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Absolute convergence of general multiple dirichlet series

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Abstract

In this paper we study the absolute convergence of general multiple Dirichlet series defined by

$$\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\frac{a_1(m_1)a_2(m_2)\cdots a_r(m_r)}{m_1^{s_1}(m_1+m_2)^{s_2}\cdots(m_1+m_2+\cdots+m_r)^{s_r}},$$

where a_j $(1 \le j \le r)$ are arithmetic functions. In particular we completely determine the region of absolute convergence under certain conditions on the arithmetic functions.

Keywords: Multiple Dirichlet series, Absolute convergence

Mathematics Subject Classification: 11M32

1 Introduction

For any integer $r \ge 1$, the multiple zeta function (Euler–Riemann–Zagier type) of depth r is defined by

$$\zeta_r(s_1, s_2, \ldots, s_r) = \sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\frac{1}{m_1^{s_1}(m_1+m_2)^{s_2}\cdots(m_1+m_2+\cdots+m_r)^{s_r}}, \tag{1}$$

where s_i $(1 \le i \le r)$ are complex variables. Throughout the article, we denote $\Re(s_i) = \sigma_i$. It is well known (see Theorem 3, [4]) that the series (1) is absolutely convergent in the region $\{(s_1, s_2, \ldots, s_r) \in \mathbb{C}^r : \sigma_r + \sigma_{r-1} + \cdots + \sigma_{r-i} > i + 1 \text{ for } 0 \le i \le r - 1\}$. For $r = 1$, it is nothing but the Riemann zeta function. So multiple zeta function of depth r is multi-variable generalization of Riemann zeta function. Zhao [5] and Akiyama et al. [1] independently have shown that $\zeta_r(s_1, s_2, \ldots, s_r)$ can be extended meromorphically to the whole $\mathbb{C}^r$.

One can consider the generalization of the series defined in (1) in the following manner

$$\Phi_r((s_j); (a_j)) = \sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\frac{a_1(m_1)a_2(m_2)\cdots a_r(m_r)}{m_1^{s_1}(m_1+m_2)^{s_2}\cdots(m_1+m_2+\cdots+m_r)^{s_r}}, \tag{2}$$

where $a_j\,(1 \le j \le r)$ are arithmetic functions. For each j, if $a_j(m) = 1$, $\forall\, m \in \mathbb{N}$, then $\Phi_r((s_j);(a_j)) = \zeta_r(s_1, s_2, \ldots, s_r)$.
The first question that one can ask, is to find the region of absolute convergence of the multiple Dirichlet series defined in (2). In this context we have the following result.

Theorem 1 *For each $j\,(1 \le j \le r)$, let $\varphi_j(s) = \sum_{m=1}^{\infty} \frac{a_j(m)}{m^s}$ be absolutely convergent for $\Re(s) > \alpha_j > 0$. Then the multiple Dirichlet series defined in (2) is absolutely convergent in the region $U_r := \{(s_1, s_2, \ldots, s_r) \in \mathbb{C}^r : \sigma_r + \sigma_{r-1} + \cdots + \sigma_{r-i} > \alpha_r + \alpha_{r-1} + \cdots + \alpha_{r-i}$ for $0 \le i \le r-1\}$.*

The series defined in (2) is already considered by Matsumoto and Tanigawa in [3], where they have mentioned that this series is absolutely convergent in the trivial region $\{(s_1, s_2, \ldots, s_r) \in \mathbb{C}^r : \sigma_i > \alpha_i$ for $1 \le i \le r\}$. Since their primary goal was to study the meromorphic continuation, so there is no need to start with exact region of absolute convergence.

Remark 1 In Proposition 3.1 of [2], Matsumoto et al. have given a region of absolute convergence for certain double Dirichlet series that region is equal to U_r for $r = 2$ in the Theorem 1.

In the following theorem we give the necessary and sufficient conditions for the series (2) to converge absolutely under certain conditions on the arithmetic functions.

