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Sign of Fourier coefficients of half-integral weight modular forms in arithmetic progressions

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Abstract

Let f be a half-integral weight cusp form of level $4N$ for odd and squarefree N and let $a(n)$ denote its nth normalized Fourier coefficient. Assuming that all the coefficients $a(n)$ are real, we study the sign of $a(n)$ when n runs through an arithmetic progression. As a consequence, we establish a lower bound for the number of integers $n \leqslant x$ such that $a(n) > n^{-\alpha}$ where x and α are positive and f is not necessarily a Hecke eigenform.

Keywords: Modular form, Fourier coefficient, Arithmetic progression, Sign

1 Introduction

Let f be an element of $S_{\ell+1/2}(4N)$, the space of cusp forms of weight $\ell + 1/2$, of level $4N$ and of trivial character modulo $4N$. Write its Fourier expansion as

$$f(z) = \sum_{n \geqslant 1} a(n) n^{\frac{\ell-1/2}{2}} e(nz)$$

for Im $z > 0$. Given positive real numbers α and x and a class a modulo a prime number p, we are interested in giving a lower bound on the number of integers $n \leqslant x$ such that $n = a\ [p]$ and $a(n) > n^{-\alpha}$ (or $a(n) < -n^{-\alpha}$ respectively). As far as we know, this specific problem for half-integral weight modular forms has not been studied before. Of course, when the weight is an integer, such a question can be partially answered using Sato-Tate equidistribution for Hecke eigenvalues (see [4, Theorem B]).

1.1 The sign of Fourier coefficients

In the recent years, the sign of coefficients of half-integral weight modular forms has drawn considerable attention. As a matter of fact, this subject comes from a question asked by Kohnen. Define f as previously and assume that it is a complete Hecke eigenform. If t is a positive squarefree integer then, by Waldspurger's formula, one knows that the value of $a(t)^2$ is essentially proportional to the central value $L(1/2, \mathrm{Sh} f \times \chi_t)$ where $\mathrm{Sh} f$ is the Shimura lift of f and χ_t is an explicit Dirichlet character depending on t.

Thus, Kohnen's question is : what squareroot of $L(1/2, \text{Sh}f \times \chi_t)$ corresponds to $a(t)$? In other words, assume that the coefficients $a(n)$ are real, then what is the sign of $a(t)$ if it is nonzero and what could we say about the other coefficients $a(n)$?

Bruinier and Kohnen [6] first showed that under some classical hypothesis, the sequence $(a(tn^2))_{n \geqslant 1}$ has infinitely many sign changes and equidistribution results were established in [15] and [1] for the sign of this sequence. In [14], the authors studied the case of the sequence $(a(t))_t$ for all squarefree t and for f of level 4. They showed that there are also infinitely many sign changes in this sequence and a lower bound for the number of positive (respectively negative) $a(t)$ for $t \leqslant x$ is given in [26].

Then, Meher and Murty [30] turned their attention to the whole sequence $(a(n))_{n \geqslant 1}$ when f is a complete eigenform in Kohnen's plus space of level 4. Detecting the sign changes, they proved in particular that

$$\left| \left\{ n \leqslant x \,\middle|\, a(n) \lessgtr 0 \right\} \right| \gg_{f,\varepsilon} x^{27/70-\varepsilon} \tag{1.1}$$

for any $\varepsilon > 0$. In [19], the authors sharpened the exponent $27/70 - \varepsilon$ to $1/2$ and generalized it when n runs through an arithmetic progression of fixed modulus.

Very recently, Lester and Radziwiłł[27] studied this problem and they showed that under the previous assumptions, there are, for any $\varepsilon > 0$ and for x large enough, at least $x^{1-\varepsilon}$ sign changes in $(a(n))_n$ where $1 \leqslant n \leqslant x$ and n is a fundamental discriminant of the form $n = 4t$ with t even and squarefree. This improves drastically the bound in (1.1) but actually they did better on this matter. Using a result of Ono and Skinner [33], they gave a rapid and elegant proof of the fact that if f is suitably normalized, then we have

$$\left| \left\{ n \leqslant x \,\middle|\, a(n) \lessgtr 0 \right\} \right| \gg_f \frac{x}{\log x}$$

and the proof can be easily adapted when $f \in S_{\ell+1/2}(4N)$ for any integer N assuming that the hypotheses of [33, Fundamental Lemma] hold and that f is a complete eigenform.

1.2 Principal results

Fix f as before. Let α be a positive real number and let a be a class modulo an integer q. We consider

$$\mathcal{T}_{a,q}^+(x; \alpha) = \left| \left\{ n \leqslant x \,\middle|\, n = a \, [q] \text{ and } a(n) > n^{-\alpha} \right\} \right|, \tag{1.2}$$

$$\mathcal{T}_{a,q}^-(x; \alpha) = \left| \left\{ n \leqslant x \,\middle|\, n = a \, [q] \text{ and } a(n) < -n^{-\alpha} \right\} \right| \tag{1.3}$$

for any $x > 0$. We also put $\mathcal{T}^{\pm}(x; \alpha) = \mathcal{T}_{0,1}^{\pm}(x; \alpha)$.

Using recent results on sums of Fourier coefficients of half-integral weight modular forms in arithmetic progressions, we prove a lower bound for $\mathcal{T}_{a,p}^{\pm}(x; \alpha)$ for some fixed α and for a positive proportion of $a \, [p]$. Here, p is a prime such that p and x are both going to infinity in a certain range.

We distinguish two cases. The first one is when f is not necessarily a Hecke eigenform. In that case, we prove that there exists a positive proportion of $a \, [p]$[1] such that at least one coefficient $a(n)$ with $n = a \, [p]$ and $n \leqslant x$ satisfies $a(n) > n^{-\alpha}$ for some positive α.

The second case is when f is a complete Hecke eigenform. Then, we establish a lower bound on the number of $a \, [p]$ such that both $\mathcal{T}_{a,p}^+(x; \alpha)$ and $\mathcal{T}_{a,p}^-(x; \alpha)$ are bigger than $\frac{x^{1-\varepsilon}}{p^{7/4}}$ for positive ε.

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[1]Here and in the rest of the paper, a positive proportion means a number of $a \, [p]$ which is $\gg p$.

We emphasize the fact that there are two novelties in this work. First, we are not just looking at the signs of the coefficients, we also provide lower bounds for $|a(n)|$. Moreover, we include the case where f is not necessarily a Hecke eigenform which is significantly different from the previous papers about the signs of coefficients of half-integral weight cusp forms. Indeed, in the works we mentioned above, the assumption that f is an eigenform is crucial since, in that case, Shimura's correspondence is quite explicit on the coefficients (see (2.1) below) and one can apply Waldspurger's formula.

Let us now state the main theorem of this paper.

Theorem 1 *Let $f \in S_{\ell+1/2}(4N)\backslash\{0\}$ where ℓ and N are two positive integers with N odd and squarefree. If $\ell = 1$, we assume that f is in the orthogonal complement of the subspace spanned by single variable theta-functions. We also assume that the Fourier coefficients of f are real.*

Then, for any $\varepsilon > 0$ and any $\alpha \in (3/14, 1/4]$, there exists a constant $x_0 = x_0(f, \varepsilon, \alpha)$ such that for all $x_0 \leqslant x^{1-2\alpha+\varepsilon} \ll p \ll x^{4/7-\varepsilon}$ with p a prime number, we have

$$\mathcal{T}_{a,p}^+(x;\alpha) \geqslant 1$$

for a positive proportion of a $[p]$ and where $\mathcal{T}_{a,p}^+(x;\alpha)$ is defined in (1.2). The same holds for $\mathcal{T}_{a,p}^-(x;\alpha)$.

If, moreover, we assume that f is a complete Hecke eigenform, then for any $\varepsilon > 0$, any $\delta > 0$ small enough and any $\alpha \in (1/8, 1/7]$, there exists a constant $x_0 = x_0(f, \varepsilon, \delta, \alpha)$ such that for all $x_0 \leqslant x^{1/2+\varepsilon} \ll p \ll x^{4\alpha-\varepsilon}$ with p a prime number, we have

$$\mathcal{T}_{a,p}^\pm(x;\alpha) \gg \frac{x^{1-2\delta}}{p^{7/4}}$$

for a number of a $[p]$ which is $\gg_{f,\delta} \frac{p^{3/4}}{x^{\delta/2}}$.

We will then deduce the following corollary.

Corollary 1 *Let $f \in S_{\ell+1/2}(4N)\backslash\{0\}$ as in Theorem 1 (but not necessarily a Hecke eigenform) with N odd and squarefree. Then,*

$$\mathcal{T}^\pm(x;3/14+\varepsilon) \gg x^{4/7-\varepsilon}$$

for any $\varepsilon > 0$ and x large enough.

Remark 1 The previous corollary implies an omega result on the absolute value of $a(n)$. However, the conclusion reached is weaker than the one established in recent papers on this subject (see [9] and [12]).

The proof of the first assertion of Theorem 1 is based on estimates on sums of Fourier coefficients over arithmetic progressions. This type of sums has drawn particular interest over the past decade, especially for integral weight modular forms (see for example [25] and [11]). The case of half-integral weights was treated in [7]. We will need the following.

Theorem 2 *Let $f \in S_{\ell+1/2}(4N)\backslash\{0\}$ and w a smooth real-valued function compactly supported in $(0, +\infty)$. Define for any $x > 0$, any prime number p and any class a modulo p*

$$E(x, p, a) = \frac{1}{\sqrt{x/p}} \sum_{n = a\,[p]} a(n)w(n/x).$$

Then, for any $\varepsilon > 0$,

$$\frac{1}{p} \sum_{a\,[p]}^{\times} |E(x, p, a)|^2 \sim c_f \|w\|_2^2$$

as long as $x^{1/2+\varepsilon} \ll p \ll x^{1-\varepsilon}$. The symbol $\sum^{\times}$ means we restrict the summation over invertible classes modulo p, $\|w\|_2$ is the L^2 norm of w and c_f is a positive constant depending only on f.

Moreover, if we assume that N is odd and squarefree, that f is in the orthogonal complement of the subspace spanned by single variable theta-functions when $\ell = 1$ and that the Fourier coefficients of f are real, then

$$\frac{1}{p} \sum_{a\,[p]}^{\times} E(x, p, a)^4 \leqslant 12(c_f \|w\|_2^2)^2 + o(1)$$

as long as $x^{1/2+\varepsilon} \ll p \ll x^{4/7-\varepsilon}$.

This is a special case of [7, Theorem 3] where the assumption that f is a complete eigenform was relaxed and the level of f is greater than 4.

