

RESEARCH

# On binary quartics and the Cassels–Tate pairing



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## Abstract

We use the invariant theory of binary quartics to give a new formula for the Cassels–Tate pairing on the 2-Selmer group of an elliptic curve. Unlike earlier methods, our formula does not require us to solve any conics. An important role in our construction is played by a certain  $K3$  surface defined by a  $(2, 2, 2)$ -form.

## 1 Introduction

Let  $E$  be an elliptic curve over a number field  $K$ . The Mordell–Weil theorem tells us that the abelian group  $E(K)$  is finitely generated, but there is no known algorithm guaranteed to compute its rank. Instead, for each integer  $n \geq 2$  there is an exact sequence of abelian groups

$$0 \rightarrow E(K)/nE(K) \rightarrow S^{(n)}(E/K) \rightarrow \text{III}(E/K)[n] \rightarrow 0.$$

The  $n$ -Selmer group  $S^{(n)}(E/K)$  is finite and effectively computable. Computing  $S^{(n)}(E/K)$  gives an upper bound for the rank of  $E(K)$ , but this will be sharp only if the  $n$ -torsion of the Tate–Shafarevich group  $\text{III}(E/K)$  is trivial.

Cassels [4] showed that there is an alternating pairing

$$\langle \cdot, \cdot \rangle_{\text{CT}} : S^{(n)}(E/K) \times S^{(n)}(E/K) \rightarrow \mathbb{Q}/\mathbb{Z}$$

whose kernel is the image of  $S^{(n^2)}(E/K)$ . By computing this pairing, our upper bound for the rank of  $E(K)$  improves from that obtained by  $n$ -descent to that obtained by  $n^2$ -descent. In view of the generalisation to abelian varieties, due to Tate, the pairing is known as the Cassels–Tate pairing.

Cassels [6] also described a method for computing the pairing in the case  $n = 2$ . His method involves solving conics over the field of definition of each 2-torsion point on  $E$ . More recently, Donnelly [10] found a method that only involves solving conics over  $K$ , and implemented this in `Magma` [3]. In this article we use the invariant theory of binary quartics to give a self-contained account of a version of his method that is relatively simple to implement.

Since this article was first written, Jiali Yan has written her PhD thesis [18], extending some of these ideas to Jacobians of genus 2 curves, and Bill Allombert has implemented

our method for computing the pairing as part of the function `ellrank` in `pari/gp` [15]. I thank them both, and also Steve Donnelly and John Cremona, for useful discussions.

## 2 Binary quartics

A *binary quartic* over a field  $K$  is a homogeneous polynomial  $g \in K[x, z]$  of degree 4. Binary quartics  $g_1$  and  $g_2$  are  $K$ -equivalent if

$$g_2(x, z) = \lambda^2 g_1(\alpha x + \gamma z, \beta x + \delta z)$$

for some  $\lambda, \alpha, \beta, \gamma, \delta \in K$  with  $\lambda(\alpha\delta - \beta\gamma) \neq 0$ . They are *properly*  $K$ -equivalent if in addition  $\lambda(\alpha\delta - \beta\gamma) = \pm 1$ . The invariants of the binary quartic

$$g(x, z) = ax^4 + bx^3z + cx^2z^2 + dxz^3 + ez^4 \tag{1}$$

are

$$I = 12ae - 3bd + c^2, \\ J = 72ace - 27ad^2 - 27b^2e + 9bcd - 2c^3.$$

The binary quartics  $g_1$  and  $g_2$  have invariants related by  $I(g_2) = \lambda^4(\alpha\delta - \beta\gamma)^4 I(g_1)$  and  $J(g_2) = \lambda^6(\alpha\delta - \beta\gamma)^6 J(g_1)$ . In particular, properly equivalent binary quartics have the same invariants. The discriminant is  $\Delta = 16(4I^3 - J^2)/27$ . We say that  $g$  is  $K$ -soluble if there exist  $x, z \in K$ , not both zero, such that  $g(x, z)$  is a square in  $K$ . The reason for this terminology is that if  $\Delta(g) \neq 0$  then there is a smooth projective curve  $C$  of genus one with affine equation  $y^2 = g(x, 1)$ , and we are asking that  $C(K) \neq \emptyset$ . As shown by Weil [17], the Jacobian of  $C$  is the elliptic curve

$$E_{I,J} : y^2 = x^3 - 27Ix - 27J. \tag{2}$$

Now let  $K$  be a number field, and  $M_K$  its set of places. A binary quartic over  $K$  is *everywhere locally soluble* if it is  $K_v$ -soluble for all places  $v \in M_K$ . We note that every elliptic curve over  $K$  can be written in the form (2) for some  $I, J \in K$  with  $4I^3 - J^2 \neq 0$ .

**Lemma 2.1** *Let  $I, J \in K$  with  $4I^3 - J^2 \neq 0$ . Then*

$$S^{(2)}(E_{I,J}/K) = \left\{ \begin{array}{l} \text{everywhere locally soluble} \\ \text{binary quartics over } K \\ \text{with invariants } I \text{ and } J \end{array} \right\} / (\text{proper } K\text{-equivalence}).$$

*Proof* The case  $K = \mathbb{Q}$  is proved in [2], the only simplification in this case being that (since the only roots of unity in  $\mathbb{Q}$  are  $\pm 1$ ) equivalent quartics with the same invariants are always properly equivalent (even in the cases where  $I = 0$  or  $J = 0$ ). The general case is similar. □

Although Lemma 2.1 specifies  $S^{(2)}(E_{I,J}/K)$  as a set, the group law is not obvious. The following description is taken from [8], [9]. Let  $L$  be the étale algebra  $K[\varphi]$  where  $\varphi$  is a root of  $X^3 - 3IX + J = 0$ . Then the binary quartic (1) has *cubic invariant*

$$z(g) = \frac{4a\varphi + 3b^2 - 8ac}{3}.$$

By a change of coordinates (that is, replacing  $g$  by a properly equivalent quartic) we may assume that  $z(g)$  is a unit in  $L$ . The group law on  $S^{(2)}(E_{I,J}/K)$  is then given by multiplying the cubic invariants in  $L^\times / (L^\times)^2$ . The method for converting an element of  $L^\times / (L^\times)^2$  back to a binary quartic does, however, involve solving a conic over  $K$ .

