

RESEARCH



Explicit zero-free regions for the Riemann zeta-function

Michael J. Mossinghoff¹, Timothy S. Trudgian^{2*} and Andrew Yang²

*Correspondence:

t.trudgian@adfa.edu.au
²School of Science The University of New South Wales Canberra, Canberra, Australia
Full list of author information is available at the end of the article
The second author was partially supported by Australian Research Council Future Fellowship FT160100094

Abstract

We prove that the Riemann zeta-function $\zeta(\sigma + it)$ has no zeros in the region $\sigma \geq 1 - 1/(55.241(\log|t|)^{2/3}(\log \log|t|)^{1/3})$ for $|t| \geq 3$. In addition, we improve the constant in the classical zero-free region, showing that the zeta-function has no zeros in the region $\sigma \geq 1 - 1/(5.558691 \log|t|)$ for $|t| \geq 2$. We also provide new bounds that are useful for intermediate values of $|t|$. Combined, our results improve the largest known zero-free region within the critical strip for $3 \cdot 10^{12} \leq |t| \leq \exp(64.1)$ and $|t| \geq \exp(1000)$.

Keywords: Riemann zeta-function, Zero-free region, Non-negative trigonometric polynomials, Simulated annealing

Mathematics Subject Classification: Primary: 11M26, 42A05, Secondary: 11Y35

1 Introduction

Let $\zeta(s)$ denote the Riemann zeta-function, where $s = \sigma + it$ is a complex variable. All non-trivial zeros of $\zeta(s)$ lie in the critical strip with $0 < \sigma < 1$. Determining regions in the critical strip that are devoid of zeros of $\zeta(s)$ is of great interest in number theory. Such regions take the shape $\sigma \geq 1 - 1/f(|t|)$ for some function $f(t)$ tending to infinity with t . The so-called classical zero-free region has $f(t) = R_0 \log t$, where R_0 is a positive constant. An asymptotically larger region, proved by Korobov [20] and Vinogradov [35], has $f(t) = R_1(\log t)^{2/3}(\log \log t)^{1/3}$, for some constant $R_1 > 0$.

Considerable effort has been made to make these results explicit, and we briefly recall the sharpest zero-free regions known. For the Korobov–Vinogradov region, in 2000 Ford [10] (see also [11] for minor corrections) established two explicit bounds, one holding for essentially all t , and a larger one holding for sufficiently large t . In the former case, Ford proved that there are no zeros of $\zeta(s)$ in the region

$$\sigma \geq 1 - \frac{1}{57.54(\log|t|)^{2/3}(\log \log|t|)^{1/3}}, \quad |t| \geq 3. \quad (1.1)$$

In the latter case, Ford showed that there are no zeros when t is sufficiently large and

$$\sigma \geq 1 - \frac{1}{49.13(\log|t|)^{2/3}(\log \log|t|)^{1/3}}. \quad (1.2)$$

This was recently slightly improved by Nielsen [22], who showed that there are no zeros with

$$\sigma \geq 1 - \frac{1}{49.08(\log|t|)^{2/3}(\log \log|t|)^{1/3}} \tag{1.3}$$

for sufficiently large $|t|$. For the classical region, in 2015 the first two authors [21] proved that the region

$$\sigma \geq 1 - \frac{1}{5.573412 \log|t|}, \quad |t| \geq 2 \tag{1.4}$$

is devoid of zeros of the zeta-function. This region is wider than that of (1.1) for all $|t| \leq \exp(10151.5)$.

We improve (1.1), (1.3) and (1.4) in this article. For the first of these, we establish the following theorem.

Theorem 1.1 *There are no zeros of $\zeta(\sigma + it)$ for $|t| \geq 3$ and*

$$\sigma \geq 1 - \frac{1}{55.241(\log|t|)^{2/3}(\log \log|t|)^{1/3}}. \tag{1.5}$$

Our improvement in Theorem 1.1 over (1.1) is a result of several ingredients.

- We employ a trigonometric polynomial of large degree: this produces our largest improvement.
- We optimize the choice of certain parameters to account for secondary error terms.
- We use improved intermediate zero-free regions to cover medium-sized t values where the argument for the asymptotic region does not perform as well.
- We employ sharper explicit estimates on the growth of the zeta-function in the critical strip.
- We use a new height up to which the Riemann hypothesis (RH) has been proved.

It appears substantially more difficult to improve the constant in the asymptotic region (1.3). We identify three potential avenues of improvement: using a better smoothing function, finding a more favorable trigonometric polynomial, and improving the constant B in Richert’s bound—see (3.1). In [13] and [37], improvements related to the first approach have been largely explored. In this work, we exploit the second method to produce the following improvement.

Theorem 1.2 *For sufficiently large $|t|$, there are no zeros of $\zeta(\sigma + it)$ with*

$$\sigma \geq 1 - \frac{1}{48.1588(\log|t|)^{2/3}(\log \log|t|)^{1/3}}. \tag{1.6}$$

We remark that Nielsen’s result (1.3) improved (1.2) by replacing a particular trigonometric polynomial that was used in Ford’s argument. Ford used a polynomial of degree 4, while Nielsen adopted one of degree 5 (see Sect. 3.3). We use polynomials of substantially larger degree here for our improvements. Also, we mention that Khale [19] recently established analogues of Theorems 1.1 and 1.2 for Dirichlet L -functions. While some of the ingredients we employ in this article do not appear to be suitable for use with Dirichlet L -functions, we note that the use of higher-degree trigonometric polynomials may lead to some improvements in [19].

For the classical region, in [21] the inequality (1.4) was established by extending some work of Kadiri [18] by constructing a more favorable trigonometric polynomial, by opti-

mizing some analytic arguments, and by employing the verification of RH up to $3.06 \cdot 10^{10}$ from [26]. It was also recorded in [21] that if RH were verified for $|t| \leq 3 \cdot 10^{11}$, then the constant in (1.7) could be replaced by 5.5666305, and this is now permissible due to the verification performed in [28]. We take the opportunity here to record a further improvement for the classical region, using two ideas. First, RH was verified for $|t| \leq 3 \cdot 10^{12}$ in [28]. Second, in 2014, Jang and Kwon [16] derived another improvement to Kadiri’s result by different means, involving the replacement of a particular smoothing function employed in [18], and we incorporate this as well. We prove the following theorem.

Theorem 1.3 *There are no zeros of $\zeta(\sigma + it)$ for $|t| \geq 2$ and*

$$\sigma \geq 1 - \frac{1}{5.558691 \log|t|}. \tag{1.7}$$

We remark that the region (1.7) is wider than that of (1.5) for $|t| < \exp(8928)$.

In order to obtain a good constant in Theorem 1.1, it is necessary to deduce another zero-free region to cover medium-sized values of t , in addition to (1.7). By employing new bounds on the growth rate of $\zeta(1/2 + it)$ given by Patel [23],¹ we establish the following theorem.

Theorem 1.4 (Intermediate zero-free region) *There are no zeros of $\zeta(\sigma + it)$ for $|t| \geq \exp(1000)$ and*

$$\sigma > 1 - \frac{0.05035}{h(t)} + \frac{0.0349}{h^2(t)},$$

where $h(t) = \frac{27}{164} \log|t| + 7.096$.

We record one additional result of Ford: from [10, Thm. 3], we see that if $t \geq 1.88 \cdot 10^{14}$ then $\zeta(\sigma + it) \neq 0$ for

$$\sigma \geq 1 - \frac{0.04962 - \frac{0.0196}{J(t)+1.15}}{J(t) + 0.685 + 0.155 \log \log t}, \tag{1.8}$$

where $J(t) = \frac{1}{6} \log t + \log \log t + \log 3$. As is common in splicing explicit bounds in number theory, we require intermediate results, such as (1.8). This is because some functions, such as the $h(t)$ in Theorem 1.4, are ultimately smaller (and hence better for our applications) than others, such as the $J(t)$ in (1.8). We also note that $J(t)$ comes from taking the logarithm of the right side of the bound $|\zeta(1/2 + it)| \leq 3t^{1/6} \log t$ ($t \geq 3$) found in [4]. The leading constant 3 in this bound has subsequently been reduced in [14, 15, 27], ultimately to 0.618 (for $t \geq 3$); this can be substituted directly in (1.8) to obtain a sharper region.

By combining these results, we may summarize the largest known zero-free region for the Riemann zeta-function within the critical strip for each height t . For $|t| \leq 3 \cdot 10^{12}$, all zeros are known to lie on the critical line. The region of Theorem 1.3 provides the best known bound for $3 \cdot 10^{12} < |t| \leq \exp(64.1)$, then for $\exp(64.1) < |t| \leq \exp(1000)$ the expression (1.8) produces the widest region. (The first value changes to $\exp(46.1)$ when we use 0.618 in place of 3 in the bound for $|\zeta(1/2 + it)|$.) After this, for $\exp(1000) < |t| \leq \exp(52242)$, Theorem 1.4 is best, and for $|t| > \exp(52242)$ the result of Theorem 1.1 produces the widest known region.

¹Improved bounds recently appeared in the preprint [24].

This article is organized in the following manner. Section 2 lists some immediate applications of our results. Section 3 reviews some results we require from the literature. Section 4 contains a number of lemmas that we require for the proofs of our main theorems. Section 5 contains the proof of Theorem 1.1. Section 6 describes the proof of Theorem 1.4, and Sect. 7 has the proof of Theorem 1.2. Section 8 summarizes our computations for determining the constants in Theorems 1.1 and 1.2. Section 9 describes the improvement in the classical region for Theorem 1.3, and Sect. 10 suggests some potential future work.

2 Some immediate applications

One application of Theorem 1.1 concerns the existence of primes between successive cubes. Dudek [7] used (1.1) to show that there exists a prime between n^3 and $(n+1)^3$ for all $n \geq \exp(\exp(33.3))$. This was subsequently improved by Cully-Hugill [5] to all $n \geq \exp(\exp(32.892))$. By substituting Theorem 1.1 into [5, §4] and choosing $\alpha = 1/3 + 10^{-13}$, we obtain the modest improvement $n \geq \exp(\exp(32.76))$.

Another application is explicit bounds on the error term of the prime number theorem—see Johnston and Yang [17], who developed work of Platt and Trudgian [29] particularly with reference to the Korobov–Vinogradov zero-free region. Other related recent works such as [2, 3, 6, 8] required explicit zero-free regions to obtain precise estimates.

As per Ford [9, p. 566], one can use Theorem 1.2 to improve on the error term in the prime number theorem. From the work of Pintz [25], a zero-free region like that in (1.6) with c in place of 48.1588 shows that

$$\pi(x) - \text{li}(x) \ll x \exp\{-d(\log x)^{3/5}(\log \log x)^{-1/5}\},$$

where

$$d = \left(\frac{5^6}{2^2 \cdot 3^4 \cdot c^3} \right)^{1/5}.$$

With $c = 48.1588$ from Theorem 1.2, we obtain $d \geq 0.2123$, improving on the current best value of 0.2098.

3 Preliminaries

To assist in our argument we review the following results from the literature.

3.1 Bounds on $\zeta(s)$ near the 1-line

The underlying philosophy in furnishing zero-free regions is to estimate $\zeta(s)$ in a region close to the line $\sigma = 1$. Richert's theorem [31] (see also [33, p. 135]), accomplishes this by proving that there exist positive constants A and B for which

$$|\zeta(\sigma + it)| \leq A|t|^{B(1-\sigma)^{3/2}}(\log|t|)^{2/3}, \quad |t| \geq 3, \quad \frac{1}{2} \leq \sigma \leq 1. \quad (3.1)$$

Ford [9] obtains a relatively small value of B while maintaining a completely explicit value of A : in (3.1) one can take $A = 76.2$ and $B = 4.45$. We remark in passing that the advances made in [30] may be applied to Ford's paper [9], and should lead to a slight reduction in the value of B in (3.1).

We also require bounds on $\zeta(s)$ slightly to the right of $\sigma = 1$. A result recorded in Bastien and Rogalski [1], originally due to O. Ramaré, states that for $\sigma > 1$, we have

$$\zeta(\sigma) \leq \frac{e^{\gamma(\sigma-1)}}{\sigma-1}, \quad (3.2)$$

where γ is Euler’s constant. Using the Maclaurin series for e^x one sees that the approximation in (3.2) is very good.

