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RESEARCH

Some mean value results related to Hardy's function

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Abstract

Let $\zeta(s)$ and $Z(t)$ be the Riemann zeta function and Hardy's function respectively. We show asymptotic formulas for $\int_0^T Z(t)\zeta(1/2+it)dt$ and $\int_0^T Z^2(t)\zeta(1/2+it)dt$. Furthermore we derive an upper bound for $\int_0^T Z^3(t)\chi^\alpha(1/2+it)dt$ for $-1/2<\alpha<1/2$, where $\chi(s)$ is the function which appears in the functional equation of the Riemann zeta function: $\zeta(s)=\chi(s)\zeta(1-s)$.

Keywords: Hardy's function, Mean value theorems, Approximate functional equation, Exponential sum and integral

Mathematics Subject Classification: 11M06, 11N07

1 Introduction

Let $Z(t)$ be Hardy's function defined by

$$Z(t)=\zeta(1/2+it)\chi^{-1/2}(1/2+it),$$

where as usual $\zeta(s)$ is the Riemann zeta-function and $\chi(s)$ is the gamma factor appearing in the functional equation of $\zeta(s)$:

$$\zeta(s)=\chi(s)\zeta(1-s). \tag{1}$$

The explicit form of $\chi(s)$ is

$$\chi(s)=2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s) \tag{2}$$

and its asymptotic behavior is given by

$$\chi(\sigma+it)=\left(\frac{|t|}{2\pi}\right)^{1/2-\sigma-it}e^{i(t\pm\frac{\pi}{4})}\left(1+O\left(\frac{1}{|t|}\right)\right) \tag{3}$$

for $|t|\ge t_0>0$, where $t\pm\frac{\pi}{4}=t+\operatorname{sgn}(t)\frac{\pi}{4}$. See Ivić [10].

From (1), it follows that $Z(t)$ is a real-valued even function for real t and $|Z(t)|=|\zeta(1/2+it)|$. Therefore the zeros of $\zeta(s)$ on the critical line $\operatorname{Re}s=1/2$ coincide with the real zeros of $Z(t)$. Historically, Hardy proved the infinity of the number of zeros of $\zeta(s)$ on the critical line in 1914. A little later Hardy and Littlewood gave another proof by showing that $\int_0^T Z(t)dt \ll T^{7/8}$ and $\int_0^T |Z(t)|dt \gg T$. See Chandrasekharan [3, Chapter II, §4 and Notes on Chapter II] or Titchmarsh [23, 10.5].

Since $Z^2(t) = |\zeta(1/2+it)|^2$, $2k$-th power moment of $Z(t)$ is equivalent to $2k$-th power moment of $|\zeta(1/2+it)|$. Hardy and Littlewood first showed the asymptotic formula in the case $k=1$. In fact they showed that

$$\int_0^T |\zeta(1/2+it)|^2 dt \sim T\log T$$

([5,6]). In 1926, Ingham [9] derived

$$\int_0^T |\zeta(1/2+it)|^2 dt = T\log\frac{T}{2\pi} + (2\gamma_0 - 1)T + E(T) \tag{4}$$

with $E(T) \ll T^{1/2}\log T$, where γ_0 is Euler's constant. There are a lot of literatures on $E(T)$ since then. For instance, Atkinson [1] gave an explicit formula for $E(T)$, which becomes the fundamental tool of further researches on $E(T)$. See Ivić [10] for more details. For $k=2$, among other things, Ingham [9] showed that

$$\int_0^T |\zeta(1/2+it)|^4 dt = \frac{1}{2\pi^4} T\log^4 T + E_2(T) \tag{5}$$

with $E_2(T) = O(T\log^3 T)$ by applying the famous approximate functional equation of $\zeta^2(s)$ of Hardy and Littlewood [7]. Ingham's result was improved by Heath-Brown [8] to $E_2(T) = T\sum_{n=0}^4 c_n \log^n T + O(T^{7/8+\varepsilon})$. Motohashi [21] studied $E_2(T)$ by the use of spectral theory of automorphic forms. See also Ivić [11] or Titchmarsh [23, 7.20]. Other mean value theorems (of even power) were studied by Hall [4] in connection with the distribution of consecutive zeros of $Z(t)$.

As for odd power moments of $Z(t)$, Ivić [12] proved in 2004 that

$$\int_0^T Z(t)dt \ll T^{1/4+\varepsilon}.$$

It shows that $Z(t)$ changes sign quite often. Ivić's result was sharpened to $\int_0^T Z(t)dt \ll T^{1/4}$ by Jutila [17,18] and Korolev [20] independently. Moreover they showed the Omega result $\int_0^T Z(t)dt = \Omega_{\pm}(T^{1/4})$ which was conjectured by Ivić [12]. It means that $T^{1/4}$ is the true order of $\int_0^T Z(t)dt$. Since there are a large amount of cancellations, it is expected that the cubic power moment has an exponent less than 1. In fact, Ivić showed that

$$\int_T^{2T} Z^3(t)dt = 2\pi\sqrt{\frac{2}{3}} \sum_{(\frac{T}{2\pi})^{3/2} \le n \le (\frac{T}{\pi})^{3/2}} \frac{d_3(n)}{n^{1/6}} \cos\left(3\pi n^{2/3} + \frac{1}{8}\pi\right) + O(T^{3/4+\varepsilon})$$

and conjectured that

$$\int_0^T Z^3(t)dt \ll T^{3/4+\varepsilon} \tag{6}$$

([14, Chapter 11]). Here $d_3(n)$ denotes the number of triples (k_1, k_2, k_3) such that $n = k_1k_2k_3$, $k_j \in \mathbb{Z}$, $k_j > 0$. If we use (4), (5) and the Cauchy-Schwarz inequality we have

$\int_0^T Z^3(t)dt \ll T(\log T)^{5/2}$. The best upper bound at present is due to Bettin, Chandee and Radziwiłł [2] who showed the second ineqality of the following:

$$\left|\int_0^T Z^3(t)dt\right| \le \int_0^T |Z(t)|^3 dt \ll T(\log T)^{9/4}. \tag{7}$$

It should be noted that $T(\log T)^{9/4}$ is the correct order of $\int_0^T |Z(t)|^3 dt$.

In this paper we shall prove several mean values of the functions combined with $Z(t)$ and $\zeta(1/2+it)$.

Theorem 1 *For large $T > 0$, we have*

$$\int_0^T Z(t)\zeta\left(\frac{1}{2}+it\right)dt = \frac{2\sqrt{2\pi}}{3}e^{\frac{\pi i}{8}}\left(\frac{T}{2\pi}\right)^{3/4}\left(\frac{1}{2}\log\frac{T}{2\pi}+2\gamma_0-2\log 2-\frac{2}{3}\right) + O(T^{1/2}\log T).$$

We recall that γ_0 is Euler's constant which coincides with the 0-th coefficient of the Laurent expansion of $\zeta(s)$ at $s=1$.

