M. Cully-Hugill, A. W. Dudek *Res. Number Theory* (2022) 8:61
https://doi.org/10.1007/s40993-022-00358-1

Research in Number Theory

RESEARCH

A conditional explicit result for the prime number theorem in short intervals

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Abstract

This paper gives an explicit bound for the prime number theorem in short intervals under the assumption of the Riemann hypothesis.

1 Introduction

The von Mangoldt function is defined as

$$\Lambda(n) = \begin{cases} \log p : n = p^m, p \text{ is prime}, m \in \mathbb{N} \\ 0 \quad\ : \text{otherwise}, \end{cases}$$

and we will consider the sum $\psi(x) = \sum_{n \le x} \Lambda(n)$. The prime number theorem (PNT) is the statement $\psi(x) \sim x$ as $x \to \infty$. For the PNT in short intervals, it is known that

$$\psi(x+h) - \psi(h) \sim h \tag{1}$$

provided that h grows suitably with respect to x. Heath-Brown [9] has shown that one can take $h = x^{\frac{7}{12}-\epsilon}$ provided that $\epsilon \to 0$ as $x \to \infty$. Assuming the Riemann hypothesis (RH), Selberg [14] showed that (1) is true for any $h = h(x)$ such that $h/(x^{1/2}\log x) \to \infty$ as $x \to \infty$. On the other hand, Maier [11] has shown that the statement is false for $h = (\log x)^{\lambda}$ for any $\lambda > 1$.

In this paper we prove the following explicit version of Selberg's result.

Theorem 1 *Assuming RH, for any h satisfying* $\sqrt{x}\log x \le h \le x^{\frac{3}{4}}$ *and all* $x \ge e^{10}$ *we have*

$$|\psi(x+h) - \psi(x) - h| < \frac{1}{\pi}\sqrt{x}\log x \log\left(\frac{h}{\sqrt{x}\log x}\right) + 2\sqrt{x}\log x. \tag{2}$$

Selberg's result follows from Theorem 1 for any $h = f(x)\sqrt{x}\log x$ with unbounded $f(x) = o(x)$, in that we would have

$$|\psi(x+h) - \psi(x) - h| \ll \sqrt{x}\log x \log(f(x)) = o(h).$$

For $h = c\sqrt{x}\log x$, Theorem 1 implies Cramér's [6] result on primes in the interval $(x, x+h)$ for all sufficiently large x and c. In an earlier paper [7], the author showed that $c = 1+\epsilon$ is suitable for any $\epsilon > 0$ and for all sufficiently large x. Carneiro, Milinovich and Soundararajan [4] have since shown that we can take $c = 22/55$ for all $x \ge 4$. The same methods used in [7] are applied to reach Theorem 1. As such, it could be possible to sharpen Theorem 1 using the techniques in [4].

The closest result to Theorem 1 is the following from Schoenfeld [13].

Theorem 2 *Assuming RH, for* $x \geq 73.2$ *we have*

$$|\psi(x) - x| < \frac{1}{8\pi}\sqrt{x}\log^2 x. \tag{3}$$

Schoenfeld's result confirms Selberg's theorem for the slightly stronger condition of $h/(\sqrt{x}\log^2 x) \to \infty$. One also has from the above

$$|\psi(x+h) - \psi(x) - h| < \frac{1}{4\pi}\sqrt{x+h}\log^2(x+h).$$

When x is sufficiently large, Theorem 1 improves the leading constant in this bound for any choice of $h \leq x^{0.735}$.

2 Proof of Theorem 1

2.1 A smooth explicit formula

The Riemann–von Mangoldt explicit formula relates $\psi(x)$ to the zeros of the Riemann zeta-function $\zeta(s)$ (e.g. see Ingham [10]). Tor all non-integer $x > 0$,

$$\psi(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \log 2\pi - \frac{1}{2}\log(1 - x^{-2}), \tag{4}$$

where the sum is over all non-trivial zeroes $\rho = \beta + i\gamma$ of $\zeta(s)$. We define the weighted sum

$$\psi_1(x) = \sum_{n \leq x}(x-n)\Lambda(n) = \int_2^x \psi(t)dt \tag{5}$$

and use the following explicit formula, proved in [7] (see also Thm. 28 of [10]).