3.2 Bounds on $\zeta(s)$ on the half-line

We recall the recent sub-Weyl bound due to Patel [23]: for $|t| \geq 3$, we have

$$|\zeta(\frac{1}{2} + it)| \leq 307.098|t|^{27/164}. \tag{3.3}$$

This is required in the proof of Theorem 1.4 in Sect. 6.

3.3 Trigonometric polynomials

For any $K \geq 2$, let

$$P_K(x) = \sum_{k=0}^K b_k \cos(kx)$$

where b_k are constants such that $b_k \geq 0$, $b_1 > b_0$ and $P_K(x) \geq 0$ for all real x . We refer to $P_K(x)$ as a K th degree non-negative trigonometric polynomial. Choosing a favorable P_K plays an integral part in determining the size of the zero-free region: see [21] for a detailed history of this problem.

In [10], Ford employed the degree 4 polynomial

$$P_4(x) = (0.225 + \cos x)^2(0.9 + \cos x)^2, \tag{3.4}$$

while Nielsen [22] adopted the degree 5 function

$$P_5(x) = (1 + \cos x)(0.1974476 \dots + \cos x)^2(0.8652559 \dots + \cos x)^2. \tag{3.5}$$

We employ simulated annealing in a large-scale search to determine favorable trigonometric polynomials of higher degree: see Sect. 8.

We utilize two polynomials in this article. Theorem 1.1 relies on a polynomial $P_{40}(x)$ with degree 40 having

$$b_0 = 1, \quad b_1 = 1.74600190914994, \quad b = \sum_{k=1}^{40} b_k = 3.56453965437134, \tag{3.6}$$

while Theorem 1.2 employs a polynomial $P_{46}(x)$ with degree 46 where

$$b_0 = 1, \quad b_1 = 1.74708744081848, \quad b = \sum_{k=1}^{46} b_k = 3.57440943022073. \tag{3.7}$$

We show the full polynomials in Tables 1 and 2 in Sect. 8, and provide details there on the process used to find these polynomials as well as a justification that they are non-negative.

3.4 Estimates of the zero-counting function

We also require bounds on the error term of estimates of $N(T)$, the number of zeros of $\zeta(s)$ with $0 \leq t < T$. For large T , the current best known explicit result is due to Hasanlizada, Shen and Wong [12, Cor. 1.2], who proved that

$$\left| N(T) - \frac{T}{2\pi} \log \frac{T}{2\pi e} \right| \leq 0.1038 \log T + 0.2573 \log \log T + 9.3675 \tag{3.8}$$

for all $T \geq e$. The bound in (3.8) can be improved in the region where RH is known to hold [28], and for some intermediate values of T as well using [27]. Nevertheless, the bound in (3.8) is the best available at present for $T \geq 10^{410}$. While that bound could be reduced using (3.3), it suffices for our purposes.

4 Required lemmas

Before proceeding to the proof of Theorem 1.1 in the next section, we review some useful lemmas. Throughout, let

$$L_1 = L_1(t) = \log(Kt + 1), \quad L_2 = L_2(t) = \log \log(Kt + 1),$$

where K is the degree of the trigonometric polynomial under consideration. Let b_0, b_1 and b be as in (3.6) or (3.7). In addition, assume throughout that (3.1) holds. Unless otherwise stated, our results will remain valid for all $A, B > 0$ for which (3.1) holds. Since $\zeta(\bar{s}) = \overline{\zeta(s)}$, it suffices to consider only $t > 0$ throughout.

As in Ford [10], the main tool used to pass from upper bounds on $\zeta(s)$ to a zero-free region is the “zero-detector”, which expresses $-\Re \frac{\zeta'}{\zeta}(s)$ as an integral involving $\zeta(s)$ over two vertical lines on either side of $\sigma = 1$ (plus a small error term). Concretely, we have the following lemma.

Lemma 4.1 (*Zero-detector for ζ*) *Let $s = 1 + it$ with $t \neq 0$ and let ρ run through the non-trivial zeros of $\zeta(z)$. For all $\eta > 0$, except for a set of Lebesgue measure 0, we have*

$$-\Re \frac{\zeta'}{\zeta}(s) = \frac{1}{4\eta} \int_{-\infty}^{\infty} \frac{\log|\zeta(s - \eta + \frac{2\eta iu}{\pi})| - \log|\zeta(s + \eta + \frac{2\eta iu}{\pi})|}{\cosh^2 u} du + \frac{\pi}{2\eta} \sum_{|\Re(s-\rho)| \leq \eta} \Re \cot\left(\frac{\pi(\rho - s)}{2\eta}\right),$$

where in the sum each zero is counted with multiplicity.

Proof Follows from using the function $\zeta(z)$ and $z_0 = s$ in [10, Lemma 2.2]. □

We remark that the sum on the right side of the corresponding expression in [10, Lemma 2.2] is somewhat more complicated, to account for poles of the function. In our application, the only pole is at $z = 1$, and because $s = 1 + it$, any contribution in the sum from this pole is 0. We also remark that the precise nature of the exceptional set for η is described in [10].

As is common practice, instead of working directly with Lemma 4.1, we consider a “mollified” version, presented in Lemma 4.2. The choice of the smoothing function f significantly influences the eventual zero-free region constant. As in Ford [10], we base our choice of f on Lemma 7.5 of Heath-Brown [13]. Jang and Kwon [16] obtained improvements for the classical zero-free region by choosing a different mollifier described by Xylouris in [37], however we find that such a choice of f did not produce significant improvements here for the Korobov–Vinogradov zero-free region.

We construct the smoothing function the same way as Ford [10], which we briefly review here for completeness. Given a qualifying trigonometric polynomial $P(x) = \sum_{k=0}^K b_k \cos(kx)$, let $\theta = \theta(b_0, b_1)$ be the unique solution to the equation

$$\sin^2 \theta = \frac{b_1}{b_0} (1 - \theta \cot \theta), \quad 0 < \theta < \frac{\pi}{2}. \tag{4.1}$$

For the choice of b_0, b_1 in (3.6), we compute

$$\theta = 1.13331020636698 \dots \tag{4.2}$$

and for (3.7) we find

$$\theta = 1.13269369969232 \dots$$

Define

$$g(u) := \begin{cases} (\cos(u \tan \theta) - \cos \theta) \sec^2 \theta, & |u| \leq \theta \cot \theta, \\ 0, & \text{otherwise,} \end{cases}$$

$$w(u) := (g * g)(u) = \int_{-\infty}^{\infty} g(t)g(u - t) dt,$$

where, in particular,

$$w(0) = (\theta \tan \theta + 3\theta \cot \theta - 3) \sec^2 \theta. \tag{4.3}$$

We then choose the smoothing function to be

$$f(u) := \lambda e^{\lambda u} w(\lambda u) \tag{4.4}$$

where λ is a positive parameter to be fixed later. Note that $f(u) \geq 0$ since $g(u) \geq 0$.

We will primarily require properties about the Laplace transform of this smoothing function. Let

$$F(z) := \int_0^{\infty} e^{-zu} f(u) du$$

denote the Laplace transform of $f(u)$, and similarly let $W(z)$ be the Laplace transform of $w(u)$. The function $W(z)$ has a closed formula, given in Ford [10],² which we state here for convenience:

$$W(z) = \frac{w(0)}{z} + W_0(z),$$

$$W_0(z) := \frac{c_0 \left(c_2 \left[(z + 1)^2 e^{-2\theta(\cot \theta)z} + z^2 - 1 \right] - c_1 z - c_3 z^3 \right)}{z^2(z^2 + \tan^2 \theta)^2}, \tag{4.5}$$

where

$$c_0 := \frac{1}{\sin \theta \cos^3 \theta}, \quad c_1 := (\theta - \sin \theta \cos \theta) \tan^4 \theta,$$

$$c_2 := \tan^3 \theta \sin^2 \theta, \quad c_3 := (\theta - \sin \theta \cos \theta) \tan^2 \theta.$$

In particular, via a direct substitution, we have

$$W'(0) = \frac{\csc \theta(3(4\theta^2 - 5) + \theta(15 - 4\theta^2) \cot \theta) - 3\theta \sec \theta}{3 \sin \theta}. \tag{4.6}$$

This will be useful later in the proof of Lemma 6.1. Meanwhile, for $R \geq 3$, and using (4.5), we have

$$|W_0(z)| \leq \frac{H(R)}{|z|^3}, \quad \Re z \geq -1, \quad |z| \geq R,$$

where

$$H(R) := \frac{c_0}{\left(1 - \frac{\tan^2 \theta}{R^2}\right)^2} \left\{ c_2 \frac{(R + 1)^2}{R^3} (e^{2\theta \cot \theta} + 1) + \frac{c_1}{R^2} + c_3 \right\}.$$

This allows us to bound $F_0(z) := F(z) - f(0)/z$, via the identity

$$F(z) = W\left(\frac{z}{\lambda} - 1\right),$$

²The expressions for $W_0(z)$, c_1 and c_3 differ from those in [10], since they account for an error, which was corrected in [11].

which is a consequence of (4.4). We obtain, as per Ford,

$$|F_0(z)| \leq C_5(R) \frac{\lambda f(0)}{|z|^2}, \quad \Re z \geq 0, |z| \geq (R + 1)\lambda,$$

where

$$C_5(R) := \frac{H(R)(R + 1)^2}{R^3 w(0)} + 1 + \frac{1}{R}. \tag{4.7}$$

The motivation for bounding $|F_0(z)|$ is shown in the next lemma.

Lemma 4.2 *Let f be a non-negative, compactly supported real function with a continuous derivative and an absolutely convergent Laplace transform $F(z)$ for $\Re z > 0$, where*

$$F(z) := \int_0^\infty f(t)e^{-zt} dt,$$

and write $F_0(z) := F(z) - \frac{f(0)}{z}$. Suppose further that for $0 < \eta \leq 3/2$ we have

$$|F_0(z)| \leq \frac{D}{|z|^2} \quad \forall \Re z \geq 0, |z| \geq \eta,$$

for some absolute constant D . If $s = 1 + it$ with $t \geq 1000$, then

$$\begin{aligned} \Re \sum_{n \geq 1} \frac{\Lambda(n)f(\log n)}{n^s} &\leq - \sum_{|1+it-\rho| \leq \eta} \Re \left\{ F(s - \rho) + f(0) \left(\frac{\pi}{2\eta} \cot \left(\frac{\pi(s - \rho)}{2\eta} \right) - \frac{1}{s - \rho} \right) \right\} \\ &\quad + \frac{f(0)}{4\eta} \left\{ \int_{-\infty}^\infty \frac{\log|\zeta(s - \eta + \frac{2\eta ui}{\pi})|}{\cosh^2 u} du - \int_{-\infty}^\infty \frac{\log|\zeta(s + \eta + \frac{2\eta ui}{\pi})|}{\cosh^2 u} du \right\} \\ &\quad + D \left\{ 1.8 + \frac{\log t}{3} + \sum_{|s-\rho| \geq \eta} \frac{1}{|s - \rho|^2} \right\} \end{aligned} \tag{4.8}$$

and

$$\sum_{n \geq 1} \frac{\Lambda(n)f(\log n)}{n} \leq F(0) + 1.8D.$$

Proof This follows by combining [10, Lemma 4.5] with [10, Lemma 4.6]. □

We remark that some negligible improvements are possible if one takes $t \geq t_0$ for some $t_0 > 1000$. These do not affect our final results, given the number of decimal places to which they are stated.

We seek to determine an upper bound on linear combinations of the right side of (4.8). We briefly outline our approach, which follows Ford [10], while incorporating some improvements from Sect. 3. The first integral,

$$\int_{-\infty}^\infty \frac{\log|\zeta(s - \eta + \frac{2\eta ui}{\pi})|}{\cosh^2 u} du,$$

is taken on a vertical line inside the critical strip, and it can be bounded using Lemma 4.3 below and (3.1). This term is by far the most significant, and highlights the sensitivity of the resulting zero-free region to the constants A and B appearing in (3.1). The contour of the second integral lies outside the critical strip, and for this we employ Ford’s trick of combining $\log|\zeta(\cdot)|$ terms, combined with (3.2). This term is the subject of Lemma 4.4.