Theorem 2 *For each $j\,(1 \le j \le r)$, let the arithmetic function $a_j(m)$ and the positive real number α_j satisfy the following conditions • $\sum_{m=1}^{\infty} \frac{a_j(m)}{m^s}$ has abscissa of absolute convergence α_j, • $\sum_{m \le t} |a_j(m)| \gg t^{\alpha_j}$ for every $t \ge 1$. Then we have that the series defined in (2) is absolutely convergent at $(s_1, s_2, \ldots, s_r) \in \mathbb{C}^r$ if and only if $\sigma_r + \sigma_{r-1} + \cdots + \sigma_{r-i} > \alpha_r + \alpha_{r-1} + \cdots + \alpha_{r-i}$ for $0 \le i \le r-1$.*

As an application of Theorem 2, we have the following result.

Corollary 1 *The region of absolute convergence of the series defined in (1) is*

$$\{(s_1, s_2, \ldots, s_r) \in \mathbb{C}^r : \sigma_r + \sigma_{r-1} + \cdots + \sigma_{r-i} > i+1 \text{ for } 0 \le i \le r-1\}.$$

Remark 2 In Proposition 2.1 of [6], Zhao et al. have derived the necessary and sufficient conditions for absolute convergence of certain generalized multiple zeta functions. Corollary 1 is also followed from this proposition.

It is well known that if the Dirichlet series $\sum_{m=1}^{\infty} \frac{1}{m^s}$ converges at $s = s_0$, then this series converges for all s such that $\Re(s) > \Re(s_0)$. In the following theorem we prove an analogues result for general multiple Dirichlet series in case of absolute convergence.

Theorem 3 *For the arithmetic functions $a_j(m)\,(1 \le j \le r)$, let the series defined in (2) be absolutely convergent at $(s_1', s_2', \ldots, s_r') \in \mathbb{C}^r$. Then the series converges absolutely at each $(s_1, s_2, \ldots, s_r) \in \mathbb{C}^r$ such that $\sigma_r + \sigma_{r-1} + \cdots + \sigma_{r-i} \ge \sigma_r' + \sigma_{r-1}' + \cdots + \sigma_{r-i}'$ for $0 \le i \le r-1$, where $\Re(s_i') = \sigma_i'$ for $1 \le i \le r$.*

2 Preliminaries

In this section we give some lemmas which are necessary ingredients to prove Theorem 2.

Lemma 1 *For $n \geq 1$ and $\sigma \in \mathbb{R}$, we have* $\sum_{m=1}^{\infty} \frac{|a(m)|}{m^{\sigma}} < \infty$ *if and only if* $\sum_{m=1}^{\infty} \frac{|a(m)|}{(n+m)^{\sigma}} < \infty$.

Proof The lemma easily follows from the fact that

$$\frac{1}{(n+m)^{\sigma}} \ll \frac{1}{m^{\sigma}} \ll \frac{1}{(n+m)^{\sigma}} \text{ for each } m \in \mathbb{N} \text{ and for any fixed } n \geq 1.$$

□

Lemma 2 *For the positive real number α, let the arithmeitc function $a(m)$ satisfies* $\sum_{m \leq t} |a(m)| \gg t^{\alpha}$ *for every $t \geq 1$. Then* $\sum_{m=1}^{\infty} \frac{|a(m)|}{m^{\alpha}}$ *diverges.*

Proof Using Abel summation formula, we have

$$\sum_{m \leq t} \frac{|a(m)|}{m^{\alpha}} = \left(\sum_{m \leq t} |a(m)| \right) \frac{1}{t^{\alpha}} + \alpha \int_1^t \left(\sum_{m \leq x} |a(m)| \right) \frac{1}{x^{\alpha+1}} dx.$$

This implies that $\sum_{m \leq t} \frac{|a(m)|}{m^{\alpha}} \gg \alpha \log t$ and hence $\sum_{m=1}^{\infty} \frac{|a(m)|}{m^{\alpha}}$ diverges. □

Lemma 3 *Let the arithmetic function $a(m)$ and positive real number α satisfy the following conditions*

- $\sum_{m=1}^{\infty} \frac{a(m)}{m^s}$ *has abscissa of absolute convergence α,*
- $\sum_{m \leq t} |a(m)| \gg t^{\alpha}$ *for every $t \geq 1$. Then for $\sigma > \alpha$ and $n \geq 1$, we have* $\sum_{m=1}^{\infty} \frac{|a(m)|}{(n+m)^{\sigma}} \gg_{\sigma} \frac{1}{n^{\sigma-\alpha}}$.

