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Recognizing Galois representations of K3 surfaces

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Abstract

Under the assumption of the Hodge, Tate and Fontaine–Mazur conjectures we give a criterion for a compatible system of ℓ-adic representations of the absolute Galois group of a number field to be isomorphic to the second cohomology of a K3 surface. This is achieved by producing a motive M realizing the compatible system, using a local to global argument for quadratic forms to produce a K3 lattice in the Betti realization of M and then applying surjectivity of the period map for K3 surfaces to obtain a complex K3 surface. Finally we use a very general descent argument to show that the complex K3 surface admits a model over a number field.

1 Introduction

This paper grew out of an attempt to answer a question on the section conjecture for moduli spaces of K3 surfaces, inspired by recent work of Patrikis et al. [13]. In this paper, the authors study the section conjecture for the moduli space of principally polarized abelian varieties. The section conjecture for a (geometrically connected) variety X over a number field K relates the set of rational points $X(K)$ with the sections of the fundamental sequence

$$1 \to \pi_1(\overline{X}) \to \pi_1(X) \to \Gamma_K \to 1$$

(we omit base points for the étale fundamental group and, for any field K, write $\Gamma_K = \mathrm{Gal}(\overline{K}/K)$ for the absolute Galois group with a fixed algebraic closure $\overline{K}$, and $\overline{X}$ the basechange of X to $\overline{K}$). Given a rational point $x\colon \mathrm{Spec}(K) \to X$, functoriality of π_1 gives a section $\Gamma_K \to \pi_1(X)$ and this defines a map $\sigma_X\colon X(K) \to H(K, X)$ where $H(K, X)$ is the set of sections up to conjugation by $\pi_1(\overline{X})$. The section conjecture for X states that the map σ_X is a bijection. Of course, for general X this map is far from a bijection, so we would want to find a class of varieties suitably determined by their fundamental groups. These are the so called anabelian varieties introduced by Grothendieck in his letter to Faltings [15], pp. 49–58. Grothendieck suggested that hyperbolic curves, moduli spaces of curves and (less emphatically) moduli spaces of abelian varieties should all be anabelian.

It is known that moduli spaces $\mathcal{A}_g$ of abelian varieties should not be anabelian by results of Ihara and Nakamura [9]. However, Theorem 1.1 of [13] shows that under the assumption of well known motivic conjectures, a large subset of sections $S_0(K, \mathcal{A}_g) \subset H(K, \mathcal{A}_g)$ is contained in the image of $\sigma_{\mathcal{A}_g}$, where the sections $S_0(K, \mathcal{A}_g)$ are those coming from points locally. The authors are able to prove this by reducing to a question about

Galois representations. More specifically there is a short exact sequence

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(\overline{\mathcal{A}_g}) & \longrightarrow & \pi_1(\mathcal{A}_g) & \longrightarrow & \Gamma_K & \longrightarrow & 1 \\
& & \downarrow{\cong} & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Sp}_{2g}(\hat{\mathbb{Z}}) & \longrightarrow & \mathrm{GSp}_{2g}(\hat{\mathbb{Z}}) & \longrightarrow & \hat{\mathbb{Z}}^{\times} & \longrightarrow & 1
\end{array}
\tag{1}
$$

Given a section $s\colon \Gamma_K \to \pi_1(\mathcal{A}_g)$ composition with the middle arrow gives a collection of ℓ-adic representations $\{\rho_\ell\colon \Gamma_K \to \mathrm{GSp}_{2g}(\mathbb{Z}_\ell)\}_\ell$. The fact that the left arrow is an isomorphism shows that the sections $H(K, \mathcal{A}_g)$ are determined by their associated ℓ-adic representations. Then the authors use well known conjectures to find conditions on a collection of ℓ-adic representations $\{\rho_\ell\}$ that ensure they are isomorphic to the ℓ-adic Tate module of an abelian variety [13, Thm 3.3]. The proof of Theorem 3.3 proceeds by using these conjectures to find a motive underlying the collection of ℓ-adic representations. Taking Betti realization of this motive gives a Hodge structure that has the Hodge weights of an abelian variety. Using Riemann's theorem one can show that this Hodge structure is isomorphic to the Hodge structure on the first homology of an abelian variety.