In order to prove the second assertion of Theorem 1, we will need the following result about the fourth moment of the Fourier coefficients. While the second moment can be easily computed using the classical theory of Rankin-Selberg transform, the fourth moment is more tricky to estimate. As it is done in [27], we will do it by using Waldspurger's formula [37] and a large-sieve type inequality by Heath-Brown for quadratic characters [13].

Proposition 1 *Let $f \in S_{\ell+1/2}(4N)$ be a complete Hecke eigenform with N odd and squarefree. If $\ell = 1$, we assume that f is in the orthogonal complement of the subspace spanned by single variable theta-functions. Then*

$$\sum_{n\leqslant x} |a(n)|^4 \ll_{f,\varepsilon} x^{1+\varepsilon}$$

for any $\varepsilon > 0$ and any $x > 0$.

Remark 2 The assumption that N is odd and squarefree in Theorem 2 comes from the fact that we need a sufficiently good theory on newforms of half-integral weight. As far as we know, such a theory doesn't exist on $S_{\ell+1/2}(4N)$ for arbitrary N.

We also need this assumption in Proposition 1 to make Waldspurger's formula a bit more explicit.

1.3 Structure of the paper

Since we are going to work with smooth sums, we will consider coefficients $a(n)$ with n/x in the compact support of a smooth function w on $(0, +\infty)$ and prove a lower bound for

$$\mathcal{T}_{a,q}^+(x, \alpha; w) = \left| \left\{ n \geqslant 1 \,\middle|\, n = a\,[q] \text{ and } a(n)w(n/x) > n^{-\alpha}w(n/x) \right\} \right|. \tag{1.4}$$

The case of $\mathcal{T}_{a,q}^-(x, \alpha; w)$ will follow easily by changing f in $-f$.

Without loss of generality, we may assume that w is supported in $(0, 1)$ and takes values in $[0, 1]$.

We will proceed as follow. After proving Theorem 2, we will combine it with Hölder's inequality to show that

$$\sum_{a\,[p]} |E(x,p,a)| \gg p$$

which yields $E(x,p,a) \gg 1$ for a positive proportion of $a\,[p]$ since $\sum_{a\,[p]} E(x,p,a)$ is small. Then, the first assertion of Theorem 1 follows from an easy counting argument. We will also prove the second assertion of Theorem 1 following the same line but we will use Proposition 1 instead of the result about the fourth moment in arithmetic progression.

We will first recap some basic facts about half-integral weight modular forms and prove Proposition 1 in Sect. 2. Section 3 is dedicated to the proof of Theorem 2 while Theorem 1 will be proved in Sect. 4.

1.4 Notations

As usual, we write $a\,[p]$ for a class a modulo p and we also put $e_p(a) = e(a/p)$ with $e(x) = e^{2i\pi x}$.

The group $\mathrm{GL}_2(\mathbb{R})^+$ (consisting of real matrices of positive determinant) acts on the Poincaré half-plane $\mathcal{H}$ by Möbius transformation and we write this action as

$$\gamma z = \frac{az+b}{cz+d}$$

for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})^+$ and any $z \in \mathcal{H}$.

We also denoted by I_2 the identity matrix in $\mathrm{GL}_2(\mathbb{R})^+$ and by $\Gamma_0(N)$ the usual congruence subgroup.

For any odd integer d, define ε_d as the normalized Gauss sum *i.e.* $\varepsilon_d = \begin{cases} 1 & \text{if } d = 1\,[4], \\ i & \text{if } d = 3\,[4], \end{cases}$

and for any fundamental discriminant D, we denote by $\left(\frac{D}{\cdot}\right)$ its associated quadratic character. More generally, any non-zero integer n, with $n = 0, 1\,[4]$, can be written in a unique way as $n = Dm^2$ where D is a fundamental discriminant and $m \in \mathbb{Z}$. Hence, we denote by $\left(\frac{n}{\cdot}\right)$ the character modulo $|n|$ induced by $\left(\frac{D}{\cdot}\right)$. If $n = 2, 3\,[4]$ then $4n$ can be written in a unique way as $4n = Dm^2$ where D is a fundamental discriminant and $m \in \mathbb{Z}$. In this case, we denote by $\left(\frac{n}{\cdot}\right)$ the character modulo $4|n|$ induced by $\left(\frac{D}{\cdot}\right)$. By convention, we also let $\left(\frac{0}{\pm 1}\right) = 1$.

If x is a square modulo an odd prime p, we denoted by $\sqrt{x}^p$ the only integer $y \in [1, (p-1)/2]$ such that $x = y^2\,[p]$.

Finally, the symbol $\sum^{\flat}$ means we restrict the summation to positive squarefree integers t and we put $\delta_p(x) = \begin{cases} 1 & \text{if } x = 0\,[p], \\ 0 & \text{otherwise.} \end{cases}$

2 Modular forms of half-integral weight

In this section we first recall the principal properties of half-integral weight modular forms that we will use in this paper. A good introduction to this theory can be found in [32] and a more complete study is done in [34] or [20]. Then, we will prove Proposition 1 and a non trivial bound for Fourier coefficients of such forms.

2.1 General setting

Let G be the set of pairs (σ, ϕ) where $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})^+$ and $\phi : \mathcal{H} \to \mathbb{C}$ is a holomorphic function such that $\phi(z)^2 = \eta(\det \sigma)^{-1/2}(cz + d)$ for all $z \in \mathcal{H}$ and with η a complex number of norm 1 not depending on z. G has a group structure with the inner law defined by

$$(\sigma, \phi)(\sigma', \phi') = (\sigma\sigma', \phi(\sigma' \cdot)\phi').$$

This group is a non-trivial central extension of $\mathrm{GL}_2(\mathbb{R})^+$ by $\mathbb{U}$ the unit circle *i.e.* the sequence

$$1 \to \mathbb{U} \to G \to \mathrm{GL}_2(\mathbb{R})^+ \to 1,$$

where $\eta \in \mathbb{U}$ is sent to (I_2, η), is exact and the center of G is the subgroup of pairs $(\alpha I_2, \eta)$ with $\alpha \in \mathbb{R}^*$ and $\eta \in \mathbb{U}$.

This sequence splits over $\Gamma_0(4)$ which means this group has a section $s_J : \Gamma_0(4) \to G$ given explicitly by $s_J(\gamma) = (\gamma, J(\gamma, z))$ with

$$J(\gamma, z) = \varepsilon_d^{-1}\left(\frac{c}{d}\right)\sqrt{cz + d}$$

for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$. For any positive integer N, we denote by $\Delta_0(4N)$ the image of $\Gamma_0(4N)$ by s_J. If χ is a Dirichlet character modulo $4N$, then we put $\chi(\xi) = \chi(d)$ for any $\xi = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \phi\right) \in \Delta_0(4N)$.

Now for any integer ℓ, any function $f : \mathcal{H} \to \mathbb{C}$ and any $\xi = (\sigma, \phi) \in G$, we define the weighted slash operator $|_{\ell+1/2}$ by

$$f|_{\ell+1/2}\xi(z) = \phi(z)^{-(2\ell+1)}f(\sigma z)$$

which gives a well-defined right action of G on such functions f.

We say that f is a modular (respectively a cusp) form of level $4N$, of weight $\ell + 1/2$ and of character χ, and we note $f \in M_{\ell+1/2}(4N, \chi)$ (respectively $f \in S_{\ell+1/2}(4N, \chi)$), if

(1) f is holomorphic on $\mathcal{H}$,
(2) $f|_{\ell+1/2}\xi = \chi(\xi)f$ for all $\xi \in \Delta_0(4N)$,
(3) f is holomorphic (respectively cuspidal) at each cusp of $\Gamma_0(4N)$.

The third point means that for all cusp $\mathfrak{a}$ of the curve $\Gamma_0(4N)\backslash\mathcal{H}$ and for all element $\xi_\mathfrak{a} = (\sigma_\mathfrak{a}, \phi_\mathfrak{a}) \in G$ with $\sigma_\mathfrak{a}\infty = \mathfrak{a}$, the function $f|_{\ell+1/2}\xi_\mathfrak{a}$ has a Fourier expansion with only non-negative (respectively positive) powers of $e(z/r_\mathfrak{a})$ for some positive integer $r_\mathfrak{a}$.

The Hecke operators T_m are defined on $M_{\ell+1/2}(4N, \chi)$ as double coset operators for $\Delta_0(4N)\xi_m\Delta_0(4N)$ with $\xi_m = \left(\begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix}, m^{1/4}\right)$. In particular (see [34]), they vanish when m is not a square and they satisfy the multiplicativity relation $T_{mn} = T_m T_n$ for $(m, n) = 1$.

Also, the T_{p^2} for $p \nmid 4N$ are normal operators (they are self-adjoint when χ is real) on $S_{\ell+1/2}(4N, \chi)$ with respect to the Petersson inner product. So there exists a basis of $S_{\ell+1/2}(4N, \chi)$ composed of common eigenfunctions of all the T_{p^2} for $p \nmid 4N$ which we call eigenforms. When N is squarefree, with a suitable theory of newforms of half-integral weight (see [29] and [28]) one can prove that some of these eigenforms are actually also eigenfunctions for T_{p^2} with $p \mid 4N$. We call them complete eigenforms.

When χ is trivial, since there exists a basis of $S_{\ell+1/2}(4N)$ composed of forms with rational coefficients (see [5]) and the Hecke operators are rational on this space, then there exists a non-trivial subspace of $S_{\ell+1/2}(4N)$ spanned by eigenforms (or even complete eigenforms if N is squarefree) which have real Fourier coefficients.

For $f \in S_{\ell+1/2}(4N, \chi)$, we denote by f_0 its image under the Fricke involution *i.e.*

$$f_0 = f|_{\ell+1/2} W_{4N}$$

where $W_{4N} = \left(\begin{pmatrix} 0 & -1 \\ 4N & 0 \end{pmatrix}, (4N)^{1/4}\sqrt{-iz} \right)$. Then $f_0 \in S_{\ell+1/2}\left(4N, \left(\frac{4N}{\cdot}\right)\overline{\chi} \right)$.

2.2 Shimura's correspondence and Waldspurger's Theorem

Let $f \in S_{\ell+1/2}(4N, \chi)$ be a complete eigenform with eigenvalues $(\lambda(p))_p$ *i.e.*

$$T_{p^2}f = \lambda(p)f$$

for any prime p. Define $\lambda(n)$ for any integer n formally by

$$\sum_{n \geqslant 1} \lambda(n)n^{-s} = \prod_p \left(1 - \lambda(p)p^{-s} + \chi^2(p)p^{2\ell-1-2s} \right)^{-1}.$$

Then, Shimura [34] and Niwa [31] showed that the function defined by

$$\mathrm{Sh}f(z) = \sum_{n \geqslant 1} \lambda(n)e(nz)$$

for $z \in \mathcal{H}$ is a complete Hecke eigenform in $S_{2\ell}(2N, \chi^2)$ whenever $\ell \geqslant 2$. For $\ell = 1$, this holds if one assumes that f is in the orthogonal complement of the subspace spanned by single variable theta-functions (we always make this assumption in the sequel).