Proof Using Abel summation formula, we have

$$\sum_{m \leq x} \frac{|a(m)|}{(n+m)^{\sigma}} = \left(\sum_{m \leq x} |a(m)| \right) \frac{1}{(n+x)^{\sigma}} + \sigma \int_1^x \left(\sum_{m \leq t} |a(m)| \right) \frac{1}{(n+t)^{\sigma+1}} dt. \quad (3)$$

Since $\sum_{m=1}^{\infty} \frac{a(m)}{m^s}$ has abscissa of absolute convergence α, it is well known that $\sum_{m \leq x} |a(m)| = o(x^{\alpha+\epsilon})$ for any $\epsilon > 0$. So the term $\left(\sum_{m \leq x} |a(m)|\right) \frac{1}{(n+x)^{\sigma}}$ tends to zero when x tends to ∞. Therefore by taking x tends to ∞ in Eq. (3), we get that

$$\sum_{m=1}^{\infty} \frac{|a(m)|}{(n+m)^{\sigma}} = \sigma \int_1^{\infty} \left(\sum_{m \leq t} |a(m)| \right) \frac{1}{(n+t)^{\sigma+1}} dt. \quad (4)$$

Now using the fact $\sum_{m \leq t} |a(m)| \gg t^{\alpha}$ for every $t \geq 1$ in Eq. (4), we have

$$\begin{aligned} \sum_{m=1}^{\infty} \frac{|a(m)|}{(n+m)^{\sigma}} &\gg \sigma \int_1^{\infty} \frac{t^{\alpha}}{(n+t)^{\sigma+1}} dt \\ &\gg \sigma \int_n^{\infty} \frac{t^{\alpha}}{(n+t)^{\sigma+1}} dt \gg_{\sigma} \frac{1}{n^{\sigma-\alpha}}. \end{aligned}$$

□

3 Proof of Theorem 1

Proof We will prove the theorem by induction on r.

The case $r = 1$ easily follows from the hypothesis.

Next we will prove the case $r = 2$. Let $(s_1, s_2) \in U_2$, then $\sigma_2 > \alpha_2$ and $\sigma_2 + \sigma_1 > \alpha_2 + \alpha_1$. Then we can choose $\epsilon_2 > 0$ such that

$$\sigma_2 > \alpha_2 + \epsilon_2 \text{ and } \sigma_2 + \sigma_1 > \alpha_2 + \alpha_1 + \epsilon_2. \tag{5}$$

Now using the hypothesis and (5), we get that

$$\begin{aligned}
&\sum_{m_1=1}^{\infty} \frac{|a_1(m_1)|}{m_1^{\sigma_1}} \sum_{m_2=1}^{\infty} \frac{|a_2(m_2)|}{(m_1+m_2)^{\sigma_2}} \\
&= \sum_{m_1=1}^{\infty} \frac{|a_1(m_1)|}{m_1^{\sigma_1+\sigma_2-(\alpha_2+\epsilon_2)}} \sum_{m_2=1}^{\infty} \frac{|a_2(m_2)|}{(m_1+m_2)^{\alpha_2+\epsilon_2}} \left(\frac{m_1}{m_1+m_2}\right)^{\sigma_2-(\alpha_2+\epsilon_2)} \\
&< \left(\sum_{m_1=1}^{\infty} \frac{|a_1(m_1)|}{m_1^{\sigma_1+\sigma_2-(\alpha_2+\epsilon_2)}}\right) \left(\sum_{m_2=1}^{\infty} \frac{|a_2(m_2)|}{m_2^{\alpha_2+\epsilon_2}}\right) < \infty.
\end{aligned}$$

Suppose the theorem is true for $r-1,\ r \geq 3$. Now we will prove for r. Let $(s_1, s_2, \ldots, s_r) \in U_r$, then

$$\sigma_r + \sigma_{r-1} + \cdots + \sigma_{r-i} > \alpha_r + \alpha_{r-1} + \cdots + \alpha_{r-i} \text{ for } 0 \leq i \leq r-1.$$