### 3 Statement of results

In this section we state our new formula for the Cassels–Tate pairing on the 2-Selmer group of an elliptic curve. First we need some more invariant theory. The binary quartic (1) has Hessian

$$h(x, z) = (3b^2 - 8ac)x^4 + 4(bc - 6ad)x^3z + 2(2c^2 - 24ae - 3bd)x^2z^2 + 4(cd - 6be)xz^3 + (3d^2 - 8ce)z^4.$$

There are exactly three linear combinations of  $g(x, z)$  and  $h(x, z)$  that are singular (i.e. have repeated roots). Following [9] this prompts us to put

$$G(x, z) = \frac{1}{3}(4\varphi g(x, z) + h(x, z)), \tag{3}$$

$$H(x, z) = \frac{1}{12} \frac{\partial^2 G}{\partial x^2} + \frac{2}{9}(I - \varphi^2)z^2, \tag{4}$$

so that  $G(1, 0)G(x, z) = H(x, z)^2$ . We note that  $z(g) = G(1, 0) = H(1, 0)$ .

**Theorem 3.1** *Let  $I, J \in K$  with  $4I^3 - J^2 \neq 0$ . Let  $g_1, g_2, g_3$  be everywhere locally soluble binary quartics over  $K$  with invariants  $I$  and  $J$ . Let  $H_1(x, z)$  be the binary quadratic form (4), with coefficients in  $L = K[\varphi]$ , associated to  $g_1$ . Suppose that  $z(g_1)z(g_2)z(g_3) = m^2$  for some  $m \in L^\times$ , and write*

$$\frac{z(g_2)z(g_3)}{m}H_1(x, z) = \alpha_1(x, z) + \beta_1(x, z)\varphi + \gamma_1(x, z)\varphi^2 \tag{5}$$

where  $\alpha_1, \beta_1, \gamma_1 \in K[x, z]$ . For each  $v \in M_K$  we choose  $x_v, z_v \in K_v$  with  $g_1(x_v, z_v)$  a square in  $K_v$  and  $\gamma_1(x_v, z_v) \neq 0$ . If  $g_2(1, 0) \neq 0$  then the Cassels–Tate pairing on  $S^{(2)}(E_{I,J}/K)$  is given by

$$\langle [g_1], [g_2] \rangle_{CT} = \prod_{v \in M_K} (g_2(1, 0), \gamma_1(x_v, z_v))_v \tag{6}$$

where  $(, )_v : K_v^\times / (K_v^\times)^2 \times K_v^\times / (K_v^\times)^2 \rightarrow \mu_2$  is the Hilbert norm residue symbol.

**Remark 3.2** (i) If we wish to compute the pairing starting only with  $g_1$  and  $g_2$ , then we first change coordinates so that  $z(g_1)$  and  $z(g_2)$  are units in  $L$ , multiply these together, and then compute  $g_3$  by solving a conic over  $K$ . This conic is the same as the one that has to be solved in Donnelly’s method [10].

- (ii) We show in Remark 8.3 that the binary quadratic form  $\gamma_1$  is not identically zero. Therefore, by our assumption that  $g_1$  is everywhere locally soluble, it is always possible to choose  $x_v, z_v \in K_v$  with the stated properties.
- (iii) The assumption that  $g_2(1, 0) \neq 0$  is no limitation, since if  $g_2(1, 0) = 0$  then  $[g_2] = 0$  in the 2-Selmer group, which certainly implies the pairing is trivial.
- (iv) By definition the Cassels–Tate pairing takes values in  $\mathbb{Q}/\mathbb{Z}$ . In our formula it takes values in  $\mu_2$ . It should be understood that we have identified  $\mu_2 = \frac{1}{2}\mathbb{Z}/\mathbb{Z}$ .
- (v) Since  $\langle , \rangle_{CT}$  is alternating and bilinear and  $[g_1] + [g_2] + [g_3] = 0$  we have  $\langle [g_1], [g_2] \rangle_{CT} = \langle [g_1], [g_3] \rangle_{CT}$ . So we may equally write  $g_2(1, 0)$  or  $g_3(1, 0)$  in (6). Notice however that the binary quartics  $g_2$  and  $g_3$  do both contribute to the pairing via (5). Moreover we must use the exact formulae for  $z(g_1)$ ,  $z(g_2)$  and  $z(g_3)$ , these

being linear in  $\varphi$ . It is not enough just to know these quantities up to squares, since this would change the left hand side of (5).

- (vi) If  $E(K)[2] = 0$  then  $m$  is uniquely determined up to sign. By the product formula for the Hilbert norm residue symbol this makes no difference to (6). If  $E(K)[2] \neq 0$  then there are more choices for  $m$ , but it turns out (see the proof of Theorem 8.2) that we may use any one of these to compute the pairing.

*Remark 3.3* The product over all places in Theorem 3.1 is a finite product. Indeed, outside an easily determined finite set of places, we have

- (i)  $v$  is a finite prime, with residue field of size at least 11.
- (ii)  $g_1$  and  $\gamma_1$  have  $v$ -adically integral coefficients, with  $v \nmid \Delta(g_1)$  content( $\gamma_1$ ).
- (iii)  $g_2(1, 0)$  is a  $v$ -adic unit and  $v \nmid 2$ .