Moreover, for any integer t which is not divisible by a square prime to $4N$, we have

$$a(tn^2)n^{\ell-1/2} = a(t)\sum_{d|n} \mu(d) \left(\frac{(-1)^\ell t}{d} \right) \chi(d)d^{\ell-1}\lambda(n/d) \tag{2.1}$$

for all integer n and where $a(n)$ is the nth normalized Fourier coefficient of f.

Therefore, by Deligne's bound for Hecke eigenvalues for integral weight modular forms [10], one has

$$|a(tn^2)| \ll_\varepsilon |a(t)|n^\varepsilon \tag{2.2}$$

for any $\varepsilon > 0$.

Waldspurger's formula relates $a(t)$ to the central value of the L-function associated to $\mathrm{Sh}f$ twisted by a character. We give a statement for such a formula which can be easily derived from [37, Théorème 1]. For more explicit formulas, see also [23], [22] and [35].

Let $f \in S_{\ell+1/2}(4N, \chi)$ as before and assume that N is odd. For squarefree t, let us consider the Dirichlet character

$$\chi_t = \left(\frac{(-1)^\ell t}{\cdot} \right) \chi \tag{2.3}$$

whose conductor divides $4Nt$.

By Shimura's correspondence and Atkin-Lehner theory [2], there exists a unique newform

$$F(z) = \sum_{n \geqslant 1} b(n)n^{\ell-1/2}e(nz)$$

in $S_{2\ell}(M, \chi^2)$ for some $M \mid 2N$ such that $b(p)p^{\ell-1/2} = \lambda(p)$ for all prime $p \nmid 2N$. Define the twisted form F_t by

$$F_t(z) = \sum_{n \geqslant 1} \overline{\chi}_t(n)b(n)n^{\ell-1/2}e(nz)$$

which is in $S_{2\ell}(16N^2t^2)$ (see [3, Proposition 3.1]). It is an eigenfunction of the pth Hecke operator for $p \nmid 2Nt$ whose eigenvalue is $\overline{\chi}_t(p)\lambda(p)$. Therefore, there exists a unique newform

$$\widetilde{F}_t(z) = \sum_{n \geqslant 1} b_t(n)n^{\ell-1/2}e(nz) \tag{2.4}$$

in $S_{2\ell}(M')$ for some $M' \mid 16N^2t^2$ such that $b_t(p)p^{\ell-1/2} = \overline{\chi}_t(p)\lambda(p)$ for all prime $p \nmid 2Nt$. We define its normalized L-function as

$$L(s, \widetilde{F}_t) = \sum_{n \geqslant 1} \frac{b_t(n)}{n^s}$$

which converges absolutely for Re $s > 1$.

Theorem 3 *[37, Théorème 1] With notations as above, there exists a bounded function Ω_f defined on squarefree integers and depending only on f such that for all squarefree t,*

$$a(t)^2 = \Omega_f(t)L(1/2, \widetilde{F}_t).$$

We will deduce from this theorem the estimate we need for the fourth moment of the coefficients $a(n)$.

2.3 The fourth moment

The goal of this subsection is to prove Proposition 1. The idea is to exploit (2.2) and Theorem 3 to reduce this problem to finding an estimation of

$$\sum_{\psi} L(1/2, g \times \psi)^2 \tag{2.5}$$

where ψ runs through quadratic characters of bounded conductors and g is some newform of integral weight.

Since the form $\widetilde{F}_t$ is not equal to F_t in general (their L-functions are equal up to a finite number of Euler factors but this number could increase with t), we will need assumptions under which $\widetilde{F}_t$ is actually the twist of F by a quadratic character. Hence, we first prove the following lemma.

Lemma 1 *Let $F(z) = \sum_{n \geqslant 1} \lambda(n)e(nz) \in S_k^{\text{new}}(N)$ be a complete Hecke eigenform of integral weight k and let ψ be a primitive quadratic character modulo M. If N is squarefree then the form*

$$F_\psi(z) = \sum_{n \geqslant 1} \psi(n)\lambda(n)e(nz)$$

is a newform and a complete Hecke eigenform.

Proof By assumption, either M is squarefree or it can be written as $M = 4t$ with t squarefree. Then, ψ decomposes as a product of primitive characters

$$\psi = \prod_{p \mid M} \psi_p$$

where $\psi_p = \left(\frac{\cdot}{p}\right)$ for odd p and ψ_2 is a primitive character of conductor 4 or 8. By [2, Theorem 6], for any newform G of level N' with p-adic valuation $v_p(N') = 0$ or 1, we have $G_{\psi_p} \in S_k^{\text{new}}(N'p^{2-v_p(N')})$. It is also suggested in [2] that this holds for quadratic characters modulo 4 and 8 but since it is not explicitly written, we prefer to refer to [3, Theorem 3.1 and Corollary 3.1] from which we can deduce that if $M = 2^\alpha M'$ with $\alpha \in \{2, 3\}$ and M' odd, then $F_{\psi_2} \in S_k^{\text{new}}(2^{2\alpha - v_2(N)}N)$. Then, it follows easily that F_ψ is a newform and since it is an eigenfunction of all but finitely many Hecke operators T_p, it must be a complete eigenform. □

We now deduce the following classical estimate for sums of type (2.5).

Proposition 2 *Let* $F(z) = \sum_{n \geqslant 1} b(n)n^{\frac{k-1}{2}} e(nz) \in S_k^{\text{new}}(N)$ *be a complete Hecke eigenform of integral weight k and with N squarefree. For $x > 0$, let $\Psi(x)$ denote the set of primitive quadratic characters of conductor at most x. Then*

$$\sum_{\psi \in \Psi(x)} |L(1/2, F_\psi)|^2 \ll_{F,\varepsilon} x^{1+\varepsilon}$$

for any $\varepsilon > 0$ and where $L(s, F_\psi) = \sum_{n \geqslant 1} \psi(n)b(n)n^{-s}$.

Proof By Lemma 1, for all $\psi \in \Psi(x)$, F_ψ is a newform whose L-function satisfies a functional equation of the form

$$\Lambda(s, F_\psi) := N_\psi^{s/2}(2\pi)^{-s}\Gamma(s + (k-1)/2)L(s, F_\psi) = \varepsilon(F_\psi)\Lambda(1 - s, F_\psi)$$

for some $\varepsilon(F_\psi) \in \{\pm 1\}$ and where $N_\psi \leqslant Nx^2$ is the level of F_ψ.

Then, using the approximate functional equation (see [18, Theorem 5.3]) one derives

$$L(1/2, F_\psi) = (1 + \varepsilon(F_\psi)) \sum_{n \geqslant 1} \frac{\psi(n)b(n)}{\sqrt{n}} V_{\frac{1}{2}}\left(\frac{n}{\sqrt{N_\psi}}\right) \tag{2.6}$$

where

$$V_{\frac{1}{2}}(y) = \frac{1}{2i\pi} \int_{(\sigma)} \frac{\Gamma(s + k/2)}{s\Gamma(k/2)}(2\pi y)^{-s}ds$$

for any $\sigma > 0$. Let $\eta > 0$. Breaking the sum in (2.6) according to $n < x^{1+\eta}$ or not and using the fact that $V_{\frac{1}{2}}(y) \ll_{k,A} y^{-A}$ for any $A > 0$, we have

$$\sum_{\psi \in \Psi(x)} |L(1/2, F_\psi)|^2 \ll_{F,\eta} \sum_{\psi \in \Psi(x)} \left| \sum_{n < x^{1+\eta}} \frac{\psi(n)b(n)}{\sqrt{n}} V_{\frac{1}{2}}\left(\frac{n}{\sqrt{N_\psi}}\right) \right|^2.$$

Now, by [13, Corollary 2], the right-hand side of the above inequality is

$$\ll_\varepsilon x^{(2+\eta)\varepsilon+1+\eta} \sum_{\substack{n_1, n_2 < x^{1+\eta} \\ \sqrt{n_1 n_2} \in \mathbb{Z}}} \frac{|b(n_1)b(n_2)|}{\sqrt{n_1 n_2}}$$

and this last sum is bounded by $\sum_{n < x^{1+\eta}} \frac{\sigma_0(n^2)}{n^{1-\eta}}$ with $\sigma_0(n^2)$ the number of divisors of n^2. Since η can be arbitrary small, the conclusion follows.

□

We can now prove Proposition 1.

Proof of Proposition 1 Write $f(z) = \sum_{n \geqslant 1} a(n) n^{\frac{\ell-1/2}{2}} e(nz) \in S_{\ell+1/2}(4N)$ as usual.

Since N is odd, by Theorem 3,

$$|a(t)|^2 \ll_f |L(1/2, \widetilde{F}_t)|$$

for any squarefree t and where $\widetilde{F}_t$ is defined by (2.4). Let D_t be the fundamental discriminant such that $\left(\frac{D_t}{\cdot}\right)$ induces χ_t (defined by (2.3) for χ principal). The discussion before Theorem 3 shows that $\widetilde{F}_t$ is an eigen-newform of integral weight whose associated eigenvalues are $\left(\frac{D_t}{p}\right) \lambda(p)$ for all but finitely many p. Hence, by Lemma 1 and [2, Theorem 4], we have $\widetilde{F}_t = F_{\left(\frac{D_t}{\cdot}\right)}$ and recall that F is a complete eigen-newform which depends only on f. Thus, by (2.2), we have for any $\varepsilon > 0$,

$$\sum_{n \leqslant x} |a(n)|^4 \ll_{f,\varepsilon} \sum_{t \leqslant x}^{\flat} |L(1/2, F_{\left(\frac{D_t}{\cdot}\right)})|^2 \sum_{m \leqslant \sqrt{\frac{x}{t}}} m^{2\varepsilon}$$

$$\ll_{f,\varepsilon} x^{1/2+\varepsilon} \sum_{t \leqslant x}^{\flat} |L(1/2, F_{\left(\frac{D_t}{\cdot}\right)})|^2 t^{-1/2-\varepsilon}$$

and since $|D_t| \leqslant 4t$, a summation by parts and Proposition 2 give the result. $\square$

2.4 Bounds for Fourier coefficients

Let $f \in S_{\ell+1/2}(4N, \chi)$ and put

$$f(z) = \sum_{n \geqslant 1} a(n) n^{\frac{\ell-1/2}{2}} e(nz)$$

for $z \in \mathcal{H}$. In the proof of Theorem 2, we will use a bound for the coefficients $a(n)$ which must hold for arbitrary f.