Ivić's conjecture (6) would follow from the bound of exponential sum

$$\sum_{N\le n\le 2\sqrt{2}N} \frac{d_3(n)}{n^{1/6}}e^{3\pi i n^{2/3}} \ll N^{1/2+\varepsilon}, \tag{8}$$

or, as Ivić noted [15, (1.6)], from

$$\sum_{N\le n\le 2N} d_3(n)e^{3\pi i n^{2/3}} \ll N^{2/3+\varepsilon}. \tag{9}$$

It seems that (9) (or (8)) is out of reach of the present method of exponential sums. However if we replace $d_3(n)$ by $d(n)$ (the divisor function $d(n) = \sum_{n=d_1 d_2} 1$), we can prove the following theorem in the frame of Theorem 1.

Theorem 2 *Let A be a parameter such that $A \gg N^{-1/4}$. Then we have*

$$\begin{aligned}\sum_{N\le k\le 2\sqrt{2}N} \frac{d(k)}{k^{1/6}}e^{3\pi i (Ak)^{2/3}} &= \sqrt{3}A^{-4/3}\sum_{A^{4/3}N^{1/3}\le k\le \sqrt{2}A^{4/3}N^{1/3}} d(k)k^{1/2}e^{-\pi i (k/A)^2} \\ &\quad + O(A^{-1/3}N^{1/2+\varepsilon}) + O(A^{1/3}N^{1/6}\log N) + O(A^{-1/9}N^{2/9+\varepsilon}) \\ &\ll A^{2/3}N^{1/2}\log N.\end{aligned}$$

For another kind of mean value of $Z(t)$ and $\zeta(1/2+it)$ we have

Theorem 3 *For large $T > 0$ we have*

$$\int_0^T Z^2(t)\zeta(1/2+it)dt = T\left\{\frac{1}{2}\left(\log\frac{T}{2\pi}\right)^2 + a_1\log\frac{T}{2\pi}+a_2\right\} + O(T^{3/4}\log^2 T),$$

where $a_1 = 3\gamma_0 - 1$, $a_2 = 3\gamma_1 + 3\gamma_0^2 - 3\gamma_0 + 1$, γ_j being the coefficients of the Laurent expansion of $\zeta(s)$ at $s=1$.

We note that the integral of the left-hand side has an asymptotic form. It may be interesting to compare with Ivić's conjecture (6).

As for another mean value, we shall prove the following

Theorem 4 *Let α be a real fixed constant such that $-1/2 < \alpha < 1/2$. Then we have*

$$\int_T^{2T} Z^3(t)\chi^\alpha(1/2+it)dt \ll \begin{cases} T^{1-\frac{\alpha}{6}+\varepsilon} & \text{if } 0 \le \alpha < 1/2, \\ T^{1+\frac{\alpha}{6}+\varepsilon} & \text{if } -1/2 < \alpha \le 0. \end{cases}$$

The cubic moment of Hardy's function corresponds to $\alpha = 0$, but unfortunately this gives only $O(T^{1+\varepsilon})$.

2 Some Lemmas

Lemma 1 *Suppose that $f(x)$ and $\varphi(x)$ are real-valued functions on the interval $[a, b]$ which satisfy the conditions*

1) $f^{(4)}(x)$ and $\varphi''(x)$ are continuous,

2) there exist numbers $H, A, U, 0 < H, A < U, 0 < b-a \le U$, such that

$$A^{-1} \ll f''(x) \ll A^{-1}, \quad f^{(3)} \ll A^{-1}U^{-1}, \quad f^{(4)}(x) \ll A^{-1}U^{-2}$$
$$\varphi(x) \ll H, \quad \varphi'(x) \ll HU^{-1}, \quad \varphi''(x) \ll HU^{-2},$$

3) $f'(c) = 0$ for some c, $a \le c \le b$.

Then

$$\int_a^b \varphi(x)\exp(2\pi i f(x))dx = \frac{1+i}{\sqrt{2}}\frac{\varphi(c)\exp(2\pi i f(c))}{\sqrt{f''(c)}} + O(HAU^{-1}) + O\left(H\min(|f'(a)|^{-1}, \sqrt{A})\right) + O\left(H\min(|f'(b)|^{-1}, \sqrt{A})\right).$$

This is Lemma 2 of Karatsuba and Voronin [19, p.71].

Remark 1 Here we give an important remark. As is noted in Ivić and Zhai [16], the proof actually shows that if there is no c which satisfies the condition 3, the term containing c does not appear in the right-hand side. Moreover if $c = a$ or $c = b$, then the main term is to be halved.

Lemma 2 *For $\frac{1}{2} \le \sigma < 1$ fixed, $1 \ll x, y \ll t^k$, $s = \sigma + it$, $xy = (\frac{t}{2\pi})^k$, $t \ge t_0$ and $k \ge 1$ a fixed integer, we have*

$$\zeta^k(s) = \sum_{m=1}^\infty \rho\left(\frac{m}{x}\right) d_k(m)m^{-s} + \chi^k(s)\sum_{m=1}^\infty \rho\left(\frac{m}{y}\right) d_k(m)m^{s-1} + O(t^{k(1-\sigma)/3-1}) + O(t^{k(1/2-\sigma)-2}y^\sigma \log^{k-1} t).$$

Here $\chi(s)$ is the function defined by (2) and $\rho(u)(\ge 0)$ is a smooth function such that $\rho(u) + \rho(1/u) = 1$ for $u > 0$ and $\rho(u) = 0$ for $u \ge 2$.

This is Lemma 4 of [16]. See also [14, Theorem 4.16].

For the proof of Theorem 4 we need the following lemma.

Lemma 3 *Let α, β, γ be fixed real numbers such that $\alpha(\alpha-1)\beta\gamma \neq 0$ and write $e(x) = e^{2\pi ix}$. Let*

$$S = \sum_{h=H+1}^{2H} \sum_{n=N+1}^{2N} \left| \sum_{M<m\le 2M} e\left(X \frac{m^\alpha h^\beta n^\gamma}{M^\alpha H^\beta N^\gamma} \right) \right|^*,$$

where $*$ means that

$$\left| \sum_{N \le n \le N'} z_n \right|^* = \max_{N \le N_1 \le N_2 \le N'} \left| \sum_{n=N_1}^{N_2} z_n \right|.$$

Then we have

$$S \ll (HNM)^{1+\varepsilon} \left\{ \left(\frac{X}{HNM^2} \right)^{1/4} + \frac{1}{M^{1/2}} + \frac{1}{X} \right\}.$$

This is Theorem 3 of Robert and Sargos [22].