Under conditions (i) and (ii) we can pick our local point (by Hensel lifting a smooth point on the reduction that is not a root of  $\gamma_1$ ) such that  $\gamma_1(x_v, z_v)$  is a unit. It follows by (iii) that the local contribution at  $v$  is trivial.

*Example 3.4* Let  $E/\mathbb{Q}$  be the elliptic curve

$$y^2 + y = x^3 - x^2 - 929x - 10595$$

labelled 571a1 in [7]. A 2-descent shows that  $S^{(2)}(E/\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^2$ , and its non-zero elements are represented by

$$\begin{aligned} g_1(x, z) &= -11x^4 + 68x^3z - 52x^2z^2 - 164xz^3 - 64z^4, \\ g_2(x, z) &= -4x^4 - 60x^3z - 232x^2z^2 - 52xz^3 - 3z^4, \\ g_3(x, z) &= -31x^4 - 78x^3z + 32x^2z^2 + 102xz^3 - 53z^4. \end{aligned}$$

Each of these binary quartics has invariants  $I = 44608$  and  $J = 18842960$ , and discriminant  $\Delta = -2^{12} \cdot 571$ . By (5), with  $m = \frac{1}{9}(20\varphi^2 - 8656\varphi + 936032)$ , we get

$$\gamma_1(x, z) = \frac{4}{9}(5x^2 - 16xz - 12z^2).$$

For each odd prime  $p$  there is a smooth  $\mathbb{F}_p$ -point on the reduction of  $y^2 = g_1(x, 1)$  mod  $p$ , whose  $x$ -coordinate is not a root of  $5x^2 - 16x - 12 = 0$ . Indeed we checked this claim directly for  $p = 3, 5, 7, 11$  and 571, and for all other primes it follows by Hasse's bound. Therefore the odd primes make no contribution to (6).

To compute the contribution at  $p = 2$  we write  $g_1(x, 1) = x^4 + 4q(x)$  where

$$q(x) = -3x^4 + 17x^3 - 13x^2 - 41x - 16.$$

By Hensel's lemma the equation  $q(x) = 0$  has a root in  $\mathbb{Z}_2$  with  $x = 2^4 + O(2^5)$ . But then  $\gamma_1(x, 1) \equiv 5 \pmod{(\mathbb{Q}_2^\times)^2}$ , and since  $(5, -1)_2 = 1$  the contribution is again trivial. Finally, since  $g_1(15, 4) > 0$  and  $\gamma_1(15, 4) < 0$ , there is a contribution from the real place. This shows that the Cassels–Tate pairing on  $S^{(2)}(E/\mathbb{Q})$  is non-trivial, and hence  $\text{rank } E(\mathbb{Q}) = 0$ .

#### 4 The Cassels–Tate pairing

There are two standard definitions of the Cassels–Tate pairing (in the case of elliptic curves) called in [11], [16] the *homogeneous space definition* and the *Weil pairing definition*.

Both definitions appear in Cassels’ original paper [4], although the method in [6] (see also [12]) is a variant of the Weil pairing definition. In this section we review the homogeneous space definition, and highlight its connection with the Brauer–Manin obstruction.

Let  $K$  be a field with separable closure  $\bar{K}$ . We write  $H^i(K, -)$  for the Galois cohomology group  $H^i(\text{Gal}(\bar{K}/K), -)$ . Let  $C/K$  be a smooth projective curve. We define

$$\text{Br}(C) = \ker \left( H^2(K, \bar{K}(C)^\times) \rightarrow H^2(K, \text{Div } C) \right). \tag{7}$$

It is shown in the Appendix to [14] that this is equivalent to the usual definition  $\text{Br}(C) = H^2_{\text{ét}}(C, \mathbb{G}_m)$ . Identifying  $\text{Br}(K) = H^2(K, \bar{K}^\times)$ , there is a natural map

$$\text{Br}(K) \rightarrow \text{Br}(C). \tag{8}$$

We will need the following two facts, whose proofs we give below.

- (i) For  $P \in C(K)$  there is an evaluation map

$$\text{Br}(C) \rightarrow \text{Br}(K); A \mapsto A(P).$$

This is a group homomorphism, and a section to the map (8). Moreover the evaluation maps behave functorially with respect to all field extensions.

- (ii) Suppose  $C$  is a smooth curve of genus one, with Jacobian elliptic curve  $E$ . If  $H^3(K, \bar{K}^\times) = 0$  then there is an isomorphism

$$\Psi_C : \frac{H^1(K, E)}{\langle [C] \rangle} \xrightarrow{\sim} \frac{\text{Br}(C)}{\text{Br}(K)}. \tag{9}$$

Now let  $E$  be an elliptic curve over a number field  $K$ . Let  $C$  and  $D$  be principal homogeneous spaces under  $E$ . Since  $H^3(K, \bar{K}^\times) = 0$  for  $K$  a number field, we have  $\Psi_C([D]) = A \pmod{\text{Br}(K)}$  for some  $A \in \text{Br}(C)$ . Now suppose that  $C$  and  $D$  are everywhere locally soluble. For each place  $v \in M_K$  we pick a local point  $P_v \in C(K_v)$ . The Cassels–Tate pairing  $\text{III}(E/K) \times \text{III}(E/K) \rightarrow \mathbb{Q}/\mathbb{Z}$  is defined by

$$\langle [C], [D] \rangle_{\text{CT}} = \sum_{v \in M_K} \text{inv}_v(A(P_v)) \tag{10}$$

where  $\text{inv}_v : \text{Br}(K_v) \rightarrow \mathbb{Q}/\mathbb{Z}$  is the local invariant map. As this form of the definition makes clear, if  $\langle [C], [D] \rangle_{\text{CT}} \neq 0$  then the genus one curve  $C$  is a counter-example to the Hasse Principle explained by the Brauer–Manin obstruction.