Next, the sum

$$\sum_{|s-\rho| \geq \eta} \frac{1}{|s - \rho|^2}$$

is bounded with the aid of $N(t, \nu)$, the number of zeros ρ with $|1 + it - \rho| \leq \nu$. This is discussed in Lemmas 4.5 and 4.6. In Lemma 4.7, we combine all these results with the trigonometric polynomial (3.6) to establish an inequality involving the real and imaginary parts of a zero.

Lemma 4.3 (Ford [10]) *Suppose that, for fixed $\frac{1}{2} \leq \sigma < 1$ and $t \geq 3$, we have*

$$|\zeta(\sigma + i\tau)| \leq X t^Y \log^Z t, \quad \forall 1 \leq |\tau| \leq t,$$

where X, Y, Z are positive constants and $Y + Z > 0.1$. If $0 < a \leq \frac{1}{2}$, $t \geq 100$ and $\frac{1}{2} \leq \sigma \leq 1 - t^{-1}$, then

$$\frac{1}{2} \int_{-\infty}^{\infty} \frac{\log|\zeta(\sigma + it + iau)|}{\cosh^2 u} du \leq \log X + Y \log t + Z \log \log t.$$

Proof See [10, Lemma 3.4]. □

Lemma 4.4 *Let $K > 1$ and b_j be the coefficients of a non-negative trigonometric polynomial of degree K . Furthermore let $t_1, t_2 \in \mathbb{R}$ and $\eta > 0$. Then*

$$\int_{-\infty}^{\infty} \frac{1}{\cosh^2 u} \sum_{j=1}^K b_j \log|\zeta(1 + \eta + ijt_1 + iut_2)| du \geq -2b_0 \log \zeta(1 + \eta).$$

Proof This is an immediate generalization of [10, Lemma 5.1] to degree K polynomials. □

Lemma 4.5 *If $0 < \eta \leq 1/4$, $t \geq 100$, and (3.1) holds for some $A > 1$ and $B > 0$, then*

$$N(t, \eta) \leq 1.3478\eta^{3/2} B \log t + 0.479 + \frac{\log A - \log \eta + \frac{2}{3} \log \log t}{1.879}.$$

Proof Same as in [10, Lemma 4.2].³ Using η in place of R in Ford’s treatment, we replace the second inequality in (4.1) of [10] with

$$\frac{1}{|\zeta(s + 2.5\eta + i\nu)|} \leq \zeta(1 + 3.1421\eta) \leq \frac{e^{3.1421\gamma\eta}}{3.1421\eta}$$

for real ν , where the last inequality follows from (3.2). Hence, since $\eta \leq 1/4$,

$$\log \left(\frac{e^{3.1421\gamma\eta}}{3.1421\eta} \right) = 3.1421\gamma\eta - \log \eta - \log 3.1421 \leq -\log \eta - 0.6914.$$

The constant to replace the 0.49 appearing in [10, Lemma 4.2] is then

$$\frac{1}{0.3758} \left(\frac{1}{3.1421} - \frac{0.6914}{5} \right) \leq 0.479, \tag{4.10}$$

and the result immediately follows from this. □

³We take this opportunity to correct a minor oversight in the corresponding statement in [10]. There, in our notation the hypothesis required $0 < \eta \leq 1/4$, and then the inequality

$$\zeta(\sigma) \leq 0.6 + \frac{1}{\sigma - 1}, \quad 1 < \sigma \leq 1.06 \tag{4.9}$$

from [10, Lemma 3.1] was used with $\sigma = 1 + 3.1421\eta$, so here one should require $0 < \eta \leq 0.019\dots$ instead. This does not affect subsequent work: this lemma was employed in the proof of [9, Thm. 2] where η was restricted by $0 < \eta \leq 0.01$. In our treatment, we retain the condition $0 < \eta \leq 1/4$, and opt to achieve a result of similar strength by using (3.2) in place of (4.9).

Lemma 4.6 *Suppose $0 < \eta \leq 1/4$, $t \geq 10000$, and (3.1) holds for some $A > 1$ and $B > 0$. Then*

$$\sum_{|1+it-\rho| \geq \eta} \frac{1}{|1+it-\rho|^2} \leq \left[5.409 + 5.392B \left(\frac{1}{\sqrt{\eta}} - 2 \right) \right] \log t + 206.7 + \frac{1}{\eta^2} \left\{ \frac{\log A - \log \eta + \frac{2}{3} \log \log t}{1.879} + 0.213 - N(t, \eta) \right\}.$$

Proof We proceed in the same manner as [10, Lemma 4.3], except we use (3.8) in place of a classical inequality of Rosser [32]. In addition, we propagate the new constant (4.10) in the bound for $N(t, \eta)$ through to this bound, where we obtain

$$0.479 - \frac{1}{2(1.879)} \leq 0.213.$$

□

Equipped with these lemmas, we are now prepared to form an inequality involving the real and imaginary parts of a zero $\rho = \beta + it$.

Lemma 4.7 *Let $0 < \eta \leq 1/4$ and $R \geq 3$ be constants, and suppose (3.1) holds for some $A > 1$ and $B > 0$. Suppose $\beta + it$ is a zero satisfying $t \geq 10000$ and $1 - \beta \leq \eta/2$. Further, suppose that there are no zeros in the rectangle*

$$1 - \lambda < \Re s \leq 1, \quad t - 1 \leq \Im s \leq Kt + 1,$$

where λ is a constant satisfying $0 < \lambda \leq \min \{1 - \beta, \eta/(R + 1)\}$. If b_0 , b_1 and b are constants associated with a degree K non-negative trigonometric polynomial, then

$$\begin{aligned} \frac{1}{\lambda} \left(\cos^2 \theta - \frac{W'(0)b_1}{w(0)b_0} \left(\frac{1 - \beta}{\lambda} - 1 \right) \right) &\leq 0.087\pi^2 \frac{b_1}{b_0} \frac{1 - \beta}{\eta^2} \\ &+ \frac{1}{2\eta} \left\{ \frac{b}{b_0} \left(\frac{2}{3}L_2 + B\eta^{3/2}L_1 + \log A \right) + \log \zeta(1 + \eta) \right\} \\ &+ C_5(R) \frac{b}{b_0} \lambda \left\{ \frac{L_1}{3} + \left[5.409 + 5.392B \left(\frac{1}{\sqrt{\eta}} - 2 \right) \right] L_1 + 209.1 \right. \\ &\quad \left. + \frac{1}{\eta^2} \left[\frac{\log(A/\eta) + \frac{2}{3}L_2}{1.879} + 0.213 \right] \right\}, \end{aligned}$$

with $C_5(R)$, $w(0)$ and $W'(0)$ are defined in (4.7), (4.3) and (4.6) respectively.

Proof We follow the argument of [10, Lemma 7.1], using our Lemma 4.6 instead of [10, Lemma 4.3], using Lemma 4.4 in place of [10, Lemma 5.1], and using the trigonometric polynomial (3.6) instead of (3.4). □

To form an explicit zero-free region for $\zeta(s)$, we apply Lemma 4.7 with η as a fixed function of t . The rate at which $\eta \rightarrow 0$ as $t \rightarrow \infty$ determines the shape and width of the zero-free region. We choose, as in [10],

$$\eta(t) := E \left(\frac{L_2}{BL_1} \right)^{2/3} \tag{4.11}$$

for some constant $E > 0$. Ultimately, we take

$$E := 1.8821259$$

in order to attain the constant of 55.241 appearing in Theorem 1.1. This choice replaces that of

$$E = \left(\frac{4}{3} \left(1 + \frac{b_0}{b} \right) \right)^{2/3}, \tag{4.12}$$

appearing in Ford [10], where b_0 and b are constants from the trigonometric polynomial in (3.6). Ford’s choice minimizes the asymptotic zero-free region constant, whereas we choose E to minimize the zero-free region constant holding for all $t \geq 3$. The difference between our choices reflects the presence of a secondary error term that decreases as E increases, and which is significant for small values of t .

5 Proof of Theorem 1.1

We divide our argument into four sections depending on the size of t . Throughout, we take

$$A = 76.2, B = 4.45$$

in (3.1) as permitted by [9, Theorem 1], and let

$$v(t) := \frac{1}{55.241(\log t)^{2/3}(\log \log t)^{1/3}}.$$

First, for $3 \leq t \leq H := 3 \cdot 10^{12}$, we use the rigorous verification of RH up to height H in [28]. Next, for $H \leq t < \exp(8928)$, the desired result follows immediately from Theorem 1.3, since

$$\frac{1}{5.558691 \log t} \geq v(t), \quad H \leq t < \exp(8928).$$

For $\exp(8928) \leq t < \exp(52238)$, we use Theorem 1.4, combined with the observation that

$$\frac{0.05035}{h(t)} - \frac{0.0349}{h^2(t)} > v(t)$$

for this range of t , where $h(t)$ is defined in Theorem 1.4. The proof for $t \geq \exp(52238)$ forms the main part of our argument. For convenience, let

$$T_0 := \exp(52238), \quad M_1 := 0.048976.$$

Define the function

$$Z(\beta, t) := (1 - \beta)B^{2/3}(\log t)^{2/3}(\log \log t)^{1/3} \tag{5.1}$$

and let

$$M := \inf_{\substack{\zeta(\beta+it)=0 \\ t \geq T_0}} Z(\beta, t). \tag{5.2}$$

If $M \geq M_1$, then we are done, since if $\beta + it$ is a zero with $t \geq T_0$, then

$$M_1 \leq M \leq (1 - \beta)B^{2/3}(\log t)^{2/3}(\log \log t)^{1/3}$$

and $B^{2/3}/M_1 < 55.241$. Assume for a contradiction that

$$M < M_1. \tag{5.3}$$

We will show that this implies $M \geq M_1$. Under assumption (5.3), there is a zero $\beta + it$ such that $t \geq T_0$ and for which $Z(\beta, t)$ is arbitrarily close to M . In particular, we may take a zero satisfying

$$M \leq Z(\beta, t) \leq M(1 + \delta), \quad \delta := \min \left\{ \frac{10^{-100}}{\log T_0}, \frac{M_1 - M}{2M} \right\}. \tag{5.4}$$

Note in particular that $Z(\beta, t) < M_1$. Next, let

$$\lambda := \lambda(t) = \frac{M}{B^{2/3}L_1^{2/3}L_2^{1/3}}.$$

There are no zeros of $\zeta(s)$ in the rectangular region

$$1 - \lambda < \Re s \leq 1, \quad t - 1 \leq \Im s \leq Kt + 1. \tag{5.5}$$

This is because if a zero $\beta' + it'$ exists in that region, then $\log t' \leq L_1(t')$ and $\log \log t' \leq L_2(t')$, so

$$1 - \beta' < \lambda(t') \leq \frac{M}{B^{2/3}(\log t')^{2/3}(\log \log t')^{1/3}},$$

i.e., $Z(\beta', t') < M$ and $t' \geq t - 1 \geq T_0 - 1$. However, if $t' \in [T_0 - 1, T_0)$ then by Theorem 1.4,

$$\begin{aligned} 1 - \beta' &\geq \frac{0.05035}{h(t')} - \frac{0.0349}{h^2(t')} \geq \frac{0.05035}{h(T_0)} - \frac{0.0349}{h^2(T_0)} \\ &> \frac{M_1}{B^{2/3}(\log(T_0 - 1))^{2/3}(\log \log(T_0 - 1))^{1/3}} \geq \frac{M_1}{B^{2/3}(\log t')^{2/3}(\log \log t')^{1/3}} \end{aligned}$$

so that $Z(\beta', t') \geq M_1 > M$, a contradiction. Note that the third inequality is verified by a numerical computation. On the other hand if $t' \in [T_0, Kt + 1]$, then by (5.2) we also arrive at a contradiction. Thus any zero $\beta + it$ must satisfy

$$\lambda \leq 1 - \beta, \tag{5.6}$$

thereby proving claim (5.5).