So there exists an $\epsilon_r > 0$ such that

$$\sigma_r + \sigma_{r-1} + \cdots + \sigma_{r-i} > \alpha_r + \alpha_{r-1} + \cdots + \alpha_{r-i} + \epsilon_r \text{ for } 0 \leq i \leq r-1. \tag{6}$$

Now consider

$$\begin{aligned}
&\sum_{m_1=1}^{\infty} \frac{|a_1(m_1)|}{m_1^{\sigma_1}} \cdots \sum_{m_{r-1}=1}^{\infty} \frac{|a_{r-1}(m_{r-1})|}{(m_1+\cdots+m_{r-1})^{\sigma_{r-1}}} \sum_{m_r=1}^{\infty} \frac{|a_r(m_r)|}{(m_1+\cdots+m_r)^{\sigma_r}} \\
&= \sum_{m_1=1}^{\infty} \frac{|a_1(m_1)|}{m_1^{\sigma_1}} \cdots \sum_{m_{r-2}=1}^{\infty} \frac{|a_{r-2}(m_{r-2})|}{(m_1+\cdots+m_{r-2})^{\sigma_{r-2}}} \\
&\quad \sum_{m_{r-1}=1}^{\infty} \frac{|a_{r-1}(m_{r-1})|}{(m_1+\cdots+m_{r-1})^{\sigma_{r-1}+\sigma_r-(\alpha_r+\epsilon_r)}} \\
&\quad \sum_{m_r=1}^{\infty} \frac{|a_r(m_r)|}{(m_1+\cdots+m_r)^{\alpha_r+\epsilon_r}} \left(\frac{m_1+\cdots+m_{r-1}}{m_1+\cdots+m_r}\right)^{\sigma_r-(\alpha_r+\epsilon_r)} \\
&< \left(\sum_{m_1=1}^{\infty} \frac{|a_1(m_1)|}{m_1^{\sigma_1}} \cdots \sum_{m_{r-1}=1}^{\infty} \frac{|a_{r-1}(m_{r-1})|}{(m_1+\cdots+m_{r-1})^{\sigma_{r-1}+\sigma_r-(\alpha_r+\epsilon_r)}}\right) \\
&\quad \left(\sum_{m_r=1}^{\infty} \frac{|a_r(m_r)|}{m_r^{\alpha_r+\epsilon_r}}\right) < \infty,
\end{aligned}$$

where the above inequality follows from induction hypothesis and (6). □

4 Proof of Theorem 2

Proof Let $\sigma_r+\sigma_{r-1}+\cdots+\sigma_{r-i} > \alpha_r+\alpha_{r-1}+\cdots+\alpha_{r-i}$ for $0 \le i \le r-1$, then it follows from hypothesis and Theorem 1 that the series defined in (2) is absolutely convergent at $(s_1, s_2, \ldots, s_r)$.

We will prove the converse part by induction on r.

The case $r = 1$ easily follows from Lemma 2.

Suppose the theorem is true for $r-1$, $r \ge 2$. Now we will prove for r. By hypothesis, we have

$$\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\frac{|a_1(m_1)||a_2(m_2)|\cdots|a_r(m_r)|}{m_1^{\sigma_1}(m_1+m_2)^{\sigma_2}\cdots(m_1+m_2+\cdots+m_r)^{\sigma_r}} < \infty. \tag{7}$$

The above inequality implies that

$$\sum_{m_r=1}^{\infty}\frac{|a_r(m_r)|}{(m_1+m_2+\cdots+m_r)^{\sigma_r}} < \infty \text{ for any } (m_1, m_2, \ldots, m_{r-1}) \in \mathbb{N}^{r-1}$$

and then by applying Lemmas 1 and 2, we have $\sigma_r > \alpha_r$. Now for any $(m_1, m_2, \ldots, m_{r-1}) \in \mathbb{N}^{r-1}$, we get from Lemma 3 that

$$\sum_{m_r=1}^{\infty}\frac{|a_r(m_r)|}{(m_1+m_2+\cdots+m_r)^{\sigma_r}} \gg_{\sigma_r} \frac{1}{(m_1+\cdots+m_{r-1})^{\sigma_r-\alpha_r}}. \tag{8}$$

