm(\alpha, \psi; s)$, $\xi_\pm(\beta, \psi; s)$, $\Xi_\pm(\beta, \psi; s)$ are defined by

$$\xi_\pm(\alpha, \psi; s) = \sum_{n=1}^{\infty} \frac{\alpha(\pm n)\tau_\psi(\pm n)}{n^s}, \quad \Xi_\pm(\alpha, \psi; s) = (2\pi)^{-s}\Gamma(s)\xi_\pm(\alpha, \psi; s),$$

$$\xi_\pm(\beta, \psi; s) = \sum_{n=1}^{\infty} \frac{\beta(\pm n)\tau_\psi(\pm n)}{n^s}, \quad \Xi_\pm(\beta, \psi; s) = (2\pi)^{-s}\Gamma(s)\xi_\pm(\beta, \psi; s),$$

where $\tau_\psi(n)$ is the Gauss sum defined by

$$\tau_\psi(n) = \sum_{\substack{m \bmod r \\ (m,r)=1}} \psi(m) e^{2\pi i m n/r}. \tag{9}$$

We put

$$\tau_\psi = \tau_\psi(1) = \sum_{\substack{m \bmod r \\ (m,r)=1}} \psi(m) e^{2\pi i m/r}$$

and denote by $\psi_{r,0}$ the principal character modulo r. Recall that the Gauss sums are calculated as follows:

$$\tau_\psi(n) = \begin{cases} \overline{\psi(n)}\tau_\psi & (n \not\equiv 0 \pmod r), \\ 0 & (n \equiv 0 \pmod r), \end{cases} \quad \text{if } \psi \neq \psi_{r,0}, \tag{10}$$

$$\tau_{\psi_{r,0}}(n) = \begin{cases} -1 & (n \not\equiv 0 \pmod r), \\ r-1 & (n \equiv 0 \pmod r). \end{cases} \tag{11}$$

Let $\mathbb{P}_N$ be a set of odd prime numbers not dividing N such that, for any positive integers a, b coprime to each other, $\mathbb{P}_N$ contains a prime number r of the form $r = am + b$ for some $m \in \mathbb{Z}_{>0}$. For an $r \in \mathbb{P}_N$, denote by X_r the set of all Dirichlet characters mod r (including the principal character $\psi_{r,0}$). For $\psi \in X_r$, we define the Dirichlet character ψ^* by

$$\psi^*(k) = \overline{\psi(k)} \left(\frac{k}{r}\right)^\ell. \tag{12}$$

We put

$$C_{\ell,r} = \begin{cases} 1 & (\ell \text{ is even}), \\ \varepsilon_r^\ell & (\ell \text{ is odd}). \end{cases} \tag{13}$$

(For the definition of ε_r, see (1).)

In the following, we fix a Dirichlet character $\chi \bmod N$ that satisfies $\chi(-1) = i^\ell$ (resp. $\chi(-1) = 1$) when ℓ is even (resp. odd).

For an $r \in \mathbb{P}_N$ and a $\psi \in X_r$, we consider the following conditions $[\mathrm{A1}]_{r,\psi} - [\mathrm{A5}]_{r,\psi}$ on $\xi_\pm(\alpha, \psi; s)$ and $\xi_\pm(\beta, \psi^*; s)$.

[A1]$_{r,\psi}$ $\xi_\pm(\alpha, \psi; s)$, $\xi_\pm(\beta, \psi^*; s)$ have meromorphic continuations to the whole s-plane, and $(s-1)(s-2+2\lambda)\xi_\pm(\alpha, \psi; s)$, $(s-1)(s-2+2\lambda)\xi_\pm(\beta, \psi^*; s)$ are entire functions, which are of finite order in any vertical strip.

[A2]$_{r,\psi}$ The residues of $\xi_\pm(\alpha, \psi; s)$ and $\xi_\pm(\beta, \psi^*; s)$ satisfy

$$\operatorname*{Res}_{s=1} \xi_+(\alpha, \psi; s) = \operatorname*{Res}_{s=1} \xi_-(\alpha, \psi; s), \ \operatorname*{Res}_{s=1} \xi_+(\beta, \psi^*; s) = \operatorname*{Res}_{s=1} \xi_-(\beta, \psi^*; s).$$

[A3]$_{r,\psi}$ $\Xi_\pm(\alpha, \psi; s)$ and $\Xi_\pm(\beta, \psi^*; s)$ satisfy the following functional equation:

$$\gamma(s) \begin{pmatrix} \Xi_+(\alpha, \psi; s) \\ \Xi_-(\alpha, \psi; s) \end{pmatrix} = \chi(r) \cdot C_{\ell,r} \cdot \psi^*(-N) \cdot r^{2\lambda-2} \cdot (Nr^2)^{2-2\lambda-s} \cdot \Sigma(\ell) \cdot \gamma(2-2\lambda-s) \begin{pmatrix} \Xi_+(\beta, \psi^*; 2-2\lambda-s) \\ \Xi_-(\beta, \psi^*; 2-2\lambda-s) \end{pmatrix},$$

where $\gamma(s)$ and $\Sigma(\ell)$ are the same as (4) in **[A3]**.

[A4]$_{r,\psi}$ If $\lambda = \frac{q}{2}$ ($q \in \mathbb{Z}_{\geq 0}$, $q \geq 4$), then

$$\xi_+(\alpha, \psi; -k) + (-1)^k \xi_-(\alpha, \psi; -k) = 0 \quad (k = 1, 2, \ldots, q-3).$$

[A5]$_{r,\psi}$ The following four relations between residues and special values hold:

- $\xi_e(\alpha, \psi; 0) = \tau_\psi(0)\xi_e(\alpha; 0)$.
- $\chi(r) \cdot \psi^*(-N) \cdot C_{\ell,r} \cdot r^{2\lambda} \operatorname*{Res}_{s=1} \xi_\pm(\beta, \psi^*; s) = \tau_\psi(0) \operatorname*{Res}_{s=1} \xi_\pm(\beta; s)$.
- $\xi_e(\beta, \psi^*; 0) = \tau_{\psi^*}(0)\xi_e(\beta; 0)$.
- $\operatorname*{Res}_{s=1} \xi_\pm(\alpha, \psi; s) = \chi(r) \cdot \psi^*(-N) \cdot C_{\ell,r} \cdot r^{-2\lambda} \cdot \tau_{\psi^*}(0) \cdot \operatorname*{Res}_{s=1} \xi_\pm(\alpha; s)$.

Lemma 1 (Converse Theorem) *We assume that $\xi_\pm(\alpha; s)$ and $\xi_\pm(\beta; s)$ satisfy the conditions* [A1] – [A4], *and define $\alpha(0), \alpha(\infty), \beta(0), \beta(\infty)$ by (5), (6), (7), (8), respectively. We assume furthermore that, for any $r \in \mathbb{P}_N$ and $\psi \in X_r$, $\xi_\pm(\alpha, \psi; s)$ and $\xi_\pm(\beta, \psi^*; s)$ satisfy the conditions* $[\mathrm{A1}]_{r,\psi} - [\mathrm{A5}]_{r,\psi}$. *Define the functions $F_\alpha(z)$ and $G_\beta(z)$ on the upper half plane $\mathcal{H}$ by*

$$F_\alpha(z) = \alpha(\infty) \cdot y^{\lambda-\ell/4} + \alpha(0) \cdot i^{-\ell/2} \cdot \frac{(2\pi)2^{1-2\lambda}\Gamma(2\lambda-1)}{\Gamma\left(\lambda+\frac{\ell}{4}\right)\Gamma\left(\lambda-\frac{\ell}{4}\right)} \cdot y^{1-\lambda-\ell/4} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \alpha(n) \cdot \frac{i^{-\ell/2} \cdot \pi^\lambda \cdot |n|^{\lambda-1}}{\Gamma\left(\lambda+\frac{\mathrm{sgn}(n)\ell}{4}\right)} \cdot y^{-\ell/4} W_{\frac{\mathrm{sgn}(n)\ell}{4},\lambda-\frac{1}{2}}(4\pi|n|y) \cdot \mathbf{e}[nx], \tag{14}$$

$$G_\beta(z) = N^\lambda \beta(\infty) \cdot y^{\lambda-\ell/4} + N^{1-\lambda}\beta(0) \cdot i^{-\ell/2} \cdot \frac{(2\pi)2^{1-2\lambda}\Gamma(2\lambda-1)}{\Gamma\left(\lambda+\frac{\ell}{4}\right)\Gamma\left(\lambda-\frac{\ell}{4}\right)} \cdot y^{1-\lambda-\ell/4} + N^{1-\lambda} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \beta(n) \cdot \frac{i^{-\ell/2} \cdot \pi^\lambda \cdot |n|^{\lambda-1}}{\Gamma\left(\lambda+\frac{\mathrm{sgn}(n)\ell}{4}\right)} \cdot y^{-\ell/4} W_{\frac{\mathrm{sgn}(n)\ell}{4},\lambda-\frac{1}{2}}(4\pi|n|y) \cdot \mathbf{e}[nx]. \tag{15}$$

Here $W_{\mu,\nu}(x)$ denotes the Whittaker function. Then $F_\alpha(z)$ (resp. $G_\beta(z)$) gives a Maass form for $\Gamma_0(N)$ of weight $\frac{\ell}{2}$ with character χ (resp. $\chi_{N,\ell}$), and eigenvalue $(\lambda-\ell/4)(1-\lambda-\ell/4)$, where

$$\chi_{N,\ell}(d)=\overline{\chi(d)}\left(\frac{N}{d}\right)^{\ell}. \tag{16}$$

Moreover, we have

$$F_\alpha\left(-\frac{1}{Nz}\right)(\sqrt{N}z)^{-\ell/2}=G_\beta(z).$$

Remark 1 In [16], we proved the converse theorem under a weaker condition $\lambda\notin\frac{1}{2}-\frac{1}{2}\mathbb{Z}_{\geq 0}$. In the present paper, since the case of $\lambda=1$ will not be treated, we assume $\lambda\notin 1-\frac{1}{2}\mathbb{Z}_{\geq 0}$, which simplifies the description of the converse theorem.

2 Prehomogeneous vector spaces

Let Y be a non-degenerate half-integral symmetric matrix of degree m, and let p be the number of positive eigenvalues of Y. *Throughout the present paper, we assume that $m\geq 5$ and $p(m-p)>0$.* We denote by $SO(Y)$ the special orthogonal group of Y defined by $SO(Y)=\{g\in SL_m(\mathbb{C})\mid {}^tgYg=Y\}$. We define the representation ρ of $G=GL_1(\mathbb{C})\times SO(Y)$ on $V=\mathbb{C}^m$ by

$$\rho(\tilde{g})v=\rho(t,g)v=tgv\qquad(\tilde{g}=(t,g)\in G, v\in V).$$

Let $P(v)$ be the quadratic form on V defined by

$$P(v)=Y[v]={}^tvYv, \tag{17}$$

where we use Siegel's notation. Then, for $\tilde{g}=(t,g)\in G$ and $v\in V$, we have

$$P(\rho(\tilde{g})v)=\chi(t,g)P(v),\quad\text{with}\quad\chi(t,g)=t^2, \tag{18}$$

and $V-S$ is a single $\rho(G)$-orbit, where S is the zero set of P:

$$S=\{v\in V\mid P(v)=0\}.$$

That is, (G,ρ,V) is a reductive regular prehomogeneous vector space. (We refer to [11,21] for the basics of the theory of prehomogeneous vector spaces.) We identify the dual space V^* of V with V itself via the inner product $\langle v,v^*\rangle={}^tvv^*$. Then the dual triplet (G,ρ^*,V^*) is given by

$$\rho^*(\tilde{g})v^*=\rho^*(t,g)v^*=t^{-1}\cdot{}^tg^{-1}v^*\qquad(\tilde{g}=(t,g)\in G, v\in V^*).$$

We define the quadratic form $P^*(v^*)$ on V^* by

$$P^*(v^*)=\frac{1}{4}Y^{-1}[v^*]=\frac{1}{4}\cdot{}^tv^*\,Y^{-1}v^*. \tag{19}$$

Then, for $\tilde{g}=(t,g)\in G$, $v^*\in V^*$, we have

$$P^*(\rho^*(\tilde{g})v^*)=\chi^*(t,g)P^*(v^*),\quad\text{with}\quad\chi^*(t,g)=t^{-2}, \tag{20}$$

and $V - S^*$ is a single $\rho^*(G)$-oribit, where S^* is the zero set of P^*:

$$S^* = \{v^* \in V^* \mid P^*(v^*) = 0\}.$$

For $\epsilon, \eta = \pm$, we put

$$V_\epsilon = \{v \in V_{\mathbb{R}} \mid \mathrm{sgn}P(v) = \epsilon\}, \qquad V_\eta^* = \{v^* \in V_{\mathbb{R}} \mid \mathrm{sgn}P^*(v^*) = \eta\}.$$

We denote by $dv = dv_1 \cdots dv_m$ the Lebesgue measure on $V_{\mathbb{R}}$, and by $\mathcal{S}(V_{\mathbb{R}})$ the space of rapidly decreasing functions on $V_{\mathbb{R}}$. Then, for $f, f^* \in \mathcal{S}(V_{\mathbb{R}})$ and $\epsilon, \eta = \pm$, we define the local zeta functions $\Phi_\epsilon(f; s)$ and $\Phi_\eta^*(f^*; s)$ by

$$\Phi_\epsilon(f; s) = \int_{V_\epsilon} f(v)|P(v)|^{s-\frac{m}{2}} dv, \quad \Phi_\eta^*(f^*; s) = \int_{V_\eta^*} f^*(v^*)|P^*(v^*)|^{s-\frac{m}{2}} dv^*. \tag{21}$$

For $\Re(s) > \frac{m}{2}$, the integrals $\Phi_\epsilon(f; s)$ and $\Phi_\eta^*(f^*; s)$ converge absolutely, and as functions of s, they can be continued analytically to the whole s-plane as meromorphic functions. Further, we define the Fourier transform $\widehat{f}(v^*)$ of $f \in \mathcal{S}(V_{\mathbb{R}})$ by

$$\widehat{f}(v^*) = \int_{V_{\mathbb{R}}} f(v)\mathbf{e}[\langle v, v^* \rangle] dv.$$

The following lemma is due to Gelfand-Shilov [5]; a detailed proof is given in Kimura [11, § 4.2].

Lemma 2 (Local Functional Equation) *Let p be the number of positive eigenvalues of Y, and put $D = \det(2Y)$. Then the following functional equation holds:*

$$\begin{pmatrix} \Phi_+^*(\widehat{f}; s) \\ \Phi_-^*(\widehat{f}; s) \end{pmatrix} = \Gamma\left(s + 1 - \frac{m}{2}\right)\Gamma(s)|D|^{\frac{1}{2}} \cdot 2^{-2s+\frac{m}{2}} \cdot \pi^{-2s+\frac{m}{2}-1} \times \begin{pmatrix} \sin\pi\left(\frac{p}{2} - s\right) & \sin\frac{\pi p}{2} \\ \sin\frac{\pi(m-p)}{2} & \sin\pi\left(\frac{m-p}{2} - s\right) \end{pmatrix} \begin{pmatrix} \Phi_+\left(f; \frac{m}{2} - s\right) \\ \Phi_-\left(f; \frac{m}{2} - s\right) \end{pmatrix}.$$

In the rest of this section, we investigate singular distributions whose supports are contained in the real points $S_{\mathbb{R}}$ of S; these distributions play an important role in the calculation of residues of Siegel's zeta functions. We decompose $S_{\mathbb{R}}$ as

$$S_{\mathbb{R}} = S_{1,\mathbb{R}} \cup S_{2,\mathbb{R}}, \quad S_{1,\mathbb{R}} = \{v \in V_{\mathbb{R}} \mid P(v) = 0, v \neq 0\}, \quad S_{2,\mathbb{R}} = \{0\}.$$

A measure on $S_{\mathbb{R}}$ that is $SO(Y)_{\mathbb{R}}$-invariant is constructed as follows. Since $P(v)$ is a non-degenerate quadratic forms, we have

$$S_{1,\mathbb{R}} = \bigcup_{i=1}^{m} U_i, \qquad U_i = \left\{ v \in S_{1,\mathbb{R}} \,\middle|\, \frac{\partial}{\partial v_i} P(v) \neq 0 \right\}.$$

For $i = 1, \ldots, m$, we define an $(m-1)$-dimensional differential form ω_i on U_i by

$$\omega_i = (-1)^{i-1} \left(\frac{\partial}{\partial v_i} P(v)\right)^{-1} dv_1 \wedge \cdots \wedge dv_{i-1} \wedge dv_{i+1} \wedge \cdots \wedge dv_m. \tag{22}$$

It is easy to see that there exists an $(m-1)$-dimensional differential form ω on $S_{\mathbb{R}}$ that satisfies

$$\omega|_{U_i} = \omega_i \qquad (i = 1, \ldots, m)$$

and

$$dP(v) \wedge \omega = dv \ (= dv_1 \wedge \cdots \wedge dv_m).$$

Since $P(gv) = P(v)$ for $g \in SO(Y)_{\mathbb{R}}$, we have

$$dv = dP(v) \wedge \omega(v) = dP(v) \wedge \omega(gv).$$

Further, $\omega(tv) = t^{m-2}\omega(v)$ for $t > 0$. Now let $|\omega(v)|_\infty$ denote the measure on $S_{1,\mathbb{R}}$ defined by ω. Then we have

$$|\omega(\rho(t,g)v)|_\infty = |\chi(t,g)|^{\frac{m}{2}-1} \cdot |\omega(v)|_\infty \tag{23}$$

for $g \in SO(Y)_{\mathbb{R}}$ and $t > 0$. Similary, for the zero set S^* of P^*, we decompose the real points $S^*_{\mathbb{R}}$ as

$$S^*_{\mathbb{R}} = S^*_{1,\mathbb{R}} \cup S^*_{2,\mathbb{R}}, \qquad S^*_{1,\mathbb{R}} = \{v^* \in V_{\mathbb{R}} \mid P^*(v^*) = 0, v^* \neq 0\}, \quad S^*_{2,\mathbb{R}} = \{0\}.$$