One might ask whether [13, Thm 3.3] can be generalized to other classes of varieties. In order for the above method to work, such a class of varieties would require an analogue of Riemann's theorem, which gives a criterion for an abstract Hodge structure to appear in the (co)homology of a variety. After abelian varieties, the most natural class of varieties with this property is K3 surfaces, where surjectivity of the period map is known. Our main theorem is precisely the analogue for K3 surfaces of [13, Thm 3.3]. We find a set of conditions on a weakly compatible system of ℓ-adic representations (for all ℓ) to relate them to the weakly compatible system $H^2(X_{\overline{K}}, \mathbb{Q}_\ell)$ for varying primes ℓ. We now make this precise.

Let K be a number field and S a finite set of rational primes. A collection of Galois representations $\{\rho_\ell\colon \Gamma_K \to \mathrm{GL}_n(\mathbb{Q}_\ell)\}_{\ell\notin S}$ is said to be weakly compatible if there exists a finite set Σ of finite places of K satisfying

I. For each $\ell \notin S$ the representation ρ_ℓ is unramified outside $\Sigma_\ell \cup \Sigma$ where Σ_ℓ is the set of places of K lying over ℓ.

II. For each $\ell \notin S$ and each place ν of K not in $\Sigma_\ell \cup \Sigma$, the characteristic polynomial of $\rho_\ell(\mathrm{Fr}_\nu)$ has rational coefficients and is independent of ℓ (here Fr_ν is a geometric Frobenius element at ν).

Now let $\Lambda = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$ be the K3 lattice. Fix a basis e, f of the first copy of the hyperbolic plane U such that $(e)^2 = (f)^2 = 0$ and $(e,f) = 1$. We prove the following

Theorem 1.1 *Let K be a number field. Assume the Hodge, Tate and Fontaine–Mazur conjectures. Let $\{\rho_\ell\colon \Gamma_K \to O(\Lambda \otimes \mathbb{Q}_\ell)\}$ be a weakly compatible system (with S empty) of semisimple representations such that*

(1) *There exist an integer $d > 0$ such that for all but finitely many primes ℓ, $(e + df) \otimes 1 \in (\Lambda \otimes \mathbb{Q}_\ell(1))^{\Gamma_K}$*

(2) *For some ℓ_0, ρ_{ℓ_0} is de Rham at all $\nu | \ell_0$.*

(3) *For some ℓ_1, $\mathrm{End}_{\Gamma_K}(\rho_{\ell_1}) = \mathbb{Q}_{\ell_1} \oplus \mathbb{Q}_{\ell_1}$.*

(4) *For some ℓ_2 and some $v|\ell_2$, $\rho_{\ell_2}|_{\Gamma_{K_v}}$ is de Rham with Hodge-Tate weights $0, 1, 2$ of multiplicities $1, 20, 1$.*

Then there is a K3 surface X over a finite extension L of K such that $\rho_\ell|_{\Gamma_L} \cong H^2(X_{\overline{L}}, \mathbb{Q}_\ell)$ for all ℓ.

The primes ℓ_0, ℓ_1, ℓ_2 could all be the same. Conditions (1), (2) and (4) are of course necessary conditions for the collection $\{\rho_\ell\}$ to be isomorphic to $H^2(X_{\overline{K}}, \mathbb{Q}_\ell)$ for X a K3 surface with a polarization of degree $2d$. Condition (3) is an irreducibility condition (similar to the hypothesis of absolute irreducibility in [13, Thm 3.3]) and is satisfied by the cohomology of the generic K3 surface, i.e. those of geometric Picard rank 1 with $\text{End}_{\Gamma_K}(H^2(X_{\overline{K}}, \mathbb{Q}_\ell)) \cong \mathbb{Q}_\ell \oplus \mathbb{Q}_\ell$. Suppose X is a K3 surface over $K \subset \mathbb{C}$. Recall that for any complex K3 surface there is a decomposition of rational $\mathbb{Q}$-Hodge structures

$$H^2(X_{\mathbb{C}}, \mathbb{Q}) = (\text{NS}(X_{\mathbb{C}}) \otimes \mathbb{Q}) \oplus (T(X_{\mathbb{C}}) \otimes \mathbb{Q})$$