We check that the pairing is well defined, i.e. it does not depend on the choices of  $A$  and of the  $P_v$ . By class field theory there is an exact sequence

$$0 \rightarrow \text{Br}(K) \rightarrow \bigoplus_{v \in M_K} \text{Br}(K_v) \xrightarrow{\sum \text{inv}_v} \mathbb{Q}/\mathbb{Z} \rightarrow 0.$$

It follows that if we change  $A$  by adding an element of  $\text{Br}(K)$  then the pairing (10) is unchanged. Next, since the class of  $D$  is trivial in  $H^1(K_v, E)$ , the analogue of (9) over  $K_v$  shows that the restriction of  $A$  to the Brauer group of  $C/K_v$  is constant, i.e. it belongs to the image of  $\text{Br}(K_v)$ . Therefore the pairing (10) does not depend on the choice of local points  $P_v$ .

We now prove the facts we quoted in (i) and (ii) above.

- (i) For  $P \in C(K)$  there is a short exact sequence of Galois modules

$$0 \rightarrow \mathcal{O}_P^\times \rightarrow \bar{K}(C)^\times \xrightarrow{\text{ord}_P} \mathbb{Z} \rightarrow 0$$

where  $\mathcal{O}_P$  is the local ring at  $P$ . Taking Galois cohomology gives an exact sequence

$$0 \longrightarrow H^2(K, \mathcal{O}_P^\times) \longrightarrow H^2(K, \overline{K}(C)^\times) \xrightarrow{\text{ord}_P} H^2(K, \mathbb{Z}).$$

It follows by (7) that each element of  $\text{Br}(C)$  can be represented by a cocycle taking values in  $\mathcal{O}_P^\times$ , and so can be evaluated at  $P$ .

(ii) There is an exact sequence of Galois modules

$$0 \rightarrow \overline{K}^\times \rightarrow \overline{K}(C)^\times \rightarrow \text{Div } C \rightarrow \text{Pic } C \rightarrow 0,$$

where  $\text{Div } C$  and  $\text{Pic } C$  are the divisor group and Picard group for  $C$  over  $\overline{K}$ . Splitting into short exact sequences, and taking Galois cohomology, gives the following exact sequences

$$\begin{array}{ccccccc} & & & & H^2(K, \overline{K}^\times) & & \\ & & & & \downarrow & & \\ & & & & H^2(K, \overline{K}(C)^\times) & & \\ & & & & \downarrow & & \\ H^1(K, \text{Div } C) & \longrightarrow & H^1(K, \text{Pic } C) & \longrightarrow & H^2(K, \overline{K}(C)^\times / \overline{K}^\times) & \longrightarrow & H^2(K, \text{Div } C) \\ & & & & \downarrow & & \\ & & & & H^3(K, \overline{K}^\times) & & \end{array}$$

By Shapiro’s lemma and the fact that  $H^1(K, \mathbb{Z}) = 0$  we have  $H^1(K, \text{Div } C) = 0$ . It follows by (7) and a diagram chase that there is an exact sequence

$$\text{Br}(K) \rightarrow \text{Br}(C) \rightarrow H^1(K, \text{Pic } C) \rightarrow H^3(K, \overline{K}^\times). \tag{11}$$

In fact, had we started from the definition  $\text{Br}(C) = H_{\text{ét}}^2(C, \mathbb{G}_m)$ , then (11) would follow from the Hochschild–Serre spectral sequence.

If  $C$  is a smooth curve of genus one with Jacobian  $E$ , then taking Galois cohomology of the exact sequence

$$0 \longrightarrow \text{Pic}^0 C \longrightarrow \text{Pic } C \xrightarrow{\text{deg}} \mathbb{Z} \longrightarrow 0$$

gives

$$\mathbb{Z} \xrightarrow{\delta} H^1(K, E) \longrightarrow H^1(K, \text{Pic } C) \longrightarrow 0 \tag{12}$$

with  $\delta(1) = [C]$ . If  $H^3(K, \overline{K}^\times) = 0$  then from (11) and (12) we obtain the isomorphism  $\Psi_C$ .

### 5 Cyclic extensions

The definition of the map  $\Psi_C$  in the last section simplifies when we evaluate it on classes split by a cyclic extension  $L/K$ . Let  $G = \text{Gal}(L/K)$  be generated by  $\sigma$  of order  $n$ . We recall that for  $A$  a  $G$ -module, the Tate cohomology groups are

$$\widehat{H}^0(G, A) = \frac{\ker(\Delta|_A)}{\text{im}(N|_A)} \quad \text{and} \quad \widehat{H}^1(G, A) = \frac{\ker(N|_A)}{\text{im}(\Delta|_A)}$$

where  $\Delta = 1 - \sigma$  and  $N = 1 + \sigma + \dots + \sigma^{n-1}$  satisfy  $\Delta N = N \Delta = 0$  in  $\mathbb{Z}[G]$ .

For  $b \in K^\times$  there is a cyclic  $K$ -algebra with basis  $1, v, \dots, v^{n-1}$  as an  $L$ -vector space, and multiplication determined by  $v^n = b$  and  $vx = \sigma(x)v$  for all  $x \in L$ . We write  $(L/K, b)$  for the class of this algebra in  $\text{Br}(K) = H^2(K, \overline{K}^\times)$ . Likewise if  $f \in K(C)^\times$  then  $(L/K, f)$  is an element of  $H^2(K, \overline{K}(C)^\times)$ .