Lemma 3 *For non-integer* $x > 0$ *we have*

$$\psi_1(x) = \frac{x^2}{2} - \sum_{\rho}\frac{x^{\rho+1}}{\rho(\rho+1)} - x\log(2\pi) + \epsilon(x) \tag{6}$$

where

$$1.545 < \epsilon(x) < 2.069.$$

The bound on $\epsilon(x)$ has been reduced from [7], as we can write

$$\begin{aligned}\epsilon(x) &= 2\log 2\pi - 2 + \sum_{\rho}\frac{2^{\rho+1}}{\rho(\rho+1)} - \frac{1}{2}\int_2^x \log(1-t^{-2})dt \\ &< 2\log 2\pi - 2 + 2^{\frac{3}{2}}(\gamma + 2 - \log 4\pi) + \log\frac{3\sqrt{3}}{4} < 2.069\end{aligned}$$

and

$$\epsilon(x) > 2\log 2\pi - 2 - 2^{\frac{3}{2}}(\gamma + 2 - \log 4\pi) > 1.545.$$

Using a linear combination of Eq. (5), we can examine the distribution of prime powers in the interval $(x, x+h)$. For $2 \leq \Delta < \sqrt{x}\log x \leq h \leq x$, let

$$w(n) = \begin{cases} (n-x+\Delta)/\Delta & : \quad x-\Delta \leq n \leq x \\ 1 & : \quad x \leq n \leq x+h \\ (x+h+\Delta-n)/\Delta & : \quad x+h \leq n \leq x+h+\Delta \\ 0 & : \quad \text{otherwise.} \end{cases}$$

This leads to the identity

$$\sum_n \Lambda(n)w(n) = \frac{1}{\Delta}(\psi_1(x+h+\Delta) - \psi_1(x+h) - \psi_1(x) + \psi_1(x-\Delta)),$$

which can be verified by expanding both sides. Notice that over $x \leq n \leq x+h$, the sum on the LHS is equal to $\psi(x+h) - \psi(x)$. We thus aim to estimate this expression by bounding the RHS of (7). Using Lemma 3 in the above equation gives the following.

Lemma 4 *Let* $2 \leq \Delta < h \leq x$ *with* $x \notin \mathbb{Z}$. *Then*

$$\sum_n \Lambda(n)w(n) = h + \Delta - \frac{1}{\Delta}\sum_\rho S(\rho) + \epsilon(\Delta)$$

where

$$S(\rho) = \frac{(x+h+\Delta)^{\rho+1} - (x+h)^{\rho+1} - x^{\rho+1} + (x-\Delta)^{\rho+1}}{\rho(\rho+1)}$$

and

$$|\epsilon(\Delta)| < \frac{21}{20\Delta}.$$

It remains to estimate the sum over zeros. We will split it into three sums,

$$\sum_\rho S(\rho) = \left(\sum_{|\gamma| \leq \alpha x/h} + \sum_{\alpha x/h < |\gamma| < \beta x/\Delta} + \sum_{|\gamma| \geq \beta x/\Delta} \right) S(\rho) \tag{7}$$

where $\alpha > 0$ and $\beta > 0$ are parameters we can later optimise over.

Lemma 5 *Let* $2 \leq \Delta < h \leq x$ *and assume RH. We have*

$$\left| \sum_{|\gamma| \geq \beta x/\Delta} S(\rho) \right| < \frac{4\Delta(x+h+\Delta)^{3/2}}{\pi \beta x} \log(\beta x/\Delta)$$

provided that $\beta x/\Delta \geq \gamma_1 = 14.13\ldots$, *the ordinate of the first zero of* $\zeta(s)$.