of the rational $\mathbb{Q}$-lattice $\text{NS}(X_{\mathbb{C}}) \otimes \mathbb{Q}$ and its orthogonal complement $T(X_{\mathbb{C}}) \otimes \mathbb{Q}$. A theorem of Zarhin [19, Thm 1.4.1] shows that $T(X_{\mathbb{C}}) \otimes \mathbb{Q}$ is an irreducible $\mathbb{Q}$-Hodge structure. The Mumford–Tate conjecture, which is known for K3 surfaces by Tankeev [16], then shows that condition (3) is satisfied by $H^2(X_{\overline{K}}, \mathbb{Q}_\ell)$ when $T(X_{\mathbb{C}}) \otimes \mathbb{Q}$ is an absolutely irreducible $\mathbb{Q}$-Hodge structure, and the Γ_K-action on

$$\text{NS}(X_{\overline{K}}) \otimes \mathbb{Q}_{\ell_1} = \text{NS}(X_{\mathbb{C}}) \otimes \mathbb{Q}_{\ell_1}$$

is absolutely irreducible.

Our final remark regards the last section of the paper. The main result of this section, Lemma 6.3, shows a rather general criterion for the rigid descent of a variety X over an algebraically closed field Ω of characteristic zero to an algebraically closed subfield $k \subset \Omega$.

1.1 Questions

In the recent preprint [2], Baldi independently proves an analogue of Theorem 1.1 above for K3 surfaces whose Picard rank ρ satisfies $12 \le \rho < 20$. This is Theorem 1.2 of [2] where he shows for representations

$$\{\rho_\ell \colon \Gamma_K \to \text{Gl}_{22-\rho}(\mathbb{Q}_\ell)\}_\ell$$

satisfying analogues of conditions (2), (3) and (4) above, there is a finite extension L of K such that $\rho_\ell|_{\Gamma_L}$ is isomorphic to $T(X_{\overline{L}})_{\mathbb{Q}_\ell}$. The main question given our result and that of Baldi is whether we can take $L = K$ for the field of definition of the K3 surface X, or at least if the degree of the finite extension can be bounded.

In both Baldi's work and this paper, the arguments of [13] are immediately extended to get a $\mathbb{Q}$-Hodge structure V of K3 type from the collection of ℓ-adic representations. The next key step, which is unique to the K3 case, is to produce a lattice inside this $\mathbb{Q}$-Hodge structure and to show that this lattice is isomorphic to a (sub-)polarized $\mathbb{Z}$-Hodge structure of $H^2(X, \mathbb{Z})$ for some complex K3 surface X. In Baldi's paper, this is done by picking any lattice $T \subset V$ and then using an embedding theorem of Nikulin to get an embedding of T into the K3 lattice $\Lambda = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$. Then one shows that the Hodge structure induced by T on Λ is of K3 type and hence, by surjectivity of the period map, isomorphic to $H^2(X, \mathbb{Z})$ for a K3 surface X over $\mathbb{C}$. Baldi's requirement that $12 \le \rho < 20$ is a consequence of the fact that this is the range for which the embedding theorem holds. For our proof,

we show that $V \cong \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ as quadratic spaces and thus produce a Hodge structure on the K3 lattice Λ from which we can apply surjectivity of the period map. In light of our theorem and Baldi's theorem it is natural to ask whether such a theorem holds for 'Picard rank' $2 \leq \rho \leq 11$, i.e. compatible system of representations $\{\rho_\ell \colon \Gamma_K \to \mathrm{O}(\Lambda \otimes \mathbb{Q}_\ell)\}$ which decompose as $\rho_\ell \cong V_\ell \oplus \mathbb{Q}_\ell(-1)^{\oplus \rho}$ with V_ℓ irreducible (and satisfying conditions (2) and (4) of Theorem 1.1). The author does not believe that the proofs of this paper or those of Baldi can be adapted to prove an analogous theorem for $2 \leq \rho \leq 11$.