Therefore from expressions (7) and (8), we have

$$\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\cdots\sum_{m_{r-1}=1}^{\infty}\frac{|a_1(m_1)||a_2(m_2)|\cdots|a_{r-1}(m_{r-1})|}{m_1^{\sigma_1}(m_1+m_2)^{\sigma_2}\cdots(m_1+\cdots+m_{r-2})^{\sigma_{r-2}}(m_1+\cdots+m_{r-1})^{\sigma_{r-1}+\sigma_r-\alpha_r}} < \infty.$$

Then from induction hypothesis, it follows that

$$\sigma_r+\sigma_{r-1}+\cdots+\sigma_{r-i} > \alpha_r+\alpha_{r-1}+\cdots+\alpha_{r-i} \text{ for } 1 \le i \le r-1.$$

□

5 Proof of Theorem 3

Proof It is enough to show that for $\sigma_r+\sigma_{r-1}+\cdots+\sigma_{r-i} \ge \sigma'_r+\sigma'_{r-1}+\cdots+\sigma'_{r-i}$, $0 \le i \le r-1$, we have

$$\begin{aligned}&\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\frac{|a_1(m_1)||a_2(m_2)|\cdots|a_r(m_r)|}{m_1^{\sigma_1}(m_1+m_2)^{\sigma_2}\cdots(m_1+m_2+\cdots+m_r)^{\sigma_r}}\\ &\le \sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\cdots\sum_{m_r=1}^{\infty}\frac{|a_1(m_1)||a_2(m_2)|\cdots|a_r(m_r)|}{m_1^{\sigma'_1}(m_1+m_2)^{\sigma'_2}\cdots(m_1+m_2+\cdots+m_r)^{\sigma'_r}}.\end{aligned} \tag{9}$$

Now consider

$$\begin{aligned}&\sum_{m_1=1}^{\infty}\frac{|a_1(m_1)|}{m_1^{\sigma_1}}\cdots\sum_{m_{r-1}=1}^{\infty}\frac{|a_{r-1}(m_{r-1})|}{(m_1+\cdots+m_{r-1})^{\sigma_{r-1}}}\sum_{m_r=1}^{\infty}\frac{|a_r(m_r)|}{(m_1+\cdots+m_r)^{\sigma_r}}\\ &=\sum_{m_1=1}^{\infty}\frac{|a_1(m_1)|}{m_1^{\sigma_1}}\cdots\sum_{m_{r-2}=1}^{\infty}\frac{|a_{r-2}(m_{r-2})|}{(m_1+\cdots+m_{r-2})^{\sigma_{r-2}}}\sum_{m_{r-1}=1}^{\infty}\frac{|a_{r-1}(m_{r-1})|}{(m_1+\cdots+m_{r-1})^{\sigma_r+\sigma_{r-1}-\sigma'_r}}\\ &\sum_{m_r=1}^{\infty}\frac{|a_r(m_r)|}{(m_1+\cdots+m_r)^{\sigma'_r}}\left(\frac{m_1+\cdots+m_{r-1}}{m_1+\cdots+m_r}\right)^{\sigma_r-\sigma'_r}.\end{aligned}$$

Here $\left(\frac{m_1+\cdots+m_{r-1}}{m_1+\cdots+m_r}\right)^{\sigma_r-\sigma_r'} \leq 1$ as $\sigma_r \geq \sigma_r'$, therefore we have