If $\ell \geqslant 2$ and t is squarefree then, by [16, Theorem 1], one has

$$|a(t)| \ll_{f,\varepsilon} t^{3/14+\varepsilon}$$

for any $\varepsilon > 0$. Actually this still holds if t is divisible by p^2 for $p \mid 4N$. Precisely, we have the following proposition.

Proposition 3 *Let $f \in S_{\ell+1/2}(4N, \chi)$ where ℓ and N are two positive integers. If $\ell = 1$, we assume that f is in the orthogonal complement of the subspace spanned by single variable theta-functions. For all integer $n = tm^2$ with squarefree t and $m \mid 4N$, we have*

$$|a(n)| \ll_{f,\varepsilon} n^{3/14+\varepsilon}$$

for any $\varepsilon > 0$.

Proof This is a straightforward consequence of [36, Theorem 1]. $\square$

From this we can deduce the following more general bound.

Proposition 4 *Assume the hypotheses of the previous proposition hold. If, moreover, N is odd and squarefree and χ is real then for all squarefree t and all positive integer n,*

$$|a(tn^2)| \ll_{f,\varepsilon} t^{3/14}(tn)^\varepsilon$$

for any $\varepsilon > 0$.

Proof By [29, Theorems 7], f can be written as a finite sum

$$f = \sum_i U(r_i^2) f_i$$

where $r_i \mid 2N$, $U(r_i^2) : \sum_{n \geqslant 1} c(n)e(nz) \mapsto \sum_{n \geqslant 1} c(r_i^2 n)e(nz)$ and f_i is either a complete eigenform of $S_{\ell+1/2}(M, \chi)$ for some M dividing $4N$ or a complete eigenform of Kohnen's plus space $S_{\ell+1/2}^+(M, \chi)$ for some M dividing $4N$. Hence,

$$a(tn^2) = \sum_i a_i(t(r_i n)^2)$$

with $a_i(m)$ the mth normalized coefficient of f_i. If f_i is a classical eigenform then, by (2.2) and Proposition 3, we have $|a_i(t(r_i n)^2)| \ll_{i,\varepsilon} t^{3/14}(tr_i n)^\varepsilon$.

If f_i is in the plus space, then relation (2.2) still holds but with $t = |D|$ where D is a fundamental discriminant (see [21]). In that case we have $|a_i(t(r_i n)^2)| \ll_{i,\varepsilon} |a_i(t)|(r_i n)^\varepsilon$ or $|a_i(t(r_i n)^2)| \ll_{i,\varepsilon} |a_i(4t)|(r_i n/2)^\varepsilon$. In both cases, we can apply Proposition 3 and get $|a_i(t(r_i n)^2)| \ll_{i,\varepsilon} t^{3/14}(tr_i n)^\varepsilon$ which is enough to conclude. $\qquad\square$

3 Fourier coefficients in arithmetic progressions

The aim of this section is to prove Theorem 2. Since we use the same tools as in [7], we will skip some details. For self-contained study of this problem, we refer to the author's Phd thesis [8].

3.1 Voronoï summation formula

Let $f(z) = \sum_{n \geqslant 1} a(n)n^{\frac{\ell-1/2}{2}}e(nz) \in S_{\ell+1/2}(4N)$ be a cusp form and let w be a smooth $[0, 1]$-valued function compactly support in $(0, 1)$. Define for any $x > 0$, any prime $p \leqslant x$ and any $a\,[p]$

$$E(x, p, a) = \frac{1}{\sqrt{x/p}} \sum_{n = a\,[p]} a(n)w(n/x). \tag{3.1}$$

It is shown in [7] that $\frac{1}{\sqrt{x/p}}$ is the right normalization of the sum above since a squareroot cancellation appears when x and p go to infinity in a certain range.

The first step consists in rearranging $E(x, p, a)$ by using the functional equation for f twisted by an additive character. Such an equation is established in [14] for the special case $N = 1$. Yet, the proof can be easily adapted to any N and one gets the following.

Proposition 5 *Let $f \in S_{\ell+1/2}(4N)$ as above. Let u and q be two coprime integers such that $(q, 4N) = 1$. Put*

$$L(s, f, u/q) = \sum_{n \geqslant 1} a(n)e_q(un)n^{-s}$$

then $L(s, f, u/q)$ converges absolutely for $\mathrm{Re}\, s > 1$ and can be extended to an entire function satisfying

$$\Lambda(s, f, u/q) := \left(\frac{\sqrt{4N}q}{2\pi}\right)^s \Gamma\left(s + \frac{\ell - 1/2}{2}\right) L(s, f, u/q) = \omega_q(u)\Lambda(1 - s, f_0, -\overline{4Nu}/q)$$

where $u\bar{u} = 1\,[q]$ and $\omega_q(u) = \varepsilon_q^{-(2\ell+1)}\left(\frac{-\bar{u}}{q}\right)$.

Moreover, this L-function has polynomial growth in vertical strips.

Next, using Mellin transform, we easily deduce the so-called Voronoï summation formula.

Proposition 6 *Let f, u and q be as above. Then for all $x > 0$,*

$$\sum_{n \geqslant 1} a(n)e_q(un)w(n/x) = \omega_q(u)\frac{x}{\sqrt{4N}q}\sum_{m \geqslant 1}a_0(m)e_q(-\overline{4Nu}\,m)B\left(\frac{m}{4Nq^2/x}\right)$$

where $a_0(m)$ is the mth normalized Fourier coefficient of f_0 and B is a smooth function of rapid decay as in [7, Sect. 3].

By Mellin transform again, we see that

$$\sum_{n \geqslant 1} a(n)w(n/x) \ll_{f,A} x^{-A} \tag{3.2}$$

for any $A > 0$ so detecting the congruence in the sum in (3.1) and applying the last proposition, we have for any $p \nmid 4N$

$$E(x, p, a) = \frac{\varepsilon_p^{-(2\ell+1)}}{\sqrt{Y}}\sum_{m \geqslant 1}a_0(m)\mathrm{Sal}_p(\overline{4N}m, a)B\left(\frac{m}{Y}\right) + O_{f,A}(x^{-A})$$

where $Y = 4Np^2/x$ and

$$\mathrm{Sal}_p(u, v) = \frac{1}{\sqrt{p}}\sum_{b\,[p]}^{\times}\left(\frac{b}{p}\right)e_p(ub + v\bar{b})$$

is the normalized Salié sum. Classically (see [24, Lemme 8.4.3]), if u and v are coprime to p then

$$\mathrm{Sal}_p(u, v) = \left(\frac{v}{p}\right)\varepsilon_p\sum_{y^2 = uv\,[p]}e_p(2y).$$

Thus, using [7, Proposition 3] (where in the proof, f does not need to be an eigenform), we infer that

$$E(x, p, a) = \varepsilon_p^{-2\ell}\left(\frac{a}{p}\right)\frac{1}{\sqrt{Y}}\sum_{1 \leqslant m \leqslant Y^{1+\eta}}a_0(m)\mathrm{Sa}_p(\overline{N}ma)B\left(\frac{m}{Y}\right) + O_{f,A}(Y^{-A}) \tag{3.3}$$

for any $\eta > 0$, provided that $Y^{1+\eta} < p$, and where

$$\mathrm{Sa}_p(y) = \begin{cases} e_p(\sqrt{y^p}) + e_p(-\sqrt{y^p}) & \text{if } \left(\frac{y}{p}\right) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

3.2 Some estimates on sums of Fourier coefficients

Before proving Theorem 2 we establish some basic facts on certain sums of Fourier coefficients.

Lemma 2 *Let $f \in S_{\ell+1/2}(4N)$ as above. Then*

$$\sum_{n \geqslant 1}|a(n)|^2 w(n/x)^2 \sim c_f\|w\|_2^2 x \quad \text{as } x \to +\infty$$

where

$$c_f = \frac{(4\pi)^{\ell+1/2}}{\Gamma(\ell+1/2)\mathrm{Vol}\,(\Gamma_0(4N)\backslash\mathcal{H})}\int_{\Gamma_0(4N)\backslash\mathcal{H}}|f(z)|^2 y^{\ell+1/2}\frac{dxdy}{y^2}.$$

Proof We have for any $\sigma > 1$,

$$\sum_{n \geqslant 1} |a(n)|^2 w(n/x)^2 = \frac{1}{2i\pi} \int_{(\sigma)} D(s, f \times \bar{f}) \widehat{w^2}(s) x^s ds$$

where $D(s, f \times \bar{f}) = \sum_{n \geqslant 1} |a(n)|^2 n^{-s}$ and $\widehat{w^2}(s) = \int_0^{+\infty} w^2(t) t^{s-1} dt$ is the Mellin transform of w^2. Because w is smooth and compactly supported in $(0, 1)$, $\widehat{w^2}(s)$ (as well as $\hat{w}(s)$) is well-defined on the whole complex plane and it is of rapid decay in vertical strips.

Classically (see [17, Sect. 13.4]), $D(s, f \times \bar{f})$ extends to a meromorphic function for $\operatorname{Re} s \geqslant 1/2$ with a finite number of poles which are simple and in the interval $1/2 < s \leqslant 1$. At $s = 1$, there is a simple pole whose residue is c_f.

Hence, moving the contour of integration to $\sigma = 1 - \varepsilon$ with $\varepsilon > 0$ small enough, we get

$$\sum_{n \geqslant 1} |a(n)|^2 w(n/x)^2 = c_f \widehat{w^2}(1) x + \frac{1}{2i\pi} \int_{(1-\varepsilon)} D(s, f \times \bar{f}) \widehat{w^2}(s) x^s ds$$

and because $D(s, f \times \bar{f})$ is of polynomial growth on vertical strips, we have the desired conclusion.

$\square$

Lemma 3 *Let $f \in S_{\ell+1/2}(4N)$ as above and take a class a $[p]$. Then*

$$\sum_{n = a \, [p]} a(n) w(n/x) = O_{f,w,\varepsilon}(x^{-\varepsilon} p^{1+2\varepsilon})$$

for any $\varepsilon > 0$.