Our original motivation was to apply Theorem 1.1 to answer a question about the section conjecture for moduli spaces of K3 surfaces, as was done in [13, Thm 1.1] for abelian varieties. The moduli space we are interested in is the space $\mathcal{F}_{2d}$, using the notation of [14], classifying primitively polarized K3 surfaces of degree 2d. Using the following diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(\overline{\mathcal{F}_{2d}}) & \longrightarrow & \pi_1(\mathcal{F}_{2d}) & \longrightarrow & \Gamma_K & \longrightarrow & 1 \\
& & & \searrow{\scriptstyle\alpha} & \downarrow & & & & \\
& & & & \mathrm{O}(\Lambda_{2d} \otimes \hat{\mathbb{Z}}) & & & &
\end{array}
\tag{2}
$$

we can associate to each section $s \in H(\mathcal{F}_{2d}, K)$ an $\mathrm{O}(\Lambda_{2d} \otimes \hat{\mathbb{Z}})$ representation. If we knew that this map was a bijection then an analogue of [13, Thm 1.1] could be proven. However, a computation of the group $\pi_1(\overline{\mathcal{F}_{2d}})$ seems difficult. One approach might be to compute the topological fundamental group of the complex analytic space $\mathcal{F}_{2d}^{\mathrm{an}}$ and then compare the profinite completion to a suitable orthogonal group. The domain $\mathcal{F}_{2d}^{\mathrm{an}}$ has an explicit description as the quotient by an orthogonal group of the complement of an infinite union of hyperplane sections in a period domain D_d, see [7, Remark 6.3.7]. Given this explicit description, one may be able to compute the topological fundamental group. This is expected to be very large, containing an infinitely generated free group generated by loops around the hyperplane sections. These are the sorts of groups that are not residually finite, and the kernel to the profinite completion can be very large, see [17]. So while $\pi_1^{\mathrm{top}}(\mathcal{F}_{2d}^{\mathrm{an}})$ is very far from any orthogonal group, it may happen that the 'non-orthogonal' part gets killed in the profinite completion.

Finally, the last question that naturally follows from this is paper is whether there are analogues of Theorem 1.1 for hyperkahler varieties. There are known results on surjectivity of the period map and global Torelli theorems are known for hyperkahler manifolds, see [6], so it may be reasonable that methods in this paper could work for such varieties.

1.2 Terminology

Throughout the paper, K will be a number field with a fixed algebraic closure $\overline{K}$. We write $\Gamma_K = \mathrm{Gal}(\overline{K}/K)$ for the absolute Galois group.

If K is a field, and F a field of characteristic 0, we denote by $\mathcal{M}_{K,F}$ the category of pure numerical motives over K with coefficients in F. If $F = \mathbb{Q}$ we simply write $\mathcal{M}_K$. The functors H_ℓ, H_B, H_{dR} are the ℓ-adic, Betti and algebraic de Rham realization functors. Implicitly whenever we write any of these functors, we are assuming the conjecture that numerical equivalence is equal to homological equivalence for that cohomology theory, and in this way the realization functors may be defined on numerical motives.

See Sect. 2 for notation about quadratic forms and lattices. If V and W are either both $\mathbb{Z}$-Hodge structures or $\mathbb{Q}$-Hodge structures equipped with pairings (e.g. polarizations) then

a map $V \to W$ is called a Hodge isometry if it is an isomorphism of Hodge structures that also respects the pairings. All K3 surfaces are smooth and projective. We write $\mathbb{Q}$-HS for the category of $\mathbb{Q}$-Hodge structures.

2 Quadratic forms and lattices

2.1 Notation

If F is a field of characteristic zero, a quadratic space over F consists of a vector space V with a non degenerate symmetric bilinear pairing $V \otimes V \to F$ (or equivalently a quadratic form on V). Of interest to us are quadratic spaces over $\mathbb{Q}$, $\mathbb{Q}_p$ and $\mathbb{R}$. When we talk about a lattice T, we mean a finitely generated free abelian group T with a pairing $(\cdot, \cdot) \colon T \times T \to \mathbb{Z}$ that is non degenerate.

2.2 Lattices associated to K3 surfaces

We write U to denote the hyperbolic plane and E_8 the lattice associated to the Dynkin diagram E_8.