The same argument as above ensures the existence of an $(m-1)$-dimensional differential form ω^* on $S^*_{1,\mathbb{R}}$ such that the restriction of ω^* on

$$U^*_i = \left\{ v^* \in S^*_{1,\mathbb{R}} \;\middle|\; \frac{\partial}{\partial v^*_i} P^*(v^*) \neq 0 \right\} \qquad (i = 1, \ldots, m)$$

is given by

$$\omega^*|_{U^*_i} = (-1)^{i-1} \left(\frac{\partial}{\partial v^*_i} P^*(v^*) \right)^{-1} dv^*_1 \wedge \cdots \wedge dv^*_{i-1} \wedge dv^*_{i+1} \wedge \cdots \wedge dv^*_m.$$

We have

$$|\omega^*(\rho^*(t,g)v^*)|_\infty = |\chi(t,g)|^{1-\frac{m}{2}} \cdot |\omega^*(v^*)|_\infty \tag{24}$$

for $g \in SO(Y)_{\mathbb{R}}$ and $t > 0$, where $|\omega^*|_\infty$ denotes the measure on $S^*_{1,\mathbb{R}}$ defined by ω^*. We refer to [5, Chap. III] for further details on the measures $|\omega|_\infty$, $|\omega^*|_\infty$. Then we have the following

Lemma 3 (1) *If $f \in C_0^\infty(V_{\mathbb{R}} - S_{\mathbb{R}})$, then we have*

$$\int_{S^*_{1,\mathbb{R}}} \widehat{f}(v^*)|\omega^*(v^*)|_\infty = \Gamma\left(\frac{m}{2} - 1\right) |D|^{\frac{1}{2}} \cdot 2^{2-\frac{m}{2}} \cdot \pi^{1-\frac{m}{2}} \times \left(\sin\frac{\pi}{2}(m-p)\,\sin\frac{\pi p}{2}\right) \left(\int_{V_+} f(v)|P(v)|^{1-\frac{m}{2}} dv \right) \left(\int_{V_-} f(v)|P(v)|^{1-\frac{m}{2}} dv \right).$$

(2) *If $\widehat{f} \in C_0^\infty(V_{\mathbb{R}} - S^*_{\mathbb{R}})$, then we have*

$$\int_{S_{1,\mathbb{R}}} f(v)|\omega(v)|_\infty = \Gamma\left(\frac{m}{2} - 1\right) |D|^{-\frac{1}{2}} \cdot 2^{2-\frac{m}{2}} \cdot \pi^{1-\frac{m}{2}} \times \left(\sin\frac{\pi}{2}(m-p)\,\sin\frac{\pi p}{2}\right) \left(\int_{V^*_+} \widehat{f}(v^*)|P^*(v^*)|^{1-\frac{m}{2}} dv^* \right) \left(\int_{V^*_-} \widehat{f}(v^*)|P^*(v^*)|^{1-\frac{m}{2}} dv^* \right).$$

This is stated, *without proof*, on p. 156 of Sato-Shintani [21] where Siegel's zeta function is picked up as an example of their theory. Since the details cannot be found in other literature, we give a proof of the lemma for convenience of readers.

Proof For $f \in C_0^\infty(V_\mathbb{R} - S_\mathbb{R})$, we consider the integral

$$\int_{S^*_{1,\mathbb{R}}} \widehat{f}(v^*)|\omega^*(v^*)|_\infty.$$

We may replace $S^*_{1,\mathbb{R}}$ by $S^*_\mathbb{R} = \{v^* \in V_\mathbb{R} \mid P^*(v^*) = 0\}$, since $S^*_{2,\mathbb{R}} = \{0\}$ has measure 0 in $S^*_\mathbb{R}$. From the identity (19) (or the first formula on p. 257) in Gelfand-Shilov [5, Chap III, §2.2], we have

$$\int_{S^*_\mathbb{R}} \widehat{f}(v^*)|\omega^*(v^*)|_\infty = \operatorname*{Res}_{s=0} \int_{P^*(v^*)>0} P^*(v^*)^{s-1} \cdot \widehat{f}(v^*)dv^*.$$

By the shift $s \mapsto s - \frac{m}{2}$, we have

$$\begin{aligned}\int_{S^*_\mathbb{R}} \widehat{f}(v^*)|\omega^*(v^*)|_\infty &= \operatorname*{Res}_{s=\frac{m}{2}} \int_{V^*_+} P^*(v^*)^{s-\frac{m}{2}-1} \cdot \widehat{f}(v^*)dv^* \\ &= \lim_{s\to+\frac{m}{2}} \left(s - \frac{m}{2}\right) \Phi^*_+\left(\widehat{f}; s-1\right).\end{aligned}$$

It then follows from the local functional equation (Lemma 2) that

$$\begin{aligned}\lim_{s\to+\frac{m}{2}} \left(s - \frac{m}{2}\right) \Phi^*_+\left(\widehat{f}; s-1\right) &= \lim_{s\to+\frac{m}{2}} \left(s - \frac{m}{2}\right) \\ &\quad \Gamma\left(s - \frac{m}{2}\right) \Gamma(s-1)|D|^{\frac{1}{2}} \cdot 2^{-2s+\frac{m}{2}+2} \cdot \pi^{-2s+\frac{m}{2}+1} \\ &\quad \times \left(\sin\pi\left(\frac{p}{2} - s + 1\right) \sin\frac{\pi p}{2}\right) \begin{pmatrix} \int_{V_+} f(v)|P(v)|^{1-s}dv \\ \int_{V_-} f(v)|P(v)|^{1-s}dv \end{pmatrix} \\ &= \Gamma\left(\frac{m}{2} - 1\right)|D|^{\frac{1}{2}} \cdot 2^{2-\frac{m}{2}} \cdot \pi^{1-\frac{m}{2}} \\ &\quad \times \left(\sin\frac{\pi}{2}(m-p) \sin\frac{\pi p}{2}\right) \begin{pmatrix} \int_{V_+} f(v)|P(v)|^{1-\frac{m}{2}}dv \\ \int_{V_-} f(v)|P(v)|^{1-\frac{m}{2}}dv \end{pmatrix},\end{aligned}$$

which proves the first assertion of Lemma 3. The second assertion can be proved in a similar fashion. □

3 Siegel's zeta functions with congruence conditions

In this section, followig Sato-Shintani [21], we define Siegel's zeta functions associated with (G, ρ, V), and give their integral representations. Moreover, we calculate the singular parts of the zeta integrals. For this calculation, we also refer to Kimura [11]. Furthermore, following Sato [20], we slightly generalize Siegel's zeta functions with using Schwartz-Bruhat functions on $\mathbb{Q}^m$ in order to treat the twisted zeta functions simultaneously. Let dx be the measure on $GL_m(\mathbb{R})$ defined by

$$dx = |\det x|^{-m} \prod_{1\le i,j\le m} dx_{ij} \qquad \text{for} \quad x = (x_{ij}) \in GL_m(\mathbb{R}),$$

and $d\lambda$ the measure on the space $\mathrm{Sym}_m(\mathbb{R})$ of symmetric matrices of degree m defined by

$$d\lambda = |\det \lambda|^{-\frac{m+1}{2}} \prod_{1\le i\le j\le m} d\lambda_{ij} \qquad \text{for} \quad \lambda = (\lambda_{ij}) \in \mathrm{Sym}_m(\mathbb{R}).$$

Then we normalize a Haar measure d^1g on the Lie group $SO(Y)_{\mathbb{R}}$ in such a way that the integration formula

$$\int_{GL_m(\mathbb{R})} F(x)dx = \int_{SO(Y)_{\mathbb{R}}\backslash GL_m(\mathbb{R})} d\lambda({}^t\dot{x}Y\dot{x}) \int_{SO(Y)_{\mathbb{R}}} F(g\dot{x})d^1g \tag{25}$$

holds for all integrable functions $F(x) \in L^1(GL_m(\mathbb{R}))$. Further, let dt be the Lebesgue measure on $\mathbb{R}$ and put

$$d^\times t = \frac{2dt}{t}. \tag{26}$$

By (18), $|P(v)|^{-\frac{m}{2}}dv$ is an $\mathbb{R}_+ \times SO(Y)_{\mathbb{R}}$-invariant measure on V_ϵ and the isotropy subgroup

$$SO(Y)_v = \{g \in SO(Y) \,|\, gv = v\}$$

at $v \in V - S$ is a reductive algebraic group. Hence, for $v \in V_\epsilon$, there exists a Haar measure $d\mu_v$ on $SO(Y)_{v,\mathbb{R}}$ such that the integration formula

$$\begin{aligned}&\int_0^\infty d^\times t \int_{SO(Y)_{\mathbb{R}}} H(t,g)d^1g \\ &= \int_0^\infty \int_{SO(Y)_{\mathbb{R}}/SO(Y)_{v,\mathbb{R}}} |P(\rho(t,\dot{g})v)|^{-\frac{m}{2}} d(\rho(t,\dot{g})v) \int_{SO(Y)_{v,\mathbb{R}}} H(t,\dot{g}h)d\mu_v(h)\end{aligned} \tag{27}$$

holds for all integrable functions $H(t,g) \in L^1(G_{\mathbb{R}})$. Similarly, for $v^* \in V_\eta^*$, we write

$$SO(Y)_{v^*,\mathbb{R}} = \{g \in SO(Y)_{\mathbb{R}} \,|\, {}^tg^{-1}v^* = v^*\}$$

and fix a Haar measure $d\mu^*_{v^*}$ on $SO(Y)_{v^*,\mathbb{R}}$ such that the integration formula

$$\begin{aligned}&\int_0^\infty d^\times t \int_{SO(Y)_{\mathbb{R}}} H(t,g)d^1g \\ &= \int_0^\infty \int_{SO(Y)_{\mathbb{R}}/SO(Y)_{v^*,\mathbb{R}}} |P^*(\rho^*(t,\dot{g})v^*)|^{-\frac{m}{2}} d(\rho^*(t,\dot{g})v^*) \int_{SO(Y)_{v^*,\mathbb{R}}} H(t,\dot{g}h)d\mu^*_{v^*}(h)\end{aligned} \tag{28}$$

holds for all integrable functions $H(t,g) \in L^1(G_{\mathbb{R}})$.

We call a function $\phi : V_{\mathbb{Q}} \to \mathbb{C}$ a *Schwartz-Bruhat* function if the following two conditions are satisfied:

(1) there exists a positive integer M such that $\phi(v) = 0$ for $v \notin \frac{1}{M}V_{\mathbb{Z}}$, and
(2) there exists a positive integer N such that if $v, w \in V_{\mathbb{Q}}$ satisfy $v - w \in NV_{\mathbb{Z}}$. then $\phi(v) = \phi(w)$.

The totality of Schwartz-functions on $V_{\mathbb{Q}}$ is denoted by $\mathcal{S}(V_{\mathbb{Q}})$. We define the Fourier transform $\widehat{\phi} \in \mathcal{S}(V_{\mathbb{Q}})$ of a Schwartz-Bruhat function $\phi \in \mathcal{S}(V_{\mathbb{Q}})$ by

$$\widehat{\phi}(v^*) = \frac{1}{[V_{\mathbb{Z}} : rV_{\mathbb{Z}}]} \sum_{v \in V_{\mathbb{Q}}/rV_{\mathbb{Z}}} \phi(v)\mathbf{e}[-\langle v, v^*\rangle], \tag{29}$$

where r is a sufficiently large positive integer such that the value $\phi(v)\mathbf{e}[-\langle v, v^*\rangle]$ depends only on the residue class v mod $rV_{\mathbb{Z}}$. Though r is not unique, the value $\widehat{\phi}(v^*)$ does not depend on the choice of r. The following lemma is essentially an adelic version of Poisson summation formula.

Lemma 4 (Poisson summation formula) *For $\phi \in \mathcal{S}(V_{\mathbb{Q}})$ and $f \in \mathcal{S}(V_{\mathbb{R}})$,*

$$\sum_{v^* \in V_{\mathbb{Q}}} \widehat{\phi}(v^*)\widehat{f}(v^*) = \sum_{v \in V_{\mathbb{Q}}} \phi(v) f(v).$$

For $\tilde{g} = (t, g) \in G_{\mathbb{R}} = \mathbb{R}^\times \times SO(Y)_{\mathbb{R}}$, we put $f_{\tilde{g}}(v) = f(\rho(t, g)v) = f(tgv)$. Since

$$\widehat{fg}(v^*) = \int_{V_{\mathbb{R}}} f(tgv)\mathbf{e}[\langle v, v^*\rangle]dv = |t|^{-m} \int_{V_{\mathbb{R}}} f(v)\mathbf{e}[\langle t^{-1}g^{-1}v, v^*\rangle]dv = |t|^{-m} \cdot \widehat{f}(\rho^*(t, g)v^*),$$

we have the following

Lemma 5 *For $\tilde{g} = (t, g) \in G_{\mathbb{R}}$, $\phi \in \mathcal{S}(V_{\mathbb{Q}})$, $f \in \mathcal{S}(V_{\mathbb{R}})$,*

$$|t|^{-m} \sum_{v^* \in V_{\mathbb{Q}}} \widehat{\phi}(v^*)\widehat{f}(\rho^*(t, g)v^*) = \sum_{v \in V_{\mathbb{Q}}} \phi(v) f(\rho(t, g)v).$$

In the following, we assume that $\phi \in \mathcal{S}(V_{\mathbb{Q}})$ is $SO(Y)_{\mathbb{Z}}$-invariant. That is, ϕ is assumed to satisfy

$$\phi(\gamma v) = \phi(v) \qquad \text{for} \quad v \in V_{\mathbb{Q}}, \gamma \in SO(Y)_{\mathbb{Z}}. \tag{30}$$

Then we define the zeta integral $Z(f, \phi; s)$ by

$$Z(f, \phi; s) = \int_0^\infty d^\times t \int_{SO(Y)_{\mathbb{R}}/SO(Y)_{\mathbb{Z}}} |\chi(t, g)|^s \sum_{v \in V_{\mathbb{Q}} - S_{\mathbb{Q}}} \phi(v) f(\rho(t, g)v) d^1 g. \tag{31}$$

Since $V_{\mathbb{Q}} - S_{\mathbb{Q}}$ can be decomposed as

$$V_{\mathbb{Q}} - S_{\mathbb{Q}} = \bigcup_{\epsilon = \pm} \bigcup_{v \in SO(Y)_{\mathbb{Z}} \backslash V_\epsilon \cap V_{\mathbb{Q}}} \bigcup_{\gamma \in SO(Y)_{\mathbb{Z}}/SO(Y)_{v,\mathbb{Z}}} \gamma v,$$

we have, by a formal calculation,

$$\begin{aligned} &Z(f, \phi; s) \\ &= \sum_{\epsilon = \pm} \sum_{v \in SO(Y)_{\mathbb{Z}} \backslash V_\epsilon \cap V_{\mathbb{Q}}} \int_0^\infty d^\times t \int_{SO(Y)_{\mathbb{R}}/SO(Y)_{\mathbb{Z}}} |\chi(t, g)|^s \\ &\quad \times \sum_{\gamma \in SO(Y)_{\mathbb{Z}}/SO(Y)_{v,\mathbb{Z}}} \phi(\gamma v) f(\rho(t, g)\gamma v) d^1 g \\ &\overset{(30)}{=} \sum_{\epsilon = \pm} \sum_{v \in SO(Y)_{\mathbb{Z}} \backslash V_\epsilon \cap V_{\mathbb{Q}}} \phi(v) \int_0^\infty d^\times t \int_{SO(Y)_{\mathbb{R}}/SO(Y)_{v,\mathbb{Z}}} |\chi(t, g)|^s f(\rho(t, g)v) d^1 g, \end{aligned}$$

and further, by applying (27) to

$$H(t, g) = |\chi(t, g)|^s \cdot f(\rho(t, g)v) = \frac{|P(\rho(t, g)v)|^s}{|P(v)|^s} \cdot f(\rho(t, g)v),$$

we have

$$Z(f,\phi;s)=\sum_{\epsilon=\pm}\sum_{v\in SO(Y)_{\mathbb{Z}}\backslash V_\epsilon\cap V_{\mathbb{Q}}}\phi(v)\int_0^\infty d^\times t \times\int_{SO(Y)_{\mathbb{R}}/SO(Y)_{v,\mathbb{R}}}\frac{|P(\rho(t,\dot g)v)|^s}{|P(v)|^s}\cdot f(\rho(t,\dot g)v)|P(\rho(t,\dot g)v|^{-\frac{m}{2}}d(\rho(t,\dot g)v) \times\int_{SO(Y)_{v,\mathbb{R}}/SO_{v,\mathbb{Z}}}d\mu_v(h).$$