Proof On RH, one has

$$|S(\rho)| \leq \frac{4(x+h+\Delta)^{3/2}}{\gamma^2}.$$

The result follows from Lemma 1(ii) of Skewes [15], that for all $T \geq \gamma_1$,

$$\sum_{\gamma \geq T} \frac{1}{\gamma^2} < \frac{1}{2\pi}\frac{\log T}{T}.$$

□

The following lemmas require estimates on the zero-counting function $N(T)$, which counts the number of zeros of $\zeta(s)$ in the critical strip $0 < \beta < 1$ with $0 < \gamma \leq T$. Backlund [1] showed that $N(T) = P(T) + Q(T)$, where

$$P(T) := \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8}$$

and $Q(T) = O(\log T)$. Hasanalizade, Shen, and Wong [8, Cor. 1.2] have given the most recent explicit version of this, of

$$|Q(T)| \le R(T) = a_1 \log T + a_2 \log\log T + a_3 \tag{8}$$

with $a_1 = 0.1038$, $a_2 = 0.2573$, and $a_3 = 9.3675$, for all $T \ge e$.

Lemma 6 *Let $2 \le \Delta < h \le x$ and assume RH. We have*

$$\left| \sum_{|\gamma| \le \alpha x/h} S(\rho) \right| < \frac{\alpha x (h+\Delta)\Delta}{\pi h \sqrt{x-\Delta}} \log(\alpha x/h).$$

Proof We can write

$$S(\rho) = \int_{x+h}^{x+h+\Delta} \int_{u-h-\Delta}^{u} t^{\rho-1} dt du,$$

so, under RH, one has

$$|S(\rho)| < \frac{(h+\Delta)\Delta}{\sqrt{x-\Delta}}.$$

With (8), we can use

$$N(T) < \frac{T \log T}{2\pi},$$

from which the result immediately follows. □

For the middle sum of (7), we will use the following lemma. It follows directly from Lemma 3 of [2], in whose notation we use $\phi(\gamma) = \gamma^{-1}$, and takes constants A_0 and A_1 from Trudgian [16, Thm. 2.2] and A_2 from [2, Lem. 2].

Lemma 7 *For $2\pi \le T_1 \le T_2$ we have*

$$\sum_{T_1 < \gamma < T_2} \frac{1}{\gamma} = \frac{1}{4\pi} \log \frac{T_2}{T_1} \log \frac{T_2 T_1}{4\pi^2} + \frac{Q(T_2)}{T_2} - \frac{Q(T_1)}{T_1} + E(T_1), \tag{9}$$

where $|Q(T)| \le R(T)$, defined in (8), and

$$|E(T)| \le \frac{2A_1 \log T + 2A_0 + A_1 + A_2}{T^2}$$

with $A_0 = 2.067$, $A_1 = 0.059$, $A_2 = 1/150$.

Lemma 8 *Let $2 \le \Delta < h \le x$ and assume RH. For $\alpha x/h \ge 15$ we have*

$$\left| \sum_{\alpha x/h < |\gamma| < \beta x/\Delta} S(\rho) \right| < \Delta (x+h+\Delta)^{1/2} \left(\frac{1}{\pi} \log\left(\frac{\beta h}{\alpha \Delta}\right) \log\left(\frac{\alpha\beta x^2}{4\pi^2 h \Delta}\right) + 5.4 \right).$$

Proof We can write

$$S(\rho) = \frac{1}{\rho}\left(\int_{x+h}^{x+h+\Delta} t^{\rho} dt - \int_{x-\Delta}^{x} t^{\rho} dt \right),$$

and so bounding trivially gives

$$|S(\rho)| \le \frac{2(x+h+\Delta)^{1/2}\Delta}{|\gamma|}.$$

It follows that

$$\left| \sum_{\alpha x/h<|\gamma|<\beta x/\Delta} S(\rho) \right| \le 4(x+h+\Delta)^{1/2}\Delta \sum_{\alpha x/h<\gamma<\beta x/\Delta} \frac{1}{\gamma},$$

on which we apply Lemma 7, and bound the smaller order terms with the assumption of $T_1 \ge 15$ to obtain the result. Note that the bound on T_1 is to reduce the constant 5.4, but not restrict α too much. □

2.2 Bounding the PNT in intervals

From Lemma 4 we can write

$$\left|\psi(x+h)-\psi(x)-h\right| < \frac{1}{\Delta}\left|\sum_{\rho} S(\rho)\right| + \Delta + \frac{21}{20\Delta} + \sum_{x-\Delta<n\le x} w(n)\Lambda(n) + \sum_{x+h<n\le x+h+\Delta} w(n)\Lambda(n)$$