Next, we choose $\eta = EB^{-2/3}(L_2/L_1)^{2/3}$, as in (4.11), with $E = 1.8821259$. This choice gives, for all $t \geq T_0$,

$$\lambda \leq \frac{M_1}{B^{2/3}L_1^{2/3}L_2^{1/3}} = \eta \cdot \frac{M_1}{EL_2} < \frac{\eta}{417.48}, \tag{5.7}$$

where the last inequality follows from $K = 40$ and $t \geq T_0$. Furthermore, using $Z(\beta, t) < M_1$ we have that for all $t \geq T_0$,

$$1 - \beta < \frac{M_1}{B^{2/3}(\log t)^{2/3}(\log \log t)^{1/3}} < \frac{EL_2^{2/3}}{417.48B^{2/3}L_1^{2/3}} < \frac{\eta}{2}, \tag{5.8}$$

Finally, by substituting the values of E, T_0 and K , we have

$$\eta \leq \frac{E}{B^{2/3}} \left(\frac{L_2(T_0)}{L_1(T_0)} \right)^{2/3} < \frac{1}{4}. \tag{5.9}$$

Collecting (5.5), (5.6), (5.7), (5.8) and (5.9), we see that all of the requirements of Lemma 4.7 are met with $R = 416$. Substituting $b_0 = 1, b_1 = 1.74600190914994, b = 3.56453965437134, K = 40$ (from (3.6)) into Lemma 4.7 and computing numerically the values of $w(0)$ and $W'(0)$, we obtain

$$\frac{1}{\lambda} \left[0.17949 - 0.20466 \left(\frac{1 - \beta}{\lambda} - 1 \right) \right] \leq 1.5T_1 + T_2 + C_5(R) \frac{b}{b_0} T_3, \tag{5.10}$$

where

$$\begin{aligned} T_1 &:= \frac{1 - \beta}{\eta^2}, \\ T_2 &:= \frac{1}{2\eta} \left[\frac{b}{b_0} \left(\frac{2}{3}L_2 + B\eta^{3/2}L_1 + \log A \right) + \log \zeta(1 + \eta) \right], \\ T_3 &:= \lambda \left[\frac{L_1}{3} + \left[5.409 + 5.392B \left(\frac{1}{\sqrt{\eta}} - 2 \right) \right] L_1 + 209.1 \right. \\ &\quad \left. + \frac{1}{\eta^2} \left\{ \frac{\log A - \log \eta + \frac{2}{3}L_2}{1.879} + 0.213 \right\} \right]. \end{aligned}$$

We proceed by bounding each of the terms T_1, T_2 and T_3 . First, using (5.1) we find

$$\begin{aligned} T_1 &= \frac{Z(\beta, t)}{B^{2/3}(\log t)^{2/3}(\log \log t)^{1/3}} \cdot \frac{B^{4/3}}{E^2} \left(\frac{L_1}{L_2}\right)^{4/3} \\ &\leq \frac{M_1 B^{2/3}}{E^2} \cdot \left(\frac{L_1}{L_2}\right)^{2/3} \cdot \left(\frac{L_1}{\log t}\right)^{2/3} \frac{1}{L_2^{2/3}(\log \log t)^{1/3}} \\ &\leq \kappa_1 \frac{M_1 B^{2/3}}{E^2} \left(\frac{L_1}{L_2}\right)^{2/3}, \end{aligned} \tag{5.11}$$

where, since $t \geq T_0$,

$$\kappa_1 = \left(\frac{L_1(T_0)}{L_2(T_0)}\right)^{2/3} \cdot \frac{1}{(\log T_0)^{2/3}(\log \log T_0)^{1/3}}.$$

Next, using (3.2) and (5.9), we have

$$\log \zeta(1 + \eta) \leq \gamma \eta - \log \eta \leq \frac{2}{3}L_2 + \frac{2}{3} \log \frac{B}{L_2} - \log E + \frac{\kappa_2 E}{B^{2/3}},$$

where

$$\kappa_2 := \gamma \left(\frac{L_2(T_0)}{BL_1(T_0)}\right)^{2/3}.$$

Hence

$$\begin{aligned} T_2 &\leq \frac{1}{2\eta} \left[\frac{b}{b_0} \left(\frac{2}{3}L_2 + B\eta^{3/2}L_1 + \log A\right) + \frac{2}{3}L_2 + \frac{2}{3} \log \frac{B}{L_2} - \log E + \frac{\kappa_2 E}{B^{2/3}} \right] \\ &= \left[\frac{1}{3E} \left(\frac{b}{b_0} + 1\right) + \frac{bE^{1/2}}{2b_0} \right] B^{2/3} L_1^{2/3} L_2^{1/3} \\ &\quad + \frac{1}{2E} \left[\frac{b}{b_0} \log A + \frac{2}{3} \log \frac{B}{L_2} - \log E + \frac{\kappa_2 E}{B^{2/3}} \right] B^{2/3} \frac{L_1^{2/3}}{L_2^{2/3}}. \end{aligned} \tag{5.12}$$

Next, substituting the definition of η , and using $\lambda \leq \eta M_1 / (EL_2)$ from (5.7), we obtain

$$\begin{aligned} T_3 &\leq \frac{M_1}{B^{2/3}L_1^{2/3}L_2^{1/3}} \left[\left(\kappa_3 + \frac{5.392B^{4/3}}{E^{1/2}} \left(\frac{L_1}{L_2}\right)^{1/3} - 10.784B \right) L_1 \right. \\ &\quad \left. + \frac{1}{E^2} \left(\frac{BL_1}{L_2}\right)^{4/3} \left\{ \frac{\log A + \frac{4}{3}L_2 + \frac{2}{3} \log \frac{B}{L_2} - \log E}{1.879} + 0.213 \right\} \right] \\ &= M_1 \left(\frac{BL_1}{L_2}\right)^{2/3} \left[\frac{5.392}{E^{1/2}} + \frac{4}{5.637E^2} + \frac{\kappa_3 - 10.784B}{B^{4/3}} \left(\frac{L_2}{L_1}\right)^{1/3} \right. \\ &\quad \left. + \frac{1}{E^2 L_2} \left\{ \frac{\log A + \frac{2}{3} \log \frac{B}{L_2} - \log E}{1.879} + 0.213 \right\} \right], \end{aligned} \tag{5.13}$$

where, since $t \geq T_0$ and $L_1(T_0) > \log(KT_0)$, we may take

$$\kappa_3 = \frac{1}{3} + 5.409 + \frac{209.1}{\log K + \log T_0}.$$

Last, we have

$$\begin{aligned} \frac{1 - \beta}{\lambda} - 1 &\leq (1 + \delta) \left(\frac{L_1}{\log t}\right)^{2/3} \left(\frac{L_2}{\log \log t}\right)^{1/3} - 1 \\ &< \left(1 + \frac{10^{-100}}{\log T_0}\right) \frac{L_1}{\log t} - 1 \leq \frac{\kappa_4}{\log T_0}, \end{aligned} \tag{5.14}$$

where, since $t \geq T_0$,

$$\kappa_4 := 10^{-100} \left(1 + \frac{\log(K + T_0^{-1})}{\log T_0}\right) + \log(K + T_0^{-1}).$$

Combining (5.10), (5.12), (5.13), (5.11) and (5.14), we conclude

$$\begin{aligned} \frac{1}{\lambda} \left(0.17949 - \frac{0.20466\kappa_4}{\log T_0} \right) &\leq \frac{1}{\lambda} \left[0.17949 - 0.20466 \left(\frac{1-\beta}{\lambda} - 1 \right) \right] \\ &\leq 1.5T_1 + T_2 + C_5(R) \frac{b}{b_0} T_3 \\ &\leq \frac{1.5\kappa_1 M_1 B^{2/3}}{E^2} \left(\frac{L_1}{L_2} \right)^{2/3} + \left[\frac{1}{3E} \left(\frac{b}{b_0} + 1 \right) + \frac{bE^{1/2}}{2b_0} \right] (BL_1)^{2/3} L_2^{1/3} \\ &\quad + \frac{1}{2E} \left[\frac{b}{b_0} \log A + \frac{2}{3} \log \frac{B}{L_2} - \log E + \frac{\kappa_2 E}{B^{2/3}} \right] \left(\frac{BL_1}{L_2} \right)^{2/3} \\ &\quad + C_5(R) \frac{b}{b_0} M_1 \left(\frac{BL_1}{L_2} \right)^{2/3} \left[\frac{5.392}{E^{1/2}} + \frac{4}{5.637E^2} + \frac{\kappa_3 - 10.784B}{B^{4/3}} \left(\frac{L_2}{L_1} \right)^{1/3} \right. \\ &\quad \left. + \frac{1}{E^2 L_2} \left\{ \frac{\log A + \frac{2}{3} \log \frac{B}{L_2} - \log E}{1.879} + 0.213 \right\} \right]. \end{aligned}$$

At this point we substitute the values $A = 76.2$, $B = 4.45$, $K = 40$, $b_0 = 1$, $b = 3.56453965437134$, $M_1 = 0.048976$, $E = 1.8821259$ and $R = 416$, and using (4.2) and (4.7) to compute $C_5(R)$, we obtain

$$\frac{1}{\lambda} \left(0.17949 - \frac{0.755}{\log T_0} \right) \leq (BL_1)^{2/3} L_2^{1/3} [3.25351 + Y(t)]$$

where

$$Y(t) := \frac{4.940431}{L_2} + \frac{0.136899}{L_2^2} - \frac{1.031863}{L_1^{1/3} L_2^{2/3}} - \frac{0.177104 \log L_2}{L_2} - \frac{0.0179076 \log L_2}{L_2^2}. \tag{5.15}$$

A short *Mathematica* computation is used to verify that $Y(t)$ is decreasing for $t \geq T_0$.⁴ Therefore, $Y(t) \leq Y(T_0) < 0.4110503$. However, from the definition of λ together with $t \geq T_0$, we have

$$\begin{aligned} M = \lambda B^{2/3} L_1^{2/3} L_2^{1/3} &\geq \frac{0.17949 - 0.755/\log T_0}{3.25351 + Y(t)} \\ &\geq \frac{0.17949 - 0.755/52238}{3.25351 + 0.4110503} > 0.04897601 > M_1, \end{aligned} \tag{5.16}$$

hence we have arrived at the desired contradiction.

6 Proof of Theorem 1.4

The proof of this theorem is similar to that of Theorem 1.1. Instead of bounding $\zeta(s)$ on $\sigma = 1 \pm \eta(t)$, we use upper bounds on $\zeta(s)$ on the lines $\sigma = 1/2$ and $\sigma = 3/2$, i.e., η is fixed at $1/2$. The main disadvantage of this new scheme is that the best known bounds of $\zeta(1/2 + it)$ are of order $t^{\theta+\epsilon}$ for some fixed $\theta > 0$, which means that the resulting zero-free region will only have width $O(1/\log t)$. Nevertheless, the resulting zero-free region has a better asymptotic constant than (1.4), so we use this result to cover the range $\exp(8928) \leq t \leq \exp(52238)$.

⁴Our monotonicity argument is used to overcome a small issue in Ford's [10] treatment (after corrections in [11]). In (8.9), the following inequality was used:

$$\frac{0.04893}{L_2} \left(\log A + \frac{2}{3} \log \frac{B}{L_2} \right) \leq 0.0048 \log A + 0.0031 \log \frac{B}{L_2}, \quad t \geq \exp(30000).$$

This should be reversed if t is sufficiently large, as both sides of the inequality are negative.

Throughout this section, let us define

$$J(t) := \frac{27}{164} \log t + \log 307.098. \tag{6.1}$$

This is the logarithm of the sub-Weyl bound in (3.3). First, we require the following lemma, which corresponds to Lemma 4.7 for this zero-free region.

Lemma 6.1 *Suppose $\zeta(\beta + it) = 0$ where $t \geq \exp(1000)$ and $\beta \geq 1 - 1712^{-1}$. Let b, b_0 and K be defined in (3.6). Suppose there exists $\lambda \in (0, 1 - \beta]$ such that there are no zeros in the rectangle*

$$1 - \lambda < \Re s \leq 1, \quad t - 1 \leq \Im s \leq Kt + 1.$$

Then

$$\begin{aligned} \frac{1}{\lambda} \left(0.17949 - 0.20466 \left(\frac{1 - \beta}{\lambda} - 1 \right) \right) &\leq 5.746(1 - \beta) + \frac{b}{b_0} J(Kt + 1) + 0.851 \\ &+ 1.0146\lambda \frac{b}{b_0} (3.5691L_1 + 5.316L_2 + 18.439), \end{aligned}$$

where $J(t)$ is defined in (6.1).