In the following, for $v\in V_{\mathbb{Q}}-S_{\mathbb{Q}}$, we put

$$\mu(v)=\int_{SO(Y)_{v,\mathbb{R}}/SO_{v,\mathbb{Z}}}d\mu_v(h). \tag{32}$$

Since it is assumed that $m\geq 5$, the generic isotropy subgroup $SO(Y)_v$ is a semisimple algebraic group, and thus we have $\mu(v)<+\infty$. (cf. [11, p. 184].) We further put $\rho(t,\dot g)v=x$ in the right hand side above. Then, since $\mathbb{R}_+\times SO(Y)_{\mathbb{R}}/SO(Y)_{v,\mathbb{R}}\cong V_\epsilon$, we have

$$Z(f,\phi;s)=\sum_{\epsilon=\pm}\left\{\sum_{v\in SO(Y)_{\mathbb{Z}}\backslash V_\epsilon\cap V_{\mathbb{Q}}}\frac{\phi(v)\mu(v)}{|P(v)|^s}\right\}\int_{V_\epsilon}f(x)|P(x)|^{s-\frac{m}{2}}dx. \tag{33}$$

The Dirichlet series

$$\zeta_\epsilon(\phi;s)=\sum_{v\in SO(Y)_{\mathbb{Z}}\backslash V_\epsilon\cap V_{\mathbb{Q}}}\frac{\phi(v)\mu(v)}{|P(v)|^s}$$

converges absolutely for $\Re(s)>\frac{m}{2}$, as will be explained in Remark 2 shortly. Hence the interchange of summation and integration, which leads to (33), can be justified under this condition. Similary, for $f^*\in\mathcal{S}(V_{\mathbb{R}})$ and $\phi^*\in\mathcal{S}(V_{\mathbb{Q}})$ that satisfies

$$\phi^*({}^t\gamma^{-1}v^*)=\phi^*(v^*)\qquad\text{for}\quad v^*\in V_{\mathbb{Q}},\gamma\in SO(Y)_{\mathbb{Z}}, \tag{34}$$

we define the zeta ingegral $Z^*(f^*,\phi^*;s)$ by

$$Z^*(f^*,\phi^*;s)=\int_0^\infty d^\times t\int_{SO(Y)_{\mathbb{R}}/SO(Y)_{\mathbb{Z}}}|\chi^*(t,g)|^s\sum_{v^*\in V_{\mathbb{Q}}-S^*_{\mathbb{Q}}}\phi^*(v^*)f^*(\rho^*(t,g)v^*)d^1g. \tag{35}$$

Furthermore, for $v^*\in V_{\mathbb{Q}}-S^*_{\mathbb{Q}}$, we put

$$\mu^*(v^*)=\int_{SO(Y)_{v^*,\mathbb{R}}/SO_{v^*,\mathbb{Z}}}d\mu^*_{v^*}(h), \tag{36}$$

where $d\mu^*_{v^*}$ is the Haar measure on $SO(Y)_{v^*,\mathbb{R}}$ defined by (28).

Definition 2 (Siegel's zeta functions with congruence conditions) Let $\epsilon, \eta = \pm$ and assume that $\phi, \phi^* \in \mathcal{S}(V_{\mathbb{Q}})$ satisfy (30), (34), respectively. Then we define $\zeta_\epsilon(\phi; s)$ and $\zeta_\eta^*(\phi^*; s)$ by

$$\zeta_\epsilon(\phi; s) = \sum_{v \in SO(Y)_{\mathbb{Z}} \backslash V_\epsilon \cap V_{\mathbb{Q}}} \frac{\phi(v)\mu(v)}{|P(v)|^s} \tag{37}$$

$$\zeta_\eta^*(\phi^*; s) = \sum_{v^* \in SO(Y)_{\mathbb{Z}} \backslash V_\eta^* \cap V_{\mathbb{Q}}} \frac{\phi^*(v^*)\mu^*(v^*)}{|P^*(v^*)|^s}. \tag{38}$$

We can summarize our argument as the following

Lemma 6 (Integral representations of the zeta functions) *Let $f, f^* \in \mathcal{S}(V_{\mathbb{R}})$ and assume that $\phi, \phi^* \in \mathcal{S}(V_{\mathbb{Q}})$ are $SO(Y)_{\mathbb{Z}}$-invariant. For $\Re(s) > \frac{m}{2}$, we have*

$$Z(f, \phi; s) = \sum_{\epsilon = \pm} \zeta_\epsilon(\phi; s)\Phi_\epsilon(f; s),$$
$$Z^*(f^*, \phi^*; s) = \sum_{\eta = \pm} \zeta_\eta^*(\phi^*; s)\Phi_\eta^*(f^*; s).$$

Remark 2 (1) The original Siegel's zeta functions are obtained by letting $\phi = \phi_0$, where ϕ_0 is the characteristic function $\mathrm{ch}_{V_{\mathbb{Z}}}$ of $V_{\mathbb{Z}}$. To apply Weil-type converse theorems, we need to examine the case where $\phi(v) = \psi(P(v))\phi_0(v)$ with Dirichlet character ψ. Since each $\phi(v)$ is a linear combination of characteristic functions of subsets of the form $a + NV_{\mathbb{Z}}$ $(a \in V_{\mathbb{Q}}, N \in \mathbb{Z}_{\geq 1})$, we call $\zeta_\epsilon(\phi; s)$, $\zeta_\eta^*(\phi^*; s)$ *Siegel's zeta functions with congruence conditions.*

(2) The absolute convergence of Siegel's zeta functions is not at all obvious, though Siegel wrote just "Die Konvergents der Reihe entnimmt man der Reduktiontheorie". A detailed proof of the convergence can be found in Tamagawa [30]. It also follows from the general theory of prehomogeneous vector spaces (Saito [18], F. Sato [19]).

(3) We can write $\zeta_\pm(\phi; s)$ as

$$\zeta_\pm(\phi; s) = \sum_{r \in \mathbb{Q}_{>0}} \frac{M(P, \phi; \pm r)}{r^s}$$

with

$$M(P, \phi; \pm r) := \sum_{\substack{v \in SO(Y)_{\mathbb{Z}} \backslash V_\pm \cap \mathrm{supp}(\phi) \\ P(v) = \pm r}} \phi(v)\mu(v).$$

Since $\phi(v) = 0$ for $v \notin \frac{1}{L}V_{\mathbb{Z}}$ with some integer L, we see that that the sum in the definition of $M(P, \phi; \pm r)$ is a finite sum (cf. Kimura [11, p.184]). In the case of $\phi = \phi_0$, we have $\mathrm{supp}(\phi_0) = V_{\mathbb{Z}}$ and $P(v) \in \mathbb{Z} \setminus \{0\}$ for $v \in V_\pm \cap V_{\mathbb{Z}}$. For $n = 1, 2, \ldots$, we put

$$M(P; \pm n) = \sum_{\substack{v \in SO(Y)_{\mathbb{Z}} \backslash V_\pm \cap V_{\mathbb{Z}} \\ P(v) = \pm n}} \mu(v). \tag{39}$$

Siegel called $M(P; n)$ the measures of representation (*Darstellungsmaß*). We have

$$\zeta_\pm(\phi_0; s) = \sum_{n=1}^{\infty} \frac{M(P; \pm n)}{n^s}.$$

To investigate analytic properties of the zeta integrals, we define measures on isotropy subgroups at singular points. We fix an arbitrary point v of $S_{1,\mathbb{R}}$. Recall that in the previous section, we have defined an $SO(Y)_{\mathbb{R}}$-invariant measure $|\omega|_\infty$ on $S_{1,\mathbb{R}} \cong SO(Y)_{\mathbb{R}}/SO(Y)_{v,\mathbb{R}}$. We can normalize a measure $d\sigma_v$ on the isotropy subgroup $SO(Y)_{v,\mathbb{R}}$ in such a way that the integration formula

$$\int_{SO(Y)_{\mathbb{R}}} \psi(g)d^1g = \int_{SO(Y)_{\mathbb{R}}/SO(Y)_{v,\mathbb{R}}} |\omega(\dot{g}v)|_\infty \int_{SO(Y)_{v,\mathbb{R}}} \psi(\dot{g}h)d\sigma_v(h) \tag{40}$$

holds for all integrable functions $\psi(g) \in L^1(SO(Y)_{\mathbb{R}})$. Similarly, for $v^* \in S^*_{1,\mathbb{R}}$, we take a measure $d\sigma^*_{v^*}$ on the isotropy subgroup $SO(Y)_{v^*,\mathbb{R}}$ such that the integration formula

$$\int_{SO(Y)_{\mathbb{R}}} \psi(g)d^1g = \int_{SO(Y)_{\mathbb{R}}/SO(Y)_{v^*,\mathbb{R}}} |\omega^*({}^t\dot{g}^{-1}v^*)|_\infty \int_{SO(Y)_{v^*,\mathbb{R}}} \psi(\dot{g}h)d\sigma^*_{v^*}(h) \tag{41}$$

holds for all integrable functions $\psi(g) \in L^1(SO(Y)_{\mathbb{R}})$. Now we put

$$\begin{aligned}
Z_+(f,\phi;s) &= \int_1^\infty d^\times t \int_{SO(Y)_{\mathbb{R}}/SO(Y)_{\mathbb{Z}}} |\chi(t,g)|^s \sum_{v\in V_{\mathbb{Q}}-S_{\mathbb{Q}}} \phi(v)f(\rho(t,g)v)d^1g,\\
Z_-(f,\phi;s) &= \int_0^1 d^\times t \int_{SO(Y)_{\mathbb{R}}/SO(Y)_{\mathbb{Z}}} |\chi(t,g)|^s \sum_{v\in V_{\mathbb{Q}}-S_{\mathbb{Q}}} \phi(v)f(\rho(t,g)v)d^1g,\\
Z^*_+(f^*,\phi^*;s) &= \int_0^1 d^\times t \int_{SO(Y)_{\mathbb{R}}/SO(Y)_{\mathbb{Z}}} |\chi^*(t,g)|^s \sum_{v^*\in V_{\mathbb{Q}}-S^*_{\mathbb{Q}}} \phi^*(v^*)f^*(\rho^*(t,g)v^*)d^1g,\\
Z^*_-(f^*,\phi^*;s) &= \int_1^\infty d^\times t \int_{SO(Y)_{\mathbb{R}}/SO(Y)_{\mathbb{Z}}} |\chi^*(t,g)|^s \sum_{v^*\in V_{\mathbb{Q}}-S^*_{\mathbb{Q}}} \phi^*(v^*)f^*(\rho^*(t,g)v^*)d^1g.
\end{aligned}$$

It is obvious that

$$Z(f,\phi;s) = Z_+(f,\phi;s) + Z_-(f,\phi;s), \quad Z^*(f^*,\phi^*;s) = Z^*_+(f^*,\phi^*;s) + Z^*_-(f^*,\phi^*;s).$$

The four integrals above converges absolutely for $\Re(s) > \frac{m}{2}$, and further, two integrals $Z_+(f,\phi;s)$ and $Z^*_+(f^*,\phi^*;s)$ are absolutely convergent for any $s \in \mathbb{C}$ and define entire functions of s. Let us calculate $Z_-(f,\phi;s)$ *formally* by using Lemma 5, the Poisson summation formula; the interchange of integral and summation will be justified later in Remark 3. Since $\chi(t,g) = \chi^*(t,g)^{-1} = t^2$, it follows from Lemma 5 that

$$\begin{aligned}
Z_-(f,\phi;s) &= \int_0^1 d^\times t \int_{SO(Y)_{\mathbb{R}}/SO(Y)_{\mathbb{Z}}} |\chi(t,g)|^s \\
&\quad \times \left\{ |t|^{-m} \sum_{v^*\in V_{\mathbb{Q}}} \widehat{\phi}(v^*)\widehat{f}(\rho^*(t,g)v^*) - \sum_{v\in S_{\mathbb{Q}}} \phi(v)f(\rho(t,g)v) \right\} d^1g \\
&= \int_0^1 d^\times t \int_{SO(Y)_{\mathbb{R}}/SO(Y)_{\mathbb{Z}}} |\chi^*(t,g)|^{\frac{m}{2}-s} \sum_{v^*\in V_{\mathbb{Q}}-S^*_{\mathbb{Q}}} \widehat{\phi}(v^*)\widehat{f}(\rho^*(t,g)v^*)d^1g \\
&\quad + \int_0^1 t^{2s-m}d^\times t \int_{SO(Y)_{\mathbb{R}}/SO(Y)_{\mathbb{Z}}} \sum_{v^*\in S^*_{\mathbb{Q}}} \widehat{\phi}(v^*)\widehat{f}(\rho^*(t,g)v^*)d^1g \\
&\quad - \int_0^1 t^{2s}d^\times t \int_{SO(Y)_{\mathbb{R}}/SO(Y)_{\mathbb{Z}}} \sum_{v\in S_{\mathbb{Q}}} \phi(v)f(\rho(t,g)v)d^1g.
\end{aligned}$$

The first term of the most right hand side is

$$\int_0^1 d^\times t \int_{SO(Y)_\mathbb{R}/SO(Y)_\mathbb{Z}} |\chi^*(t,g)|^{\frac{m}{2}-s} \sum_{v^*\in V_\mathbb{Q}-S^*_\mathbb{Q}} \widehat{\phi}(v^*)\widehat{f}(\rho^*(t,g)v^*)d^1g = Z^*_+\left(\widehat{f},\widehat{\phi};\frac{m}{2}-s\right).$$

Using (40) and (41), we calculate the second and third terms following the method of Sato-Shintani [21, Theorem 2]. Put

$$S_{1,\mathbb{Q}} = \{v \in V_\mathbb{Q} \mid P(v) = 0, v \neq 0\}, \qquad S^*_{1,\mathbb{Q}} = \{v^* \in V_\mathbb{Q} \mid P^*(v^*) = 0, v^* \neq 0\}$$

By the interchange of summation and integration, the third term above becomes

$$\begin{aligned}
&\int_0^1 t^{2s} d^\times t \int_{SO(Y)_\mathbb{R}/SO(Y)_\mathbb{Z}} \sum_{v\in S_\mathbb{Q}} \phi(v) f(\rho(t,g)v) d^1g \\
&= \sum_{v\in SO(Y)_\mathbb{Z}\backslash S_{1,\mathbb{Q}}} \phi(v) \int_0^1 t^{2s} d^\times t \int_{SO(Y)_\mathbb{R}/SO(Y)_{v,\mathbb{Z}}} f(\rho(t,g)v) d^1g \\
&\quad + \phi(0)f(0) \int_0^1 t^{2s} d^\times t \int_{SO(Y)_\mathbb{R}/SO(Y)_\mathbb{Z}} d^1g. \qquad (42)
\end{aligned}$$

By applying (40) to $\psi(g) = f(\rho(t,g)v) = f(tgv)$, we have

$$\begin{aligned}
\int_{SO(Y)_\mathbb{R}/SO(Y)_{v,\mathbb{Z}}} f(\rho(t,g)v) d^1g &= \int_{SO(Y)_\mathbb{R}/SO(Y)_{v,\mathbb{R}}} |\omega(\dot{g}v)|_\infty \\
&\quad \times \int_{SO(Y)_{v,\mathbb{R}}/SO(Y)_{v,\mathbb{Z}}} f(t\dot{g}hv) d\sigma_v(h) \\
&= \int_{S_{1,\mathbb{R}}} f(tz)|\omega(z)|_\infty \int_{SO(Y)_{v,\mathbb{R}}/SO(Y)_{v,\mathbb{Z}}} d\sigma_v(h) \\
&= t^{2-m} \int_{S_{1,\mathbb{R}}} f(z)|\omega(z)|_\infty \int_{SO(Y)_{v,\mathbb{R}}/SO(Y)_{v,\mathbb{Z}}} d\sigma_v(h).
\end{aligned}$$

Here we have used (23) in the third equality. Hence the integral (42) is calculated as

$$\begin{aligned}
&\int_0^1 t^{2s} d^\times t \int_{SO(Y)_\mathbb{R}/SO(Y)_\mathbb{Z}} \sum_{v\in S_\mathbb{Q}} \phi(v) f(\rho(t,g)v) d^1g \\
&= \sum_{v\in SO(Y)_\mathbb{Z}\backslash S_{1,\mathbb{Q}}} \phi(v) \int_0^1 t^{2s+2-m} \frac{2dt}{t} \int_{S_{1,\mathbb{R}}} f(z)|\omega(z)|_\infty \int_{SO(Y)_{v,\mathbb{R}}/SO(Y)_{v,\mathbb{Z}}} d\sigma_v(h) \\
&\quad + \phi(0)f(0) \int_0^1 t^{2s} \frac{2dt}{t} \int_{SO(Y)_\mathbb{R}/SO(Y)_\mathbb{Z}} d^1g \\
&= \frac{1}{s+1-\frac{m}{2}} \int_{S_{1,\mathbb{R}}} f(z)|\omega(z)|_\infty \sum_{v\in SO(Y)_\mathbb{Z}\backslash S_{1,\mathbb{Q}}} \phi(v) \int_{SO(Y)_{v,\mathbb{R}}/SO(Y)_{v,\mathbb{Z}}} d\sigma_v(h) \\
&\quad + \frac{\phi(0)f(0)}{s} \int_{SO(Y)_\mathbb{R}/SO(Y)_\mathbb{Z}} d^1g.
\end{aligned}$$