**Lemma 5.1** *Suppose  $\Xi \in \text{Div}_L^0 C$  with  $N_{L/K}(\Xi) = \text{div}(f)$  for some  $f \in K(C)^\times$ . If  $\xi$  is the image of  $\Xi$  under*

$$\widehat{H}^1(G, \text{Pic}_L^0 C) \cong H^1(G, \text{Pic}_L^0 C) \xrightarrow{\text{inf}} H^1(K, E)$$

then  $\Psi_C(\xi) = (L/K, f)$ .

*Proof* We follow the construction of  $\Psi_C$  in Sect. 4. We start with the exact sequence of  $G$ -modules

$$0 \rightarrow L^\times \rightarrow L(C)^\times \rightarrow \text{Div}_L C \rightarrow \text{Pic}_L C \rightarrow 0.$$

Splitting into short exact sequences, and taking Galois cohomology, gives a diagram as before. The connecting map

$$\widehat{H}^1(G, \text{Pic}_L C) \rightarrow \widehat{H}^0(G, L(C)^\times / L^\times)$$

is now given by  $\Xi \mapsto f$ . Therefore  $\Psi_C(\xi)$  is the image of  $f$  under the map

$$\frac{K(C)^\times}{N_{L/K}(L(C)^\times)} = \widehat{H}^0(G, L(C)^\times) \cong H^2(G, L(C)^\times) \xrightarrow{\text{inf}} H^2(K, \overline{K}(C)^\times).$$

This is the cyclic algebra  $(L/K, f)$  as required. □

### 6 Pairs of binary quartics and (2, 2)-forms

Let  $C$  be a smooth curve of genus one. First suppose, as in Sect. 2, that  $C$  is defined by a binary quartic  $g$ . Then  $C \rightarrow \mathbb{P}^1$  is a double cover ramified over the 4 roots of  $g$ . We write  $H$  for the hyperplane section (i.e., fibre of the map  $C \rightarrow \mathbb{P}^1$ ), and  $\iota$  for the involution on  $C$  with  $Q + \iota(Q) \sim H$  for all  $Q \in C$ .

Next we suppose that  $C \subset \mathbb{P}^1 \times \mathbb{P}^1$  is defined by a (2, 2)-form, i.e., a polynomial  $f(x_1, z_1; x_2, z_2)$  that is homogeneous of degree 2 in each of the sets of variables  $x_1, z_1$  and  $x_2, z_2$ . Projecting  $C$  to either factor gives a double cover of  $\mathbb{P}^1$ . The corresponding binary quartics are obtained by writing  $f$  as a binary quadratic form in one of the sets of variables, and taking its discriminant. We write  $\text{pr}_1, \text{pr}_2 : C \rightarrow \mathbb{P}^1$  for the projection maps. Let  $H_1, H_2$  and  $\iota_1, \iota_2$  be the corresponding hyperplane sections and involutions.

**Lemma 6.1** *Let  $C = \{f(x_1, z_1; x_2, z_2) = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^1$  as above.*

- (i) *The composite  $\iota_1 \iota_2$  is translation by some  $P \in E = \text{Jac}(C)$ . Moreover the isomorphism  $\text{Pic}^0(C) \cong E$  sends  $[H_1 - H_2] \mapsto P$ .*
- (ii) *If  $H_i = \text{pr}_i^*(1 : 0)$  then*

$$\text{div}(f(x_1, z_1; 1, 0)/z_1^2) = H_2 + \iota_1^* H_2 - 2H_1.$$

*Proof* (i) If  $Q \in C$  then  $\iota_2 Q + \iota_1 \iota_2 Q \sim H_1$  and  $Q + \iota_2 Q \sim H_2$ . Subtracting one from the other gives  $[\iota_1 \iota_2 Q - Q] = [H_1 - H_2]$  as required.

- (ii) The specified rational function on  $C$  factors via  $\text{pr}_1$  and is therefore invariant under pull back by  $\iota_1$ . It has a zero at each point in the support of  $H_2$ , and a double pole at

each point in the support of  $H_1$ . Since there are no other poles, and the divisor has degree 0, it must therefore be as stated.  $\square$

*Remark 6.2* Lemma 6.1(i) is closely related to Poncelet’s Porism, as described in [13]. Our use of (2, 2)-forms was inspired by the treatment in [1].

We write  $\text{disc}_k(f)$  for the discriminant of  $f$  when it is viewed as a binary quadratic form in the  $k$ th set of variables.

**Lemma 6.3** *Let  $C = \{f(x_1, z_1; x_2, z_2) = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^1$  as above. Let  $a \in K^\times$  and let  $C_1, C_2$  be the following quadratic twists of  $C$ .*

$$\begin{aligned} C_1 : \quad & ay^2 = \text{disc}_2(f) \\ C_2 : \quad & ay^2 = \text{disc}_1(f) \end{aligned}$$

Then  $\Psi_{C_1}([C_2]) = (K(\sqrt{a})/K, f(x_1, z_1; 1, 0)/z_1^2)$ .

Proof. If  $a \in (K^\times)^2$  then  $C_1$  and  $C_2$  are isomorphic over  $K$  and so by (9) we have  $\Psi_{C_1}([C_2]) = 0$ . We may therefore suppose that  $a \notin (K^\times)^2$ . Let  $L = K(\sqrt{a})$  and  $G = \text{Gal}(L/K) = \{1, \sigma\}$ . We claim there is a divisor  $\Xi \in \text{Div}_L^0(C_1)$  such that

- (i)  $C_2$  is the twist of  $C_1$  by the class of  $\Xi$  in  $\widehat{H}^1(G, \text{Pic}_L^0(C_1))$ , and
- (ii)  $N_{L/K}(\Xi) = \text{div}(f(x_1, z_1; 1, 0)/z_1^2)$ .