Proof We largely follow the argument of [10, Lemma 9.2]. The main changes are to replace Ford’s definition of $J(t)$ with (6.1), use a better trigonometric polynomial, increase the height from which the zero-free region takes effect (to improve its constants) and more carefully bound some secondary terms that arise.

Let $R := \frac{1}{2(1-\beta)} - 1 \geq 855$. Using the trigonometric polynomial (3.6), and with $C_5(R)$ as defined in (4.7), we have $C_5(R) \leq 1.0146$. Suppose that $|z| \geq (R + 1)\lambda$. By the lemma’s assumptions, this implies $|z| \geq 1/2$ and hence, as in Sect. 4,

$$|F_0(z)| \leq C_5(R) \frac{\lambda f(0)}{|z|^2} \leq \frac{1.0146\lambda f(0)}{|z|^2}, \quad \Re z \geq 0, |z| \geq (R + 1)\lambda.$$

Therefore the conditions of Lemma 4.2 are satisfied with $D = 1.0146\lambda f(0)$ and $\eta = 1/2$. Hence for $\Re s = 1$ and $\Im s \geq 1000$ we have

$$\begin{aligned} \Re \sum_{n \geq 1} \frac{\Lambda(n)f(\log n)}{n^s} &\leq - \sum_{|s-\rho| \leq \frac{1}{2}} \Re \left\{ F(s - \rho) + f(0) \left(\pi \cot(\pi(s - \rho)) - \frac{1}{s - \rho} \right) \right\} \\ &+ \frac{f(0)}{2} \int_{-\infty}^{\infty} \frac{\log|\zeta(s - \frac{1}{2} + \frac{ui}{\pi})| - \log|\zeta(s + \frac{1}{2} + \frac{ui}{\pi})|}{\cosh^2 u} du \\ &+ D \left\{ 1.8 + \frac{\log t}{3} + \sum_{|s-\rho| \geq \frac{1}{2}} \frac{1}{|s - \rho|^2} \right\} \end{aligned} \tag{6.2}$$

and

$$\sum_{n=1}^{\infty} \frac{\Lambda(n)f(\log n)}{n} \leq F(0) + 1.8D.$$

We sum (6.2) for $s = 1 + jt$ for $0 \leq j \leq K$, and using $\log(jt) \leq L_1$ and the sub-Weyl bound (3.3) on the line $\sigma = 1/2$, along with the fact that $J(t)$ is increasing, we find

$$0 \leq \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n} f(\log n) P_K(t \log n) = \Re \sum_{j=0}^K b_j \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{1+ijt}} f(\log n)$$

$$\begin{aligned} &\leq -\Re \sum_{j=0}^K b_j \sum_{|1+ijt-\rho| \leq \frac{1}{2}} \left\{ F(1+ijt-\rho) + f(0) \left(\pi \cot(\pi(1+ijt-\rho)) - \frac{1}{1+ijt-\rho} \right) \right\} \\ &\quad + bf(0)J(Kt+1) - \frac{1}{2} \sum_{j=1}^K b_j \int_{-\infty}^{\infty} \frac{\log|\zeta(\frac{3}{2}+ijt+\frac{iu}{\pi})|}{\cosh^2 u} du + b_0F(0) \\ &\quad + D \left\{ b \left(1.8 + \frac{L_1}{3} \right) + 1.8b_0 + \sum_{j=1}^K b_j \sum_{|1+ijt-\rho| \geq \frac{1}{2}} \frac{1}{|1+ijt-\rho|^2} \right\}, \end{aligned} \tag{6.3}$$

where $P_K(x)$ is any degree K non-negative trigonometric polynomial as defined in § 3.3. Note that the first inequality above depends on the non-negativity of f , which is guaranteed by (4.4) and the fact that $w(u)$ is an autoconvolution.

We now seek an upper bound for the right side of (6.3), following largely the same arguments as Ford. First, as in the proof of [10, Thm. 3], we have

$$-\frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{\cosh^2 u} \sum_{j=1}^K b_j \log \left| \zeta \left(\frac{3}{2} + ijt + \frac{iu}{\pi} \right) \right| du \leq 0.851b_0.$$

Note that this bound relies on special properties of $\zeta(s)$ on the line $\sigma = 3/2$, and is sharper than what Lemma 4.4 implies. Next, by [10, eq. (9.2)], we have

$$\begin{aligned} &\sum_{j=1}^K b_j \sum_{|1+ijt-\rho| \geq \frac{1}{2}} \frac{1}{|1+ijt-\rho|^2} \\ &\leq \sum_{j=1}^K b_j \left[3.2357 \log(jt) + 5.316 \log \log(jt) + 16.134 - 4N \left(jt, \frac{1}{2} \right) \right] \\ &\leq b [3.2357L_1 + 5.316L_2 + 16.134] - 4 \sum_{j=1}^K b_j N \left(jt, \frac{1}{2} \right). \end{aligned} \tag{6.4}$$

In [10, eq. (7.7) of Lemma 7.1 and eq. (9.5)] it is established, via the maximum modulus principle, that if $\Re z \geq \pi\lambda$ and $|1+ijt-\rho| \leq \frac{1}{2}$, then

$$-\Re \left\{ F(1+ijt-\rho) + f(0) \left(\pi \cot(\pi(1+ijt-\rho)) - \frac{1}{1+ijt-\rho} \right) \right\} \leq c\pi^2\lambda f(0), \tag{6.5}$$

where

$$c := \frac{4}{\pi^2} + \frac{\pi(1-\beta)H(R)}{w(0)(\frac{\pi}{2} - \pi(1-\beta))^2}.$$

We use (6.5) for $j \neq 1$. If $j = 1$, then $1+ijt-\rho = 1-\beta$ and $\cot x - x^{-1} \geq -0.3334x$ for $0 < x \leq \pi(1-\beta)$, so

$$\begin{aligned} &-\Re \left\{ F(1+it-\rho) + f(0) \left(\pi \cot(\pi(1+it-\rho)) - \frac{1}{1+it-\rho} \right) \right\} \\ &\leq F(1-\beta) - 0.3334\pi^2f(0)(1-\beta). \end{aligned} \tag{6.6}$$

Combining (6.5) and (6.6), and noting that $c\pi^2\lambda f(0)b_1N(t, \frac{1}{2}) \geq 0$, we find

$$\begin{aligned} &-\Re \sum_{j=0}^K b_j \sum_{|1+ijt-\rho| \leq \frac{1}{2}} \left\{ F(1+ijt-\rho) + f(0) \left(\pi \cot(\pi(1+ijt-\rho)) - \frac{1}{1+ijt-\rho} \right) \right\} \\ &\leq -b_1 [F(1-\beta) - 0.3334\pi^2f(0)(1-\beta)] + c\pi^2\lambda f(0) \sum_{j=0}^K b_j N \left(jt, \frac{1}{2} \right). \end{aligned} \tag{6.7}$$

Substituting (6.4), (6.5) and (6.7) into (6.3), we obtain

$$\begin{aligned}
 0 \leq & -b_1 [F(1 - \beta) - 0.3334\pi^2 f(0)(1 - \beta)] + (c\pi^2 \lambda f(0) - 4D) \sum_{j=1}^K b_j N \left(jt, \frac{1}{2} \right) \\
 & + f(0) [bJ(Kt + 1) + 0.851b_0] + b_0 F(0) \\
 & + D \left\{ b \left(1.8 + \frac{L_1}{3} + 3.2357L_1 + 5.316L_2 + 16.134 \right) + 1.8b_0 \right\}.
 \end{aligned} \tag{6.8}$$

However, from the definition of D and combined with the estimates $H(R) \leq 134.87$, $w(0) \geq 5.64531$, we have

$$c\pi^2 \lambda f(0) - 4D = \left[4 + \frac{\pi(1 - \beta)H(R)}{w(0)(\beta - \frac{1}{2})^2} - 4.0584 \right] \lambda f(0) \leq 0, \tag{6.9}$$

so we may drop the sum in (6.8). Furthermore,

$$\begin{aligned}
 b_0 F(0) - b_1 F(1 - \beta) &= b_0 W(-1) - b_1 W \left(\frac{1 - \beta}{\lambda} - 1 \right) \\
 &= b_1 \left(W(0) - W \left(\frac{1 - \beta}{\lambda} - 1 \right) \right) - (b_1 W(0) - b_0 W(-1)) \\
 &= b_1 \left(W(0) - W \left(\frac{1 - \beta}{\lambda} - 1 \right) \right) - \frac{b_0 f(0) \cos^2 \theta}{\lambda}.
 \end{aligned} \tag{6.10}$$

Note that $W(z)$ is decreasing, since

$$W'(z) = \frac{d}{dz} \int_0^\infty e^{-zu} w(u) du = \int_0^\infty \frac{d}{dz} e^{-zu} w(u) du = - \int_0^\infty u e^{-zu} w(u) du < 0,$$

and similarly $W'(z)$ is increasing. Hence by the mean-value theorem, for some $0 < \xi < (1 - \beta)/\lambda - 1$ we have

$$\begin{aligned}
 W(0) - W \left(\frac{1 - \beta}{\lambda} - 1 \right) &= \left(1 - \frac{1 - \beta}{\lambda} \right) W'(\xi) \\
 &\leq \left(1 - \frac{1 - \beta}{\lambda} \right) W'(0) \\
 &\leq 0.6617195 \left(\frac{1 - \beta}{\lambda} - 1 \right),
 \end{aligned} \tag{6.11}$$

where the last inequality follows from (4.6). Substituting (6.9), (6.10) and (6.11) into (6.8), we obtain the desired result. □

We now proceed to the proof of Theorem 1.4. Let $J(t)$ be as defined in (6.1), and assume throughout that $t \geq t_0 := \exp(1000)$. Observe that

$$J(Kt + 1) - J(t) = \frac{27}{164} \log \left(\frac{Kt + 1}{t} \right) \leq 0.60732.$$

We will show that if $\zeta(\beta + it) = 0$ then $1 - \beta > c(t)(0.05035 - 0.0349c(t))$, where

$$c(t) := \frac{1}{J(t) + 1.3686}.$$

Define $Y(\beta, t)$ implicitly via the equation

$$1 - \beta = \frac{0.05035 - 0.0349c(t)}{J(t) + Y(\beta, t)}. \tag{6.12}$$

As $t \rightarrow \infty$, we have $Y(\beta, t) \rightarrow -\infty$, and $M := \max_{t \geq t_0} Y(\beta, t)$ is well-defined and finite. If $M \leq 1.3686$ then we are done, so assume that $M > 1.3686$, i.e., that there exists a zero $\rho = \beta + it$ such that $Y(\beta, t) = M > 1.3686$. We pursue a contradiction argument as before. By our assumption, there is a rectangular region near ρ that is zero-free, specifically $\zeta(s) \neq 0$ for all s satisfying

$$1 - \lambda < \Re s \leq 1 \quad \text{and} \quad t - 1 \leq \Im s \leq Kt + 1,$$

with

$$\lambda = \frac{0.05035 - 0.0349c(t)}{J(Kt + 1) + M}.$$

Combining this with the definition of $Y(\beta, t)$, we obtain

$$\frac{1 - \beta}{\lambda} - 1 = \frac{J(Kt + 1) + M}{J(t) + M} - 1 = \frac{J(Kt + 1) - J(t)}{J(t) + M} \leq \frac{0.60732}{J(t) + M}.$$

Applying Lemma 6.1, we obtain

$$\begin{aligned} J(Kt + 1) + M &= \frac{0.05035 - 0.0349c(t)}{\lambda} \\ &\leq \frac{b_0}{b} \frac{1}{\lambda} \left(0.17949 - 0.20466 \left(\frac{0.60732}{J(t) + M} \right) \right) \\ &\leq 5.746 \frac{b_0}{b} (1 - \beta) + J(Kt + 1) + 0.851 \frac{b_0}{b} \\ &\quad + 1.0146\lambda(3.5691L_1 + 5.316L_2 + 18.439), \end{aligned}$$

and hence, substituting the values of b_0 and b , and using (6.12) with $Y(\beta, t) > 1.3686$, we obtain

$$\begin{aligned} M &\leq 1.612(1 - \beta) + 0.23875 + 1.0146\lambda(3.5691L_1 + 5.316L_2 + 18.439) \\ &\leq 0.23875 + c(t)(0.05035 - 0.0349c(t))(34.384 + 3.6227 \log t + 5.3958 \log \log t). \end{aligned}$$

Now for $t \geq t_0 = \exp(1000)$, we have

$$\begin{aligned} 34.384 + 3.6227 \log t + 5.3958 \log \log t \\ \leq 3.69436 \log t \leq 22.4399(J(t) + 1.3686) = \frac{22.4399}{c(t)}, \end{aligned}$$

hence $M \leq 0.23875 + 22.4399(0.05035 - 0.0349c(t)) \leq 1.3686$. This achieves our desired contradiction. To complete the proof, we simply note that

$$J(t) + 1.3686 \leq h(t),$$

where $h(t)$ is defined in the statement of Theorem 1.4.