Example 2.1 Let Λ be the lattice $U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$. The disciminant $d(\Lambda) = -1$ so Λ is even, unimodular and has signature $(3, 19)$. This is called the K3 lattice, because if X is a K3 surface over $\mathbb{C}$, then the singular cohomology $H^2(X, \mathbb{Z})$ with the cup product pairing is isomorphic to Λ. See [7].

We will need the following lemma for the proof of the main theorem.

Lemma 2.2 *Let Λ be the K3 lattice. Then there exists a $\mathbb{Q}$-quadratic space V of signature (r, s) such that $V \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}_p$ for all finite primes p if and only if (r, s) is one of the following pairs:*

$$(19, 3), (15, 7), (11, 11), (7, 15), (3, 19)$$

Proof By [4, Thm 1.3, p. 77] and the fact that we know that the local data, we compute

$$\prod_{p \neq \infty} c_p(V) = \prod_{p \neq \infty} c_p(\Lambda) = c_\infty(\Lambda) = -1$$

(here we use Theorem 1.3, p. 77; Theorem 1.2, p. 56, loc. cit.). Again those same two theorems and the fact that we know the local data imply that $(-1)^{s(s-1)/2} = -1$, so $s \equiv 2, 3 \mod 4$. By the Grunwald-Wang theorem, we can assume that $d(V) = d(\Lambda)$. Therefore by [4, Thm 1.2, p. 56] we have $(-1)^s = -1$. We conclude from Theorem 1.3, p. 77, loc. cit. that the required V will exist if and only if $s \equiv 3 \mod 4$. Therefore the possible signatures are $(19, 3), (15, 7), (11, 11), (7, 15), (3, 19)$.   □

3 K3 surfaces

3.1 Facts about K3 surfaces

For convenience of the reader, we recall facts about K3 surfaces that we will use. All proofs may be found in the book [7] whose terminology we use.

Let V be a finite free $\mathbb{Z}$ or $\mathbb{Q}$-module. A Hodge structure of K3 type on V is a weight 2 Hodge structure such that $V^{2,0}$, $V^{0,2}$ are 1-dimensional and $V^{i,j} = 0$ for $|i - j| > 2$. Let Λ be the K3 lattice (see Example 2.1). The period domain D is defined as

$$D := \{x \in \mathbb{P}(\Lambda_{\mathbb{C}}) \colon (x, \bar{x}) > 0, \quad (x)^2 = 0\}$$

Given an element $x \in D$ we get a unique polarizable Hodge structure of K3 type on Λ satisfying $\Lambda^{2,0} = x$ and $x \perp \Lambda^{1,1}$. The key fact that we use about K3 surfaces is the surjectivity of the period map, which we recall here.

Theorem 3.1 (Surjectivity of the period map [7, Ch. 6, Rmk 3.3]) *For any $x \in D$ there exists a K3 surface X and a Hodge isometry $\varphi \colon H^2(X, \mathbb{Z}) \xrightarrow{\sim} \Lambda$ such that $\varphi^{-1}(x)$ spans $H^{2,0}(X)$.*

Following the definition of [8], two complex K3 surfaces X and X' are isogenous if there exists a $\mathbb{Q}$-Hodge isometry $H^2(X, \mathbb{Q}) \xrightarrow{\sim} H^2(X', \mathbb{Q})$. We record the following lemma to be used later.

Lemma 3.2 *Let X be a complex K3 surface, and let $\mathcal{S}(X)$ be the set of isomorphism classes of complex K3 surfaces Y that are isogenous to X. Then $\mathcal{S}(X)$ is countable.*

Proof The lemma will follow from a theorem of Huybrechts along with the countability of the Brauer group of a surface and finiteness results on Fourier-Mukai partners of twisted K3 surfaces. Recall that a twisted K3 surface (S, α) consists of a K3 surface S and an element α of the Brauer group of S. The set of twisted Fourier-Mukai partners of (S, α) is

$$\mathrm{FM}(S, \alpha) = \{(S', \alpha') \mid \text{there is an equivalence } D^b(S, \alpha) \cong D^b(S', \alpha')\}/\cong$$

where $D^b(S, \alpha)$ is the derived category of α-twisted coherent sheaves on S. All that is important for us is that for any K3 surface S and any $\alpha \in \mathrm{Br}(X)$ the set $\mathrm{FM}(S, \alpha)$ is countable, see [11, Prop 4.3]