Proof Write

$$\sum_{n = a \, [p]} a(n) w(n/x) = \frac{1}{p} \sum_{b \, [p]} e_p(-ba) \sum_{n \geqslant 1} a(n) e_p(bn) w(n/x)$$

$$= \frac{1}{p} \sum_{b \, [p]} e_p(-ba) \frac{1}{2i\pi} \int_{(\sigma)} L(s, f, b/p) \hat{w}(s) x^s ds$$

for any $\sigma > 1$. Moving the contour of the integral to $\sigma = -\varepsilon$, we get

$$\sum_{n = a \, [p]} a(n) w(n/x) = \frac{1}{p} \sum_{b \, [p]} e_p(-ba) \frac{1}{2i\pi} \int_{(-\varepsilon)} L(s, f, b/p) \hat{w}(s) x^s ds$$

since the integrated functions are entire. Using the functional equation given in Proposition 5 (for $b \neq 0$ $[p]$), the result follows.

$\square$

We will also need to compare sums of Fourier coefficients to Dirichlet series over arithmetic progressions.

Lemma 4 *Let $0 < \alpha < 1/2$ and take a class a $[p]$. Then*

$$\sum_{n = a \, [p]} n^{-\alpha} w(n/x) = O\left(\frac{x^{1-\alpha}}{p}\right)$$

Proof Since w is supported in $(0, 1)$ and takes values in $[0, 1]$, we have

$$
\left| \sum_{n=a\,[p]} n^{-\alpha} w(n/x) \right| \leqslant \sum_{n \leqslant x/p} \frac{1}{(a+np)^{\alpha}}
$$

$$
\ll \int_{1}^{x/p} \frac{dt}{(a+tp)^{\alpha}}
$$

$$
\ll \frac{x^{1-\alpha}}{p}.
$$

$\square$

We are now ready to prove Theorem 2.

3.3 Proof of Theorem 2

Since the computation of the second moment is the same whether the $a(n)$'s (or equivalently the $a_0(n)$'s) are real or not, we will assume from now that they are. Let $\nu \in \{2, 4\}$ and write for $p \nmid 4N$

$$
\frac{1}{p} \sum_{a\,[p]}^{\times} E(x, p, a)^{\nu} = \frac{1}{2} M_{\nu}^{+} + \frac{1}{2} M_{\nu}^{-}
$$

with

$$
M_{\nu}^{\pm} = \frac{2}{p} \sum_{\left(\frac{a}{p}\right)=1} E(x, p, N\mu_{\pm}a)^{\nu} = \frac{1}{p} \sum_{b\,[p]}^{\times} E(x, p, N\mu_{\pm}b^2)^{\nu}
$$

and $\mu_{\pm}$ is any positive integer such that $\left(\frac{\mu_{\pm}}{p}\right) = \pm 1$.

Then, by (3.3) and [7, Lemma 5], we have

$$
M_{\nu}^{\pm} = \frac{1}{Y^{\nu/2}} \sum_{\substack{1 \leqslant m_i \leqslant Y^{1+\eta} \\ 1 \leqslant i \leqslant \nu, \left(\frac{m_i}{p}\right)=\pm 1}} \prod_{i=1}^{\nu} a_0(m_i) B\left(\frac{m_i}{Y}\right) \sum_{\mathbf{e}\in\{\pm 1\}^{\nu}} \delta_p\left(\sum_{i=1}^{\nu} e_i \sqrt{\mu_{\pm}m_i}^{p}\right)
$$

$$
+ O_{f,A}\left(\frac{Y^{\nu/2}}{p} + Y^{-A}\right)
$$

with $Y = 4Np^2/x$. If $\nu = 2$ then notice that

$$
e_1 \sqrt{\mu_{\pm}m_1}^{p} + e_2 \sqrt{\mu_{\pm}m_2}^{p} = 0\,[p] \iff \begin{cases} m_1 = m_2 \\ e_1 e_2 = -1 \end{cases}
$$

since $1 \leqslant \sqrt{\mu_{\pm}m_i}^{p} < p/2$ and $1 \leqslant m_i < p$. Therefore,

$$
M_2^{\pm} = \frac{2}{Y} \sum_{\substack{1 \leqslant m \leqslant Y^{1+\eta} \\ \left(\frac{m}{p}\right)=\pm 1}} a_0(m)^2 B^2\left(\frac{m}{Y}\right) + O_{f,A}\left(\frac{Y}{p} + Y^{-A}\right)
$$

and

$$
\frac{1}{p} \sum_{a\,[p]}^{\times} E(x, p, a)^2 = \frac{1}{Y} \sum_{1 \leqslant m \leqslant Y^{1+\eta}} a_0(m)^2 B^2\left(\frac{m}{Y}\right) + O_{f,A}\left(\frac{Y}{p} + Y^{-A}\right)
$$

so, if $Y \to +\infty$ with $Y^{1+\eta} < p$, we get the first assertion of Theorem 2 since, by Lemma 2 or simply [7, Sect. 6], we have

$$
\frac{1}{Y} \sum_{1 \leqslant m \leqslant Y^{1+\eta}} a_0(m)^2 B^2\left(\frac{m}{Y}\right) \sim c_f \|w\|_2^2 \quad \text{as } Y \to +\infty \tag{3.4}
$$

and the constant c_f is the same as in Lemma 2 because a change of variable shows that f and f_0 have the same Petersson norm.

Things are a bit trickier when $\nu = 4$. Put

$$Q_4(\mathbf{x}) = \prod_{\substack{\mathbf{e} \in \{\pm 1\}^4 \\ e_1 = 1}} \sum_{i=1}^{4} e_i \sqrt{x_i}$$

for any positive x_1, x_2, x_3, x_4. Because of the parity in the variables $\sqrt{x_i}$'s of the right-hand side of the above equality, we may view $Q_4(\mathbf{x})$ as a homogeneous polynomial of $\mathbb{Z}[x_1, x_2, x_3, x_4]$.

For $1 \leqslant m_1, m_2, m_3, m_4 \leqslant Y^{1+\eta}$ we have

$$|Q_4(m_1, m_2, m_3, m_4)| \leqslant \prod_{\substack{\mathbf{e} \in \{\pm 1\}^4 \\ e_1 = 1}} 4Y^{\frac{1+\eta}{2}} \leqslant 2^{16} Y^{4(1+\eta)}.$$

Thus, if $p \ll x^{4/7-\varepsilon}$ then there exists $\varepsilon' > 0$ such that $p^{7/4+2\varepsilon'} \ll x$ which implies that $p^{2(1+\varepsilon')}/x \ll p^{1/4}$ so, taking $0 < \eta < \varepsilon'$, we have

$$2^{16} Y^{4(1+\eta)} < p/2$$

for Y large enough. Assume this is the case, then for any $\mathbf{e} \in \{\pm 1\}^4$ and any $\mathbf{m} = (m_1, m_2, m_3, m_4)$ with $1 \leqslant m_i \leqslant Y^{1+\eta}$ and $\left(\frac{m_i}{p} \right) = \pm 1$, one has

$$\sum_{i=1}^{4} e_i \sqrt{\mu_{\pm} m_i}^p = 0 \ [p] \ \Rightarrow \ Q_4(\mu_{\pm} \mathbf{m}) = 0 \ [p]$$

$$\Rightarrow \ Q_4(\mathbf{m}) = 0 \ \Rightarrow \ \exists \mathbf{e}' \in \{\pm 1\}^4, \sum_{i=1}^{4} e_i' \sqrt{m_i} = 0.$$

For any $1 \leqslant i \leqslant 4$ and $1 \leqslant m_i \leqslant Y^{1+\eta}$, write $m_i = t_i r_i^2$ where t_i is squarefree and $r_i \geqslant 1$. Since the different values of the $\sqrt{t_i}$'s are linearly independent over $\mathbb{Q}$, then $\sum_{i=1}^{4} e_i' \sqrt{m_i} = 0$ only if $|\{t_1, t_2, t_3, t_4\}| = 1$ or 2. In the second case, say $t_1 = t_2 \neq t_3 = t_4$,

$$\sum_{i=1}^{4} e_i' \sqrt{m_i} = 0 \ \Rightarrow \ (e_1' r_1 + e_2' r_2)\sqrt{t_1} + (e_3' r_3 + e_4' r_4)\sqrt{t_3} = 0$$

$$\Rightarrow \ \begin{cases} r_1 = r_2 \\ e_1' e_2' = -1 \\ r_3 = r_4 \\ e_3' e_4' = -1 \end{cases} \ \Rightarrow \ \begin{cases} m_1 = m_2 \\ m_3 = m_4 \end{cases}$$

since $r_i \geqslant 1$ for all i. But if $m_1 = m_2 \neq m_3 = m_4$ then

$$(e_1 + e_2)\sqrt{\mu_{\pm} m_1}^p + (e_3 + e_4)\sqrt{\mu_{\pm} m_3}^p = 0 \ [p] \ \Longleftrightarrow \ e_1 = -e_2 \text{ and } e_3 = -e_4.$$

Indeed, if, for example, $e_1 + e_2 \neq 0$ *i.e.* $e_1 + e_2 \in \{\pm 2\}$, then $e_3 + e_4 \neq 0$ (otherwise $\sqrt{\mu_{\pm} m_1}^p = 0 \ [p]$) and $\mu_{\pm} m_1 = \mu_{\pm} m_3 \ [p]$ which implies $m_1 = m_3$ but we have excluded this case. This proves the necessary condition of the above equivalence and the sufficient condition is trivial. Therefore,

$$\sum_{\mathbf{e} \in \{\pm 1\}^4} \delta_p \left(\sum_{i=1}^{4} e_i \sqrt{\mu_{\pm} m_i}^p \right) = 4.$$

Since this discussion is the same if $m_1 = m_3 \neq m_2 = m_4$ or $m_1 = m_4 \neq m_2 = m_3$, it allows us to write

$$M_4^{\pm} = \frac{12}{Y^2} \sum_{\substack{1 \leqslant m_1, m_2 \leqslant Y^{1+\eta} \\ t_1 \neq t_2, \left(\frac{m_i}{p}\right) = \pm 1}} a_0(m_1)^2 B^2\left(\frac{m_1}{Y}\right) a_0(m_2)^2 B^2\left(\frac{m_2}{Y}\right)$$

$$+ \frac{R}{Y^2} + O_{f,A}\left(\frac{Y^2}{p} + Y^{-A}\right) \tag{3.5}$$

where

$$R = \sum_{\substack{t \leqslant Y^{1+\eta} \\ \left(\frac{t}{p}\right) = \pm 1}}^{\flat} \sum_{\substack{1 \leqslant tr_i^2 \leqslant Y^{1+\eta} \\ 1 \leqslant i \leqslant 4}} \prod_{i=1}^{4} a_0(tr_i^2) B\left(\frac{tr_i^2}{Y}\right) \sum_{\mathbf{e} \in \{\pm 1\}^4} \delta_p\left(\sum_{i=1}^{4} e_i \sqrt{\mu_{\pm} tr_i^2}^p\right). \tag{3.6}$$

We are going to show that this last term is negligible. Precisely we have the following.