3 Proofs of Theorem 1 and 2

Proof of Theorem 1 We consider the integral

$$J = \int_T^{2T} Z(t)\zeta\left(\frac{1}{2}+it\right) dt. \tag{10}$$

By the definition of $Z(t)$ and applying Lemma 2 we have

$$\begin{aligned} Z(t)\zeta\left(\frac{1}{2}+it\right) &= \zeta^2\left(\frac{1}{2}+it\right)\chi^{-1/2}\left(\frac{1}{2}+it\right) \\ &= \Bigg(\sum_{k=1}^{\infty} \rho\left(\frac{k}{x}\right) \frac{d(k)}{k^{1/2+it}} + \chi^2\left(\frac{1}{2}+it\right) \sum_{k=1}^{\infty} \rho\left(\frac{k}{y}\right) \frac{d(k)}{k^{1/2-it}} \\ &\quad + O\left(t^{-2/3}\right) + O\left(t^{-2} y^{1/2} \log t\right) \Bigg) \chi^{-1/2}\left(\frac{1}{2}+it\right), \end{aligned}$$

where $xy = (t/2\pi)^2$. Substituting this expression to (10), we have

$$J = J_1 + J_2 + O(T^{1/3}), \tag{11}$$

where

$$J_1 = \sum_{k=1}^{\infty} \frac{d(k)}{k^{1/2}} \int_T^{2T} \rho\left(\frac{k}{x}\right) k^{-it} \chi^{-1/2}\left(\frac{1}{2}+it\right) dt \tag{12}$$

and

$$J_2 = \sum_{k=1}^{\infty} \frac{d(k)}{k^{1/2}} \int_T^{2T} \rho\left(\frac{k}{y}\right) k^{it} \chi^{3/2}\left(\frac{1}{2}+it\right) dt. \tag{13}$$

We take

$$x = 2\left(\frac{t}{2\pi}\right), \quad y = \frac{1}{2}\left(\frac{t}{2\pi}\right),$$

and put $K = \frac{T}{\pi}$. Then the ranges of k in the sums in (12) and (13) are, in fact, $k \le 4K$ and $k \le K$, respectively.

We first consider J_1. Using (3) we find that

$$k^{-it}\chi^{-1/2}\left(\frac{1}{2}+it\right) = e^{-\frac{\pi i}{8}} e^{\frac{i}{2}(t\log\frac{t}{2\pi}-t-t\log k^2)} + O(1/t),$$

hence we have

$$J_1 = e^{-\frac{\pi i}{8}} \sum_{k\le 4K} \frac{d(k)}{k^{1/2}} \int_T^{2T} \rho\left(\frac{k}{x}\right) e^{\frac{i}{2}(t\log\frac{t}{2\pi}-t-t\log k^2)} dt + O(T^{1/2}\log T).$$

We evaluate the above integral by applying Lemma 1 with $\varphi(t) = \rho\left(k\left(\frac{\pi}{t}\right)\right)$ and $f(t) = \frac{1}{4\pi}(t\log\frac{t}{2\pi} - t - t\log k^2)$. Note that $\varphi(t)$ satisfies the conditions of Lemma 1 with $H = 1, U = T$. Since $f'(t_0) = 0$ if and only if $t_0 = 2\pi k^2$, the main term of the integral appears for k such that

$$\left(\frac{T}{2\pi}\right)^{1/2} \le k \le \left(\frac{T}{\pi}\right)^{1/2}. \tag{14}$$

Thus we get

$$\begin{aligned}&\int_T^{2T} \rho\left(\frac{k}{x}\right) e^{\frac{i}{2}(t\log\frac{t}{2\pi}-t-t\log k^2)} dt \\ &\quad = M(k) + O\left(1 + \min\Big(\sqrt{T}, \frac{1}{|\log(\frac{T}{2\pi k^2})|}\Big) + \min\Big(\sqrt{T}, \frac{1}{|\log(\frac{T}{\pi k^2})|}\Big)\right),\end{aligned}$$

where

$$M(k) = e^{\frac{\pi i}{4}} \rho\left(\frac{1}{2k}\right) 2\sqrt{2}\pi k e^{-\pi i k^2} = 2\sqrt{2}\pi e^{\frac{\pi i}{4}} k(-1)^k$$

for k satisfying the condition (14) and 0 otherwise. This yields that

$$\begin{aligned}J_1 &= 2\sqrt{2}\pi e^{\frac{\pi i}{8}} \sideset{}{'}\sum_{(\frac{T}{2\pi})^{1/2}\le k\le(\frac{T}{\pi})^{1/2}} (-1)^k d(k) k^{1/2} \\ &\quad + \sum_{k\le 4K} \frac{d(k)}{k^{1/2}} O\left(1 + \min\Big(\sqrt{T}, \frac{1}{|\log(\frac{T}{2\pi k^2})|}\Big) + \min\Big(\sqrt{T}, \frac{1}{|\log(\frac{T}{\pi k^2})|}\Big)\right) \\ &\quad + O(T^{1/2}\log T) \\ &=: R_0 + R_1 + R_2 + R_3 + O(T^{1/2}\log T),\end{aligned}$$

where $\sum'$ means that the terms for $k = (T/2\pi)^{1/2}$ and $k = (T/\pi)^{1/2}$ are to be halved if they are integers. It is clear that $R_1 \ll T^{1/2}\log T$. To estimate R_2, we divide the sum into four parts:

$$\begin{aligned}\sum_{k\le 4K} &= \sum_{1\le k<\frac{1}{2}\left(\frac{T}{2\pi}\right)^{1/2}} + \sum_{\frac{1}{2}\left(\frac{T}{2\pi}\right)^{1/2}\le k<\left(\frac{T}{2\pi}\right)^{1/2}} + \sum_{\left(\frac{T}{2\pi}\right)^{1/2}\le k\le 2\left(\frac{T}{2\pi}\right)^{1/2}} + \sum_{2\left(\frac{T}{2\pi}\right)^{1/2}<k\le 4K} \\ &=: S_1 + S_2 + S_3 + S_4.\end{aligned}$$