Similarly, by term-by-term integration, we have

$$\begin{aligned}
&\int_0^1 t^{2s-m} d^\times t \int_{SO(Y)_{\mathbb{R}}/SO(Y)_{\mathbb{Z}}} \sum_{v^* \in S^*_{\mathbb{Q}}} \widehat{\phi}(v^*)\widehat{f}(\rho^*(t,g)v^*) d^1 g \\
&= \sum_{v^* \in SO(Y)_{\mathbb{Z}} \backslash S^*_{1,\mathbb{Q}}} \widehat{\phi}(v^*) \int_0^1 t^{2s-m} d^\times t \int_{SO(Y)_{\mathbb{R}}/SO(Y)_{v^*,\mathbb{Z}}} \widehat{f}(\rho^*(t,g)v^*) d^1 g \\
&\quad + \widehat{\phi}(0)\widehat{f}(0) \int_0^1 t^{2s-m} d^\times t \int_{SO(Y)_{\mathbb{R}}/SO(Y)_{\mathbb{Z}}} d^1 g, \qquad (43)
\end{aligned}$$

and by using (24) and (41), we obtain

$$\begin{aligned}
&\int_{SO(Y)_{\mathbb{R}}/SO(Y)_{v^*,\mathbb{Z}}} \widehat{f}(\rho^*(t,g)v^*) d^1 g \\
&= \int_{SO(Y)_{\mathbb{R}}/SO(Y)_{v^*,\mathbb{R}}} \widehat{f}(t^{-1} \cdot {}^t\dot{g}^{-1} v^*) |\omega^*({}^t\dot{g}^{-1} v^*)|_\infty \int_{SO(Y)_{v^*,\mathbb{R}}/SO(Y)_{v^*,\mathbb{Z}}} d\sigma^*_{v^*}(h) \\
&= \int_{S^*_{1,\mathbb{R}}} \widehat{f}(t^{-1} z^*) |\omega^*(z^*)|_\infty \int_{SO(Y)_{v^*,\mathbb{R}}/SO(Y)_{v^*,\mathbb{Z}}} d\sigma^*_{v^*}(h) \\
&= t^{m-2} \int_{S^*_{1,\mathbb{R}}} \widehat{f}(z^*) |\omega^*(z^*)|_\infty \int_{SO(Y)_{v^*,\mathbb{R}}/SO(Y)_{v^*,\mathbb{Z}}} d\sigma^*_{v^*}(h).
\end{aligned}$$

Hence we see that

$$\begin{aligned}
&\int_0^1 t^{2s-m} d^\times t \int_{SO(Y)_{\mathbb{R}}/SO(Y)_{\mathbb{Z}}} \sum_{v^* \in S^*_{\mathbb{Q}}} \widehat{\phi}(v^*)\widehat{f}(\rho^*(t,g)v^*) d^1 g \\
&= \frac{1}{s-1} \int_{S^*_{1,\mathbb{R}}} \widehat{f}(z^*) |\omega^*(z^*)|_\infty \sum_{v^* \in SO(Y)_{\mathbb{Z}} \backslash S^*_{1,\mathbb{Q}}} \widehat{\phi}(v^*) \int_{SO(Y)_{v^*,\mathbb{R}}/SO(Y)_{v^*,\mathbb{Z}}} d\sigma^*_{v^*}(h) \\
&\quad + \frac{\widehat{\phi}(0)\widehat{f}(0)}{s - \frac{m}{2}} \int_{SO(Y)_{\mathbb{R}}/SO(Y)_{\mathbb{Z}}} d^1 g
\end{aligned}$$

Now we put

$$\sigma(v) := \int_{SO(Y)_{v,\mathbb{R}}/SO(Y)_{v,\mathbb{Z}}} d\sigma_v(h), \qquad (44)$$

$$\sigma^*(v^*) := \int_{SO(Y)_{v^*,\mathbb{R}}/SO(Y)_{v^*,\mathbb{Z}}} d\sigma^*_{v^*}(h). \qquad (45)$$

Then we have the first assertion of the following lemma; the second assertion can be proved similarly as the first assertion, and then the third assertion follows immediately from the first and second assertions.

Lemma 7 (1) *For $\Re(s) > \frac{m}{2}$, we have*

$$\begin{aligned} Z(f,\phi;s) &= Z_+(f,\phi;s) + Z_+^*\left(\widehat{f},\widehat{\phi};\frac{m}{2}-s\right) \\ &\quad + \frac{1}{s-1}\int_{S_{1,\mathbb{R}}^*} \widehat{f}(z^*)|\omega^*(z^*)|_\infty \sum_{v^*\in SO(Y)_\mathbb{Z}\backslash S_{1,\mathbb{Q}}^*} \widehat{\phi}(v^*)\sigma^*(v^*) \\ &\quad + \frac{\widehat{\phi}(0)\widehat{f}(0)}{s-\frac{m}{2}}\int_{SO(Y)_\mathbb{R}/SO(Y)_\mathbb{Z}} d^1g \\ &\quad - \frac{1}{s+1-\frac{m}{2}}\int_{S_{1,\mathbb{R}}} f(z)|\omega(z)|_\infty \sum_{v\in SO(Y)_\mathbb{Z}\backslash S_{1,\mathbb{Q}}} \phi(v)\sigma(v) \\ &\quad - \frac{\phi(0)f(0)}{s}\int_{SO(Y)_\mathbb{R}/SO(Y)_\mathbb{Z}} d^1g. \end{aligned}$$

(2) *For $\Re(s) > \frac{m}{2}$, we have*

$$\begin{aligned} Z^*(\widehat{f},\widehat{\phi};s) &= Z_+^*(\widehat{f},\widehat{\phi};s) + Z_+\left(f,\phi,\frac{m}{2}-s\right) \\ &\quad + \frac{1}{s-1}\int_{S_{1,\mathbb{R}}} f(z)|\omega(z)|_\infty \sum_{v\in SO(Y)_\mathbb{Z}\backslash S_{1,\mathbb{Q}}} \phi(v)\sigma(v) \\ &\quad + \frac{\phi(0)f(0)}{s-\frac{m}{2}}\int_{SO(Y)_\mathbb{R}/SO(Y)_\mathbb{Z}} d^1g \\ &\quad - \frac{1}{s+1-\frac{m}{2}}\int_{S_{1,\mathbb{R}}^*} \widehat{f}(z^*)|\omega^*(z^*)|_\infty \sum_{v^*\in SO(Y)_\mathbb{Z}\backslash S_{1,\mathbb{Q}}^*} \widehat{\phi}(v^*)\sigma^*(v^*) \\ &\quad - \frac{\widehat{\phi}(0)\widehat{f}(0)}{s}\int_{SO(Y)_\mathbb{R}/SO(Y)_\mathbb{Z}} d^1g. \end{aligned}$$

(3) *As functions of s, the integrals $Z(f,\phi,s)$ and $Z^*(\widehat{f},\widehat{\phi};s)$ can be continued analytically to the whole s-plane, and satisfy the following functional equation:*

$$Z^*(\widehat{f},\widehat{\phi};s) = Z\left(f,\phi;\frac{m}{2}-s\right).$$

Remark 3 In [10], Igusa studied the so-called admissible representations related to the Siegel-Weil formula [33]. According to his classification, our prehomogeneous vector space $(GL_1(\mathbb{C})\times SO(Y), \mathbb{C}^m)$ gives an admissible representation if $m \geq 5$, and this implies that the integrals

$$\int_{SO(Y)_\mathbb{R}/SO(Y)_\mathbb{Z}} \sum_{v\in V_\mathbb{Q}} \phi(v)f(gv)d^1g, \qquad \int_{SO(Y)_\mathbb{R}/SO(Y)_\mathbb{Z}} \sum_{v^*\in V_\mathbb{Q}} \phi^*(v^*)f^*({}^tg^{-1}v^*)d^1g$$

are absolutely convergent for all Schwartz-Bruhat functions $f, f^* \in \mathcal{S}(V_\mathbb{R})$ and $\phi, \phi^* \in \mathcal{S}(V_\mathbb{Q})$. Hence the integrals

$$\begin{aligned} \int_{SO(Y)_\mathbb{R}/SO(Y)_\mathbb{Z}} \sum_{v\in S_{1,\mathbb{Q}}} \phi(v)f(gv)d^1g &= \int_{S_{1,\mathbb{R}}} f(z)|\omega(z)|_\infty \sum_{v\in SO(Y)_\mathbb{Z}\backslash S_{1,\mathbb{Q}}} \phi(v)\sigma(v), \\ \int_{SO(Y)_\mathbb{R}/SO(Y)_\mathbb{Z}} \sum_{v^*\in S_{1,\mathbb{Q}}^*} \widehat{\phi}(v^*)\widehat{f}({}^tg^{-1}v^*)d^1g &= \int_{S_{1,\mathbb{R}}^*} \widehat{f}(z^*)|\omega^*(z^*)|_\infty \sum_{v^*\in SO(Y)_\mathbb{Z}\backslash S_{1,\mathbb{Q}}^*} \widehat{\phi}(v^*)\sigma^*(v^*), \end{aligned}$$

which appear in Lemma 7, are absolutely convergent, and the interchange of integral and summation can be justified by Fubini's theorem.

4 Analytic properties of Siegel's zeta functions

Theorem 1 *Assume that $\phi \in \mathcal{S}(V_{\mathbb{Q}})$ is $SO(Y)_{\mathbb{Z}}$-invariant.*

(1) *The zeta functions $\zeta_\epsilon(\phi;s)$ and $\zeta_\eta^*(\widehat{\phi};s)$ have analytic continuations of s in $\mathbb{C}$, and the zeta functions multiplied by $(s-1)(s-\frac{m}{2})$ are entire functions of s of finite order in any vertical strip.*

(2) *Th zeta functions $\zeta_\epsilon(\phi;s)$ and $\zeta_\eta^*(\widehat{\phi};s)$ satisfy the following functional equation:*

$$\begin{aligned}\begin{pmatrix}\zeta_+\left(\phi;\frac{m}{2}-s\right)\\ \zeta_-\left(\phi;\frac{m}{2}-s\right)\end{pmatrix} &= \Gamma\left(s+1-\frac{m}{2}\right)\Gamma(s)|D|^{\frac{1}{2}}\cdot 2^{-2s+\frac{m}{2}}\cdot\pi^{-2s+\frac{m}{2}-1}\\ &\times\begin{pmatrix}\sin\pi\left(\frac{p}{2}-s\right) & \sin\frac{\pi(m-p)}{2}\\ \sin\frac{\pi p}{2} & \sin\pi\left(\frac{m-p}{2}-s\right)\end{pmatrix}\begin{pmatrix}\zeta_+^*(\widehat{\phi};s)\\ \zeta_-^*(\widehat{\phi};s)\end{pmatrix}.\end{aligned} \tag{46}$$

(3) *The residues of $\zeta_\epsilon(\phi;s)$, $\zeta_\eta^*(\widehat{\phi};s)$ at $s=1$ and $s=\frac{m}{2}$ are given by*

$$\operatorname*{Res}_{s=\frac{m}{2}}\zeta_\epsilon(\phi;s) = \widehat{\phi}(0)\int_{SO(Y)_{\mathbb{R}}/SO(Y)_{\mathbb{Z}}} d^1g, \tag{47}$$

$$\operatorname*{Res}_{s=\frac{m}{2}}\zeta_\eta^*(\widehat{\phi};s) = \phi(0)\int_{SO(Y)_{\mathbb{R}}/SO(Y)_{\mathbb{Z}}} d^1g, \tag{48}$$

$$\begin{aligned}\operatorname*{Res}_{s=1}\zeta_\epsilon(\phi;s) &= \Gamma\left(\frac{m}{2}-1\right)|D|^{\frac{1}{2}}\cdot 2^{2-\frac{m}{2}}\cdot\pi^{1-\frac{m}{2}}\sum_{v^*\in SO(Y)_{\mathbb{Z}}\backslash S_{1,\mathbb{Q}}^*}\widehat{\phi}(v^*)\sigma^*(v^*)\\ &\times\begin{cases}\sin\dfrac{\pi}{2}(m-p)\ (\epsilon=+)\\ \sin\dfrac{\pi p}{2}\qquad\ (\epsilon=-)\end{cases},\end{aligned} \tag{49}$$

$$\begin{aligned}\operatorname*{Res}_{s=1}\zeta_\eta^*(\widehat{\phi};s) &= \Gamma\left(\frac{m}{2}-1\right)|D|^{-\frac{1}{2}}\cdot 2^{2-\frac{m}{2}}\cdot\pi^{1-\frac{m}{2}}\sum_{v\in SO(Y)_{\mathbb{Z}}\backslash S_{1,\mathbb{Q}}}\phi(v)\sigma(v)\\ &\times\begin{cases}\sin\dfrac{\pi}{2}(m-p)\ (\eta=+)\\ \sin\dfrac{\pi p}{2}\qquad\ (\eta=-)\end{cases}.\end{aligned} \tag{50}$$

(3) *The following relations hold:*

$$\zeta_+\left(\phi;\frac{m}{2}-1\right)+\zeta_-\left(\phi;\frac{m}{2}-1\right) = -\sum_{v\in SO(Y)_{\mathbb{Z}}\backslash S_{1,\mathbb{Q}}}\phi(v)\sigma(v), \tag{51}$$

$$\zeta_+^*\left(\widehat{\phi};\frac{m}{2}-1\right)+\zeta_-^*\left(\widehat{\phi};\frac{m}{2}-1\right) = -\sum_{v^*\in SO(Y)_{\mathbb{Z}}\backslash S_{1,\mathbb{Q}}^*}\widehat{\phi}(v^*)\sigma^*(v^*). \tag{52}$$

Proof Let $f\in C_0^\infty(V_{\mathbb{R}})$ in Lemma 7 (1). Then we see that

$$\begin{aligned}Z(f,\phi;s) &= Z_+(f,\phi;s)+Z_+^*\left(\widehat{f},\widehat{\phi};\frac{m}{2}-s\right)\\ &+\frac{1}{s-1}\int_{S_{1,\mathbb{R}}^*}\widehat{f}(z^*)|\omega^*(z^*)|_\infty\sum_{v^*\in SO(Y)_{\mathbb{Z}}\backslash S_{1,\mathbb{Q}}^*}\widehat{\phi}(v^*)\sigma^*(v^*)\\ &+\frac{\widehat{\phi}(0)\widehat{f}(0)}{s-\frac{m}{2}}\int_{SO(Y)_{\mathbb{R}}/SO(Y)_{\mathbb{Z}}}d^1g,\end{aligned}$$

and thus the integral $Z(f,\phi;s)$ can be continued to a meromorphic function on the whole $\mathbb{C}$, and $(s-1)(s-\frac{m}{2})Z(f,\phi;s)$ is an entire function of s. Further, for any $s\in\mathbb{C}$, we take $f_\epsilon\in C_0^\infty(V_\epsilon)$ such that $\Phi_\epsilon(f;s)\neq 0$. Then Lemma 6 implies that

$$\zeta_\epsilon(\phi;s)=\frac{Z(\phi,f_\epsilon;s)}{\Phi_\epsilon(f;s)},$$

and hence $\zeta_\epsilon(\phi;s)$ also can be continued to a meromorphic function on the whole $\mathbb{C}$, and $(s-1)(s-\frac{m}{2})\zeta_\epsilon(\phi;s)$ is an entire function of s. The analytic continuation of $\zeta_\eta^*(\widehat{\phi};s)$ can be proved in a similar fashion. Further, one can prove the boundedness of $\zeta_\epsilon(\phi;s)$ and $\zeta_\eta^*(\widehat{\phi};s)$ in the same method as in Ueno [32, § 4]. By Lemma 6 and Lemma 7 (3), we have

$$\left(\Phi_+^*(\widehat{f};s)\ \Phi_-^*(\widehat{f};s)\right)\begin{pmatrix}\zeta_+^*(\widehat{\phi};s)\\ \zeta_-^*(\widehat{\phi};s)\end{pmatrix}=\left(\Phi_+\left(f;\tfrac{m}{2}-s\right)\ \Phi_-\left(f;\tfrac{m}{2}-s\right)\right)\begin{pmatrix}\zeta_+\left(\phi;\frac{m}{2}-s\right)\\ \zeta_-\left(\phi;\frac{m}{2}-s\right)\end{pmatrix},$$

and by Lemma 2, we have

$$\left(\Phi_+^*(\widehat{f};s)\ \Phi_-^*(\widehat{f};s)\right)=\left(\Phi_+\left(f;\tfrac{m}{2}-s\right)\ \Phi_-\left(f;\tfrac{m}{2}-s\right)\right)\cdot {}^tA(s),$$

where $A(s)$ is given by

$$A(s)=\Gamma\left(s+1-\frac{m}{2}\right)\Gamma(s)|D|^{\frac{1}{2}}\cdot 2^{-2s+\frac{m}{2}}\cdot\pi^{-2s+\frac{m}{2}-1}\begin{pmatrix}\sin\pi\left(\frac{p}{2}-s\right) & \sin\frac{\pi p}{2}\\ \sin\frac{\pi(m-p)}{2} & \sin\pi\left(\frac{m-p}{2}-s\right)\end{pmatrix}.$$