Proposition 7 *Define R as in (3.6). Then for any $\varepsilon > 0$ and η sufficiently small*

$$|R| \ll_{f,\varepsilon} Y^{2-1/7+\varepsilon}.$$

Proof Following the previous discussion or simply by [7, Lemma 6],

$$\sum_{\mathbf{e} \in \{\pm 1\}^4} \delta_p\left(\sum_{i=1}^{4} e_i \sqrt{\mu_{\pm} tr_i^2}^p\right) \ll \sum_{\mathbf{e} \in \{\pm 1\}^4} \delta_0\left(\sum_{i=1}^{4} e_i r_i\right)$$

so, since B is bounded, it suffices to prove that

$$R' := \sum_{t \leqslant Y^{1+\eta}}^{\flat} \sum_{\substack{1 \leqslant tr_i^2 \leqslant Y^{1+\eta} \\ \sum_{i=1}^{4} e_i r_i = 0}} \prod_{i=1}^{4} |a_0(tr_i^2)| \ll_{f,\varepsilon} Y^{2-1/7+\varepsilon} \tag{3.7}$$

for any $\mathbf{e} \in \{\pm 1\}^4$ and any $\varepsilon > 0$. Fix such $\mathbf{e}$ and ε. For any t, the inner sum in R' becomes

$$\sum_{\substack{1 \leqslant tr_i^2 \leqslant Y^{1+\eta} \\ 1 \leqslant i \leqslant 3}} \left| a_0\left(t\left(\sum_{i=1}^{3} e_i r_i\right)^2\right) \right| \prod_{i=1}^{3} |a_0(tr_i^2)| \ll_{f,\varepsilon} t^{6/7+\varepsilon} Y^{\varepsilon(1+\eta)} \sum_{\substack{1 \leqslant tr_i^2 \leqslant Y^{1+\eta} \\ 1 \leqslant i \leqslant 3}} \prod_{i=1}^{3} r_i^{\varepsilon}$$

by Proposition 4. Thus, for η sufficiently small,

$$R' \ll_{f,\varepsilon} Y^{\varepsilon(1+\eta)} \sum_{t \leqslant Y^{1+\eta}}^{\flat} t^{6/7+\varepsilon} \left(\frac{Y^{1+\eta}}{t}\right)^{\frac{3}{2}(1+\varepsilon)}$$

$$\ll_{f,\varepsilon} Y^{3/2+6\varepsilon} \sum_{t \leqslant Y^{1+\eta}}^{\flat} t^{5/14-1}$$

$$\ll_{f,\varepsilon} Y^{2-1/7+7\varepsilon}.$$

$\square$

Remark 3 If one has a better exponent in Proposition 4, say $a(tr^2) \ll_{f,\varepsilon} t^{\theta}(tr)^{\varepsilon}$ for some $\theta > 0$, then the exponent $2-1/7$ in Proposition 7 is replaced by $1+4\theta$. This only improves slightly the error term in (3.5) but it does not extend the range of convergence for the fourth moment.

To finish the proof of Theorem 2, note that

$$\sum_{\substack{1 \leqslant m_1, m_2 \leqslant Y^{1+\eta} \\ t_1 = t_2, \left(\frac{m_i}{p}\right) = \pm 1}} a_0(m_1)^2 B^2 \left(\frac{m_1}{Y}\right) a_0(m_2)^2 B^2 \left(\frac{m_2}{Y}\right)$$

is bounded by R' defined in (3.7) (with $\mathbf{e} = (1, -1, 1, -1)$ for example) so we have from (3.5) and Proposition 7,

$$M_4^{\pm} = \frac{12}{Y^2} \sum_{\substack{1 \leqslant m_1, m_2 \leqslant Y^{1+\eta} \\ \left(\frac{m_i}{p}\right) = \pm 1}} a_0(m_1)^2 B^2 \left(\frac{m_1}{Y}\right) a_0(m_2)^2 B^2 \left(\frac{m_2}{Y}\right) + O_{f,\varepsilon}\left(Y^{-1/7+\varepsilon} + \frac{Y^2}{p}\right)$$

$$= 12 \left(\frac{1}{Y} \sum_{\substack{1 \leqslant m \leqslant Y^{1+\eta} \\ \left(\frac{m}{p}\right) = \pm 1}} a_0(m)^2 B^2 \left(\frac{m}{Y}\right)\right)^2 + O_{f,\varepsilon}\left(Y^{-1/7+\varepsilon} + \frac{Y^2}{p}\right)$$

for any $\varepsilon > 0$. Recall that $x^{1/2+\varepsilon} \ll p \ll x^{4/7-\varepsilon}$ for some $\varepsilon > 0$ so the error term above is $o(1)$ as $x \to +\infty$. Hence

$$|M_4^{\pm}| \leqslant 12 \left(\frac{1}{Y} \sum_{1 \leqslant m \leqslant Y^{1+\eta}} a_0(m)^2 B^2 \left(\frac{m}{Y}\right)\right)^2 + o(1)$$

and again, using (3.4), we get the conclusion.

4 Proof of Theorem 1

We are now going to prove an analog of Theorem 1 for $\mathcal{T}_{a,q}^+(x, \alpha; w)$ defined in (1.4). This result will be even stronger than Theorem 1 since it counts the number of positive coefficients $a(n)$ with n/x in the support of w.

4.1 Preliminary lemmas

We first prove two elementary lemmas that we will use several times.

Lemma 5 *Let $(b(n))_n$ be a sequence of real numbers such that*

$$\sum_{n \leqslant x} b(n) = o\left(\sum_{n \leqslant x} |b(n)|\right)$$

as $x \to +\infty$. Put

$$\sum\nolimits^+(x) = \sum_{\substack{n \leqslant x \\ b(n) > 0}} b(n) \quad and \quad \sum\nolimits^-(x) = -\sum_{\substack{n \leqslant x \\ b(n) < 0}} b(n).$$

Then,

$$\sum\nolimits^{\pm}(x) \sim \frac{1}{2} \sum_{n \leqslant x} |b(n)|$$

as $x \to +\infty$.

Proof We have

$$\sum{}^+(x) = \frac{1}{2}\left(\sum{}^+(x) + \sum{}^-(x)\right) + \frac{1}{2}\left(\sum{}^+(x) - \sum{}^-(x)\right)$$

$$= \frac{1}{2}\sum_{n\leqslant x}|b(n)| + \frac{1}{2}\sum_{n\leqslant x}b(n)$$

$$\sim \frac{1}{2}\sum_{n\leqslant x}|b(n)|$$

by assumptions. The proof is the same for $\sum{}^-(x)$. □

Lemma 6 *Let X be a finite set of positive integers and for any $n \in X$, let $b(n)$ and $c(n)$ be two real numbers with $c(n) \geqslant 0$. Assume there exists $M > 0$ and $V > 0$ such that*

$$\sum_{n\in X}c(n) \leqslant M \leqslant \sum_{n\in X}b(n)$$

and

$$\sum_{n\in X}b(n)^2 \leqslant V.$$

Then,

$$\left|\{n \in X \mid b(n) > c(n)\}\right| \geqslant \left(M - \sum_{n\in X}c(n)\right)^2 V^{-1}$$

Proof One has

$$M \leqslant \sum_{\substack{n \in X \\ b(n) \leqslant c(n)}}b(n) + \sum_{\substack{n \in X \\ b(n) > c(n)}}b(n)$$

$$\leqslant \sum_{n\in X}c(n) + \left(\sum_{\substack{n \in X \\ b(n) > c(n)}}1\right)^{1/2}\left(\sum_{n\in X}b(n)^2\right)^{1/2}$$

using Cauchy–Schwarz inequality in the second sum.

Since $\sum_{n\in X}c(n) \leqslant M$ and $\sum_{n\in X}b(n)^2 \leqslant V$, the result follows easily.

□

4.2 Case where f is arbitrary

Fix f as in Theorem 1 (but not necessarily an eigenform). For $x > 0$ and a prime number p, we always assume that $x^{1/2+\varepsilon} \ll p \ll x^{4/7-\varepsilon}$ for some fixed $\varepsilon > 0$. Hence, if x goes to infinity then so does p but restricted in this range. We can first establish the following proposition.

Proposition 8 *If $0 < m < \frac{\|w\|_2\sqrt{c_f}}{4\sqrt{3}}$ then*

$$\left|\{a\,[p] \mid E(x, p, a) > m\}\right| \geqslant \left(\frac{1}{4\sqrt{3}} - \frac{m}{\|w\|_2\sqrt{c_f}}\right)^2 p + o(p)$$

as $x \to +\infty$.

Proof By Hölder's inequality, we have

$$\frac{1}{p}\sideset{}{^\times}\sum_{a\,[p]} E(x,p,a)^2 \leqslant \left(\frac{1}{p}\sideset{}{^\times}\sum_{a\,[p]} |E(x,p,a)|\right)^{2/3} \left(\frac{1}{p}\sideset{}{^\times}\sum_{a\,[p]} E(x,p,a)^4\right)^{1/3}$$

so using Theorem 2, we get

$$c_f\|w\|_2^2 + o(1) \leqslant \left(\frac{1}{p}\sideset{}{^\times}\sum_{a\,[p]} |E(x,p,a)|\right)^{2/3} \left(12(c_f\|w\|_2^2)^2 + o(1)\right)^{1/3}$$

and then

$$\sideset{}{^\times}\sum_{a\,[p]} |E(x,p,a)| \geqslant \frac{\|w\|_2\sqrt{c_f}}{2\sqrt{3}} p + o(p).$$

Also, by (3.2) and Lemma 3,

$$\sideset{}{^\times}\sum_{a\,[p]} E(x,p,a) = \frac{1}{\sqrt{x/p}} \sum_{n\geqslant 1} a(n)w(n/x)$$

$$-\frac{1}{\sqrt{x/p}} \sum_{n=0\,[p]} a(n)w(n/x) = O_f(x^{-1/2}p^{3/2+\varepsilon}) = O_f(p^{1-\delta})$$

for some $\delta > 0$ because $p \ll x^{4/7-\varepsilon}$.

Thus, Lemma 5 yields

$$\sideset{}{^+}\sum_{a\,[p]} E(x,p,a) \geqslant \frac{\|w\|_2\sqrt{c_f}}{4\sqrt{3}} p + o(p) \tag{4.1}$$

where $\sum_{a\,[p]}^+$ means that we restrict the sum to invertible classes $a\,[p]$ such that $E(x,p,a) > 0$.