As the smooth weight has $|w(n)| \le 1$, the above bound is no greater than

$$\frac{1}{\Delta}\left|\sum_{\rho} S(\rho)\right| + \Delta + \frac{21}{20\Delta} + 2\sum_{\substack{x+h<p^k\le x+h+\Delta \\ k\ge 1}} \log p. \tag{10}$$

The largest term in this bound comes from the sum over ρ, in particular, the section estimated in Lemma 8. Larger Δ results in a smaller main-term constant, so we will set $\Delta = C\sqrt{x}\log x$ and later choose an optimal value of $C \in (0,1)$. The reason for not taking larger Δ is two-fold: to keep $\Delta < h$ and ensure the smaller terms in (10) are $O(\sqrt{x}\log x)$.

To bound the sum over prime powers we can use Montgomery and Vaughan's version of the Brun–Titchmarsh theorem for primes in intervals [12, Eq. 1.12]. Defining $\theta(x) = \sum_{p\le x}\log p$, Eq. (1.12) of [12] implies

$$\theta(x+h)-\theta(x) = \sum_{x<p\le x+h} \log p \le \frac{2h\log(x+h)}{\log h}.$$

The contribution from higher prime powers is relatively small, and can be bounded with explicit estimates on the difference between the Chebyshev functions $\psi(x)$ and $\theta(x)$. Costa Pereira [5, Thm. 2,4,5] gives lower bounds for different ranges of x. These can be combined into

$$\psi(x)-\theta(x) > 0.999x^{\frac{1}{2}} + \frac{2}{3}x^{\frac{1}{3}} \tag{11}$$

for all $x \ge 2187$. Broadbent *et al.* [3, Cor. 5.1] give

$$\psi(x)-\theta(x) < \alpha_1 x^{\frac{1}{2}} + \alpha_2 x^{\frac{1}{3}} \tag{12}$$

with $\alpha_1 = 1 + 1.93378\cdot 10^{-8}$ and $\alpha_2 = 2.69$ for all $x \ge e^{10}$. Thus, we have

$$\begin{aligned}\psi(x+h+\Delta)-\psi(x+h) &\le \theta(x+h+\Delta)-\theta(x+h)+E_1(x)\\ &\le \frac{2\Delta\log(x+h+\Delta)}{\log\Delta} + E_1(x)\end{aligned}$$

where $E_1(x) = \alpha_1(x+h+\Delta)^{\frac{1}{2}} + \alpha_2(x+h+\Delta)^{\frac{1}{3}} - 0.999(x+h)^{\frac{1}{2}} - \frac{2}{3}(x+h)^{\frac{1}{3}}$, and is bounded by $E_1(x) \le \beta_1 x^{\frac{1}{2}} + \beta_2 x^{\frac{1}{3}}$ with

$$\beta_1 = \sqrt{3}\alpha_1 - 0.999 \quad \text{and} \quad \beta_2 = 3^{\frac{1}{3}}\alpha_2 - \frac{2}{3}.$$

Here and hereafter, let $x_0 = e^{10}$. For $x \ge x_0$ we can bound the smaller order terms in (10),

$$\Delta + \frac{21}{20\Delta} + 2\sum_{\substack{x+h<p^k\le x+h+\Delta \\ k\ge 1}} \log p < K_1\sqrt{x}\log x$$

where, for $h \le x^t$ with $t < 1$,

$$K_1 = C + \frac{4C\log(x_0 + 2x_0^t)}{\log(C\sqrt{x_0}\log x_0)} + \frac{2\beta_1}{\log x_0} + \frac{2\beta_2}{x_0^{\frac{1}{6}}\log x_0} + \frac{21}{20Cx_0\log^2 x_0}.$$

This, along with Lemmas 5 and 6, allow us to bound

$$\left|\psi(x+h) - \psi(x) - h\right| < \frac{1}{\Delta}\left|\sum_{\alpha x/h<|\gamma|<\beta x/\Delta} S(\rho)\right| + E(x,h,\Delta) \tag{13}$$

where

$$E(x,h,\Delta) = K_1\sqrt{x} + \frac{\alpha x(h+\Delta)}{\pi h\sqrt{x-\Delta}}\log\left(\frac{\alpha x}{h}\right) + \frac{4(x+h+\Delta)^{3/2}}{\pi\beta x}\log\left(\frac{\beta x}{\Delta}\right).$$