7 Proof of Theorem 1.2

Let $Y(t)$ be as defined in (6.12) and let $Y_1(t) := 0.755/\log t$. Observe that the contradiction argument in the proof of Theorem 1.1 relies on the inequality

$$M_1 \leq \min_{t \geq T_0} \left\{ \frac{\cos^2 \theta - Y_1(T_0)}{\frac{1}{3E} \left(\frac{b}{b_0} + 1 \right) + \frac{bE^{1/2}}{2b_0} + Y(t)} \right\} \tag{7.1}$$

which is simply (5.16) once the right side is evaluated with specific choices of θ, E, b and b_0 . Note in particular that $Y_1(t), Y(t) \rightarrow 0$ as $t \rightarrow \infty$. If we choose E as in (4.12), then (7.1) is satisfied for any

$$M_1 < \cos^2 \theta \left(\frac{4}{3} \right)^{2/3} \frac{b_0}{b} \left(1 + \frac{b_0}{b} \right)^{-1/3}$$

so long as we take T_0 sufficiently large. Also, as in [10, §1], by appealing to zero-density theorems, the number of zeros of $\zeta(s)$ in the rectangle $3/4 \leq \Re s \leq 1, T_0 - 1 \leq \Im s \leq T_0$ is $O(T_0^{1-\delta})$ for some $\delta > 0$, so for sufficiently large T_0 , most such rectangles are free of zeros. Therefore, following the argument in Sect. 5, there are no zeros in the region

$$\sigma > 1 - \frac{M_1 B^{-2/3}}{(\log|t|)^{2/3}(\log \log|t|)^{1/3}}, \quad |t| \geq T_0$$

for sufficiently large T_0 . Using the polynomial $P_{46}(x)$ with values shown in (3.7), we conclude by noting that

$$\cos^2 \theta \left(\frac{4}{3}\right)^{2/3} \frac{b_0}{b} \left(1 + \frac{b_0}{b}\right)^{-1/3} B^{-2/3} \geq 0.05617776 B^{-2/3} > (48.1588)^{-1}. \quad (7.2)$$

Remark The value of 0.05617776 improves on 0.05507 appearing in [10] and 0.055127 appearing in [22].

8 Computations

The work of Ford [10] relied in part on the selection of a trigonometric polynomial $P_K(x)$ with certain properties. Recall that

$$P_K(x) = \sum_{k=0}^K b_k \cos(kx),$$

and we require that each $b_k \geq 0$, that $b_1 > b_0$, and that $P_K(x) \geq 0$ for all real x . Any such polynomial gives rise to an asymptotic zero-free region of the Riemann zeta-function having the form

$$\sigma \geq 1 - \frac{1}{R_2(\log|t|)^{2/3}(\log \log|t|)^{1/3}}$$

when $|t|$ is sufficiently large, and a value for the constant R_2 can be computed using the polynomial. Let θ be the unique solution in $(0, \pi/2)$ to

$$b_0 \sin^2 \theta = b_1(1 - \theta \cot \theta), \quad (8.1)$$

and as in (3.6) and (3.7) let

$$b = \sum_{k=1}^K b_k = P_K(0) - b_0.$$

Using (7.2) we may take

$$R_2 = \frac{1}{\cos^2 \theta} \left(\frac{3}{4}\right)^{2/3} \frac{b}{b_0} \left(1 + \frac{b_0}{b}\right)^{1/3} B^{2/3}. \quad (8.2)$$

In [10] and [22] respectively, the polynomials (3.4) and (3.5) with degrees $K = 4$ and 5 were used. Here we employ a heuristic optimization technique to determine good polynomials $P_K(x)$ with $K > 4$ that produce an improved constant R_2 .

As in [21], we apply the technique of simulated annealing to find favorable trigonometric polynomials $P_K(x)$ having the required properties. Simulated annealing is a randomized algorithm for minimizing a cost function C over a configuration space S . In general terms, from a given state $s_0 \in S$ one selects another state s_1 (typically s_1 is near s_0 in some sense), and then computes the change in the cost function when moving to the new state: $\Delta C = C(s_1) - C(s_0)$. If $\Delta C < 0$, that is, if the change reduced the value of the cost function, then one keeps the change, and sets s_0 to the value of s_1 . On the other hand, if $\Delta C > 0$,

so the cost function is worse at s_1 , then one accepts the new state as the new current state only with a particular probability. This probability depends on a real parameter T known the *temperature*. The threshold for accepting a worse state decays exponentially with ΔC , but the decay is gentler when T is larger: the threshold value is $\exp(-\Delta C/T)$. This process is repeated many times, with the parameter T gradually decreasing, culminating with a number of trials where the temperature is effectively 0, so a simple greedy descent. This optimization procedure is inspired by the physical process of annealing in metallurgy, which involves heating and slowly cooling a material to improve its crystalline structure. See [34] for more information on this method and its mathematical properties.

In some optimization problems, simple gradient descent or greedy descent suffices to determine a good value for the cost function, but if this function achieves many non-optimal local minima over the configuration space, then something more complicated is required to allow a search to escape one local minimum and find a better configuration. Simulated annealing is well-suited for such problems, since its stochastic feature allows greater exploration of the configuration space. We chose this method for the optimization problem over trigonometric polynomials that we require here, since the landscape of our cost function is quite complicated over the underlying space.

Suppose we have selected a degree K . We must guarantee that any candidate polynomial $P_K(x)$ has the property that $P_K(x) \geq 0$ for all real x . For this, instead of manipulating the coefficients b_k of $P_K(x)$ directly, we maintain a list of coefficients $c_0 = 1, c_1, c_2, \dots, c_K$. We set

$$g(x) = \sum_{k=0}^K c_k e^{ikx}$$

and let

$$P_K(x) = \frac{|g(x)|^2}{\sum_{j=0}^K c_j^2}.$$

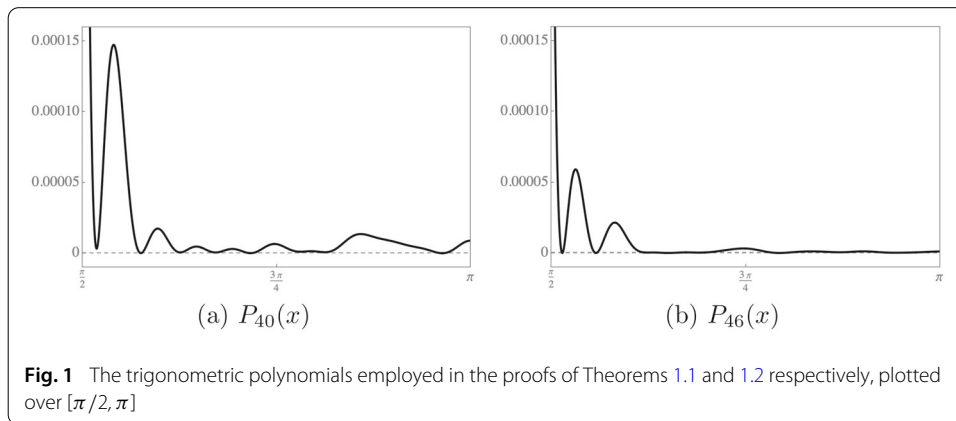
Thus, $b_0 = 1$ and for $k > 0$ the coefficient b_k is the k th autocorrelation of the sequence of coefficients c_k , suitably scaled:

$$b_k = \frac{2 \sum_{j=0}^{K-k} c_j c_{j+k}}{\sum_{j=0}^K c_j^2}.$$

With this construction, we are assured that $P_K(x) \geq 0$ for all x .

Given K and a positive real number H , our procedure begins by setting $c_0 = 1$ and selecting each c_k with $1 \leq k \leq K$ uniformly at random from the interval $[0, H]$, and then it computes the associated coefficients b_k . Then $P_K(x) \geq 0$ and each $b_k \geq 0$ by construction, and we need only check if $b_1 > b_0$. If this does not hold we simply pick a new list of values c_k and restart. We typically find a qualifying polynomial in short order, and we compute its associated value R_2 using (8.2). We then employ simulated annealing to search for better polynomials from this starting point.

In this process, we maintain a current maximum step size S and temperature T . Given these values, we perform an adjustment to our current polynomial for a certain number of iterations N . In each iteration, we select a positive integer $k < K$ at random, and a random real value $s \in [-S, S]$, then add s to c_k and perform the $O(K)$ operations required to update the b_k values. If any $b_k < 0$, or if $b_1 \leq b_0$, then we reject this adjustment and return c_k to its prior value. Otherwise, we compute the value R_2 for the adjusted polynomial, using



Newton’s method to determine θ in (8.1). If the new value is smaller than our prior value, then we keep this adjustment and move to the next iteration. If the new value is larger than our prior value, then we keep the adjustment with probability $\exp(-\Delta R_2/T)$, where ΔR_2 denotes the change in the value of R_2 , otherwise we reject it and return c_k to its prior value. For a fixed value of S , our method executes N iterations for each value of a decreasing sequence of temperature values T , culminating with the effective selection $T = 0$, so where only improvements to R_2 are allowed. We repeat this for several values of S , which decay exponentially.

We implemented this procedure in C++, and used it to search for favorable polynomials with degree from $K = 10$ to $K = 72$. In each case we typically selected $H \in [100, 200]$, step values S decreasing gradually from 50 or 60 and slowly decreasing to approximately 2, using about twelve temperature values T , and selecting N near 8000. We found many polynomials with $R_2 < 48.18$, and our best polynomial has degree 46 and is recorded in Table 1. It produces

$$R_2 = 48.1587921551117,$$

which we employ in the statement of Theorem 1.2. Figure 1b shows a plot of this polynomial over the interval $[\pi/2, \pi]$.

For the result in Theorem 1.1, we employed two similar procedures. First, we amended the objective function to compute a value R_1 in the zero-free region for all $|t| \geq 3$. However, each iteration of this computation was much slower, so our computations here were limited. Indeed, we restricted our searches to degrees $K \leq 28$ in this case due to the greater computational complexities. Second, we used the objective function for the asymptotic constant, and reset our parameters to allow the step size to decrease more rapidly while greatly increasing the number of iterations N per round (taking N between 10^5 and 10^6), as well as the number of total number searches performed per degree selection. This allowed for a much larger search, and we tested degrees $K \leq 55$ with this strategy. Each search recorded polynomials optimized for the asymptotic constant R_2 , then these polynomials were tested to determine their R_1 value for Theorem 1.1. One degree 40 polynomial found with this procedure had an asymptotic constant of $R_2 \approx 48.162$, which is inferior to the polynomial displayed in Table 1, but it had the best value for R_1 . We used this polynomial as our initial state in a further annealing procedure that optimized for R_1 to determine an additional small improvement. Our final polynomial is listed in Table 2.