Then by Lemma 5.1 we have  $\Psi_{C_1}([C_2] - [C_1]) = (L/K, f(x_1, z_1; 1, 0)/z_1^2)$ . Since  $\Psi_{C_1}$  is a group homomorphism and  $\Psi_{C_1}([C_1]) = 0$  this proves the lemma.

We construct  $\Xi$  as follows. We factor the projection map  $\text{pr}_i : C \rightarrow \mathbb{P}^1$  as

$$C \xrightarrow{\phi_i} C_i \xrightarrow{\xi_i} \mathbb{P}^1$$

where  $\phi_i$  is the quadratic twist map (an isomorphism defined over  $L$ ), and  $\xi_i = (x_i : z_i)$  is the natural double cover. Let  $D_i = \xi_i^*(1 : 0)$  and  $H_i = \phi_i^*D_i = \text{pr}_i^*(1 : 0)$ . We put  $\phi = \phi_1\phi_2^{-1}$  and  $\Xi = \phi_*D_2 - D_1$ . We now prove (i) and (ii).

- (i) Let  $\iota_1$  and  $\iota_2$  be the involutions on  $C$  defined before Lemma 6.1. Since  $\sigma(\phi_1) = \phi_1\iota_1$  and  $\sigma(\phi_2) = \phi_2\iota_2$  it follows that  $\sigma(\phi)\phi^{-1} = \phi_1\iota_1\iota_2\phi_1^{-1}$ . Identifying  $C$  and  $C_1$  via  $\phi_1$ , and hence  $\Xi$  with  $H_2 - H_1$ , it follows by Lemma 6.1(i) that  $\sigma(\phi)\phi^{-1}$  is translation by some  $P \in E = \text{Jac}(C_1)$ , and the isomorphism  $\text{Pic}^0(C_1) \cong E$  sends  $[\Xi] \mapsto -P$ . The minus sign does not matter since  $|G| = 2$ .
- (ii) By Lemma 6.1(ii) with  $H_1 = \phi_1^*D_1$  and  $H_2 = \phi_2^*D_2$  we have

$$\begin{aligned} \text{div}(f(x_1, z_1; 1, 0)/z_1^2) &= \phi_{1*}(\phi_2^*D_2 + \iota_1^*\phi_2^*D_2 - 2\phi_1^*D_1) \\ &= \phi_*D_2 + \sigma(\phi_*D_2) - 2D_1 \\ &= N_{L/K}(\Xi). \end{aligned} \quad \square$$

### 7 Triples of binary quartics and (2, 2, 2)-forms

Let  $E/K$  be an elliptic curve. An  $n$ -covering of  $E$  is a pair  $(C, \nu)$  where  $C$  is a smooth curve of genus one, and  $\nu : C \rightarrow E$  is a morphism, such that, for some choice of isomorphism



$\psi : C \rightarrow E$  defined over  $\overline{K}$ , there is a commutative diagram

$$\begin{array}{ccc} C & & \\ \psi \downarrow & \searrow v & \\ E & \xrightarrow{\times n} & E \end{array}$$

The  $n$ -coverings of  $E$  are parametrised by  $H^1(K, E[n])$ .

Suppose that  $C_1, C_2, C_3$  are 2-coverings of  $E$  that sum to zero in  $H^1(K, E[2])$ . We pick isomorphisms  $\psi_i : C_i \rightarrow E$  as above, and let  $\varepsilon_i = (\sigma \mapsto \sigma(\psi_i)\psi_i^{-1})$  be the corresponding cocycle in  $Z^1(\text{Gal}(\overline{K}/K), E[2])$ . Our hypothesis is that  $\varepsilon_1 + \varepsilon_2 + \varepsilon_3$  is a coboundary. However, by adjusting the choice of  $\psi_3$ , we may suppose that  $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$ . It may then be checked that the morphism

$$\begin{aligned} \mu : C_1 \times C_2 \times C_3 &\rightarrow E \\ (P_1, P_2, P_3) &\mapsto \psi_1(P_1) + \psi_2(P_2) + \psi_3(P_3) \end{aligned}$$

is defined over  $K$ .

*Remark 7.1* We are still free to replace  $\psi_3$  by  $P \mapsto \psi_3(P) + T$  for  $T \in E(K)[2]$ , and for this reason there are  $\#E(K)[2]$  choices for the map  $\mu$ .

Suppose further that  $C_1, C_2, C_3$  are defined by binary quartics  $g_1, g_2, g_3$  with the same invariants  $I$  and  $J$ . Let  $\pi : C_1 \times C_2 \times C_3 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$  be the map that projects to the  $x$ -coordinates. Then  $S = \pi(\mu^{-1}(0_E))$  is a surface in  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ . Geometrically it is the Kummer surface  $(E \times E)/\{\pm 1\}$ .

We write  $\text{disc}_k(F)$  for the discriminant of a  $(2, 2, 2)$  form  $F$  when it is viewed as a binary quadratic form in the  $k$ th set of variables.

**Proposition 7.2** *The surface  $S \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$  is defined by a  $(2, 2, 2)$ -form  $F$ . Moreover we may scale  $F$  so that it has coefficients in  $K$ , and for all permutations  $i, j, k$  of  $1, 2, 3$  we have  $\text{disc}_k(F) = g_i g_j$ .*

*Proof* We first consider the special case where  $C_1 = C_2 = C_3 = E$ . Suppose that  $P_1, P_2, P_3 \in E$  satisfy  $P_1 + P_2 + P_3 = 0_E$ . If we specify the  $x$ -coordinates of  $P_1$  and  $P_2$ , then in general this leaves two possibilities for the  $x$ -coordinate of  $P_3$ . The exceptional cases are when either  $P_1$  or  $P_2$  is a 2-torsion point.