This implies that the vector

$$\begin{pmatrix}\zeta_+\left(\phi;\frac{m}{2}-s\right)\\ \zeta_-\left(\phi;\frac{m}{2}-s\right)\end{pmatrix}-{}^tA(s)\begin{pmatrix}\zeta_+^*(\widehat{\phi};s)\\ \zeta_-^*(\widehat{\phi};s)\end{pmatrix}\tag{53}$$

is orthogonal to the vector

$$\left(\Phi_+\left(f;\tfrac{m}{2}-s\right)\ \Phi_-\left(f;\tfrac{m}{2}-s\right)\right)$$

for arbitrary $f\in\mathcal{S}(V_\mathbb{R})$. For any $s\in\mathbb{C}$, there exists an $f_\epsilon\in C_0^\infty(V_\epsilon)$ such that $\Phi_\epsilon(f;\frac{m}{2}-s)\neq 0$, and hence (53) is the zero vector. This proves the functional equation (46). Next we calculate the residues. For the simple pole at $s=\frac{m}{2}$, we have

$$\operatorname*{Res}_{s=\frac{m}{2}}Z(\phi,f;s)=\widehat{\phi}(0)\widehat{f}(0)\cdot\int_{SO(Y)_\mathbb{R}/SO(Y)_\mathbb{Z}}d^1g$$

by Lemma 7 (1). For $f\in C_0^\infty(V_\epsilon)$, Lemma 6 implies $Z(\phi,f;s)=\zeta_\epsilon(\phi;s)\cdot\Phi_\epsilon(f;s)$, and $\Phi_\epsilon\left(f;\frac{m}{2}\right)$ is meaningful:

$$\Phi_\epsilon\left(f;\tfrac{m}{2}\right)=\lim_{s\to\frac{m}{2}}\int_{V_\epsilon}f(x)|P(x)|^{s-\frac{m}{2}}dx=\int_{V_\mathbb{R}}f(x)dx=\widehat{f}(0).$$

Hence we have

$$\operatorname*{Res}_{s=\frac{m}{2}}\zeta_\epsilon(\phi;s)=\widehat{\phi}(0)\int_{SO(Y)_\mathbb{R}/SO(Y)_\mathbb{Z}}d^1g,$$

and similarly

$$\operatorname*{Res}_{s=\frac{m}{2}}\zeta_\eta^*(\widehat{\phi};s)=\phi(0)\int_{SO(Y)_\mathbb{R}/SO(Y)_\mathbb{Z}}d^1g.$$

By Lemma 7 (1), it is easy to pick up the residue of $Z(f,\phi;s)$ at the simple pole $s=1$, and together with Lemma 6, it implies that for $f\in C_0^\infty(V_\epsilon)$,

$$\operatorname*{Res}_{s=1}\zeta_\epsilon(\phi;s)\cdot\Phi_\epsilon(f;1)=\int_{S_{1,\mathbb{R}}^*}\widehat{f}(z^*)|\omega^*(z^*)|_\infty\sum_{\nu^*\in SO(Y)_\mathbb{Z}\backslash S_{1,\mathbb{Q}}^*}\widehat{\phi}(\nu^*)\sigma^*(\nu^*).$$

Here the value $\Phi_\epsilon(f;1)$ is meaningful, and

$$\Phi_\epsilon(f;1) = \lim_{s\to 1}\int_{V_\epsilon} f(x)|P(x)|^{s-\frac{m}{2}}dx = \int_{V_\epsilon} f(x)|P(x)|^{1-\frac{m}{2}}dx.$$

Furthermore, by Lemma 3 (1), we have

$$\int_{S^*_{1,\mathbb{R}}} \widehat{f}(v^*)|\omega^*(v^*)|_\infty = \Gamma\left(\frac{m}{2}-1\right)|D|^{\frac{1}{2}}\cdot 2^{2-\frac{m}{2}}\cdot\pi^{1-\frac{m}{2}} \times \begin{cases} \sin\dfrac{\pi}{2}(m-p)\displaystyle\int_{V_+} f(v)|P(v)|^{1-\frac{m}{2}}dv & (f\in C_0^\infty(V_+) \\ \sin\dfrac{\pi p}{2}\displaystyle\int_{V_-} f(v)|P(v)|^{1-\frac{m}{2}}dv & (f\in C_0^\infty(V_-) \end{cases},$$

and hence we obtain the residue formula (49). Similarly, the residue formula (50) can be proved with Lemma 3 (2); the detail is omitted. To prove the relation (51), we let $s=1$ in the functiona equation (46):

$$\begin{aligned}&\Gamma\left(s+1-\frac{m}{2}\right)^{-1}\left(\zeta_+\left(\phi;\frac{m}{2}-s\right)+\zeta_-\left(\phi;\frac{m}{2}-s\right)\right)\\ &\quad=\Gamma(s)|D|^{\frac{1}{2}}\cdot 2^{-2s+\frac{m}{2}}\cdot\pi^{-2s+\frac{m}{2}-1}\\ &\qquad\times\left(\sin\pi\left(\tfrac{p}{2}-s\right)+\sin\tfrac{\pi p}{2}\,\sin\tfrac{\pi(m-p)}{2}+\sin\pi\left(\tfrac{m-p}{2}-s\right)\right)\begin{pmatrix}\zeta_+^*(\widehat{\phi};s)\\ \zeta_-^*(\widehat{\phi};s)\end{pmatrix}.\end{aligned}$$

Since

$$\left(\sin\pi\left(\frac{p}{2}-s\right)+\sin\frac{\pi p}{2}\right)\Big|_{s=1}=0,\ \left(\sin\frac{\pi(m-p)}{2}+\sin\pi\left(\frac{m-p}{2}-s\right)\right)\Big|_{s=1}=0,$$

we have

$$\begin{aligned}&\lim_{s\to 1}\Gamma\left(s+1-\frac{m}{2}\right)^{-1}\left(\zeta_+\left(\phi;\frac{m}{2}-s\right)+\zeta_-\left(\phi;\frac{m}{2}-s\right)\right)\\ &\quad=|D|^{\frac{1}{2}}\cdot 2^{-2+\frac{m}{2}}\cdot\pi^{-3+\frac{m}{2}}\\ &\qquad\times\left(\tfrac{d}{ds}\sin\pi\left(\tfrac{p}{2}-s\right)\Big|_{s=1}\ \tfrac{d}{ds}\sin\pi\left(\tfrac{m-p}{2}-s\right)\Big|_{s=1}\right)\begin{pmatrix}\operatorname{Res}_{s=1}\zeta_+^*(\widehat{\phi};s)\\ \operatorname{Res}_{s=1}\zeta_-^*(\widehat{\phi};s)\end{pmatrix}.\end{aligned}$$

By using (50) and

$$\begin{aligned}\frac{d}{ds}\sin\pi\left(\frac{p}{2}-s\right)\Big|_{s=1}&=-\pi\cos\pi\left(\frac{p}{2}-1\right)=\pi\cos\frac{\pi p}{2},\\ \frac{d}{ds}\sin\pi\left(\frac{m-p}{2}-s\right)\Big|_{s=1}&=-\pi\cos\pi\left(\frac{m-p}{2}-1\right)=\pi\cos\frac{\pi(m-p)}{2},\end{aligned}$$

we see that

$$\begin{aligned}&\lim_{s\to 1}\Gamma\left(s+1-\frac{m}{2}\right)^{-1}\left(\zeta_+\left(\phi;\frac{m}{2}-s\right)+\zeta_-\left(\phi;\frac{m}{2}-s\right)\right)\\ &\quad=\Gamma\left(\frac{m}{2}-1\right)\cdot\pi^{-2}\sum_{v\in SO(Y)_{\mathbb{Z}}\backslash S_{1,\mathbb{Q}}}\phi(v)\sigma(v)\cdot\left(\pi\cos\tfrac{\pi p}{2}\ \pi\cos\tfrac{\pi(m-p)}{2}\right)\begin{pmatrix}\sin\frac{\pi(m-p)}{2}\\ \sin\frac{\pi p}{2}\end{pmatrix}\\ &\quad=-\Gamma\left(\frac{m}{2}-1\right)\sin\pi\left(\frac{m}{2}-1\right)\cdot\pi^{-1}\cdot\sum_{v\in SO(Y)_{\mathbb{Z}}\backslash S_{1,\mathbb{Q}}}\phi(v)\sigma(v).\end{aligned}$$

Since

$$\lim_{s\to 1}\Gamma\left(s+1-\frac{m}{2}\right)^{-1}=\Gamma\left(\frac{m}{2}-1\right)\sin\pi\left(\frac{m}{2}-1\right)\cdot\pi^{-1},$$

we obtain the desired relation

$$\zeta_+\left(\phi;\frac{m}{2}-1\right)+\zeta_-\left(\phi;\frac{m}{2}-1\right)=-\sum_{v\in SO(Y)_{\mathbb{Z}}\backslash S_{1,\mathbb{Q}}}\phi(v)\sigma(v).$$

Finally, let $s=\frac{m}{2}-1$ in the functional equation (46). We have

$$\begin{aligned}-\operatorname*{Res}_{s=1}\begin{pmatrix}\zeta_+(\phi;s)\\ \zeta_-(\phi;s)\end{pmatrix}&=\operatorname*{Res}_{s=0}\Gamma(s)\cdot\Gamma\left(\frac{m}{2}-1\right)|D|^{\frac{1}{2}}\cdot 2^{-\frac{m}{2}+2}\cdot\pi^{-\frac{m}{2}+1}\\&\times\begin{pmatrix}\sin\pi\left(\frac{p}{2}-\frac{m}{2}+1\right)&\sin\frac{\pi(m-p)}{2}\\ \sin\frac{\pi p}{2}&\sin\pi\left(-\frac{p}{2}+1\right)\end{pmatrix}\begin{pmatrix}\zeta_+^*\left(\widehat{\phi};\frac{m}{2}-1\right)\\ \zeta_-^*\left(\widehat{\phi};\frac{m}{2}-1\right)\end{pmatrix},\end{aligned}$$

and by using (49), we obtain

$$\zeta_+^*\left(\widehat{\phi};\frac{m}{2}-1\right)+\zeta_-^*\left(\widehat{\phi};\frac{m}{2}-1\right)=-\sum_{v^*\in SO(Y)_{\mathbb{Z}}\backslash S^*_{1,\mathbb{Q}}}\widehat{\phi}(v^*)\sigma^*(v^*).$$

□

Let N be the level of $2Y$. By definition, N is the smallest positive integer such that $N(2Y)^{-1}$ is an even matrix (a matrix whose entries are integers and even along the diagonal). We normalize the zeta functions $\zeta_\epsilon(\phi;s)$, $\zeta_\eta^*(\widehat{\phi};s)$ as follows:

$$\widetilde{\zeta}_\epsilon(\phi;s)=|D|^{-\frac{1}{2}}\cdot e^{\frac{\pi i}{4}(2p-m)}\cdot\zeta_\epsilon\left(\phi;s+\frac{m}{2}-1\right),\tag{54}$$

$$\widetilde{\zeta}_\eta^*(\widehat{\phi};s)=N^{-s}\cdot\zeta_\eta^*\left(\widehat{\phi};s+\frac{m}{2}-1\right).\tag{55}$$

Lemma 8 *The normalized zeta functions $\widetilde{\zeta}_\epsilon(\phi;s)$, $\widetilde{\zeta}_\eta^*(\widehat{\phi};s)$ satisfy the following functional equation:*

$$\begin{aligned}&(2\pi)^{-s}\Gamma(s)\gamma(s)\begin{pmatrix}\widetilde{\zeta}_+(\phi;s)\\ \widetilde{\zeta}_-(\phi;s)\end{pmatrix}\\&\quad=N^{2-\frac{m}{2}-s}\cdot(2\pi)^{-(2-\frac{m}{2}-s)}\Gamma\left(2-\frac{m}{2}-s\right)\\&\qquad\times\Sigma(2p-m)\gamma\left(2-\frac{m}{2}-s\right)\begin{pmatrix}\widetilde{\zeta}_+^*\left(\widehat{\phi};2-\frac{m}{2}-s\right)\\ \widetilde{\zeta}_-^*\left(\widehat{\phi};2-\frac{m}{2}-s\right)\end{pmatrix},\end{aligned}\tag{56}$$

where $\gamma(s)$ and $\Sigma(\ell)$ are matrices defined by (4).

Proof Let $s\mapsto 1-s$ in the functional equation (46):

$$\begin{aligned}\begin{pmatrix}\zeta_+\left(\phi;s+\frac{m}{2}-1\right)\\ \zeta_-\left(\phi;s+\frac{m}{2}-1\right)\end{pmatrix}&=\Gamma\left(2-\frac{m}{2}-s\right)\Gamma(1-s)|D|^{\frac{1}{2}}\cdot 2^{2s+\frac{m}{2}-2}\cdot\pi^{2s+\frac{m}{2}-3}\\&\times\begin{pmatrix}\sin\pi\left(s+\frac{p}{2}-1\right)&\sin\frac{\pi(m-p)}{2}\\ \sin\frac{\pi p}{2}&\sin\pi\left(s+\frac{m-p}{2}-1\right)\end{pmatrix}\begin{pmatrix}\zeta_+^*(\widehat{\phi};1-s)\\ \zeta_-^*(\widehat{\phi};1-s)\end{pmatrix}.\end{aligned}\tag{57}$$

By (54), (55) and $\Gamma(1-s)=\dfrac{\pi}{\Gamma(s)\sin\pi s}$, we see that this relation can be written as

$$\begin{aligned}(2\pi)^{-s}\Gamma(s)\begin{pmatrix}\widetilde{\zeta}_+(\phi;s)\\ \widetilde{\zeta}_-(\phi;s)\end{pmatrix} &= N^{2-\frac{m}{2}-s}\cdot(2\pi)^{-(2-\frac{m}{2}-s)}\Gamma\left(2-\frac{m}{2}-s\right)\\ &\quad\times\frac{e^{\frac{\pi i}{4}(2p-m)}}{\sin\pi s}\begin{pmatrix}\sin\pi\left(s+\frac{p}{2}-1\right) & \sin\frac{\pi(m-p)}{2}\\ \sin\frac{\pi p}{2} & \sin\pi\left(s+\frac{m-p}{2}-1\right)\end{pmatrix}\\ &\quad\times\begin{pmatrix}\widetilde{\zeta}^*_+\left(\widehat{\phi};2-\frac{m}{2}-s\right)\\ \widetilde{\zeta}^*_-\left(\widehat{\phi};2-\frac{m}{2}-s\right)\end{pmatrix}.\end{aligned}$$

An elementary calculation with $\det\gamma(s)=2i\sin\pi s$ shows that

$$\frac{e^{\frac{\pi i}{4}(2p-m)}}{\sin\pi s}\begin{pmatrix}\sin\pi\left(s+\frac{p}{2}-1\right) & \sin\frac{\pi(m-p)}{2}\\ \sin\frac{\pi p}{2} & \sin\pi\left(s+\frac{m-p}{2}-1\right)\end{pmatrix}=\gamma(s)^{-1}\cdot\Sigma(2p-m)\gamma\left(2-\frac{m}{2}-s\right),$$

which completes the proof of the lemma. □

The functional equation (56) is quite the same as the functional equation of the condition **[A3]** in Sect. 1 with $\frac{m}{2}=2\lambda$, $\ell\equiv 2p-m \pmod 4$. Hence it is reasonable to expect that our converse theorem (Lemma 1) can apply to the normalized zeta functions $\widetilde{\zeta}_\epsilon(\phi;s)$, $\widetilde{\zeta}^*_\eta(\widehat{\phi};s)$ to obtain Maass forms. The following lemma is indispensable for the application.

Lemma 9 (1) *If m is odd, then we have*

$$\widetilde{\zeta}_+(\phi;-k)+(-1)^k\cdot\widetilde{\zeta}_-(\phi;-k)=0$$

for $k=1,2,3,\dots$.

(2) *Assume that m is even and p is odd. Let $q=\frac{m}{2}$. Then we have*

$$\widetilde{\zeta}_+(\phi;-k)+(-1)^k\cdot\widetilde{\zeta}_-(\phi;-k)=0$$

for $k=1,2,\dots,q-2$.