For S_1 and S_4 we have $\min(\sqrt{T}, \frac{1}{|\log(\frac{T}{2\pi k^2})|}) \ll \frac{1}{\log 4}$, hence we get $S_1 \ll T^{1/4}\log T$ and $S_4 \ll T^{1/2}\log T$. For S_2, we write $k = [(\frac{T}{2\pi})^{1/2}] - j$ for k in this range and divide the sum over j as $S_2 = S_{2,1} + S_{2,2}$, where $S_{2,1}$ is the sum for $j = 0, 1, 2$ and $S_{2,2}$ is the sum for $3 \le j \le [(\frac{T}{2\pi})^{1/2}] - \frac{1}{2}(\frac{T}{2\pi})^{1/2}$. For $S_{2,1}$ we use $\min(\sqrt{T}, \frac{1}{|\log(\frac{T}{2\pi k^2})|}) \le \sqrt{T}$ and hence $S_{2,1} \ll T^{1/4+\varepsilon}$ since the sum is finite. As for $S_{2,2}$, from

$$\log \frac{\left(\frac{T}{2\pi}\right)^{1/2}}{k} = \left| \log \frac{\left[\left(\frac{T}{2\pi}\right)^{1/2}\right] - j}{\left(\frac{T}{2\pi}\right)^{1/2}} \right| \asymp \frac{j}{\left(\frac{T}{2\pi}\right)^{1/2}},$$

we have

$$S_{2,2} \ll T^{-1/4+\varepsilon} \sum_j \frac{\left(\frac{T}{2\pi}\right)^{1/2}}{j} \ll T^{1/4+\varepsilon}.$$

Thus we get $S_2 \ll T^{1/4+\varepsilon}$. It is the same for S_3. Combining these estimates we find that $R_2 \ll T^{1/2} \log T$. Similarly we have $R_3 \ll T^{1/2} \log T$. As a result, we get

$$J_1 = 2\sqrt{2}\pi e^{\frac{\pi i}{8}} \sum_{\left(\frac{T}{2\pi}\right)^{1/2} \le k \le \left(\frac{T}{\pi}\right)^{1/2}}{}' (-1)^k d(k) k^{1/2} + O(T^{1/2} \log T). \tag{15}$$

Next we consider J_2. Similarly to the case of J_1, we have by (3) that

$$J_2 = e^{\frac{3\pi i}{8}} \sum_{k \le K} \frac{d(k)}{k^{1/2}} \int_T^{2T} \rho\left(\frac{k}{y}\right) e^{-\frac{3}{2} i (t \log \frac{t}{2\pi} - t - t \log k^{2/3})} dt$$
$$+ O(T^{1/2} \log T). \tag{16}$$

We apply Lemma 1 to the above integral with $\varphi(t) = \rho(2k(2\pi/t))$ and $f(t) = -\frac{3}{4\pi}(t \log \frac{t}{2\pi} - t - t \log k^{2/3})$. In this case $f'(t_0) = 0$ if and only if $t_0 = 2\pi k^{2/3}$ and t_0 is contained in the interval $[T, 2T]$ if and only if

$$\left(\frac{T}{2\pi}\right)^{3/2} \le k \le \left(\frac{T}{\pi}\right)^{3/2}.$$

Since the range of the sum over k is $1 \le k \le K$, there are no such k, that is, the integral in (16) does not have a main term. Considering the error term by Lemma 1 we find that

$$J_2 \ll \sum_{k \le K} \frac{d(k)}{k^{1/2}} \left(1 + \min\left((\sqrt{T}, \frac{1}{|\log \frac{T}{2\pi k^{2/3}}|}\right) + \min\left(\sqrt{T}, \frac{1}{|\log \frac{T}{\pi k^{2/3}}|}\right)\right)$$
$$=: R_1' + R_2' + R_3'.$$

We have clearly $R_1' \ll T^{1/2} \log T$. For R_2' and R_3' we note that $|\log \frac{T}{k^{2/3}}| \gg 1$ since $k \le K$, which implies that $R_2', R_3' \ll T^{1/2} \log T$. Hence

$$J_2 \ll T^{1/2} \log T. \tag{17}$$

From (11), (15) and (17), we get

$$J = 2\sqrt{2}\pi e^{\frac{\pi i}{8}} \sum_{(\frac{T}{2\pi})^{1/2} \le k \le (\frac{T}{\pi})^{1/2}}{}' (-1)^k d(k) k^{1/2} + O(T^{1/2} \log T).$$

Now dividing the interval $[0, T]$ as $\cup_j [T/2^j, T/2^{j-1}]$ and summing the above evaluations we have

$$\int_0^T Z(t)\zeta\left(\frac{1}{2}+it\right)dt = 2\sqrt{2}\pi e^{\frac{\pi i}{8}} \sum_{k\le(\frac{T}{2\pi})^{1/2}} (-1)^k d(k) k^{1/2} + O(T^{1/2}\log T). \tag{18}$$

To evaluate the sum on the right-hand side of (18) we recall that

$$\sum_{k\le x}(-1)^k d(k) = \frac{x}{2}(\log x + 2\gamma_0 - 1 - 2\log 2) + O(x^{1/3+\varepsilon})$$

for $x \gg 1$ (see, e.g., Ivić [13]), so by partial summation we have

$$\sum_{k\le x}(-1)^k d(k)k^{1/2} = \frac{1}{3}x^{3/2}\left(\log x + 2\gamma_0 - 2\log 2 - \frac{2}{3}\right) + O(x^{5/6+\varepsilon}).$$

Substituting this form to (18) we finally get

$$\int_0^T Z(t)\zeta\left(\frac{1}{2}+it\right)dt = \frac{2\sqrt{2}\pi}{3}e^{\frac{\pi i}{8}}\left(\frac{T}{2\pi}\right)^{3/4}\left(\frac{1}{2}\log\frac{T}{2\pi} + 2\gamma_0 - 2\log 2 - \frac{2}{3}\right) + O(T^{1/2}\log T).$$

This proves the assertion of Theorem 1. □

Proof of Theorem 2 Let A be a parameter such that $T^{-1/2} \ll A \ll T^{3/2}$. We shall consider the integral

$$J_A = \int_T^{2T} Z(t)\zeta\left(\frac{1}{2}+it\right)A^{it}dt$$

by the same way as in the proof of Theorem 1. Applying Lemma 2 we get

$$J_A = J_{A,1} + J_{A,2} + O(T^{1/3}), \tag{19}$$

where

$$J_{A,1} = \int_T^{2T} \chi^{-1/2}\left(\frac{1}{2}+it\right)\sum_{k=1}^{\infty}\rho\left(\frac{k}{x}\right)\frac{d(k)}{k^{1/2+it}}A^{it}dt \tag{20}$$

and

$$J_{A,2} = \int_T^{2T} \chi^{3/2}\left(\frac{1}{2}+it\right)\sum_{k=1}^{\infty}\rho\left(\frac{k}{y}\right)\frac{d(k)}{k^{1/2-it}}A^{it}dt, \tag{21}$$

where $xy = (\frac{t}{2\pi})^2$. Hereafter we put $K_0 = \left(\frac{T}{\pi}\right)^{1/2}$. □

Now we shall evaluate $J_{A,1}$ and $J_{A,2}$ by taking two different choices of x and y, that is,
Case 1 : we take $x = 8A(\frac{t}{2\pi})^{1/2}$ and $y = \frac{1}{8A}(\frac{t}{2\pi})^{3/2}$,
Case 2 : we take $x = \frac{A}{4}(\frac{t}{2\pi})^{1/2}$ and $y = \frac{4}{A}(\frac{t}{2\pi})^{3/2}$.