$$\sum_{m_1=1}^{\infty} \frac{|a_1(m_1)|}{m_1^{\sigma_1}} \cdots \sum_{m_{r-1}=1}^{\infty} \frac{|a_{r-1}(m_{r-1})|}{(m_1+\cdots+m_{r-1})^{\sigma_{r-1}}} \sum_{m_r=1}^{\infty} \frac{|a_r(m_r)|}{(m_1+\cdots+m_r)^{\sigma_r}}$$
$$\leq \sum_{m_1=1}^{\infty} \frac{|a_1(m_1)|}{m_1^{\sigma_1}} \cdots \sum_{m_{r-2}=1}^{\infty} \frac{|a_{r-2}(m_{r-2})|}{(m_1+\cdots+m_{r-2})^{\sigma_{r-2}}} \sum_{m_{r-1}=1}^{\infty} \frac{|a_{r-1}(m_{r-1})|}{(m_1+\cdots+m_{r-1})^{\sigma_r+\sigma_{r-1}-\sigma_r'}}$$
$$\sum_{m_r=1}^{\infty} \frac{|a_r(m_r)|}{(m_1+\cdots+m_r)^{\sigma_r'}}$$
$$= \sum_{m_1=1}^{\infty} \frac{|a_1(m_1)|}{m_1^{\sigma_1}} \cdots \sum_{m_{r-3}=1}^{\infty} \frac{|a_{r-3}(m_{r-3})|}{(m_1+\cdots+m_{r-3})^{\sigma_{r-3}}} \sum_{m_{r-2}=1}^{\infty} \frac{|a_{r-2}(m_{r-2})|}{(m_1+\cdots+m_{r-2})^{\sigma_r+\sigma_{r-1}+\sigma_{r-2}-\sigma_r'-\sigma_{r-1}'}}$$
$$\sum_{m_{r-1}=1}^{\infty} \frac{|a_{r-1}(m_{r-1})|}{(m_1+\cdots+m_{r-1})^{\sigma_{r-1}'}} \left(\frac{m_1+\cdots+m_{r-2}}{m_1+\cdots+m_{r-1}}\right)^{\sigma_r+\sigma_{r-1}-\sigma_r'-\sigma_{r-1}'} \sum_{m_r=1}^{\infty} \frac{|a_r(m_r)|}{(m_1+\cdots+m_r)^{\sigma_r'}}$$
$$\leq \sum_{m_1=1}^{\infty} \frac{|a_1(m_1)|}{m_1^{\sigma_1}} \cdots \sum_{m_{r-3}=1}^{\infty} \frac{|a_{r-3}(m_{r-3})|}{(m_1+\cdots+m_{r-3})^{\sigma_{r-3}}} \sum_{m_{r-2}=1}^{\infty} \frac{|a_{r-2}(m_{r-2})|}{(m_1+\cdots+m_{r-2})^{\sigma_r+\sigma_{r-1}+\sigma_{r-2}-\sigma_r'-\sigma_{r-1}'}}$$
$$\sum_{m_{r-1}=1}^{\infty} \frac{|a_{r-1}(m_{r-1})|}{(m_1+\cdots+m_{r-1})^{\sigma_{r-1}'}} \sum_{m_r=1}^{\infty} \frac{|a_r(m_r)|}{(m_1+\cdots+m_r)^{\sigma_r'}},$$

where the last inequality follows from the fact $\left(\frac{m_1+\cdots+m_{r-2}}{m_1+\cdots+m_{r-1}}\right)^{\sigma_r+\sigma_{r-1}-\sigma_r'-\sigma_{r-1}'} \leq 1$ as $\sigma_r + \sigma_{r-1} \geq \sigma_r' + \sigma_{r-1}'$.

Continuing in the same manner, we obtain (9). □

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References

1. Akiyama, S., Egami, S., Tanigawa, Y.: Analytic continuation of multiple zeta functions and their values at non-positive integers. Acta Arith. **98**(2), 107–116 (2001)
2. Matsumoto, K., Nawashiro, A., Tsumura, H.: Double Dirichlet series associated with arithmetic functions. Kodai Math. J. **44**(3), 437–456 (2021)
3. Matsumoto, K., Tanigawa, Y.: The analytic continuation and the order estimate of multiple Dirichlet series. J. Théor. Nombres Bordeaux **15**(1), 267–274 (2003)
4. Matsumoto, K.: On the Analytic Continuation of Various Multiple Zeta Functions, Number Theory for the Millennium, II (Urbana, IL, 2000), 417–440. A K Peters, Natick (2002)
5. Zhao, J.: Analytic continuation of multiple zeta functions. Proc. Am. Math. Soc. **128**(5), 1275–1283 (2000)
6. Zhao, J., Zhou, X.: Written multiple zeta values attached to $\mathfrak{sl}(4)$. Tokyo J. Math. **34**(1), 135–152 (2011)

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