Proof By a little calculation, we obtain

$$\begin{aligned}\begin{pmatrix}\zeta^*_+\left(\widehat{\phi};1-s\right)\\ \zeta^*_-\left(\widehat{\phi};1-s\right)\end{pmatrix} &= \Gamma(s)\Gamma\left(s+\frac{m}{2}-1\right)|D|^{-\frac{1}{2}}\cdot 2^{-2s-\frac{m}{2}+2}\cdot\pi^{-2s-\frac{m}{2}+1}\\ &\quad\times\begin{pmatrix}\sin\pi\left(s+\frac{m-p}{2}\right) & \sin\frac{\pi(m-p)}{2}\\ \sin\frac{\pi p}{2} & \sin\pi\left(s+\frac{p}{2}\right)\end{pmatrix}\begin{pmatrix}\zeta_+\left(\phi;s+\frac{m}{2}-1\right)\\ \zeta_-\left(\phi;s+\frac{m}{2}-1\right)\end{pmatrix}.\end{aligned}\tag{58}$$

Let us consider the values of both sides at $s=-k$ ($k\in\mathbb{Z}_{>0}$). On the left hand side, $\zeta^*_\eta\left(\widehat{\phi};1-s\right)$ is holomorphic at $s=-k$ except when m is even and $k=\frac{m}{2}-1=q-1$. On the right hand side, if m is odd, then $\Gamma(s)\Gamma\left(s+\frac{m}{2}-1\right)$ has a simple pole at $s=-k$ ($k\in\mathbb{Z}_{>0}$), and if m is even, then $\Gamma(s)\Gamma\left(s+\frac{m}{2}-1\right)=\Gamma(s)\Gamma(s+q-1)$ has a simple pole at $s=-k$ ($1\le k\le q-2$). We assume that $1\le k\le q-2$ in the case of even m. Then, since $\Gamma(s)\Gamma\left(s+\frac{m}{2}-1\right)$ has a simple pole at $s=-k$, we see that

$$\begin{aligned}&\begin{pmatrix}\sin\pi\left(s+\frac{m-p}{2}\right)\cdot\zeta_+\left(\phi;s+\frac{m}{2}-1\right)+\sin\frac{\pi(m-p)}{2}\cdot\zeta_-\left(\phi;s+\frac{m}{2}-1\right)\\ \sin\frac{\pi p}{2}\cdot\zeta_+\left(\phi;s+\frac{m}{2}-1\right)+\sin\pi\left(s+\frac{p}{2}\right)\cdot\zeta_-\left(\phi;s+\frac{m}{2}-1\right)\end{pmatrix}\\ &=\begin{pmatrix}\sin\pi\left(s+\frac{m-p}{2}\right)\cdot\widetilde{\zeta}_+(\phi;s)+\sin\frac{\pi(m-p)}{2}\cdot\widetilde{\zeta}_-(\phi;s)\\ \sin\frac{\pi p}{2}\cdot\widetilde{\zeta}_+(\phi;s)+\sin\pi\left(s+\frac{p}{2}\right)\cdot\widetilde{\zeta}_-(\phi;s)\end{pmatrix}\end{aligned}$$

becomes the zero vector at $s = -k$. Since $\sin \pi(-k) = 0$, $\cos \pi(-k) = (-1)^k$, we have

$$(-1)^k \sin \frac{\pi(m-p)}{2} \cdot \widetilde{\zeta}_+(\phi; -k) + \sin \frac{\pi(m-p)}{2} \cdot \widetilde{\zeta}_-(\phi; -k) = 0,$$
$$\sin \frac{\pi p}{2} \cdot \widetilde{\zeta}_+(\phi; -k) + (-1)^k \sin \frac{\pi p}{2} \cdot \widetilde{\zeta}_-(\phi; -k) = 0.$$

If m is odd, then either p or $m-p$ is odd, and thus we have

$$\widetilde{\zeta}_+(\phi; -k) + (-1)^k \cdot \widetilde{\zeta}_-(\phi; -k) = 0.$$

In the case of even m, if p is odd, then the relation above should hold. In the case that both of p and $m-p$ are even, this argument can not apply since $\sin \frac{\pi p}{2} = \sin \frac{\pi(m-p)}{2} = 0$. □

The following lemma follows immediately from the relations (51) and (52).

Lemma 10 *We have the following relations:*

$$-\left(\widetilde{\zeta}_+(\phi; 0) + \widetilde{\zeta}_-(\phi; 0)\right) = |D|^{-\frac{1}{2}} \cdot e^{\frac{\pi i}{4}(2p-m)} \cdot \sum_{v \in SO(Y)_{\mathbb{Z}} \backslash S_{1,\mathbb{Q}}} \phi(v)\sigma(v), \tag{59}$$

$$-\left(\widetilde{\zeta}^*_+(\widehat{\phi}\,; 0) + \widetilde{\zeta}^*_-(\widehat{\phi}\,; 0)\right) = \sum_{v^* \in SO(Y)_{\mathbb{Z}} \backslash S^*_{1,\mathbb{Q}}} \widehat{\phi}(v^*)\sigma^*(v^*). \tag{60}$$

In the rest of this section, we discuss the invariance of volumes with respect to scalar multiplications.

Lemma 11 (1) *For $v \in V_{\mathbb{Q}} - S_{\mathbb{Q}}$, $v^* \in V_{\mathbb{Q}} - S^*_{\mathbb{Q}}$, we define the volumes $\mu(v)$ and $\mu^*(v^*)$ by (32) and (36), respectively. For $r > 0$, we have*

$$\mu(rv) = \mu(v), \qquad \mu^*(rv^*) = \mu^*(v^*).$$

(2) *For $v \in S_{1,\mathbb{Q}}$, $v^* \in S^*_{1,\mathbb{Q}}$, we define the volumes $\sigma(v)$ and $\sigma^*(v^*)$ by (44) and (45), respectively. For $r > 0$, we have*

$$\sigma(rv) = r^{2-m} \cdot \sigma(v), \qquad \sigma^*(rv^*) = r^{2-m} \cdot \sigma^*(v^*).$$

Proof (1) We prove the second formula $\mu^*(rv^*) = \mu^*(v^*)$, which will be used later. Let $F \in C_0^\infty(V^*_\eta)$. Then, by (28) and (36), we have

$$\begin{aligned}
&\int_0^\infty d^\times t \int_{SO(Y)_{\mathbb{R}}/SO(Y)_{v^*,\mathbb{Z}}} F(\rho^*(t,g)v^*) d^1 g \\
&= \int_{V^*_\eta} F(x^*)|P^*(x^*)|^{-\frac{m}{2}} dx^* \int_{SO(Y)_{v^*,\mathbb{R}}/SO(Y)_{v^*,\mathbb{Z}}} d\mu^*_{v^*}(h) \\
&= \mu^*(v^*) \cdot \int_{V^*_\eta} F(x^*)|P^*(x^*)|^{-\frac{m}{2}} dx^*.
\end{aligned}$$

By the substitution $v^* \mapsto rv^*$, we have

$$\int_0^\infty d^\times t \int_{SO(Y)_{\mathbb{R}}/SO(Y)_{rv^*,\mathbb{Z}}} F(\rho^*(t,g) \cdot rv^*) d^1 g = \mu^*(rv^*) \cdot \int_{V^*_\eta} F(x^*)|P^*(x^*)|^{-\frac{m}{2}} dx^*.$$

Put $F_r(v^*) := F(rv^*)$. Since $SO(Y)_{rv^*} = SO(Y)_{v^*}$, we have

$$\begin{aligned}\int_0^\infty d^\times t \int_{SO(Y)_\mathbb{R}/SO(Y)_{rv^*,\mathbb{Z}}} F(\rho^*(t,g)\cdot rv^*)d^1g &= \int_0^\infty d^\times t \int_{SO(Y)_\mathbb{R}/SO(Y)_{v^*,\mathbb{Z}}} F_r(\rho^*(t,g)v^*)d^1g \\ &= \mu^*(v^*)\cdot\int_{V_\eta^*} F_r(x^*)|P^*(x^*)|^{-\frac{m}{2}}dx^* \\ &= \mu^*(v^*)\cdot\int_{V_\eta^*} F(rx^*)|P^*(x^*)|^{-\frac{m}{2}}dx^* \\ &= \mu^*(v^*)\cdot\int_{V_\eta^*} F(x^*)\underbrace{|P^*(r^{-1}x^*)|^{-\frac{m}{2}}d(r^{-1}x^*)}_{=|P^*(x^*)|^{-\frac{m}{2}}dx^*} \\ &= \mu^*(v^*)\cdot\int_{V_\eta^*} F(x^*)|P^*(x^*)|^{-\frac{m}{2}}dx^*.\end{aligned}$$

This proves $\mu^*(rv^*) = \mu^*(v^*)$. The first formula can be proved similarly.

(2) Let us show that $\sigma(rv) = r^{2-m}\cdot\sigma(v)$. Take an $f \in \mathcal{S}(V_\mathbb{R})$ and put $\psi(g) = f(gv)$. By using (40), we have

$$\begin{aligned}\int_{SO(Y)_\mathbb{R}/SO(Y)_{v,\mathbb{Z}}} f(gv)d^1g &= \int_{SO(Y)_\mathbb{R}/SO(Y)_{v,\mathbb{R}}} f(gv)|\omega(\dot{g}v)|_\infty \int_{SO(Y)_{v,\mathbb{R}}/SO(Y)_{v,\mathbb{Z}}} d\sigma_v(h) \\ &= \sigma(v)\cdot\int_{S_{1,\mathbb{R}}} f(z)|\omega(z)|_\infty.\end{aligned}$$

By the substitution $v \mapsto rv$, we have

$$\int_{SO(Y)_\mathbb{R}/SO(Y)_{rv,\mathbb{Z}}} f(grv)d^1g = \sigma(rv)\cdot\int_{S_{1,\mathbb{R}}} f(z)|\omega(z)|_\infty.$$

Put $f_r(v) := f(rv)$. Since $SO(Y)_{rv} = SO(Y)_v$, we have

$$\begin{aligned}\int_{SO(Y)_\mathbb{R}/SO(Y)_{rv,\mathbb{Z}}} f(grv)d^1g &= \int_{SO(Y)_\mathbb{R}/SO(Y)_{v,\mathbb{Z}}} f_r(gv)d^1g \\ &= \sigma(v)\cdot\int_{S_{1,\mathbb{R}}} f_r(z)|\omega(z)|_\infty \\ &= \sigma(v)\cdot\int_{S_{1,\mathbb{R}}} f(rz)|\omega(z)|_\infty \\ &= \sigma(v)\cdot\int_{S_{1,\mathbb{R}}} f(z)|\omega(r^{-1}z)|_\infty \\ &= \sigma(v)\cdot\int_{S_{1,\mathbb{R}}} f(z)r^{-(m-2)}|\omega(z)|_\infty \\ &= r^{2-m}\cdot\sigma(v)\cdot\int_{S_{1,\mathbb{R}}} f(z)|\omega(z)|_\infty,\end{aligned}$$

where we have used (23) on the fifth equality. This proves $\sigma(rv) = r^{2-m}\cdot\sigma(v)$. The second formula can be proved in a similar fashion. □

5 The main theorem

To prove the functional equation of twisted zeta functions, we quote a result of Stark [28]. Let Y be a non-degenerate half-integral symmetric matrix of degree m. Let $D = \det(2Y)$ and N be the level of $2Y$. We define a half-integral symmetric matrix $\widehat{Y}$ by

$$\widehat{Y} = \frac{1}{4}NY^{-1}.$$

We define the quadratic form $P(v)$ on V by $P(v) = Y[v] = {}^t vYv$, and the quadratic form $\widehat{P}(v^*)$ on V^* by

$$\widehat{P}(v^*) = \widehat{Y}[v^*] = NP^*(v^*), \tag{61}$$

where P^* is defined by (19). For this $\widehat{P}$, we define the measure $M^*(\widehat{P}; n)$ of representation by

$$M^*(\widehat{P}; \pm n) = \sum_{\substack{v^* \in SO(Y)_{\mathbb{Z}} \backslash V_{\pm}^* \cap V_{\mathbb{Z}} \\ \widehat{P}(v^*) = \pm n}} \mu^*(v^*). \tag{62}$$

For an odd prime r with $(r, N) = 1$ and a Dirichlet character ψ of modulus r, we define the function $\phi_{\psi,P}(v)$ on $V_{\mathbb{Q}}$ by

$$\phi_{\psi,P}(v) = \tau_{\psi}(P(v)) \cdot \phi_0(v),$$

where $\tau_{\psi}(P(v))$ is the Gauss sum defined by (9), and $\phi_0(v)$ is the characteristic function of $\mathbb{Z}^m$. It is easy to see that $\phi_{\psi,P}(v)$ is a Schwartz-Bruhat function on $V_{\mathbb{Q}}$. We define a field K by

$$K = \begin{cases} \mathbb{Q}(\sqrt{(-1)^{m/2}D}) & (m \equiv 0 \pmod 2) \\ \mathbb{Q}(\sqrt{2|D|}) & (m \equiv 1 \pmod 2) \end{cases},$$

and χ_K be the Kronecker symbol associated to K. (If $K = \mathbb{Q}$, we regard χ_K as the principal character.) Furthermore, we define a Dirichlet character $\psi^* \bmod r$ by

$$\psi^*(k) = \overline{\psi(k)} \left(\frac{k}{r}\right)^m,$$

and put

$$C_{2p-m,r} = \begin{cases} 1 & (m \equiv 0 \pmod 2) \\ \varepsilon_r^{2p-m} & (m \equiv 1 \pmod 2) \end{cases}$$

as (13). Then the following lemma follows from Stark [28, Lemmas 5 and 6].

Lemma 12 *Let $\widehat{\phi_{\psi,P}}(v^*)$ be the Fourier transform of $\phi_{\psi,P}$ defined by (29). Then the support of $\widehat{\phi_{\psi,P}}(v^*)$ is contained in $r^{-1}\mathbb{Z}^m$, and for $v^* \in \mathbb{Z}^m$, we have*

$$\widehat{\phi_{\psi,P}}(r^{-1}v^*) = r^{-m/2}\chi_K(r) \cdot C_{2p-m,r} \cdot \psi^*(-N) \cdot \tau_{\psi^*}(\widehat{P}(v^*)).$$

Let $\phi = \phi_0$ in the normalized zeta function $\widetilde{\zeta}_{\pm}(\phi; s)$ of (54). For $v \in V_{\epsilon} \cap V_{\mathbb{Z}}$, we have $P(v) = \epsilon n$ for some $n = 1, 2, 3, \ldots$, and hence $\widetilde{\zeta}_{\pm}(\phi_0; s)$ can be transformed as

$$\begin{aligned} \widetilde{\zeta}_{\pm}(\phi_0; s) &= |D|^{-\frac{1}{2}} \cdot e^{\frac{\pi i}{4}(2p-m)} \cdot \zeta_{\pm}\left(\phi_0; s + \frac{m}{2} - 1\right) \\ &= |D|^{-\frac{1}{2}} \cdot e^{\frac{\pi i}{4}(2p-m)} \cdot \sum_{v \in SO(Y)_{\mathbb{Z}} \backslash V_{\pm} \cap V_{\mathbb{Z}}} \frac{\mu(v)}{|P(v)|^{s+\frac{m}{2}-1}} \\ &= |D|^{-\frac{1}{2}} \cdot e^{\frac{\pi i}{4}(2p-m)} \cdot \sum_{n=1}^{\infty} \left\{ \sum_{\substack{v \in SO(Y)_{\mathbb{Z}} \backslash V_{\pm} \cap V_{\mathbb{Z}} \\ P(v) = \pm n}} \mu(v) \right\} n^{-s-\frac{m}{2}-1} \\ &= \sum_{n=1}^{\infty} \frac{\mathbf{a}(\pm n)}{n^s}, \end{aligned}$$

where $\mathbf{a}(\pm n)$ $(n = 1, 2, 3, \dots)$ is defined by

$$\begin{aligned}\mathbf{a}(\pm n) &= |D|^{-\frac{1}{2}} \cdot e^{\frac{\pi i}{4}(2p-m)} \cdot n^{1-\frac{m}{2}} \sum_{\substack{v \in SO(Y)_{\mathbb{Z}} \backslash V_{\pm} \cap V_{\mathbb{Z}} \\ P(v) = \pm n}} \mu(v) \\ &= |D|^{-\frac{1}{2}} \cdot e^{\frac{\pi i}{4}(2p-m)} \cdot n^{1-\frac{m}{2}} \cdot M(P; \pm n), \end{aligned} \tag{63}$$

where $M(P; n)$ is the measure of representation defined as (39). Further, by plugging $\phi_{\psi,P}(v) = \tau_{\psi}(P(v)) \cdot \phi_0(v)$ in (54), we have

$$\begin{aligned}\widetilde{\zeta}_{\pm}(\phi_{\psi,P}; s) &= |D|^{-\frac{1}{2}} \cdot e^{\frac{\pi i}{4}(2p-m)} \cdot \sum_{v \in SO(Y)_{\mathbb{Z}} \backslash V_{\pm} \cap V_{\mathbb{Z}}} \frac{\tau_{\psi}(P(v))\mu(v)}{|P(v)|^{s+\frac{m}{2}-1}} \\ &= \sum_{n=1}^{\infty} \frac{\tau_{\psi}(\pm n)\mathbf{a}(\pm n)}{n^s}. \end{aligned}$$

On the other hand, let $\phi = \phi_0$ in the normalized zeta function $\widetilde{\zeta}^*_{\eta}(\widehat{\phi}; s)$ of (55). Since $\widehat{\phi_0} = \phi_0$, we have

$$\begin{aligned}\widetilde{\zeta}^*_{\pm}(\widehat{\phi_0}; s) &= N^{-s} \cdot \zeta^*_{\pm}\left(\phi_0; s + \frac{m}{2} - 1\right) \\ &= N^{-s} \cdot \sum_{v^* \in SO(Y)_{\mathbb{Z}} \backslash V^*_{\pm} \cap V_{\mathbb{Z}}} \frac{\mu^*(v^*)}{|P^*(v^*)|^{s+\frac{m}{2}-1}} \\ &= N^{\frac{m}{2}-1} \cdot \sum_{v^* \in SO(Y)_{\mathbb{Z}} \backslash V^*_{\pm} \cap V_{\mathbb{Z}}} \frac{\mu^*(v^*)}{|NP^*(v^*)|^{s+\frac{m}{2}-1}}. \end{aligned}$$