3.1 Case 1

The ranges of the sums in $J_{A,1}$ and $J_{A,2}$ are at most $k \le 16AK_0$ and $k \le \frac{1}{4A}K_0^3$, respectively. By (3) and the trivial estimate for the error term we get

$$J_{A,1} = e^{-\frac{\pi i}{8}} \sum_{k \le 16AK_0} \frac{d(k)}{k^{1/2}} \int_T^{2T} \rho\left(\frac{k}{x}\right) e^{\frac{i}{2}(t\log\frac{t}{2\pi} - t - t\log(\frac{k}{A})^2)} dt + O\left(A^{1/2}T^{1/4+\varepsilon}\right). \tag{22}$$

We shall evaluate the integral by Lemma 1. Let $f(t) = \frac{1}{4\pi}(t\log\frac{t}{2\pi} - t - t\log(\frac{k}{A})^2)$. Then $f'(t_0) = 0$ if and only if $t_0 = 2\pi(\frac{k}{A})^2$ and $T \le t_0 \le 2T$ if and only if

$$A\left(\frac{T}{2\pi}\right)^{1/2} \le k \le A\left(\frac{T}{\pi}\right)^{1/2}. \tag{23}$$

We find that all k satisfing (23) are contained in the range $k \le 16AK_0$. Therefore the integral in (22) has a main term which is given by

$$M_A(k) = e^{\frac{\pi i}{4}} \rho\left(\frac{1}{8}\right) 2\sqrt{2}\pi \frac{k}{A} e^{-\pi i(k/A)^2}$$

for $A(\frac{T}{2\pi})^{1/2} \le k \le A(\frac{T}{\pi})^{1/2}$ and $M_A(k) = 0$ otherwise. We note that $\rho(1/8) = 1$ in the above formula. It follows from Lemma 1 and (22) that

$$\begin{aligned} J_{A,1} &= e^{-\frac{\pi i}{8}} \sum_{A\left(\frac{T}{2\pi}\right)^{1/2} \le k \le A\left(\frac{T}{\pi}\right)^{1/2}} \frac{d(k)}{k^{1/2}} M_A(k) \\ &\quad + \sum_{k \le 4AK_0} \frac{d(k)}{k^{1/2}} O\left(1 + \min\left(\sqrt{T}, \frac{1}{\left|\log\left(\frac{(\frac{T}{2\pi})^{\frac{1}{2}}}{k/A}\right)\right|}\right) + \min\left(\sqrt{T}, \frac{1}{\left|\log\left(\frac{(\frac{T}{\pi})^{\frac{1}{2}}}{k/A}\right)\right|}\right)\right) \\ &\quad + O(A^{1/2}T^{1/4+\varepsilon}). \end{aligned}$$

Similarly to the proof of Theorem 1, we see that the contributions from the O-terms are bounded by $O(A^{1/2}T^{1/4+\varepsilon} + A^{-1/2}T^{1/4+\varepsilon})$. Hence we get

$$J_{A,1} = e^{\frac{\pi i}{8}} \frac{2\sqrt{2}\pi}{A} \sum_{A\left(\frac{T}{2\pi}\right)^{\frac{1}{2}} \le k \le A\left(\frac{T}{\pi}\right)^{\frac{1}{2}}} d(k)k^{1/2} e^{-\pi i(k/A)^2} + O(A^{1/2}T^{1/4+\varepsilon}) + O(A^{-1/2}T^{1/4+\varepsilon}). \tag{24}$$

Next we consider $J_{A,2}$. Similarly to $J_{A,1}$ we have

$$J_{A,2} = e^{\frac{3\pi i}{8}} \sum_{k \le \frac{1}{4A}K_0^3} \frac{d(k)}{k^{1/2}} \int_T^{2T} \rho\left(\frac{k}{y}\right) e^{-\frac{3}{2}i(t\log\frac{t}{2\pi} - t - t\log(Ak)^{2/3})} dt + O(A^{-1/2}T^{3/4+\varepsilon}).$$

If we put $f(t) = -\frac{3}{4\pi}(t\log\frac{t}{2\pi} - t - t\log(Ak)^{2/3})$ this time, $f'(t_0) = 0$ if and only if $t_0 = 2\pi(Ak)^{2/3}$ and so $T \le t_0 \le 2T$ if and only if

$$\frac{1}{A}\left(\frac{T}{2\pi}\right)^{3/2} \le k \le \frac{1}{A}\left(\frac{T}{\pi}\right)^{3/2}. \tag{25}$$

Since k runs in the range $1 \le k \le \frac{1}{4A}K_0^3$, there is no main term in the integral of $J_{A,2}$. Hence by Lemma 1, we get similarly that

$$J_{A,2} \ll \sum_{k\le \frac{1}{4A}K_0^3} \frac{d(k)}{k^{1/2}} \Bigg(1+\min\Bigg(\sqrt{T}, \frac{1}{|\log\left(\frac{(T/2\pi)^{3/2}}{Ak}\right)|}\Bigg) + \min\left(\sqrt{T}, \frac{1}{|\log\left(\frac{(T/\pi)^{3/2}}{Ak}\right)|}\right)\Bigg) \ll A^{-1/2}T^{3/4+\varepsilon} + A^{1/2}T^{-1/4+\varepsilon}. \tag{26}$$

From (19), (24) and (26), we obtain

$$J_A = e^{\frac{\pi i}{8}} \frac{2\sqrt{2}\pi}{A} \sum_{A(\frac{T}{2\pi})^{\frac12} \le k \le A(\frac{T}{\pi})^{\frac12}} d(k)k^{1/2}e^{-\pi i(k/A)^2} + O(A^{1/2}T^{1/4+\varepsilon}) + O(A^{-1/2}T^{3/4+\varepsilon}) + O(T^{1/3}). \tag{27}$$