For $\sqrt{x}\log x \le h \le x^t$ we have

$$E(x,h,\Delta) \le K_1\sqrt{x} + \frac{2\alpha x}{\pi\sqrt{x - C\sqrt{x}\log x}}\log\left(\frac{\alpha\sqrt{x}}{\log x}\right) + \frac{4(x+x^t+C\sqrt{x}\log x)^{3/2}}{\pi\beta x}\log\left(\frac{\beta\sqrt{x}}{C\log x}\right) \le K_2\sqrt{x}\log x,$$

where, for $x \ge x_0 \ge e^{\beta/C}$ and $0 < \alpha \le 5$, we can take

$$K_2 = \frac{K_1}{\log x_0} + \frac{\alpha}{\pi} + \frac{2(x_0 + x_0^t + C\sqrt{x_0}\log x_0)^{3/2}}{\pi\beta x_0^{3/2}}.$$

The first term in (13) can be estimated with Lemma 8, so that

$$\frac{1}{\Delta}\left|\sum_{\alpha x/h<|\gamma|<\beta x/\Delta} S(\rho)\right| < (x+h+\Delta)^{1/2}\left(\frac{1}{\pi}\log\left(\frac{\beta h}{\alpha\Delta}\right)\log\left(\frac{\alpha\beta x^2}{4\pi^2 h\Delta}\right) + 5.4\right)$$
$$< \frac{\sqrt{x}}{\pi}\log x\log\left(\frac{h}{\sqrt{x}\log x}\right) + K_3\sqrt{x}\log x,$$

in which, assuming $100e^{-10} \leq \frac{\alpha\beta}{4\pi^2 C} \leq 100$, we can take

$$\begin{aligned} K_3 = & \frac{1}{\pi} \log\left(\frac{\beta}{\alpha C}\right) \log\left(\frac{\alpha\beta x_0}{4\pi^2 C \log^2 x_0}\right) \frac{1}{\log x_0} \\ & + \frac{x_0^{t/2-1/2}}{\pi \log x_0} \log\left(\frac{\beta x_0^{t-1/2}}{\alpha C \log x_0}\right) \log\left(\frac{\alpha\beta x_0}{4\pi^2 C \log^2 x_0}\right) \\ & + \frac{\sqrt{C}}{\pi x_0^{1/4} \sqrt{\log x_0}} \log\left(\frac{\beta x_0^{t-1/2}}{\alpha C \log x_0}\right) \log\left(\frac{\alpha\beta x_0}{4\pi^2 C \log^2 x_0}\right) \\ & + \frac{5.4}{\log x_0} \left(1 + x_0^{t-1} + \frac{C \log x_0}{\sqrt{x_0}}\right)^{1/2}. \end{aligned}$$

Note that the assumption for α and β is to ensure certain terms are bounded for all $x \geq x_0$. Combining estimates, we have

$$\left|\psi(x+h) - \psi(x) - h\right| < \frac{\sqrt{x}}{\pi} \log x \log\left(\frac{h}{\sqrt{x}\log x}\right) + K_4 \sqrt{x} \log x, \tag{14}$$

where $K_4 = K_3 + K_2$. It remains to optimise over the parameters. Before deciding these values, recall that we have made the assumptions $\beta \leq 10C$,

$$\frac{15h}{x} \leq \alpha \leq 5, \quad \beta \geq \gamma_1 \frac{C \log x}{\sqrt{x}}, \quad C\alpha < \beta \leq \alpha, \quad \text{and} \quad \frac{100}{e^{10}} \leq \frac{\alpha\beta}{4\pi^2 C} \leq 100.$$

The restriction on α will be satisfied for all $\sqrt{x}\log x \leq h \leq x^{\frac{3}{4}}$ if we take $\alpha \geq 15x_0^{-\frac{1}{4}}$. Optimising over C, α, and β to minimise K_4, we find that choosing $C = 0.25$ and $\alpha = \beta = 1.35$ allows us to take $K_4 = 2$ for all $x \geq x_0$.

Data Availibility Statement Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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Received: 24 April 2022 Accepted: 18 July 2022 Published online: 12 August 2022

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