By the definition (61), we have $\widehat{P}(v^*) = NP^*(v^*) \in \mathbb{Z} \setminus \{0\}$ for $v^* \in V^*_{\pm} \cap V_{\mathbb{Z}}$ and hence

$$\begin{aligned}\widetilde{\zeta}^*_{\pm}(\widehat{\phi_0}; s) &= N^{\frac{m}{2}-1} \cdot \sum_{n=1}^{\infty} \left\{ \sum_{\substack{v^* \in SO(Y)_{\mathbb{Z}} \backslash V^*_{\pm} \cap V_{\mathbb{Z}} \\ \widehat{P}(v^*) = \pm n}} \mu^*(v^*) \right\} n^{-s-\frac{m}{2}-1} \\ &= \sum_{n=1}^{\infty} \frac{\mathbf{b}(\pm n)}{n^s}, \end{aligned}$$

where $\mathbf{b}(\pm n)$ $(n = 1, 2, 3, \dots)$ is defined by

$$\mathbf{b}(\pm n) = \left(\frac{n}{N}\right)^{1-\frac{m}{2}} \sum_{\substack{v^* \in SO(Y)_{\mathbb{Z}} \backslash V^*_{\pm} \cap V_{\mathbb{Z}} \\ \widehat{P}(v^*) = \pm n}} \mu^*(v^*) = \left(\frac{n}{N}\right)^{1-\frac{m}{2}} \cdot M^*(\widehat{P}; \pm n), \tag{64}$$

where $M^*(\widehat{P};n)$ is defined as (62). Finally, let $\phi = \phi_{\psi,P}(v) = \tau_\psi(P(v)) \cdot \phi_0(v)$ in (55). It then follows from Lemmas 11 (1) and 12, and also $\widehat{P}(r^{-1}v^*) = r^{-2} \cdot \widehat{P}(v^*)$ that

$$\begin{aligned}\widetilde{\zeta}^*_\eta(\widehat{\phi_{\psi,P}};s) &= N^{\frac{m}{2}-1} \cdot \sum_{v^* \in SO(Y)_{\mathbb{Z}} \backslash V^*_{\pm} \cap V_{\mathbb{Z}}} \frac{\widehat{\phi_{\psi,P}}(r^{-1}v^*)\mu^*(r^{-1}v^*)}{|NP^*(r^{-1}v^*)|^{s+\frac{m}{2}-1}} \\ &= N^{\frac{m}{2}-1} \cdot r^{-m/2}\chi_K(r) \cdot C_{2p-m,r} \cdot \psi^*(-N) \\ &\quad \cdot r^{2(s+\frac{m}{2}-1)} \cdot \sum_{v^* \in SO(Y)_{\mathbb{Z}} \backslash V^*_{\pm} \cap V_{\mathbb{Z}}} \frac{\tau_{\psi^*}(\widehat{P}(v^*))\mu^*(v^*)}{|\widehat{P}(v^*)|^{s+\frac{m}{2}-1}} \\ &= r^{2s+\frac{m}{2}-2}\chi_K(r) \cdot C_{2p-m,r} \cdot \psi^*(-N) \cdot \sum_{n=1}^{\infty} \frac{\tau_{\psi^*}(\pm n)\mathbf{b}(\pm n)}{n^s}.\end{aligned}$$

We thus obtain the first assertion of the following

Lemma 13 *For $n = 1, 2, 3, \ldots$, we define $\mathbf{a}(\pm n)$ and $\mathbf{b}(\pm n)$ by (63) and (64) respectively, and let*

$$\begin{aligned}\zeta_\pm(\mathbf{a};s) &= \sum_{n=1}^{\infty} \frac{\mathbf{a}(\pm n)}{n^s}, \quad \zeta_\pm(\mathbf{a},\psi;s) = \sum_{n=1}^{\infty} \frac{\tau_\psi(\pm n)\mathbf{a}(\pm n)}{n^s}, \\ \zeta_\pm(\mathbf{b};s) &= \sum_{n=1}^{\infty} \frac{\mathbf{b}(\pm n)}{n^s}, \quad \zeta_\pm(\mathbf{b},\psi^*;s) = \sum_{n=1}^{\infty} \frac{\tau_{\psi^*}(\pm n)\mathbf{b}(\pm n)}{n^s}.\end{aligned}$$

(1) *We have*

$$\begin{aligned}\widetilde{\zeta}_\pm(\phi_0;s) &= \zeta_\pm(\mathbf{a};s), \\ \widetilde{\zeta}_\pm(\phi_{\psi,P};s) &= \zeta_\pm(\mathbf{a},\psi;s), \\ \widetilde{\zeta}^*_\eta(\widehat{\phi_0};s) &= \zeta_\pm(\mathbf{b};s), \\ \widetilde{\zeta}^*_\eta(\widehat{\phi_{\psi,P}};s) &= r^{2s+\frac{m}{2}-2}\chi_K(r) \cdot C_{2p-m,r} \cdot \psi^*(-N) \cdot \zeta_\pm(\mathbf{b},\psi^*;s).\end{aligned}$$

(2) *On residues and special values of zeta functions, the following four relations hold:*

$$\zeta_+(\mathbf{a},\psi;0) + \zeta_-(\mathbf{a},\psi;0) = \tau_\psi(0) \cdot (\zeta_+(\mathbf{a};0) + \zeta_-(\mathbf{a};0)), \tag{65}$$

$$r^{\frac{m}{2}} \cdot \chi_K(r) \cdot C_{2p-m,r} \cdot \psi^*(-N) \cdot \operatorname*{Res}_{s=1} \zeta_\pm(\mathbf{b},\psi^*;s) = \tau_\psi(0) \operatorname*{Res}_{s=1} \zeta_\pm(\mathbf{b};s), \tag{66}$$

$$\zeta_+(\mathbf{b},\psi^*;0) + \zeta_-(\mathbf{b},\psi^*;0) = \tau_{\psi^*}(0) \cdot (\zeta_+(\mathbf{b};0) + \zeta_-(\mathbf{b};0)), \tag{67}$$

$$\operatorname*{Res}_{s=1} \zeta_\pm(\mathbf{a},\psi;s) = r^{-\frac{m}{2}} \cdot \chi_K(r) \cdot C_{2p-m,r} \cdot \psi^*(-N) \cdot \operatorname*{Res}_{s=1} \zeta_\pm(\mathbf{a};s). \tag{68}$$

(3) *Assume that at least one of m or p is an odd integer. Let $\lambda = \frac{m}{4}$ and take an integer ℓ with $\ell \equiv 2p - m \pmod 4$. Then $\zeta_\pm(\mathbf{a};s)$ and $\zeta_\pm(\mathbf{b};s)$ satisfy the assumptions **[A1]–[A4]** of Sect. 1, and further, $\zeta_\pm(\mathbf{a},\psi;s)$ and $\zeta_\pm(\mathbf{b},\psi^*;s)$ satisfy the assumptions $\mathbf{[A1]}_{r,\psi}$–$\mathbf{[A5]}_{r,\psi}$ of Sect. 1.*

Proof (2) By letting $\phi = \phi_0$ in (59), we have

$$-(\zeta_+(\mathbf{a};0) + \zeta_-(\mathbf{a};0)) = |D|^{-\frac{1}{2}} \cdot e^{\frac{\pi i}{4}(2p-m)} \cdot \sum_{v \in SO(Y)_{\mathbb{Z}} \backslash S_{1,\mathbb{Z}}} \sigma(v), \tag{69}$$

and by letting $\phi = \phi_{\psi,P}$ in (59), we have

$$\begin{aligned} -(\zeta_+(\mathbf{a},\psi;0)+\zeta_-(\mathbf{a},\psi;0)) &= |D|^{-\frac{1}{2}}\cdot e^{\frac{\pi i}{4}(2p-m)}\cdot\tau_\psi(0)\sum_{\nu\in SO(Y)_{\mathbb{Z}}\backslash S_{1,\mathbb{Z}}}\sigma(\nu)\\ &= -\tau_\psi(0)\cdot(\zeta_+(\mathbf{a};0)+\zeta_-(\mathbf{a};0)), \end{aligned}$$

which proves (65). By (55) and Theorem 1 (3), we have

$$\begin{aligned} \operatorname*{Res}_{s=1}\zeta_\pm(\mathbf{b};s) &= \operatorname*{Res}_{s=1}\widetilde{\zeta}^*_\pm(\widehat{\phi_0};s)\\ &= \operatorname*{Res}_{s=1}\left(N^{-s}\cdot\zeta^*_\pm\left(\widehat{\phi_0};s+\frac{m}{2}-1\right)\right)\\ &= N^{-1}\cdot\operatorname*{Res}_{s=\frac{m}{2}}\zeta^*_\pm(\widehat{\phi_0};s)\\ &= N^{-1}\int_{SO(Y)_{\mathbb{R}}/SO(Y)_{\mathbb{Z}}}d^1g, \end{aligned}$$

and we thus obtain

$$\operatorname*{Res}_{s=1}\zeta_\pm(\mathbf{b};s) = N^{-1}\int_{SO(Y)_{\mathbb{R}}/SO(Y)_{\mathbb{Z}}}d^1g. \tag{70}$$

Let us consider the residues at $s=1$ of the both sides of

$$\widetilde{\zeta}^*_\pm(\widehat{\phi_{\psi,P}};s) = r^{2s+\frac{m}{2}-2}\chi_K(r)\cdot C_{2p-m,r}\cdot\psi^*(-N)\cdot\zeta_\pm(\mathbf{b},\psi^*;s).$$

The residue at $s=1$ of the left hand side is

$$\operatorname*{Res}_{s=1}\widetilde{\zeta}^*_\pm(\widehat{\phi_{\psi,P}};s) = N^{-1}\cdot\phi_{\psi,P}(0)\int_{SO(Y)_{\mathbb{R}}/SO(Y)_{\mathbb{Z}}}d^1g = \tau_\psi(0)\operatorname*{Res}_{s=1}\zeta_\pm(\mathbf{b};s),$$

and that of the right hand side is

$$\begin{aligned} &\operatorname*{Res}_{s=1}\left\{r^{2s+\frac{m}{2}-2}\chi_K(r)\cdot C_{2p-m,r}\cdot\psi^*(-N)\cdot\zeta_\pm(\mathbf{b},\psi^*;s)\right\}\\ &\quad= r^{\frac{m}{2}}\cdot\chi_K(r)\cdot C_{2p-m,r}\cdot\psi^*(-N)\cdot\operatorname*{Res}_{s=1}\zeta_\pm(\mathbf{b},\psi^*;s), \end{aligned}$$

by which we obtain (66). Next let $\phi=\phi_0$ in the relation (60). Then we have

$$-(\zeta_+(\mathbf{b};0)+\zeta_-(\mathbf{b};0)) = \sum_{\nu^*\in SO(Y)_{\mathbb{Z}}\backslash S^*_{1,\mathbb{Z}}}\sigma^*(\nu^*). \tag{71}$$

By letting $\phi=\phi_{\psi,P}$ in (60) and using Lemmas 12 and 11 (2), we have

$$\begin{aligned} &-r^{\frac{m}{2}-2}\chi_K(r)\cdot C_{2p-m,r}\cdot\psi^*(-N)\cdot\left(\zeta_+(\mathbf{b},\psi^*;0)+\zeta_-(\mathbf{b},\psi^*;0)\right)\\ &\quad= \sum_{\nu^*\in SO(Y)_{\mathbb{Z}}\backslash S^*_{1,\mathbb{Z}}}\widehat{\phi_{\psi,P}}(r^{-1}\nu^*)\underbrace{\sigma^*(r^{-1}\nu^*)}_{=r^{m-2}\cdot\sigma^*(\nu^*)}\\ &\quad= r^{-\frac{m}{2}}\chi_K(r)\cdot C_{2p-m,r}\cdot\psi^*(-N)\cdot\tau_{\psi^*}(0)\cdot r^{m-2}\sum_{\nu^*\in SO(Y)_{\mathbb{Z}}\backslash S^*_{1,\mathbb{Z}}}\sigma^*(\nu^*)\\ &\quad= -r^{\frac{m}{2}-2}\chi_K(r)\cdot C_{2p-m,r}\cdot\psi^*(-N)\cdot\tau_{\psi^*}(0)\,(\zeta_+(\mathbf{b};0)+\zeta_-(\mathbf{b};0)), \end{aligned}$$

and this proves

$$\zeta_+(\mathbf{b},\psi^*;0)+\zeta_-(\mathbf{b},\psi^*;0) = \tau_{\psi^*}(0)\cdot(\zeta_+(\mathbf{b};0)+\zeta_-(\mathbf{b};0)),$$

which is the relation (67). By (54) and Theorem 1 (3), we have

$$\begin{aligned}\operatorname*{Res}_{s=1}\zeta_{\pm}(\mathbf{a};s) &= \operatorname*{Res}_{s=1}\widetilde{\zeta}_{\pm}(\phi_0;s)\\ &= \operatorname*{Res}_{s=1}\left(|D|^{-\frac{1}{2}}\cdot e^{\frac{\pi i}{4}(2p-m)}\cdot\zeta_{\pm}\left(\phi_0;s+\frac{m}{2}-1\right)\right)\\ &= |D|^{-\frac{1}{2}}\cdot e^{\frac{\pi i}{4}(2p-m)}\cdot\operatorname*{Res}_{s=\frac{m}{2}}\zeta_{\pm}\left(\phi_0;s\right)\\ &= |D|^{-\frac{1}{2}}\cdot e^{\frac{\pi i}{4}(2p-m)}\int_{SO(Y)_{\mathbb{R}}/SO(Y)_{\mathbb{Z}}}d^1g,\end{aligned}$$

and we thus obtain

$$\operatorname*{Res}_{s=1}\zeta_{\pm}(\mathbf{a};s) = |D|^{-\frac{1}{2}}\cdot e^{\frac{\pi i}{4}(2p-m)}\int_{SO(Y)_{\mathbb{R}}/SO(Y)_{\mathbb{Z}}}d^1g. \tag{72}$$

Furthermore, it follows from Lemma 12 that the residue of $\zeta_{\pm}(\mathbf{a},\psi;s)=\widetilde{\zeta}_{\pm}(\phi_{\psi,P};s)$ at $s=1$ is given by

$$\begin{aligned}\operatorname*{Res}_{s=1}\zeta_{\pm}(\mathbf{a},\psi;s) &= |D|^{-\frac{1}{2}}\cdot e^{\frac{\pi i}{4}(2p-m)}\cdot\widehat{\phi_{\psi,P}}(0)\int_{SO(Y)_{\mathbb{R}}/SO(Y)_{\mathbb{Z}}}d^1g\\ &= |D|^{-\frac{1}{2}}\cdot e^{\frac{\pi i}{4}(2p-m)}\cdot r^{-\frac{m}{2}}\chi_K(r)\cdot C_{2p-m,r}\cdot\psi^*(-N)\int_{SO(Y)_{\mathbb{R}}/SO(Y)_{\mathbb{Z}}}d^1g\\ &= r^{-\frac{m}{2}}\chi_K(r)\cdot C_{2p-m,r}\cdot\psi^*(-N)\cdot\operatorname*{Res}_{s=1}\zeta_{\pm}(\mathbf{a};s),\end{aligned}$$

by which we obtain the relation (68).

(3) By Theorem 1 (1), (3), we see that our zeta functions satisfy the assumptions **[A1]**, $\mathbf{[A1]}_{r,\psi}$, **[A2]**, and $\mathbf{[A2]}_{r,\psi}$. The functional equation of **[A3]** is nothing but the equation (56) with $\phi=\phi_0$. Let $\phi=\phi_{\psi,P}$ in (56); then the first assertion of the lemma implies that

$$\begin{aligned}&(2\pi)^{-s}\Gamma(s)\gamma(s)\begin{pmatrix}\zeta_+(\mathbf{a},\psi;s)\\ \zeta_-(\mathbf{a},\psi;s)\end{pmatrix}\\ &\quad=\chi_K(r)\cdot C_{2p-m,r}\cdot\psi^*(-N)\cdot r^{\frac{m}{2}-2}\cdot(Nr^2)^{2-\frac{m}{2}-s}\\ &\qquad\cdot(2\pi)^{-(2-\frac{m}{2}-s)}\Gamma\left(2-\frac{m}{2}-s\right)\\ &\qquad\cdot\Sigma(2p-m)\gamma\left(2-\frac{m}{2}-s\right)\begin{pmatrix}\zeta_+\left(\mathbf{b},\psi^*;2-\frac{m}{2}-s\right)\\ \zeta_-\left(\mathbf{b},\psi^*;2-\frac{m}{2}-s\right)\end{pmatrix},\end{aligned}$$

which shows that the functional equation of $\mathbf{[A3]}_{r,\psi}$ holds. Lemma 9 implies that our zeta functions satisfy the assumptions **[A4]** and $\mathbf{[A4]}_{r,\psi}$. Finally, the compatibility condition $\mathbf{[A5]}_{r,\psi}$ on residues and special values follows from (65), (66), (67) and (68). □

In general, $SO(Y)_{\mathbb{Z}}\backslash S_{1,\mathbb{Z}}$ is always an infinite set, since for $v\in S_{1,\mathbb{Z}}$, any two of $v, 2v, 3v,\ldots$ can not lie in the same $SO(Y)_{\mathbb{Z}}$-orbit. However, as is seen in the following lemma that is taken from [11, pp.188–189], the number of $SO(Y)_{\mathbb{Z}}$-orbits in *primitive* vectors in $S_{1,\mathbb{Z}}$ and $S^*_{1,\mathbb{Z}}$ is finite.