3.2 Case 2

In this choice of x and y, the sums in (20) and (21) are actually over $k \le \frac{1}{2}AK_0$ and $k \le \frac{8}{A}{K_0}^3$, respectively. Thus

$$J_{A,1} = e^{-\frac{\pi i}{8}} \sum_{k \le \frac{A}{2}K_0} \frac{d(k)}{k^{1/2}} \int_T^{2T} \rho\left(\frac{k}{x}\right) e^{\frac{i}{2}(t\log\frac{t}{2\pi} - t - t\log(\frac{k}{A})^2)} dt + O\left(A^{1/2}T^{1/4+\varepsilon}\right)$$

and

$$J_{A,2} = e^{\frac{3\pi i}{8}} \sum_{k\le \frac{8}{A}K_0^3} \frac{d(k)}{k^{1/2}} \int_T^{2T} \rho\left(\frac{k}{y}\right) e^{-\frac{3}{2}i(t\log\frac{t}{2\pi} - t - t\log(Ak)^{2/3})} dt + O(A^{-1/2}T^{3/4+\varepsilon}).$$

As for $J_{A,1}$, the integral has a main term if and only if k satisfies (23). Since k runs over $1 \le k \le \frac{A}{2}K_0$, there are no such k. The contribution from the error term of the integral is the same as in the previous case since the range of the sum has the same order, hence we get

$$J_{A,1} \ll A^{1/2}T^{1/4+\varepsilon} + A^{-1/2}T^{1/4+\varepsilon}. \tag{28}$$

On the other hand, the integral of $J_{A,2}$ has a main term if and only if k satisfies (25), and in fact all k are in the range $k \le \frac{8}{A}K_0^3$. Hence by Lemma 1, $J_{A,2}$ has the following form:

$$J_{A,2} = e^{\frac{3\pi i}{8}} \sum_{\frac{1}{A}\left(\frac{T}{2\pi}\right)^{3/2} \le k \le \frac{1}{A}\left(\frac{T}{\pi}\right)^{3/2}} \frac{d(k)}{k^{1/2}} \widetilde{M}_A(k) + \sum_{k\le\frac{8}{A}K_0^3} \frac{d(k)}{k^{1/2}} O\Bigg(1+\min\Bigg(\sqrt{T}, \frac{1}{|\log\left(\frac{(T/2\pi)^{3/2}}{Ak}\right)|}\Bigg)$$

$$+\min\left(\sqrt{T}, \frac{1}{|\log\left(\frac{(T/\pi)^{3/2}}{Ak}\right)|}\right)\Bigg)$$
$$+ O(A^{-1/2}T^{3/4+\varepsilon}),$$

where

$$\widetilde{M}_A(k) = e^{-\frac{\pi i}{4}}\rho\left(\frac{1}{4}\right)\frac{2\sqrt{2}\pi}{\sqrt{3}}(Ak)^{1/3}e^{3\pi i(Ak)^{2/3}}$$

for $\frac{1}{A}(\frac{T}{2\pi})^{3/2} \leq k \leq \frac{1}{A}(\frac{T}{\pi})^{3/2}$ and 0 otherwise. We see that the contribution from the O-term is the same as the previous case, therefore

$$J_{A,2} = e^{\frac{\pi i}{8}}\frac{2\sqrt{2}\pi}{\sqrt{3}}A^{1/3}\sum_{\frac{1}{A}\left(\frac{T}{2\pi}\right)^{3/2}\leq k\leq\frac{1}{A}\left(\frac{T}{\pi}\right)^{3/2}}\frac{d(k)}{k^{1/6}}e^{3\pi i(Ak)^{2/3}}$$
$$+ O(A^{-1/2}T^{3/4+\varepsilon}) + O(A^{1/2}T^{-1/4+\varepsilon}). \quad (29)$$

From (28) and (29) we obtain that

$$J_A = e^{\frac{\pi i}{8}}\frac{2\sqrt{2}\pi}{\sqrt{3}}A^{1/3}\sum_{\frac{1}{A}\left(\frac{T}{2\pi}\right)^{3/2}\leq k\leq\frac{1}{A}\left(\frac{T}{\pi}\right)^{3/2}}\frac{d(k)}{k^{1/6}}e^{3\pi i(Ak)^{2/3}}$$
$$+ O(A^{-1/2}T^{3/4+\varepsilon}) + O(A^{1/2}T^{1/4+\varepsilon}) + O(T^{1/3}). \quad (30)$$

Now we have two expressions of J_A: (27) and (30). Comparing these expressions we obtain

$$\sum_{\frac{1}{A}\left(\frac{T}{2\pi}\right)^{3/2}\leq k\leq\frac{1}{A}\left(\frac{T}{\pi}\right)^{3/2}}\frac{d(k)}{k^{1/6}}e^{3\pi i(Ak)^{2/3}}$$
$$= \sqrt{3}A^{-4/3}\sum_{A\left(\frac{T}{2\pi}\right)^{\frac{1}{2}}\leq k\leq A\left(\frac{T}{\pi}\right)^{\frac{1}{2}}} d(k)k^{1/2}e^{-\pi i(k/A)^2}$$
$$+ O(A^{-5/6}T^{3/4+\varepsilon}) + O(A^{1/6}T^{1/4+\varepsilon}) + O(A^{-1/3}T^{1/3+\varepsilon})$$
$$\ll A^{1/6}T^{3/4}\log T, \quad (31)$$

where the last inequality is obtained by the trivial estimate. In (31), we take $T = 2\pi(AN)^{2/3}$. Then (31) is transformed to

$$\begin{aligned}&\sum_{N\le k\le 2\sqrt{2}N}\frac{d(k)}{k^{1/6}}e^{3\pi i(Ak)^{2/3}}\\&=\sqrt{3}A^{-4/3}\sum_{A^{4/3}N^{1/3}\le k\le\sqrt{2}A^{4/3}N^{1/3}}d(k)k^{1/2}e^{-\pi i(k/A)^2}\\&+O(A^{-1/3}N^{1/2+\varepsilon})+O(A^{1/3}N^{1/6+\varepsilon})+O(A^{-1/9}N^{2/9+\varepsilon})\\&\ll A^{2/3}N^{1/2}\log N\end{aligned}$$

for $A\gg N^{-1/4}$. This proves the assertion of Theorem 2.

4 Proof of Theorem 3

Since the method is similar to Theorem 1, we shall only give an outline of proof. Let

$$I=\int_T^{2T}Z^2(t)\zeta\left(\frac{1}{2}+it\right)dt.$$

This time we have

$$\begin{aligned}Z^2(t)\zeta\left(\frac{1}{2}+it\right)&=\zeta^3\left(\frac{1}{2}+it\right)\chi^{-1}\left(\frac{1}{2}+it\right)\\&=\chi^{-1}\left(\frac{1}{2}+it\right)\sum_{k=1}^{\infty}\rho\left(\frac{k}{x}\right)\frac{d_3(k)}{k^{1/2+it}}+\chi^2\left(\frac{1}{2}+it\right)\sum_{k=1}^{\infty}\rho\left(\frac{k}{y}\right)\frac{d_3(k)}{k^{1/2-it}}\\&+O\left(t^{-1/2}\right)+O\left(t^{-2}y^{1/2}\log^2 t\right),\end{aligned}\tag{32}$$

where $xy=(\frac{t}{2\pi})^3$.