Let  $\Delta_i = \psi_i^{-1}(E[2])$  be the set of ramification points for  $C_i \rightarrow \mathbb{P}^1$ . We identify  $\Delta_i$  with its image in  $\mathbb{P}^1$ , i.e., the set of roots of  $g_i$ . The observations in the last paragraph show that when we project onto the  $i$ th and  $j$ th factors,  $S \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  is a double cover ramified over  $\Delta_i \times \mathbb{P}^1$  and  $\mathbb{P}^1 \times \Delta_j$ . This shows that  $S$  is defined by a  $(2, 2, 2)$ -form  $F$ . Moreover  $\text{disc}_k(F) = \lambda_k g_i g_j$  for some  $\lambda_1, \lambda_2, \lambda_3 \in K^\times$ . We claim that (i)  $\lambda_3 \in (K^\times)^2$  and (ii)  $\lambda_1 = \lambda_2 = \lambda_3$ . It is then clear we may rescale  $F$  so that  $\lambda_1 = \lambda_2 = \lambda_3 = 1$ .

(i) Let  $C_i$  have equation  $y_i^2 = g_i(x_i, z_i)$ . We note that  $K(S) \subset K(C_1 \times C_2)$  is a quadratic extension of  $K(\mathbb{P}^1 \times \mathbb{P}^1)$  with Kummer generator

$$\frac{\text{disc}_3(F)}{z_1^4 z_2^4} = \frac{\lambda_3 g_1(x_1, z_1) g_2(x_2, z_2)}{z_1^4 z_2^4} = \lambda_3 \left( \frac{y_1 y_2}{z_1^2 z_2^2} \right)^2.$$

Since this is a square in  $K(C_1 \times C_2)$  it follows that  $\lambda_3 \in (K^\times)^2$ .

(ii) Since  $g_1, g_2, g_3$  have the same invariants  $I$  and  $J$ , we may reduce by the action of  $SL_2(\bar{K}) \times SL_2(\bar{K}) \times SL_2(\bar{K})$  to the case

$$g_1(x, z) = g_2(x, z) = g_3(x, z) = x^3z - \frac{1}{3}Ixz^3 - \frac{1}{27}Jz^4.$$

The result then follows by symmetry. □

**Corollary 7.3** *Let  $C_1, C_2, C_3$  and  $F$  be as above. If  $a = g_3(1, 0) \neq 0$  then*

$$\Psi_{C_1}([C_2]) = (K(\sqrt{a})/K, F(x_1, z_1; 1, 0; 1, 0)/z_1^2).$$

*Proof* We put  $f(x_1, z_1; x_2, z_2) = F(x_1, z_1; x_2, z_2; 1, 0)$ . By Proposition 7.2 we have  $\text{disc}_1(f) = ag_2(x_2, z_2)$  and  $\text{disc}_2(f) = ag_1(x_1, z_1)$ . The curves  $C_1$  and  $C_2$  are therefore isomorphic to those considered in Lemma 6.3. Applying Lemma 6.3 gives the result. □

### 8 Computing the (2, 2, 2)-forms

To complete the proof of Theorem 3.1 we must explain how to compute the (2, 2, 2)-form  $F$ . As before it is helpful to first consider the special case where  $C_1 = C_2 = C_3 = E$ .

Let  $E$  be the elliptic curve  $y^2 = x^3 + ax + b$ . We consider the maps

$$\begin{array}{ccc} E \times E \times E & \xrightarrow{\mu} & E \\ \downarrow \pi & & \\ \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 & & \end{array}$$

where  $\mu(P_1, P_2, P_3) = P_1 + P_2 + P_3$  and  $\pi$  is the map taking the  $x$ -coordinate of each point. An equation for  $S = \pi(\mu^{-1}(0_E))$  is computed as follows.

Let  $P_i = (x_i, y_i)$  for  $i = 1, 2, 3$  be points on  $E$  with  $P_1 + P_2 + P_3 = 0_E$ . These points lie on a line, say  $y = \lambda x + v$ . Then as polynomials in  $x$  we have

$$x^3 + ax + b - (\lambda x + v)^2 = (x - x_1)(x - x_2)(x - x_3).$$

Comparing the coefficients of the powers of  $x$  we obtain

$$\begin{aligned} \lambda^2 &= s_1, \\ 2\lambda v &= a - s_2, \\ v^2 &= b + s_3, \end{aligned}$$

where  $s_1, s_2, s_3$  are the elementary symmetric polynomials in  $x_1, x_2, x_3$ . Eliminating  $\lambda$  and  $v$  gives the equation

$$(a - s_2)^2 - 4s_1(b + s_3) = 0.$$

The required (2, 2, 2)-form  $F$  is obtained by homogenising this equation, i.e. we replace  $x_i$  by  $x_i/z_i$  and multiply through by  $z_1^2 z_2^2 z_3^2$ .

*Remark 8.1* We have  $F(x_1, 1; x_2, 1; x_3, 1) = W_0 x_3^2 - W_1 x_3 + W_2$  where

$$\begin{aligned} W_0 &= (x_1 - x_2)^2, \\ W_1 &= 2(x_1 x_2 + a)(x_1 + x_2) + 4b, \\ W_2 &= x_1^2 x_2^2 - 2ax_1 x_2 - 4b(x_1 + x_2) + a^2. \end{aligned}$$

These are the formulae used in [5, Chapter 17] to show that the height on an elliptic curve is a quadratic form.

We now turn to the general case. So let  $S \subset \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$  be as in Sect. 7. Let  $z(g_i)$  be the cubic invariant, and write  $H_1, H_2, H_3$  for the binary quadratic forms (4) over  $L = K[\varphi]$  associated to  $g_1, g_2, g_3$ .