Table 1 $P_{46}(x) = \sum_{k=0}^{46} b_k \cos(kx) = |\sum_{k=0}^{46} c_k e^{ikx}|^2 / \sum_{k=0}^{46} c_k^2$, with $b = \sum_{k=1}^{46} b_k$

k	c_k	b_k	k	c_k	b_k
0	1	1	24	49282.888742825	0.000127104592072581
1	338.377844758599	1.74708744081848	25	-72469.9665724928	$1.74058423843506 \cdot 10^{-7}$
2	-219.537480547081	1.14338015090023	26	-80343.7855839228	$6.156980223188 \cdot 10^{-9}$
3	-736.781312848966	0.521864216745001	27	130557.454262211	$7.4923012998548 \cdot 10^{-5}$
4	914.902037465737	0.132187571762225	28	456655.665589724	$6.29610657045172 \cdot 10^{-5}$
5	1915.78694475716	$1.44250682908725 \cdot 10^{-7}$	29	686366.255781866	$4.51492091998615 \cdot 10^{-7}$
6	-1310.28600595906	$4.69075278525482 \cdot 10^{-9}$	30	690091.748824027	$1.76696516341167 \cdot 10^{-8}$
7	-3389.853917904	0.0141904926848435	31	504386.928024044	$3.57616762286565 \cdot 10^{-5}$
8	1732.46060916218	0.00859097729886965	32	256781.756010027	$2.9356535048273 \cdot 10^{-5}$
9	6943.01235038993	$5.05758761820625 \cdot 10^{-7}$	33	60405.4597040306	$2.6547976338407 \cdot 10^{-7}$
10	-278.171504957099	$4.42284301054098 \cdot 10^{-10}$	34	-37039.4291423529	$7.39578841754684 \cdot 10^{-7}$
11	-11594.9052445657	0.00262452919575262	35	-49829.9664619879	$1.5703528751761 \cdot 10^{-5}$
12	-4279.8222109347	0.0018969952017721	36	-22696.5925525196	$1.16349907747152 \cdot 10^{-5}$
13	14539.7736361703	$4.69472495111911 \cdot 10^{-10}$	37	1689.57285600626	$1.01423339047177 \cdot 10^{-7}$
14	11710.3298598379	$2.18058618368512 \cdot 10^{-7}$	38	9780.98700327532	$1.71248131672039 \cdot 10^{-6}$
15	-18824.0950949349	0.000818384876659817	39	10336.0633101459	$7.84636117271159 \cdot 10^{-6}$
16	-33323.9900467912	0.000639651965532567	40	9993.04428459519	$5.93829512034697 \cdot 10^{-6}$
17	-663.769351563045	$3.11262094946825 \cdot 10^{-8}$	41	9558.78229646887	$9.47232309558493 \cdot 10^{-7}$
18	34162.7992046244	$7.74994211145798 \cdot 10^{-7}$	42	7861.68784142526	$4.84440446543232 \cdot 10^{-8}$
19	7425.01374396162	0.000329183630974004	43	6657.72906076572	$9.72548049252508 \cdot 10^{-7}$
20	-56820.1949038606	0.000268358318561904	44	4736.89926522741	$8.45180184576162 \cdot 10^{-7}$
21	-60583.1989268389	$4.43747297378809 \cdot 10^{-7}$	45	2233.04706685592	$2.25111200007826 \cdot 10^{-7}$
22	27278.3371854473	$1.87358718910571 \cdot 10^{-7}$	46	504.683217557847	$6.56678999833493 \cdot 10^{-10}$
23	101206.908417904	0.000151428354073652			$b = 3.57440943022073$

This polynomial $P_{40}(x)$ produced the value $R_1 = 55.241$ used for Theorem 1.1. Figure 1a displays a plot of this polynomial over $[\pi/2, \pi]$.

It is likely that better polynomials exist in these optimization problems, but perhaps not much better, given the diminishing returns we witnessed in our searches. It seems likely at least that finding better polynomials would require substantially more computation if simulated annealing is employed. For example, while we performed searches using degrees as large as $K = 72$, the configuration space at the higher degrees is so large that it was difficult to explore extensively without more computations.

We remark that all numerical values in Tables 1 and 2 are rounded at fifteen significant digits.

9 The classical zero-free region

In 2005, Kadiri [18] established the value $R_0 = 5.69693$ in the classical zero-free region of the Riemann zeta-function by means of a clever iterative procedure. This method relied in part on a particular smoothing function $f(z)$ having certain properties: it was required that $f \in C^2[0, 1]$, with compact support, and having a Laplace transform $F(z)$ that is non-negative on the positive real axis. Kadiri noted that Heath-Brown’s work on Linnik’s theorem [13] developed four families of such functions, and that these were well-adapted for application to the problem of the classical zero-free region. We denote these four families of functions by $f_{\eta, \lambda, \theta}^{(i)}(t)$. Each is defined by

$$f_{\eta, \lambda, \theta}^{(i)}(t) = \eta h_{\lambda, \theta}^{(i)}(\eta t)$$

Table 2 $P_{40}(x) = \sum_{k=0}^{40} b_k \cos(kx) = |\sum_{k=0}^{40} c_k e^{ikx}|^2 / \sum_{k=0}^{40} c_k^2$, with $b = \sum_{k=1}^{40} b_k$

k	c_k	b_k	k	c_k	b_k
0	1	1	21	14616.1664568754	$4.66702819061453 \cdot 10^{-7}$
1	8.70590487645377	1.74600190914994	22	15112.6306248979	$8.88183754657211 \cdot 10^{-7}$
2	253.542513581082	1.14055431833244	23	3281.48150931095	$6.61799442215331 \cdot 10^{-5}$
3	538.912985014916	0.518966962914028	24	-9858.76392710328	$3.70153227317542 \cdot 10^{-5}$
4	1421.76588050758	0.130885859164882	25	-11913.1717506499	$6.2332255794641 \cdot 10^{-8}$
5	3062.1230018832	$8.86418531143308 \cdot 10^{-8}$	26	-2607.30174667086	$3.29243016002061 \cdot 10^{-5}$
6	5755.1498181548	$1.79787121328335 \cdot 10^{-6}$	27	6649.42849986177	$4.89938220699415 \cdot 10^{-5}$
7	9653.05616924715	0.0137716529944408	28	6689.88754688983	$1.50988491954013 \cdot 10^{-5}$
8	14967.239037407	0.00825900683475376	29	193.678093993709	$1.13051732969427 \cdot 10^{-7}$
9	21237.398416925	$4.91544374578637 \cdot 10^{-6}$	30	-3912.86637215382	$2.11823533257304 \cdot 10^{-5}$
10	27168.5781338032	$2.20263007866541 \cdot 10^{-6}$	31	-2318.83016640653	$2.13859401551174 \cdot 10^{-5}$
11	31408.8257398599	0.00243120523137902	32	911.79644433382	$1.55071932288034 \cdot 10^{-6}$
12	32409.0713030987	0.00172926530269636	33	1499.03441911128	$1.51812185041036 \cdot 10^{-6}$
13	28642.8233658012	$1.35500078722447 \cdot 10^{-6}$	34	159.800369623307	$1.67615806595912 \cdot 10^{-5}$
14	19217.7742754807	$2.20879127662495 \cdot 10^{-6}$	35	-551.30680615611	$1.60031224178442 \cdot 10^{-5}$
15	6084.93971979693	0.00069712400164774	36	-146.185445028008	$3.94634065729451 \cdot 10^{-6}$
16	-6971.7133423118	0.000530583559753362	37	160.626530894317	$4.08859029078879 \cdot 10^{-7}$
17	-16051.2777747034	$6.3973072524226 \cdot 10^{-7}$	38	9.7531801403406	$1.77819241241605 \cdot 10^{-6}$
18	-17900.8974008674	$5.37323136636712 \cdot 10^{-7}$	39	-46.7104974975636	$5.06885733758335 \cdot 10^{-8}$
19	-10944.9022767045	0.000234320877800568	40	23.9407317021713	$7.50406436813653 \cdot 10^{-9}$
20	2745.65474520683	0.000177364641910045			$b = 3.56453965437134$

for $1 \leq i \leq 4$, where $\lambda > 0, \theta$, and η are real parameters, and $h_{\lambda, \theta}^{(i)}(u)$ are certain functions. Kadiri employed the fourth of these, where

$$h_{\lambda, \theta}^{(4)}(u) = \lambda \sec^2 \theta \left\{ \lambda \sec^2 \theta \left(\frac{-\theta}{\lambda \tan \theta} - \frac{u}{2} \right) \cos(\lambda u \tan \theta) - \frac{2\theta}{\tan \theta} - \lambda u - \frac{\sin(2\theta + \lambda u \tan \theta)}{\sin 2\theta} + 2 \left(1 + \frac{\sin(\theta + \lambda u \tan \theta)}{\sin \theta} \right) \right\}, \tag{9.1}$$

and one requires $\pi/2 < \theta < \pi$. Kadiri set $\lambda = 1$ and selected $\theta = 1.848$ in her analysis.

In 2014, Jang and Kwon [16] applied all four families of functions $f_{\eta, \lambda, \theta}^{(i)}(u)$ inherited from Heath-Brown’s list to this problem, and optimized over λ and θ in each case. They found that some of Heath-Brown’s other functions performed slightly better than (9.1), and obtained a better value for R_0 . Most of their improvement was due to the use of a larger height T_0 for which RH had been verified: they used $T_0 = 3.06 \cdot 10^{10}$ while Kadiri employed the best value known at the time, approximately $3.3 \cdot 10^9$. However, investigation of other auxiliary functions allowed them to reduce their R_0 value further. They found that the four functions $h_{\lambda, \theta}^{(i)}(u)$ produced in turn 5.68372, 5.68483, 5.68484, and 5.68486 with the new T_0 value, so among these their best result arose from $h_{\lambda, \theta}^{(1)}$, which is defined by

$$h_{\lambda, \theta}^{(1)}(u) = \lambda \sec^2 \theta \left\{ \lambda \sec^2 \theta \left(\frac{\theta}{\lambda \tan \theta} - \frac{u}{2} \right) \cos(\lambda u \tan \theta) + \frac{2\theta}{\tan \theta} - \lambda u + \frac{\sin(2\theta - \lambda u \tan \theta)}{\sin 2\theta} - 2 \left(1 + \frac{\sin(\theta - \lambda u \tan \theta)}{\sin \theta} \right) \right\}, \tag{9.2}$$

and one requires $0 < \theta < \pi/2$. Jang and Kwon chose $\lambda = 1.03669$ and $\theta = 1.13537$, selecting these values in concert with their choice of a non-negative trigonometric polynomial, in order to optimize the constant with this function using Kadiri’s method. As in

[18], Jang and Kwon selected a favorable non-negative trigonometric polynomial of degree 4. Jang and Kwon also employed a function $h^{(5)}$ from Xylouris [36], which out-performed $h_{\lambda,\theta}^{(1)}$ just slightly, producing $R_0 = 5.68371$. This was the final value established in [16].

Independent of [16], in 2015 the first two authors [21] determined an improved value for the constant R_0 in the classical region by amending Kadiri's method in different ways, and showed that $R_0 = 5.573412$ is permissible. A small part of that improvement arose by employing the larger value for $T_0 = 3.06 \cdot 10^{10}$. Most of the gain resulted from two other changes: optimizing over a particular error term, and investigating admissible non-negative trigonometric polynomials of larger degree. A polynomial of degree 16 was constructed there by using simulated annealing with an appropriate objective function, and the constant $R_0 = 5.574312$ was computed using $h_{1,\theta}^{(4)}$ as in [18], with an appropriate value of θ .

We take the opportunity here to combine the ideas from [16] and [21], together with the recent work [29] establishing RH to the height $T_0 = 3 \cdot 10^{12}$, to record an improved value for the constant R_0 in the classical zero-free region of the zeta-function. We employ the admissible non-negative trigonometric polynomial of degree 16 from [21, Table 5], the auxiliary function $h_{\lambda,\theta}^{(1)}(u)$ from (9.2), and the new value for T_0 . We set $\lambda = 1$ since adjusting this value did not produce any further gains of significance, and choose $\theta = 1.13489$. We follow the method detailed in [21], with the following adjustments owing to the use of (9.2) rather than (9.1):

- Since θ is now restricted to $(0, \pi/2)$, we set $d_1(\theta) = 2\theta \cot \theta$.
- We now have $g_1(\theta) = h_{1,\theta}^{(1)}(0) = (\theta \tan \theta + 3\theta \cot \theta - 3) \sec^2 \theta$.
- We use the inequality $e^y \leq 1 + y + y^2/2 + y^3/3.47$, which is valid for $d_1(\theta) \leq 1.89355$, so for $0 \leq \theta \leq 1.13544$. In [21], the value 3.45 was used in place of 3.47.
- For the iteration, we use $r = 5$ for our lower bound on the constant we aim to achieve, as in [21], but now set the initial upper bound to $R_0 = 5.573412$.
- We verify that the error term $C(\eta)$ as defined in [18, §2.4] satisfies $C(\eta) \leq 0$ over the interval $[0, 1/(5 \log(3 \cdot 10^{12}))]$, and that the function $K(w)$ from the same source with $\theta = 1.13489$ is increasing on $[0, 1]$, as required by the method.