We take $x=2(\frac{t}{2\pi})^{3/2}$ and $y=\frac{1}{2}(\frac{t}{2\pi})^{3/2}$ in (32) and put $K_3=(T/\pi)^{3/2}$. Then the ranges of k in the above two sums are at most $k\le 4K_3$ and $k\le K_3$, respectively. Hence

$$\begin{aligned}I&=\sum_{k\le 4K_3}\frac{d_3(k)}{k^{1/2}}\int_T^{2T}\rho\left(\frac{k}{x}\right)k^{-it}\chi^{-1}\left(\frac{1}{2}+it\right)dt\\&+\sum_{k\le K_3}\frac{d_3(k)}{k^{1/2}}\int_T^{2T}\rho\left(\frac{k}{y}\right)k^{it}\chi^2\left(\frac{1}{2}+it\right)dt+O(T^{1/2})\\&=:I_1+I_2+O(T^{1/2}).\end{aligned}\tag{33}$$

As for I_2, using (3), we get

$$I_2=e^{\frac{\pi i}{2}}\sum_{k\le K_3}\frac{d_3(k)}{k^{1/2}}\int_T^{2T}\rho\left(\frac{k}{y}\right)e^{-2i(t\log\frac{t}{2\pi}-t-t\log\sqrt{k})}dt+O(T^{3/4}\log^2 T).$$

As in the pevious case, we apply Lemma 1 to the above integral with $\varphi(t)=\rho\left(2k\left(\frac{2\pi}{t}\right)^{3/2}\right)$ and $f(t)=-\frac{1}{\pi}(t\log\frac{t}{2\pi}-t-t\log\sqrt{k})$. Then we find that $f'(t_0)=0$ if and only if $t_0=2\pi\sqrt{k}$, and this t_0 is contained in the interval $[T,2T]$ if and only if

$$\left(\frac{T}{2\pi}\right)^2\le k\le\left(\frac{T}{\pi}\right)^2.\tag{34}$$

Since k runs over the range $1 \le k \le K_3$, there is no k which satisfies (34), hence the main term does not appear in this integral. On the other hand, the error term of this integral is given by

$$1 + \min\left(\sqrt{T}, \frac{1}{|\log \frac{T}{2\pi\sqrt{k}}|}\right) + \min\left(\sqrt{T}, \frac{1}{|\log \frac{T}{\pi\sqrt{k}}|}\right) \ll 1,$$

hence

$$I_2 \ll \sum_{k \le K_3} \frac{d_3(k)}{k^{1/2}} + T^{3/4}\log^2 T \ll T^{3/4}\log^2 T. \tag{35}$$

Next we treat I_1. By (3) again, we have

$$I_1 = e^{-\frac{\pi i}{4}} \sum_{k \le 4K_3} \frac{d_3(k)}{k^{1/2}} \int_T^{2T} \rho\left(\frac{k}{x}\right) e^{i(t\log\frac{t}{2\pi} - t - t\log k)} dt + O(T^{3/4}\log^2 T).$$

In this case $\varphi(t) = \rho(k(\frac{2\pi}{t})^{3/2}/2)$ and $f(t) = \frac{1}{2\pi}(t\log\frac{t}{2\pi} - t - t\log k)$. We see that $f'(t_0) = 0$ if and only if $t_0 = 2\pi k$ and this t_0 is contained in $[T, 2T]$ if and only if

$$\frac{T}{2\pi} \le k \le \frac{T}{\pi}.$$

Therefore we have

$$\begin{aligned}&\int_T^{2T} \rho\left(\frac{k}{x}\right) e^{i(t\log\frac{t}{2\pi} - t - t\log k)} dt \\ &\quad = M(k) + O\left(1 + \min\left(\sqrt{T}, \frac{1}{|\log\frac{T}{2\pi k}|}\right) + \min\left(\sqrt{T}, \frac{1}{|\log\frac{T}{\pi k}|}\right)\right),\end{aligned}$$

where $M(k)$ is the main term given by

$$M(k) = e^{\frac{\pi i}{4}} \rho\left(\frac{1}{2\sqrt{k}}\right)(2\pi t_0)^{1/2} e^{-2\pi i k} = 2\pi e^{\frac{\pi i}{4}} k^{1/2}$$

for k such that $\frac{T}{2\pi} \le k \le \frac{T}{\pi}$ and 0 otherwise. Therefore we get

$$\begin{aligned} I_1 &= 2\pi \sideset{}{'}\sum_{\frac{T}{2\pi} \le k \le \frac{T}{\pi}} d_3(k) \\ &\quad + \sum_{k \le 4K_3} \frac{d_3(k)}{k^{1/2}} \left(1 + \min\left(\sqrt{T}, \frac{1}{|\log\frac{T}{2\pi k}|}\right) + \min\left(\sqrt{T}, \frac{1}{|\log\frac{T}{\pi k}|}\right)\right) \\ &= 2\pi \sideset{}{'}\sum_{\frac{T}{2\pi} \le k \le \frac{T}{\pi}} d_3(k) + O(T^{3/4}\log^2 T). \end{aligned} \tag{36}$$

Here we can get the last O-term by the same way as in the previous case. Combining (33), (35) and (36), we obtain

$$I = 2\pi \sideset{}{'}\sum_{\frac{T}{2\pi} \le k \le \frac{T}{\pi}} d_3(k) + O(T^{3/4}\log^2 T).$$

Now dividing the interval $[0, T]$ as $\cup_j[T/2^j, T/2^{j-1}]$ and summing the above estimate we obtain that

$$\int_0^T Z^2(t)\zeta\left(\frac{1}{2}+it\right)dt = 2\pi\sum_{k\le\frac{T}{2\pi}} d_3(k) + O(T^{3/4}\log^2 T).$$

Theorem 3 follows from the well-known formula:

$$\sum_{n\le x} d_3(n) = x\left(\frac{1}{2}\log^2 x + (3\gamma_0-1)\log x + 3\gamma_1 + 3\gamma_0^2 - 3\gamma_0 + 1\right) + O(x^{1/2}),$$

where γ_j is the coefficients of the Laurent expansion of $\zeta(s)$ at $s=1$.