**Theorem 8.2** *If  $z(g_1)z(g_2)z(g_3) = m^2$  for some  $m \in L^\times$ , and*

$$\frac{H_1H_2H_3}{m} = F_0 + F_1\varphi + F_2\varphi^2 \tag{13}$$

*where  $F_0, F_1, F_2$  are (2, 2, 2)-forms defined over  $K$ , then  $S$  has equation  $F_2 = 0$ .*

*Proof* Let  $P_i = (x_i : y_i : z_i) \in C_i$  for  $i = 1, 2, 3$ , with  $\mu(P_1, P_2, P_3) = 0_E$ . Let  $Q_i$  be the image of  $P_i$  under the covering map  $C_i \rightarrow E$ . By the formulae for the covering map coming from classical invariant theory (see for example [8, Proposition 4.2]), the  $x$ -coordinate of  $Q_i$  is

$$\xi_i = \frac{3h_i(x_i, z_i)}{4g_i(x_i, z_i)}. \tag{14}$$

We recall from Sect. 3 that

$$z(g_i) \frac{4\varphi g_i + h_i}{3} = H_i^2. \tag{15}$$

By (14), (15) and the equation  $y_i^2 = g_i(x_i, z_i)$  for  $C_i$  we have

$$\xi_i + 3\varphi = \frac{9H_i^2}{4z(g_i)y_i^2}$$

and hence

$$\prod_{i=1}^3 (\xi_i + 3\varphi) = \left( \frac{27H_1H_2H_3}{8my_1y_2y_3} \right)^2. \tag{16}$$

Since  $Q_1 + Q_2 + Q_3 = 0_E$  these points lie on a line, say  $y = \lambda x + \nu$  for some  $\lambda, \nu \in K$ . Then as a polynomial in  $x$  we have

$$x^3 - 27Ix - 27J - (\lambda x + \nu)^2 = (x - \xi_1)(x - \xi_2)(x - \xi_3).$$

Putting  $x = -3\varphi$  gives

$$\prod_{i=1}^3 (\xi_i + 3\varphi) = (\nu - 3\lambda\varphi)^2. \tag{17}$$

We first suppose  $E(K)[2] = 0$ . In this case  $L$  is a field, so comparing (16) and (17) we have

$$\frac{27H_1H_2H_3}{8my_1y_2y_3} = \pm(\nu - 3\lambda\varphi),$$

in  $L(S)$ . Taking the coefficient of  $\varphi^2$  we see that  $F_2$  vanishes on  $S$ . In general there are  $\#E(K)[2]$  choices for the square root, up to sign, and these correspond to the  $\#E(K)[2]$  choices in Remark 7.1.

It remains to check that  $F_2$  is not identically zero. For this we may work over an algebraically closed field. Then by a change of coordinates we may suppose that  $g_i$  and  $h_i$  are

linear combinations of  $x_i^4 + z_i^4$  and  $x_i^2 z_i^2$ . The singular quartics in this pencil are  $(x_i^2 - z_i^2)^2$ ,  $(x_i^2 + z_i^2)^2$  and  $(x_i z_i)^2$ . Since  $L \cong K \times K \times K$  we may identify  $H_i$  as a triple of binary quadratic forms. These are non-zero multiples of  $x_i^2 - z_i^2$ ,  $x_i^2 + z_i^2$  and  $x_i z_i$ , in this order if we made a suitable change of coordinates. (This last claim may be checked without any calculation if we use stereographic projection to identify the roots of the binary quadratic forms with the vertices of an octahedron, and then rotate the octahedron.) Therefore the space of  $(2, 2, 2)$ -forms spanned by  $F_0, F_1, F_2$  contains the forms

$$(x_1^2 - z_1^2)(x_2^2 - z_2^2)(x_3^2 - z_3^2), \quad (x_1^2 + z_1^2)(x_2^2 + z_2^2)(x_3^2 + z_3^2), \quad x_1 z_1 x_2 z_2 x_3 z_3.$$

Since these are linearly independent, it follows that  $F_2$  is non-zero.  $\square$

*Proof of Theorem 3.1* Let  $F = F_2$  be the equation for  $S$  in Theorem 8.2. We specialise the last two sets of variables in (13) to  $(1, 0)$ . Then comparing with (5) we have  $F(x, z; 1, 0; 1, 0) = \gamma_1(x, z)$ . By Corollary 7.3 we have

$$\Psi_{C_1}([C_2]) = (K(\sqrt{a})/K, \gamma_1(x, z)/z^2),$$

where  $a = g_3(1, 0)$ . Then by (10) we have

$$\langle [C_1], [C_2] \rangle_{CT} = \sum_{v \in M_K} \text{inv}_v(K_v(\sqrt{a})/K_v, \gamma_1(x_v, z_v)/z_v^2).$$

Subject to identifying  $\mu_2 = \frac{1}{2}\mathbb{Z}/\mathbb{Z}$ , the Hilbert norm residue symbol is given by

$$(a, b)_v = \text{inv}_v(K_v(\sqrt{a})/K_v, b).$$

This gives the formula in Theorem 3.1, except that we have  $g_3(1, 0)$  in place of  $g_2(1, 0)$ . As noted in Remark 3.2(v), this change does not matter.  $\square$

*Remark 8.3* To show that  $\gamma_1(x, z)$  is not identically zero we show more generally that  $F$  cannot be made to vanish by specialising two of the sets of variables. Indeed, by considering  $F$  as given in Remark 8.1, it suffices to show that the polynomials  $W_0, W_1, W_2$  never simultaneously vanish. This may be checked by setting  $x_1 = x_2 = x$  and computing that the resultant of  $W_1$  and  $W_2$  is  $2^8(4a^3 + 27b^2)^2$ . This last expression is non-zero, by definition of an elliptic curve.

**Data availability** Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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