Now, use Lemma 6 with $X = \{0 < a < p \mid E(x,p,a) > 0\}$ and, for $a \in X$, with $b(a) = E(x,p,a)$ and $c(a) = m < \frac{\|w\|_2\sqrt{c_f}}{4\sqrt{3}}$. By (4.1) and Theorem 2, we obtain

$$\left|\{a\,[p] \mid E(x,p,a) > m\}\right| \geqslant \left(\frac{\|w\|_2\sqrt{c_f}}{4\sqrt{3}} p - mp + o(p)\right)^2 \left(c_f\|w\|_2^2 p + o(p)\right)^{-1}$$

$$\geqslant \left(\frac{1}{4\sqrt{3}} - \frac{m}{\|w\|_2\sqrt{c_f}}\right)^2 p + o(p).$$

$\square$

Proposition 8 allows us to give a lower bound for $\sum_{n=a\,[p]} a(n)w(n/x)$ for a certain number of $a\,[p]$. We are now going to upper bound $\sum_{n=a\,[p]} a(n)^2 w(n/x)^2$ for a large number of $a\,[p]$ in order to apply Lemma 6 once again.

Proposition 9 *Let $m > 0$. Then*

$$\left|\{a\,[p] \mid \sum_{n=a\,[p]} a(n)^2 w(n/x)^2 > mx/p\}\right| \leqslant \left(\frac{c_f\|w\|_2^2}{m} + o(1)\right)p.$$

Proof This is a straightforward consequence Markov's inequality and Lemma 2.

$\square$

Now, let us prove the main result of this subsection.

Theorem 4 *Let f, x and p be as above. For any $a\,[p]$, define $\mathcal{T}_{a,p}^{+}(x,\alpha;w)$ as in (1.4). Let $\alpha \in (3/14, 1/4]$ and $r < 1/48$. Then, for x large enough*

$$\left|\left\{ a\,[p]\,\middle|\, \mathcal{T}_{a,p}^{+}(x,\alpha;w) \geqslant 1 \right\}\right| \geqslant rp$$

as long as $x^{1-2\alpha+\varepsilon} \ll p \ll x^{4/7-\varepsilon}$ for some $\varepsilon > 0$.

Proof Let $m_1 > 0$ and $m_2 > 0$ such that

$$\sqrt{r} < \frac{1}{4\sqrt{3}} - \frac{m_1}{\|w\|_2 \sqrt{c_f}}$$

and

$$\frac{c_f \|w\|_2^2}{m_2} < \left(\frac{1}{4\sqrt{3}} - \frac{m_1}{\|w\|_2 \sqrt{c_f}} \right)^2 - r.$$

Apply Propositions 8 and 9 to see that

$$\sum_{n=a\,[p]} a(n)w(n/x) \geqslant m_1\sqrt{x/p} \quad \text{and} \quad \sum_{n=a\,[p]} a(n)^2 w(n/x)^2 \leqslant m_2 x/p \tag{4.2}$$

for a certain number of invertible $a\,[p]$ greater than rp for p large enough *i.e.* x large enough.

Also, by Lemma 4,

$$\sum_{n=a\,[p]} n^{-\alpha} w(n/x) \ll \frac{x^{1-\alpha}}{p}.$$

The right-hand side of the above inequality is $o\left(\sqrt{x/p}\right)$ because $\frac{x^{1-\alpha}}{p} = \frac{x^{1/2-\alpha}}{p^{1/2}}\sqrt{x/p}$ and $x^{1-2\alpha+\varepsilon} \ll p$.

Hence, for these invertible $a\,[p]$ satisfying (4.2), using Lemma 6 with $X = \{n = a\,[p] \mid w(n/x) \neq 0\}$, $b(n) = a(n)w(n/x)$ and $c(n) = n^{-\alpha}w(n/x)$, we have for x large enough,

$$\mathcal{T}_{a,p}^{+}(x,\alpha;w) \geqslant \frac{1}{m_2 x/p}\left(m_1\sqrt{x/p} + o\left(\sqrt{x/p}\right) \right)^2 > 0$$

and since $\mathcal{T}_{a,p}^{+}(x,\alpha;w)$ is an integer, we get the result.

$\square$

Theorem 4 easily implies the first assertion of Theorem 1. Unfortunately, the lower bound for $\mathcal{T}_{a,p}^{+}(x,\alpha)$ cannot be improved with our method since it only gives

$$\mathcal{T}_{a,p}^{+}(x,\alpha) \geqslant \frac{m_1^2}{m_2} = \frac{m_1^2}{c_f \|w\|_2} \frac{c_f \|w\|_2}{m_2}$$

with $\frac{m_1^2}{c_f\|w\|_2}$ and $\frac{c_f\|w\|_2}{m_2}$ both less than $\frac{1}{4\sqrt{3}}$ so the right-hand side of the above inequality cannot be greater than one.

We also deduce Corollary 1 from Theorem 4.

Proof of Corollary 1 Let $\varepsilon > 0$ and $x > 0$. For x large enough, there always exists a prime p in the interval $[x^{4/7-2\varepsilon}, x^{4/7-\varepsilon}]$ by Bertrand's postulate. Then, applying Theorem 4 with $\alpha = 3/14 + 2\varepsilon$ and ε small enough, we see that the number of $n \in [1,x]$ such that $a(n) > n^{-\alpha}$ is greater than $rp \geqslant rx^{4/7-2\varepsilon}$ for fixed $r < 1/48$.

$\square$

We now turn our attention to the second assertion of Theorem 1, that we will prove using the same technics as previously.

4.3 Case where *f* is a complete eigenform

From now on, assume that f is a complete eigenform and that x and p still satisfy $x^{1/2+\varepsilon} \ll p \ll x^{4/7-\varepsilon}$ for some $\varepsilon > 0$. We start by proving the following proposition.

Proposition 10 *For $m > 0$ and $\delta > 0$, put*

$$
\mathcal{A}(x, p, m, \delta) = \left\{ a\ [p] \left| \begin{array}{l} \sum\limits_{n=a\ [p]} a(n)^2 w(n/x)^2 > mx/p \\ \sum\limits_{n=a\ [p]} a(n)^4 w(n/x)^4 \leqslant \frac{x^{1+\delta}}{\sqrt{p}} \end{array} \right. \right\}. \tag{4.3}
$$

Then, for m sufficiently small and x large enough, one has

$$
\left| \mathcal{A}(x, p, m, \delta) \right| \gg_{f,\delta} x^{-\delta/2} p^{3/4}.
$$

Proof First note that, by Cauchy–Schwarz inequality,

$$
\sum_{n=a\ [p]} a(n)^2 w(n/x)^2 \leqslant \sqrt{x/p} \left(\sum_{n=a\ [p]} a(n)^4 w(n/x)^4 \right)^{1/2}
$$

since w is compactly supported in $(0, 1)$. It is also $[0, 1]$-valued, so using Proposition 1, one gets

$$
\sum_{n=a\ [p]} a(n)^2 w(n/x)^2 \ll_{f,\delta_1} \frac{x^{1+\delta_1}}{\sqrt{p}}
$$

for any $\delta_1 > 0$ and any $a\ [p]$. However, if $a \in \mathcal{A}(x, p, m, \delta)$ then we even have

$$
\sum_{n=a\ [p]} a(n)^2 w(n/x)^2 \ll_{f,\delta} \frac{x^{1+\delta/2}}{p^{3/4}}.
$$

By Markov's inequality and Proposition 1, we also have that

$$
\left| \left\{ a\ [p] \left| \sum_{n=a\ [p]} a(n)^4 w(n/x)^4 > \frac{x^{1+\delta}}{\sqrt{p}} \right. \right\} \right| \leqslant \frac{\sqrt{p}}{x^{\delta_2}}
$$

for any $0 < \delta_2 < \delta$.

Therefore, using Lemma 2,

$$
x \ll_f \sum_{a \notin \mathcal{A}(x,p,m,\delta)} \sum_{n=a\ [p]} a(n)^2 w(n/x)^2 + \sum_{a \in \mathcal{A}(x,p,m,\delta)} \sum_{n=a\ [p]} a(n)^2 w(n/x)^2
$$

and splitting the first sum according to $\sum\limits_{n=a\ [p]} a(n)^2 w(n/x)^2 \leqslant mx/p$ or not, we get

$$
x \ll_{f,\delta,\delta_1} mx + \frac{x^{1+\delta_1}}{\sqrt{p}} \frac{\sqrt{p}}{x^{\delta_2}} + \frac{x^{1+\delta/2}}{p^{3/4}} \left| \mathcal{A}(x, p, m, \delta) \right|
$$

and the result follows by choosing $\delta_1 < \delta_2$ and m small enough.

□

We will prove that for most $a \in \mathcal{A}(x, p, m, \delta)$, the coefficients $a(n)$'s with $n = a\ [p]$ have a certain number of positive and negative signs. To do so, we need to bound the number of $a\ [p]$ such that $\left| \sum\limits_{n=a\ [p]} a(n)w(n/x) \right|$ or $\sum\limits_{n=a\ [p]} a(n)^2 w(n/x)^2$ is too big.

Proposition 11 *For $\delta > 0$, put*

$$\mathcal{B}(x, p, \delta) = \left\{ a\,[p] \;\middle|\; \left| \sum_{n=a\,[p]} a(n)w(n/x) \right| > \frac{x^{1-\delta}}{p^{5/4}} \text{ or } \sum_{n=a\,[p]} a(n)^2 w(n/x)^2 > \frac{x^{1+\delta}}{p^{3/4}} \right\}.$$

$$(4.4)$$

Then, for $m > 0$ and $\delta > 0$ small enough,

$$\left| \mathcal{B}(x, p, \delta) \right| = o\left(\left| \mathcal{A}(x, p, m, \delta) \right| \right)$$

as long as $x^{1/2+\varepsilon} \ll p \ll x^{4/7-\varepsilon}$ for some $\varepsilon > 0$

Proof By Chebychev's inequality and Theorem 2, the number of $a\,[p]$ such that

$$\left| \sum_{n=a\,[p]} a(n)w(n/x) \right| > \frac{x^{1-\delta}}{p^{5/4}}$$

is less than

$$\frac{(c_f \|w\|_2^2 + o(1))x}{x^{2-2\delta}p^{-5/2}} \ll_f \frac{p^{5/2}}{x^{1-2\delta}} = x^{-\delta/2}p^{3/4} \frac{p^{7/4}}{x^{1-5\delta/2}}$$

and $\frac{p^{7/4}}{x^{1-5\delta/2}} = o(1)$ for δ small enough since $p \ll x^{4/7-\varepsilon}$.