5 Proof of Theorem 4

To prove Theorem 4, we put

$$I(\alpha) = \int_T^{2T} Z^3\left(\frac{1}{2}+it\right)\chi^\alpha\left(\frac{1}{2}+it\right)dt,$$

where α is a fixed constant such that $-1/2 < \alpha < 1/2$. This time we have

$$I = I_1 + I_2 + O(T^{1/2}),$$

where

$$I_1(\alpha) = \sum_{k=1}^\infty \frac{d_3(k)}{k^{1/2}}\int_T^{2T}\rho\left(\frac{k}{x}\right)k^{-it}\chi^{\alpha-\frac{3}{2}}\left(\frac{1}{2}+it\right)dt \tag{37}$$

and

$$I_2(\alpha) = \sum_{k=1}^\infty \frac{d_3(k)}{k^{1/2}}\int_T^{2T}\rho\left(\frac{k}{y}\right)k^{it}\chi^{\alpha+\frac{3}{2}}\left(\frac{1}{2}+it\right)dt, \tag{38}$$

where $xy = (\frac{t}{2\pi})^3$. We only sketch an outline of evaluations of $I_j(\alpha)$.

Assume that $0\le\alpha<\frac{1}{2}$. We take $x = 2(\frac{t}{2\pi})^{1/2}$ and $y = \frac{1}{2}(\frac{t}{2\pi})^{1/2}$ and put $K_4 = (\frac{T}{\pi})^{3/2}$. Then the range of k in the sum of (37) and (38) are at most $1\le k\le 4K_4$ and $1\le k\le K_4$, respectively.

The integral in (37) becomes

$$e^{\frac{\pi i}{4}\left(\alpha-\frac{3}{2}\right)}\int_T^{2T}\rho\left(\frac{k}{x}\right)e^{\left(\frac{3}{2}-\alpha\right)i\left(t\log\frac{t}{2\pi}-t-t\log k^{\frac{1}{3/2-\alpha}}\right)}dt + O(1).$$

The main term of this integral appears only when

$$\left(\frac{T}{2\pi}\right)^{\frac{3}{2}-\alpha}\le k\le\left(\frac{T}{\pi}\right)^{\frac{3}{2}-\alpha},$$

in which case it is given by

$$M_\alpha(k) = e^{\frac{\pi i}{4}}\rho\left(k^{\frac{-2\alpha}{3-2\alpha}}/2\right)\frac{2\pi}{\sqrt{3/2-\alpha}}k^{\frac{1}{3-2\alpha}}e^{-\left(\frac{3}{2}-\alpha\right)ik^{\frac{1}{3/2-\alpha}}}.$$

By Lemma 1 again, we get

$$I_1(\alpha) = e^{\frac{\pi i}{4}(\alpha-\frac{1}{2})}\frac{2\pi}{\sqrt{3/2-\alpha}}\sum_{\left(\frac{T}{2\pi}\right)^{3/2-\alpha}\le k\le\left(\frac{T}{\pi}\right)^{3/2-\alpha}}\frac{d_3(k)}{k^{1/2}}k^{\frac{1}{3-2\alpha}}e^{-\left(\frac{3}{2}-\alpha\right)ik^{\frac{1}{3/2-\alpha}}}$$
$$+\sum_{k\le K_4}\frac{d_3(k)}{k^{1/2}}O\left(1+\min\left(\sqrt{T},\frac{1}{|\log\frac{(T/2\pi)^{3/2-\alpha}}{k}|}\right)\right.$$
$$\left.+\min\left(\sqrt{T},\frac{1}{|\log\frac{(T/\pi)^{3/2-\alpha}}{k}|}\right)\right). \tag{39}$$

Just in the same way as the previous cases, we can see easily that the above O-term is estimated as $O(T^{3/4}\log^2 T)$.

On the other hand, for $I_2(\alpha)$, the main term does not appear from the integral by the assumption $0\le\alpha<1/2$ and the sum over k is estimated as $O(T^{3/4}\log^2 T)$.

Now it remains to evaluate the sum over k in (39). Let

$$S=\sum_{\left(\frac{T}{2\pi}\right)^{3/2-\alpha}\le k\le\left(\frac{T}{\pi}\right)^{3/2-\alpha}}\frac{d_3(k)}{k^{1/2}}k^{\frac{1}{3-2\alpha}}e^{-\left(\frac{3}{2}-\alpha\right)ik^{\frac{1}{3/2-\alpha}}}.$$

By partial summation we may have

$$S\ll T^{\frac{\alpha}{2}-\frac{1}{4}}\max_{\left(\frac{T}{2\pi}\right)^{3/2-\alpha}\le T'\le\left(\frac{T}{\pi}\right)^{3/2-\alpha}}\left|\sum_{\left(\frac{T}{2\pi}\right)^{3/2-\alpha}\le k\le T'}d_3(k)e^{-(3/2-\alpha)ik^{\frac{1}{3/2-\alpha}}}\right|. \tag{40}$$

Considering the definition of $d_3(k)$, it is reduced to the estimate of the sum of the form

$$S_1:=\sum_{T_1\le k_1k_2k_3\le 2T_1}e^{2\pi ic(k_1k_2k_3)^{\delta}},$$

where $\delta=\frac{1}{3/2-\alpha}$, c is a real constant and $(\frac{T}{2\pi})^{3/2-\alpha}\le T_1\le\frac{1}{2}(\frac{T}{\pi})^{3/2-\alpha}$. Since $\delta\ne 0,1$ we can apply Lemma 3. Divide the summation condition in S_1 into $O(\log^3 T)$ subintervals of the form $(k_1,k_2,k_3)\in[H,2H]\times[N,2N]\times[M,2M]$. By symmetry of k_j, we can assume that M is the largest, hence $M\gg T_1^{1/3}$. Now applying Lemma 3 to the sum S_1 by taking $X=(HNM)^{\delta}\asymp T_1^{\delta}$, we find that

$$S_1\ll T_1^{1+\varepsilon}(T_1^{(\delta-\frac{4}{3})/4}+T_1^{-1/6}+T_1^{-\delta})\ll T_1^{2/3+\delta/4+\varepsilon}. \tag{41}$$

Here the last inequality follows from the assumption $0\le\alpha<1/2$. By (40), (41) and $T_1\asymp T^{3/2-\alpha}$, $\delta=\frac{1}{3/2-\alpha}$ we find that

$$S\ll T^{1-\frac{\alpha}{6}+\varepsilon}.$$

This proves the assertion in the case $0\le\alpha<1/2$.

In the case $-1/2<\alpha\le 0$, we take $x=\frac{1}{2}(\frac{t}{2\pi})^{3/2}$ and $y=2(\frac{t}{2\pi})^{3/2}$. Then the main term arises from the integral corresponding $I_2(\alpha)$ and the assertion is proved similarly. We omit the details in this case.

Authors' contributions

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