We refer the reader to [21] for full details on the method. Using this strategy, after seven iterations we compute the value $R_0 = 5.5586904517$ and establish Theorem 1.3. The successive values of R_0 determined after each iteration are displayed in Table 3, together with the values of other parameters that arise during the calculation—we include these to mirror the data shown in [21, Table 3].

We remark that with $h_{1,\theta}^{(4)}$ in place of $h_{1,\theta}^{(1)}$ we produced a slightly larger constant with the method, $R_0 = 5.5608403$, so swapping the smoothing function in this way allowed us to reduce the value of R_0 by approximately an additional $2.15 \cdot 10^{-3}$. This is about twice the marginal gain reported in [16] for the same swap of auxiliary functions. We did not investigate the function of Xylouris, as this had a very small marginal benefit in [16].

Finally, using this method, and again employing $h_{1,\theta}^{(1)}(u)$ with $\theta = 1.13489$ and using the same degree 16 trigonometric polynomial, we computed the value for R_0 that would be achieved by this method if RH were verified up to height T_0 in the future, for several values of $T_0 \leq 10^{15}$. These values are exhibited in Table 4.

Table 3 Values of parameters from [21] in successive iterations when $T_0 = 3 \cdot 10^{12}$, using $r = 5$ and $\theta = 1.13489$

R_0	$\eta_0 \cdot 10^3$	$\eta_1 \cdot 10^3$	κ	δ	r
5.5734120	6.259945	0.7565797	0.4410554	0.61994498	5.5603156
5.5603156	6.261572	0.7581420	0.4410415	0.61994923	5.5588702
5.5588702	6.261752	0.7583148	0.4410400	0.61994971	5.5587103
5.5587103	6.261772	0.7583339	0.4410399	0.61994976	5.5586927
5.5586927	6.261774	0.7583360	0.4410398	0.61994976	5.5586907
5.5586907	6.261775	0.7583362	0.4410398	0.61994976	5.5586905
5.5586905	6.261775	0.7583363	0.4410398	0.61994976	5.5586905

All values are rounded at the last recorded decimal place

Table 4 Allowable values for the constant R_0 in the classical zero-free region if RH were to be verified to height T_0

T_0	R_0
10^{13}	5.5559836
$3 \cdot 10^{13}$	5.5536904
10^{14}	5.5513505
$3 \cdot 10^{14}$	5.5493579
10^{15}	5.5473149

All values are rounded up at the last recorded decimal place

10 Future work

The approach taken in the proof of Theorem 1.1 may be summarized as follows. If a zero-free region $\sigma \geq 1 - \nu(t)$ can be established over a small finite region, say for $t \in [T_0 - 1, T_0)$, then under appropriate conditions the same zero-free region holds for all $t \geq T_0$. This suggests an inductive argument may be used—given a sequence of suitable functions $\nu_1(t)$, $\nu_2(t)$, ..., $\nu_N(t)$, we may use the zero-free region $\sigma \geq 1 - \nu_j(t)$ to show that there are no zeros in a small finite region, which then implies the next zero-free region $\sigma \geq 1 - \nu_{j+1}(t)$, and so on. If in addition the first zero-free region can be established unconditionally, this produces a method of iteratively constructing a zero-free region as a union of small zero-free regions.

One possible choice for $\nu_j(t)$ is given by

$$\nu_j(t) := \frac{1}{r_j(\log t)^{\phi_j}(\log \log t)^{1-\phi_j}}, \quad t \geq T_0^{(j)},$$

for some $\phi_j > 2/3$, which can be easily established by choosing

$$\eta = \frac{E}{B^{2/3}} \left(\frac{L_2}{L_1} \right)^{\phi_j}$$

in place of (4.11), then following the rest of the proof of Theorem 1.1. If r_j is small enough, then the resulting zero-free region will be sharper than Theorem 1.1 over some finite interval $t \in [T_1^{(j)}, T_2^{(j)}]$. By combining multiple such results, we create an envelope of zero-free regions whose union covers the interval $[3, T_0]$ for some large T_0 . We then use the same argument as Theorem 1.1 to cover the range $[T_0, \infty)$. In particular, this allows us to take T_0 much larger than is otherwise possible, which reduces the size of R_1 , the zero-free region constant.

To attain a non-trivial result via this method, we need to take r_j small enough that $\nu_j(T_0^{(j)}) > \nu_{j-1}(T_0^{(j)} - 1)$, so there is a small region

$$T_0 - 1 \leq t \leq T_0, \quad \nu_j(t) \leq \sigma \leq \nu_{j-1}(t), \tag{10.1}$$

in which zeros may exist, hence invalidating the inductive argument. Therefore, if we have tools to exclude the possibility of zeros in small, finite regions at known locations within the critical strip, immediate improvements to Theorem 1.1 are possible. By judiciously choosing ϕ_j , we find that using 355 such regions suffice to improve the constant of Theorem 1.1 to 52.74, provided that no zeros exist in regions of the form (10.1) for each j . Conventional arguments, such as raw computation or zero-density estimates, are currently insufficient to completely exclude zeros in these regions due to their large height, ranging from $t \approx \exp(40000)$ to $t \approx \exp(5 \cdot 10^7)$. However, if a new method was developed to exclude zeros in small finite regions, then immediate improvements to Theorem 1.1 are possible.

Acknowledgements

We are grateful to the referee for a thorough reading of our manuscript, and for supplying some very useful feedback.

Funding Open Access funding enabled and organized by CAUL and its Member Institutions

Author details

¹Center for Communications Research, Princeton, NJ, USA, ²School of Science The University of New South Wales Canberra, Canberra, Australia.

Received: 14 December 2022 Accepted: 11 December 2023 Published online: 12 January 2024

References

- Bastien, G., Rogalski, M.: Convexité, complète monotonie et inégalités sur les fonctions zêta et gamma, sur les fonctions des opérateurs de Baskakov et sur des fonctions arithmétiques. *Can. J. Math.* **54**(5), 916–944 (2002)
- Bennett, M.A., Martin, G., O’Byrant, K., Rechnitzer, A.: Explicit bounds for primes in arithmetic progressions. III. *J. Math.* **62**(1–4), 427–532 (2018)
- Broadbent, S., Kadiri, H., Lumley, A., Ng, N., Wilk, K.: Sharper bounds for the Chebyshev function $\theta(x)$. *Math. Comput.* **90**(331), 2281–2315 (2021)
- Cheng, Y.F., Graham, S.W.: Explicit estimates for the Riemann zeta function. *Rocky Mt. J. Math.* **34**(4), 1261–1280 (2004)
- Cully-Hugill, M.: Primes between consecutive powers. *J. Number Theory* **247**, 100–117 (2023)
- Cully-Hugill, M., Lee, E.S.: Explicit interval estimates for prime numbers. *Math. Comput.* **91**(336), 1955–1970 (2022)
- Dudek, A.W.: An explicit result for primes between cubes. *Funct. et Approx.* **55**(2), 177–197 (2016)
- Fiori, A., Kadiri, H., Swidinsky, J.: Sharper bounds for the error term in the prime number theorem. *Res. Number Theory* **9**(63), 19 (2023)
- Ford, K.: Vinogradov’s integral and bounds for the Riemann zeta function. *Proc. Lond. Math. Soc.* **85**(3), 565–633 (2002)
- Ford, K.: Zero-free regions for the Riemann zeta function. *Number Theory for the Millennium, II* (Urbana, IL, 2000), pp. 25–56. A K Peters, Natick, MA (2002)
- Ford, K.: Zero-free regions for the Riemann zeta function. [arXiv:1910.08205v4](https://arxiv.org/abs/1910.08205v4) [math.NT] (22 Sep 2022)
- Hasanalizade, E., Shen, Q., Wong, P.J.: Counting zeros of the Riemann zeta function. *J. Number Theory* **235**, 219–241 (2021)
- Heath-Brown, D.R.: Zero-free regions for Dirichlet L -functions, and the least prime in an arithmetic progression. *Proc. Lond. Math. Soc.* **64**(2), 265–338 (1992)
- Hiary, G.A.: An explicit van der Corput estimate for $\zeta(1/2 + it)$. *Indag. Math.* **27**(2), 524–533 (2016)
- Hiary, G.A., Patel, D., Yang, A.: An improved explicit estimate for $\zeta(1/2 + it)$. *J. Number Theory* **256**, 195–217 (2024)
- Jang, W.-J., Kwon, S.-H.: A note on Kadiri’s explicit zero free region for Riemann zeta function. *J. Korean Math. Soc.* **51**(6), 1291–1304 (2014)
- Johnston, D.R., Yang, A.: Some explicit estimates for the error term in the prime number theorem. *J. Math. Anal. Appl.* **527**(2), 127460
- Kadiri, H.: Une région explicite sans zéros pour la fonction ζ de Riemann. *Acta Arith.* **117**(4), 303–339 (2005)
- Khale, T.: An explicit Vinogradov–Korobov zero-free region for Dirichlet L -functions. [arXiv:2210.06457v1](https://arxiv.org/abs/2210.06457v1) [math.NT] (12 Oct 2022)
- Korobov, N.M.: Estimates of trigonometric sums and their applications. *Uspehi Mat. Nauk* **13**, 185–192 (1958)
- Mossinghoff, M.J., Trudgian, T.S.: Nonnegative trigonometric polynomials and a zero-free region for the Riemann zeta-function. *J. Number Theory* **157**(1–2), 329–349 (2015)
- Nielsen, P.P.: Trigonometric inequalities and the Riemann zeta-function. [arXiv:2210.14130v1](https://arxiv.org/abs/2210.14130v1) [math.NT] (25 Oct 2022)
- Patel, D.: Explicit sub-Weyl bound for the Riemann zeta function. Ph.D. Thesis, The Ohio State University (2021)
- Patel, D., Yang, A.: An explicit sub-Weyl bound for $\zeta(1/2 + it)$. [arXiv:2302.13444v1](https://arxiv.org/abs/2302.13444v1) [math.NT] (27 Feb 2023)
- Pintz, J.: On the remainder term of the prime number formula II. On a theorem of Ingham. *Acta Arith.* **37**, 209–220 (1980)
- Platt, D.J.: Computing $\pi(x)$ analytically. *Math. Comput.* **84**(293), 1521–1535 (2015)
- Platt, D.J., Trudgian, T.S.: An improved explicit bound on $|\zeta(\frac{1}{2} + it)|$. *J. Number Theory* **147**, 842–851 (2015)
- Platt, D.J., Trudgian, T.S.: The Riemann hypothesis is true up to $3 \cdot 10^{12}$. *Bull. Lond. Math. Soc.* **53**(3), 792–797 (2021)

29. Platt, D.J., Trudgian, T.S.: The error term in the prime number theorem. *Math. Comput.* **90**(328), 871–881 (2021)
30. Preobrazhenskii, S.N.: A new estimate in I.M. Vinogradov's mean value theorem. *Math. Notes* **2**, 285–299 (2011)
31. Richert, H.E.: Zur Abschätzung der Riemannschen Zetafunktion in der Nähe der Vertikalen $\sigma = 1$. *Math. Ann.* **169**(1), 97–101 (1967)
32. Rosser, B.: Explicit bounds for some functions of prime numbers. *Am. J. Math.* **63**(1), 211–232 (1941)
33. Titchmarsh, E.C.: *The Theory of the Riemann Zeta-Function*, 2nd edn. Oxford University Press, Oxford (1986)
34. van Laarhoven, P.J.M., Aarts, E.H.L.: *Simulated Annealing: Theory and Applications*. D. Reidel, Dordrecht (1987)
35. Vinogradov, I.M.: A new estimate for $\zeta(1 + it)$. *Izv. Akad. Nauk SSSR Ser. Mat.* **22**, 161–164 (1958)
36. Xylouris, T.: Über die Linniksche Konstante. Diplomarbeit, Universität Bonn (2009). [arXiv:0906.2749v1](https://arxiv.org/abs/0906.2749v1) [math.NT]
37. Xylouris, T.: On the least prime in an arithmetic progression and estimates for the zeros of Dirichlet L -functions. *Acta Arith.* **150**(1), 65–91 (2011)

Publisher's Note

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.