Lemma 14 (1) *We call a vector $v=(v_1,\ldots,v_m)\in V_{\mathbb{Z}}$ primitive if the greatest common divisor of $v_1,\ldots,v_m$ is* 1. *Then*

$$\{v\in SO(Y)_{\mathbb{Z}}\backslash S_{1,\mathbb{Z}}\,;\ v \textit{ is primitive}\}$$

is a finite set. Let $a_1, \dots, a_h$ be a complete system of representatives of this set. Then we have

$$\sum_{v \in SO(Y)_{\mathbb{Z}} \backslash S_{1,\mathbb{Z}}} \sigma(v) = \zeta(m-2) \sum_{i=1}^{h} \sigma(a_i).$$

(2) *Let $b_1, \dots, b_k$ be a complete system of the finite set*

$$\{v^* \in SO(Y)_{\mathbb{Z}} \backslash S^*_{1,\mathbb{Z}}\,;\ v^* \text{ is primitive}\}.$$

Then we have

$$\sum_{v \in SO(Y)_{\mathbb{Z}} \backslash S^*_{1,\mathbb{Z}}} \sigma^*(v^*) = \zeta(m-2) \sum_{i=1}^{k} \sigma^*(b_i).$$

Now we are in a position to state

Theorem 2 *Assume that at least one of m or p is an odd integer. Take an integer ℓ with $\ell \equiv 2p - m \pmod 4$. Define C^∞-functions $F(z)$ and $G(z)$ on $\mathcal{H}$ by*

$$\begin{aligned}
F(z) &= y^{(m-\ell)/4} \cdot \int_{SO(Y)_{\mathbb{R}}/SO(Y)_{\mathbb{Z}}} d^1 g \\
&\quad + (-1)^{(2p-m-\ell)/4} \zeta(m-2) \sum_{i=1}^{h} \frac{\sigma(a_i)}{|D|^{\frac{1}{2}}} \cdot \frac{(2\pi) 2^{1-\frac{m}{2}} \Gamma\left(\frac{m}{2}-1\right)}{\Gamma\left(\frac{m+\ell}{4}\right) \Gamma\left(\frac{m-\ell}{4}\right)} \cdot y^{1-(m+\ell)/4} \\
&\quad + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} (-1)^{(2p-m-\ell)/4} \cdot \frac{M(P;n)}{|D|^{\frac{1}{2}}} \frac{\pi^{\frac{m}{4}} \cdot |n|^{-\frac{m}{4}}}{\Gamma\left(\frac{m+\operatorname{sgn}(n)\ell}{4}\right)} \cdot y^{-\frac{\ell}{4}} W_{\frac{\operatorname{sgn}(n)\ell}{4}, \frac{m}{4}-\frac{1}{2}}(4\pi |n| y) \mathbf{e}[nx], \\
G(z) &= N^{\frac{m}{4}} \cdot |D|^{-\frac{1}{2}} e^{\frac{\pi i}{4}(2p-m)} \cdot y^{(m-\ell)/4} \cdot \int_{SO(Y)_{\mathbb{R}}/SO(Y)_{\mathbb{Z}}} d^1 g \\
&\quad + i^{-\frac{\ell}{2}} N^{1-\frac{m}{4}} \zeta(m-2) \sum_{i=1}^{k} \sigma^*(b_i) \frac{(2\pi) 2^{1-\frac{m}{2}} \Gamma(\frac{m}{2}-1)}{\Gamma\left(\frac{m+\ell}{4}\right) \Gamma\left(\frac{m-\ell}{4}\right)} \cdot y^{1-(m+\ell)/4} \\
&\quad + i^{-\frac{\ell}{2}} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left(\frac{|n|}{N}\right)^{-\frac{m}{4}} M^*(\widehat{P};n) \frac{\pi^{\frac{m}{4}}}{\Gamma\left(\frac{m+\operatorname{sgn}(n)\ell}{4}\right)} \cdot y^{-\frac{\ell}{4}} W_{\frac{\operatorname{sgn}(n)\ell}{4}, \frac{m}{4}-\frac{1}{2}}(4\pi |n| y) \mathbf{e}[nx].
\end{aligned}$$

Then, $F(z)$ (resp. $G(z)$) is a Maass form for $\Gamma_0(N)$ of weight $\ell/2$ with eigenvalue $(m-\ell)(4-m-\ell)/16$ and character χ_K (resp. χ_{K_N}). Here we denote by χ_K and χ_{K_N} the Kronecker characters associated to the fields

$$K = \begin{cases} \mathbb{Q}(\sqrt{(-1)^{m/2} D}) & (m \equiv 0 \pmod 2) \\ \mathbb{Q}(\sqrt{2|D|}) & (m \equiv 1 \pmod 2) \end{cases},$$

and

$$K_N = \begin{cases} \mathbb{Q}(\sqrt{(-1)^{m/2} D}) & (m \equiv 0 \pmod 2) \\ \mathbb{Q}(\sqrt{2|D|N}) & (m \equiv 1 \pmod 2) \end{cases},$$

respectively. Further we have

$$F\left(-\frac{1}{Nz}\right) (\sqrt{N} z)^{-\ell/2} = G(z).$$

Proof We apply the converse theorem (Lemma 1) to the normalized zeta functions $\zeta_\pm(\mathbf{a}; s)$ and $\zeta_\pm(\mathbf{b}; s)$ of Lemma 13. It remains to calculate the constant terms

$\mathbf{a}(0)$, $\mathbf{a}(\infty)$, $\mathbf{b}(0)$, $\mathbf{b}(\infty)$ along with the definitions (5), (6), (7), (8). First, by (69) and Lemma 14 (1), we have

$$\begin{aligned}\mathbf{a}(0) &= -\left(\zeta_+(\mathbf{a};0)+\zeta_-(\mathbf{a};0)\right)\\ &= |D|^{-\frac{1}{2}}\cdot e^{\frac{\pi i}{4}(2p-m)}\cdot\zeta(m-2)\sum_{i=1}^{h}\sigma(a_i).\end{aligned}$$

Second, by (70), we have

$$\begin{aligned}\mathbf{a}(\infty) &= \frac{N}{2}\left(\operatorname*{Res}_{s=1}\zeta_+(\mathbf{b};s)+\operatorname*{Res}_{s=1}\zeta_-(\mathbf{b};s)\right)\\ &= \int_{SO(Y)_{\mathbb{R}}/SO(Y)_{\mathbb{Z}}} d^1g.\end{aligned}$$

Third, by (71) and Lemma 14 (2), we have

$$\begin{aligned}\mathbf{b}(0) &= -\left(\zeta_+(\mathbf{b};0)+\zeta_-(\mathbf{b};0)\right)\\ &= \zeta(m-2)\sum_{i=1}^{k}\sigma^*(b_i).\end{aligned}$$

Finally, by (72), we have

$$\begin{aligned}\mathbf{b}(\infty) &= \frac{i^{-\ell}}{2}\left(\operatorname*{Res}_{s=1}\zeta_+(\mathbf{a};s)+\operatorname*{Res}_{s=1}\zeta_-(\mathbf{a};s)\right)\\ &= i^{-\ell}\cdot|D|^{-\frac{1}{2}}e^{\frac{\pi i}{4}(2p-m)}\int_{SO(Y)_{\mathbb{R}}/SO(Y)_{\mathbb{Z}}} d^1g\\ &= |D|^{-\frac{1}{2}}e^{-\frac{\pi i}{4}(2p-m)}\int_{SO(Y)_{\mathbb{R}}/SO(Y)_{\mathbb{Z}}} d^1g.\end{aligned}$$

□

Remark 4 One can verify that our $M(P;n)$ is identical to $M(\mathfrak{S},\mathfrak{a},t)$ ($\mathfrak{a}=0$), which is defined as the formula (14) of Siegel [26]. Moreover, up to a power of y, our $F(z)$ coincides with the integral $\int_F f_{\mathfrak{a}}(z,\mathfrak{P})dv$ ($\mathfrak{a}=0$) of the indefinite theta series $f_{\mathfrak{a}}(z,\mathfrak{P})$ over some fundamental domain F. See Siegel [26, Hilfssatz 4], [27] for the detail. We also note that Funke [4] calculated the Mellin transform of some indefinite theta series and obtained Siegel's zeta functions associated with ternary zero forms.

6 Holomorphic modular forms arising from Siegel's zeta functions

Under some conditions, the γ-matrix in Siegel's functional equation (46) can be an upper or lower triangular matrix. In such a case, we obtain a single functional equation. More precisely,

• Assume that the number of negative eigenvalues of Y is even; that is, $m-p$ is an even integer. Then the first row of (46) is of the following form:

$$\begin{aligned}\zeta_+\left(\phi;\frac{m}{2}-s\right) = \Gamma\left(s+1-\frac{m}{2}\right)\Gamma(s)|D|^{\frac{1}{2}}\cdot 2^{-2s+\frac{m}{2}}\cdot\pi^{-2s+\frac{m}{2}-1}\\ \times\sin\pi\left(\frac{p}{2}-s\right)\zeta_+^*(\widehat{\phi};s).\end{aligned}$$

This suggests that $\zeta_+(\phi;s)$ and $\zeta_+^*(\phi;s)$ satisfy the functional equation of Hecke type.

• Assume that the number of positive eigenvalues of Y is even; that is, p is an even integer. Then the second row of (46) is of the following form:

$$\zeta_-\left(\phi;\frac{m}{2}-s\right)=\Gamma\left(s+1-\frac{m}{2}\right)\Gamma(s)|D|^{\frac{1}{2}}\cdot 2^{-2s+\frac{m}{2}}\cdot\pi^{-2s+\frac{m}{2}-1}\\ \times\sin\pi\left(\frac{m-p}{2}-s\right)\zeta_-^*(\widehat{\phi};s).$$

This suggests that $\zeta_-(\phi;s)$ and $\zeta_-^*(\phi;s)$ satisfy the functional equation of Hecke type.

In the following, we assume that $m-p$ is even; if p is even, we replace P with $-P$. We introduce Dirichlet series $L(M;s)$ and $L(M^*;s)$ as follows:

$$L(M;s)=\sum_{n=1}^{\infty}\frac{a(n)}{n^s}\quad\text{with}\quad a(n):=|D|^{-1/2}\cdot M(P;n), \tag{73}$$

$$L(M^*;s)=\sum_{n=1}^{\infty}\frac{b(n)}{n^s}\quad\text{with}\quad b(n):=(-1)^{\frac{m-2p}{4}}\cdot N^{\frac{m}{4}}\cdot M^*(\widehat{P};n). \tag{74}$$

Further, we put

$$\Lambda_N(s;M)=\left(\frac{2\pi}{\sqrt{N}}\right)^{-s}\cdot\Gamma(s)\cdot L(M;s),$$
$$\Lambda_N(s;M^*)=\left(\frac{2\pi}{\sqrt{N}}\right)^{-s}\cdot\Gamma(s)\cdot L(M^*;s),$$

and

$$a(0)=(-1)^{\frac{m-p}{2}}(2\pi)^{-\frac{m}{2}}\cdot\Gamma\left(\frac{m}{2}\right)\int_{SO(Y)_{\mathbb{R}}/SO(Y)_{\mathbb{Z}}}d^1g, \tag{75}$$

$$b(0)=i^{-\frac{m}{2}}\cdot(2\pi)^{-\frac{m}{2}}\cdot\Gamma\left(\frac{m}{2}\right)N^{\frac{m}{4}}|D|^{-1/2}\int_{SO(Y)_{\mathbb{R}}/SO(Y)_{\mathbb{Z}}}d^1g. \tag{76}$$

Then Theorem 1 implies that the following lemma holds:

Lemma 15 *Assume that $m-p$ is even. Both $\Lambda_N(s;M)$ and $\Lambda_N(s;M^*)$ can be continued analytically to the whole s-plane, satisfy the functional equation*

$$\Lambda_N(s;M)=i^{\frac{m}{2}}\Lambda_N\left(\frac{m}{2}-s;M^*\right),$$

and the function

$$\Lambda_N(s;M)+\frac{a(0)}{s}+\frac{i^{\frac{m}{2}}\cdot b(0)}{\frac{m}{2}-s}$$

is holomorphic on the whole s-plane and bounded on any vertical strip.

Let r be an odd prime with $(N,r)=1$. We denote by $\varphi=\left(\frac{*}{r}\right)$ the Dirichlet character defined by the quadratic residue symbol. For a *primitive* Dirichlet character $\psi \bmod r$, we define Dirichlet series $L(M;s,\psi)$ and $L(M^*;s,\psi)$ by

$$L(M;s,\psi)=\sum_{n=1}^{\infty}\frac{\psi(n)a(n)}{n^s},$$

$$L(M^*;s,\psi)=\begin{cases} r\displaystyle\sum_{n=1}^{\infty}\frac{b(rn)}{(rn)^s}-\sum_{n=1}^{\infty}\frac{b(n)}{n^s} & \text{if } m \text{ is odd and } \psi=\varphi=\left(\frac{*}{r}\right),\\ \displaystyle\sum_{n=1}^{\infty}\frac{\psi(n)b(n)}{n^s} & \text{otherwise,}\end{cases}$$

where $a(n)$ and $b(n)$ are defined by (73) and (74), respectively. Furthermore, we set

$$\Lambda_N(s;M,\psi)=\left(\frac{2\pi}{r\sqrt{N}}\right)^{-s}\cdot\Gamma(s)\cdot L(M;s,\psi),$$
$$\Lambda_N(s;M^*,\psi)=\left(\frac{2\pi}{r\sqrt{N}}\right)^{-s}\cdot\Gamma(s)\cdot L(M^*;s,\psi).$$

Then, by using Lemma 12, the formulas (10) and (11), we can prove the following

Lemma 16 *Assume that $m-p$ is even.*

(1) *In the case of even m, for any primitive Dirichlet character $\psi \bmod r$, $\Lambda_N(s;M,\psi)$ can be holomorphically continued to the whole s-plane, bounded on any vertical strip, and satisfies the following functional equation*

$$\Lambda_N(s;M,\psi)=i^{\frac{m}{2}}C_\psi\Lambda_N\left(\frac{m}{2}-s;M^*,\overline{\psi}\right)$$

with the constant

$$C_\psi=\chi_K(r)\psi(-N)\tau_\psi/\tau_{\overline{\psi}}.$$

(2) *In the case of odd m, for any primitive Dirichlet character $\psi \bmod r$ with $\psi\neq\varphi=\left(\frac{*}{r}\right)$, $\Lambda_N(s;M,\psi)$ can be holomorphically continued to the whole s-plane, bounded on any vertical strip, and satisfies the following functional equation*

$$\Lambda_N(s;M,\psi)=i^{\frac{m}{2}}C_\psi^{(1)}\Lambda_N\left(\frac{m}{2}-s;M^*,\overline{\psi}\varphi\right)$$

with the constant

$$C_\psi^{(1)}=\left(\frac{-1}{m}\right)^{\frac{m-1}{2}}\cdot\chi_K(r)\left(\frac{N}{r}\right)\psi(-N)\epsilon_r^{-1}\tau_{\psi\varphi}/\tau_{\overline{\psi}}.$$

(3) *In the case that m is odd and $\psi=\varphi=\left(\frac{*}{r}\right)$,*

$$\Lambda_N(s;M,\psi)+C_\psi^{(2)}\frac{(r^{1/2}-r^{-1/2})b(0)}{\frac{m}{2}-s}$$

can be holomorphically continued to the whole s-plane, bounded on any vertical strip, and satisfies the following functional equation

$$\Lambda_N(s;M,\varphi)=i^{\frac{m}{2}}C_\psi^{(2)}\Lambda_N\left(\frac{m}{2}-s;M^*,\varphi\right)$$

with the constant

$$C_\psi^{(2)}=\left(\frac{-1}{m}\right)^{\frac{m-1}{2}}\cdot\chi_K(r).$$

These lemmas show that Weil's converse theorems for holomorphic modular forms can apply to $L(M;s)$ and $L(M^*;s)$. We refer to Miyake [15, Theorem 4.3.15] for Weil's converse theorem for the case of integral weight. For the case of half-integral weight, Shimura [22] stated a similar converse theorem. Although the details were not given in [22], the proof is roughly identical to the case of integral weight, and can be found in Bruinier [1]. We therefore obtain the following

Theorem 3 *Assume that $m-p$ is even. We define holomorphic functions $F(z)$ and $G(z)$ on $\mathcal{H}$ by*

$$\begin{aligned} F(z) &= (-1)^{\frac{m-p}{2}}(2\pi)^{-\frac{m}{2}}\cdot\Gamma\left(\frac{m}{2}\right)\int_{SO(Y)_{\mathbb{R}}/SO(Y)_{\mathbb{Z}}} d^1g \\ &\quad + |D|^{-1/2}\cdot\sum_{n=1}^{\infty} M(P;n)\mathbf{e}[nz], \\ G(z) &= i^{-\frac{m}{2}}\cdot(2\pi)^{-\frac{m}{2}}\cdot\Gamma\left(\frac{m}{2}\right)N^{\frac{m}{4}}|D|^{-1/2}\int_{SO(Y)_{\mathbb{R}}/SO(Y)_{\mathbb{Z}}} d^1g \\ &\quad + (-1)^{\frac{m-2p}{4}}\cdot N^{\frac{m}{4}}\cdot\sum_{n=1}^{\infty} M^*(\widehat{P};n)\mathbf{e}[nz]. \end{aligned}$$

Then, $F(z)$ (resp. $G(z)$) is a holomorphic modular form for $\Gamma_0(N)$ of weight $m/2$ with character χ_K (resp. χ_{K_N}). Further we have

$$F\left(-\frac{1}{Nz}\right)(\sqrt{N}z)^{-m/2} = G(z).$$

Remark 5 If p is even, we can prove the same assertion for $M(P;-n)$. Theorem 2 excludes the case where both m and p are even, but Theorem 3 shows that both ζ_+ and ζ_- correspond to holomorphic modular forms in this case.

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