Similarly, by Markov's inequality and Lemma 2, the number of $a\,[p]$ such that

$$\sum_{n=a\,[p]} a(n)^2 w(n/x)^2 > \frac{x^{1+\delta}}{p^{3/4}}$$

is less than

$$\frac{(c_f \|w\|_2^2 + o(1))x}{x^{1+\delta}p^{-3/4}} \ll_f \frac{p^{3/4}}{x^{\delta}}$$

which is $o(x^{-\delta/2}p^{3/4})$.

$\square$

As previously, when $a \in \mathcal{A}(x, p, m, \delta)$, we use Hölder's inequality to give a lower bound on $\sum_{n=a\,[p]} |a(n)|w(n/x)$.

Lemma 7 *Let $a \in \mathcal{A}(x, p, m, \delta)$ defined in (4.3). Then*

$$\sum_{n=a\,[p]} |a(n)|w(n/x) \geqslant m^{3/2}\frac{x^{1-\delta/2}}{p^{5/4}}.$$

Proof Hölder's inequality yields

$$mx/p < \sum_{n=a\,[p]} a(n)^2 w(n/x)^2 \leqslant \left(\sum_{n=a\,[p]} |a(n)|w(n/x) \right)^{2/3} \left(\sum_{n=a\,[p]} a(n)^4 w(n/x)^4 \right)^{1/3}.$$

Hence

$$\sum_{n=a\,[p]} |a(n)|w(n/x) \geqslant (mx/p)^{3/2} \left(\frac{x^{1+\delta}}{\sqrt{p}} \right)^{-1/2} \geqslant m^{3/2}\frac{x^{1-\delta/2}}{p^{5/4}}.$$

$\square$

We can now prove the main Theorem of this subsection which implies the second assertion of Theorem 1.

Theorem 5 *Let f, x and p be as above. Assume that f is a complete eigenform. For any $a\,[p]$, define $\mathcal{T}_{a,p}^{\pm}(x,\alpha;w)$ as in (1.4). Let $\alpha \in (1/8, 1/7]$. Then, for any $\delta > 0$ small enough and any x large enough*

$$\left| \left\{ a\,[p] \ \middle| \ \min\left(\mathcal{T}_{a,p}^{+}(x,\alpha;w), \mathcal{T}_{a,p}^{-}(x,\alpha;w) \right) \gg \frac{x^{1-2\delta}}{p^{7/4}} \right\} \right| \gg_{f,\delta} \frac{p^{3/4}}{x^{\delta/2}}$$

as long as $x^{1/2+\varepsilon} \ll p \ll x^{4\alpha-\varepsilon}$ for some $\varepsilon > 0$.

Proof Let $a \in \mathcal{A}(x,p,m,\delta) \backslash \mathcal{B}(x,p,\delta)$. By Propositions 10 and 11, such $a\,[p]$ exists for m and δ small enough and there are $\gg_{f,\delta} x^{-\delta/2} p^{3/4}$ of them.

Lemma 7 implies that

$$\left| \sum_{n=a\,[p]} a(n)w(n/x) \right| \leqslant \frac{x^{1-\delta}}{p^{5/4}} = o\left(\sum_{n=a\,[p]} |a(n)|w(n/x) \right)$$

so, by Lemma 5 and for x large enough,

$$\pm \sum_{n=a\,[p]}^{\pm} a(n)w(n/x) \gg \frac{x^{1-\delta/2}}{p^{5/4}}$$

where $\displaystyle\sum_{n=a\,[p]}^{\pm}$ means that we restrict the sum over $n = a\,[p]$ such that $a(n) > 0$ or $a(n) < 0$ respectively.

Also, by Lemma 4,

$$\sum_{n=a\,[p]} n^{-\alpha} w(n/x) \ll \frac{x^{1-\alpha}}{p} = \frac{x^{1-\delta/2}}{p^{5/4}} \frac{p^{1/4}}{x^{\alpha-\delta/2}} = o\left(\frac{x^{1-\delta/2}}{p^{5/4}} \right)$$

for δ small enough because $p \ll x^{4\alpha-\varepsilon}$. Hence, recalling that $a \notin \mathcal{B}(x,p,\delta)$, we can apply Lemma 6 and obtain

$$\mathcal{T}_{a,p}^{\pm}(x,\alpha;w) \gg \frac{x^{2-\delta}p^{-5/2}}{x^{1+\delta}p^{-3/4}} = \frac{x^{1-2\delta}}{p^{7/4}}.$$

$\square$

References
1. Arias-de-Reyna, S., Inam, I., Wiese, G.: On conjectures of Sato-Tate and Bruinier–Kohnen. Ramanujan J. **36**(3), 455–481 (2015)
2. Atkin, A.O.L., Lehner, J.: Hecke operators on $\Gamma_0(m)$. Math. Ann. **185**, 134–160 (1970)
3. Atkin, A.O.L., Li, Wen Ch'ing Winnie: Twists of newforms and pseudo-eigenvalues of W-operators. Invent. Math. **48**(3), 221–243 (1978)
4. Barnet-Lamb, T., Geraghty, D., Harris, M., Taylor, R.: A family of Calabi–Yau varieties and potential automorphy II. Publ. Res. Inst. Math. Sci. **47**(1), 29–98 (2011)
5. Belabas, K., Cohen, H.: Modular forms in Pari/GP. Res. Math. Sci. **5**(3), 19 (2018)
6. Bruinier, J.H., Kohnen, W.: Sign changes of coefficients of half integral weight modular forms. Modular forms on Schiermonnikoog, pp. 57–65. Cambridge University Press, Cambridge (2008)
7. Darreye, C.: Fourier coefficients of modular forms of half-integral weight in arithmetic progressions. Int. Math. Res. Not. IMRN (2020). https://doi.org/10.1093/imrn/rnaa105

8. Darreye, C.: Sur la répartition des coefficients des formes modulaires de poids demi-entier. Thèse de doctorat de l'Université de Bordeaux (2020)
9. Das, S.: Omega result for Fourier coefficients of half-integral weight and Siegel modular forms, Preprint (2019)
10. Deligne, P.: La conjecture de Weil. I. Publ. Math. Inst. Hautes Études Sci. **43**, 273–307 (1974). (French)
11. Fouvry, E., Ganguly, S., Kowalski, E., Michel, P.: Gaussian distribution for the divisor function and Hecke eigenvalues in arithmetic progressions. Comment. Math. Helv. **89**(4), 979–1014 (2014)
12. Gun, S., Kohnen, W., Soundararajan, K.: Large Fourier coefficients of half-integer weight modular forms (2020). https://arxiv.org/pdf/2004.14450.pdf
13. Heath-Brown, D.R.: A mean value estimate for real character sums. Acta Arith. **72**(3), 235–275 (1995)
14. Hulse, T.A., Kiral, E.M., Kuan, C.I., Lim, L.-M.: The sign of Fourier coefficients of half-integral weight cusp forms. Int. J. Number Theory **8**(3), 749–762 (2012)
15. Inam, I., Wiese, G.: Equidistribution of signs for modular eigenforms of half integral weight. Arch. Math. (Basel) **101**(4), 331–339 (2013)
16. Iwaniec, H.: Fourier coefficients of modular forms of half-integral weight. Invent. Math. **87**(2), 385–401 (1987)
17. Iwaniec, H.: Topics in classical automorphic forms. Graduate Studies in Mathematics, vol. 17. American Mathematical Society, Providence (1997)
18. Iwaniec, H., Kowalski, E.: Analytic number theory. American Mathematical Society Colloquium Publications, vol. 53. American Mathematical Society, Providence (2004)
19. Jiang, Y.-J., Lau, Yuk-Kam, Lü, G.-S., Royer, E., Wu, J.: Sign changes of Fourier coefficients of modular forms of half integral weight **2** (2018). arXiv:1602.08922
20. Knapp, A.W.: Elliptic Curves, Mathematical Notes, vol. 40. Princeton University Press, Princeton (1992)
21. Kohnen, W.: Newforms of half-integral weight. J. Reine Angew. Math. **333**, 32–72 (1982)
22. Kohnen, W.: Fourier coefficients of modular forms of half-integral weight. Math. Ann. **271**(2), 237–268 (1985)
23. Kohnen, W., Zagier, D.: Values of *L*-series of modular forms at the center of the critical strip. Invent. Math. **64**(2), 175–198 (1981)
24. Kowalski, E.: Un cours de théorie analytique des nombres, Cours Spécialisés [Specialized Courses], vol. 13, Société Mathématique de France, Paris, 2004 (French)
25. Lau, Y.-K., Zhao, L.: On a variance of Hecke eigenvalues in arithmetic progressions. J. Number Theory **132**(5), 869–887 (2012)
26. Lau, Y., Royer, E., Jie, W.: Sign of Fourier coefficients of modular forms of half-integral weight. Mathematika **62**(3), 866–883 (2016)
27. Lester, S., Radziwiłł, M.: Signs of Fourier coefficients of half-integral weight modular forms (2019). https://arxiv.org/pdf/1903.05811.pdf
28. Manickam, M.: Newforms of half-integral weight on $\Gamma_0(8N)$, Number theory. Lecture Notes Series, vol. 15, pp. 63–71. Ramanujan Mathematical Society, Mysore (2011)
29. Manickam, M., Ramakrishnan, B., Vasudevan, T.C.: On the theory of newforms of half-integral weight. J. Number Theory **34**, 210–224 (1990)
30. Meher, J., Murty, M.: Rj: Sign changes of Fourier coefficients of half-integral weight cusp forms. Int. J. Number Theory **10**(4), 905–914 (2014)
31. Niwa, S.: Modular forms of half integral weight and the integral of certain theta-functions. Nagoya Math. J. **56**, 147–161 (1975)
32. Ono, K.: The web of modularity: arithmetic of the coefficients of modular forms and *q*-series, CBMS Regional Conference Series in Mathematics, vol. 102, Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI (2004)
33. Ono, K., Skinner, C.: Non-vanishing of quadratic twists of modular *L*-functions. Invent. Math. **134**(3), 651–660 (1998)
34. Shimura, G.: On modular forms of half integral weight. Ann. Math. **97**, 440–481 (1973)
35. Shimura, G.: On the Fourier coefficients of Hilbert modular forms of half-integral weight. Duke Math. J. **71**(2), 501–557 (1993)
36. Waibel, F.: Fourier coefficients of half-integral weight cusp forms and Waring's problem. Ramanujan J. **47**(1), 185–200 (2018)
37. Waldspurger, J.-L.: Sur les coefficients de Fourier des formes modulaires de poids demi-entier. J. Math. Pures Appl. **60**(4), 375–484 (1981). (French)

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