Let $Y \in \mathcal{S}(X)$. Then by [8], we can find Brauer classes $\alpha \in \mathrm{Br}(X)$ and $\beta \in \mathrm{Br}(Y)$, complex K3 surfaces $S_1, \ldots, S_n$ and Brauer classes $\alpha_i, \beta_i \in \mathrm{Br}(S_i)$ for $i = 1, \ldots, n$ such that there is a chain of equivalences

$$(X, \alpha) \sim_{\mathrm{FM}} (S_1, \alpha_1), (S_1, \beta_1) \sim_{\mathrm{FM}} (S_2, \alpha_2), \ldots, (S_{n-1}, \beta_{n-1}) \sim_{\mathrm{FM}}$$
$$(S_n, \alpha_n), (S_n, \beta_n) \sim_{\mathrm{FM}} (Y, \beta)$$

Where we use $\sim_{\mathrm{FM}}$ to denote the relation of being a twisted Fourier-Mukai partner. Further, we may assume $n \leq 22$. Recall that $\mathrm{Br}(S_i) = H^2(S_i, \mathcal{O}_{S_i}^\times)_{\mathrm{tors}}$. From the exponential sequence, we get an exact sequence

$$0 \to H^1(X, \mathcal{O}_{S_i}^\times) \to H^2(S_i, \mathbb{Z}) \to H^2(S_i, \mathcal{O}_{S_i}) \to H^2(S_i, \mathcal{O}_{S_i}^\times) \to 0$$

and hence $H^2(S_i, \mathcal{O}_{S_i})$ is isomorphic as groups to $\mathbb{C}/\mathbb{Z}^{\oplus 22 - \rho(S_i)}$. It follows that $\mathrm{Br}(S_i)$ is isomorphic to $(\mathbb{Q}/\mathbb{Z})^{\oplus 22 - \rho(S_i)}$ which is countable. Hence, given X there are only countably many options for α and by countability of $\mathrm{FM}(X, \alpha)$ only countably many options for (S_1, α_1). Likewise, there are only countably many options for β_1 and hence only countably many options for (S_2, α_2). Proceeding in this fashion we conclude there are only countably many options for Y. $\square$

4 Motivic setup

We recall basics facts about motives and refer the reader to [1] for details. For a field K, write $\mathcal{P}_K$ for the category of smooth projective varieties over K. If F is a field of characteristic zero we write $\mathcal{M}_{K,F}$ for the category of pure homological motives over K with coefficients in F. There is a functor $h \colon \mathcal{P}_K \to \mathcal{M}_{K,F}$ that functions as a universal

cohomology theory, meaning that if $H^*\colon \mathcal{P}_K \to F\text{-Alg}$ is a Weil cohomology theory, then H^* extends uniquely through h to a functor $\mathcal{M}_{K,F} \to F\text{-Alg}$. Under the hom = num conjecture (which says that numerical equivalence is the same as homological equivalence) then the category $\mathcal{M}_{K,F}$ is a semisimple rigid abelian tensor category by [10] (for all of the categories of motives appearing in this paper, the hom = num conjecture is a consequence of the Hodge or Tate conjecture, which we will assume). A choice of a Weil cohomology theory H^* that extends to $\mathcal{M}_{K,F}$ is a fiber functor, making $\mathcal{M}_{K,F}$ a neutral Tannakian category. Thus by general theory, we have an equivalence between $\mathcal{M}_{K,F}$ and $\mathrm{Rep}_{\mathcal{G}_{K,F}}(F)$, the category of F representations of the pro-reductive algebraic group $\mathcal{G}_{K,F} = \mathrm{Aut}^{\otimes} H^*$. We recall the most basic examples of fiber functors, and the extra structures they carry.

Example 4.1 Let K be any field, K^{sep} be a separable closure and ℓ a prime. For X smooth projective over K, the ℓ-adic cohomology $H_\ell(X) = H^*_{\mathrm{et}}(X_{K^{\mathrm{sep}}}, \mathbb{Q}_\ell)$ is a Weil cohomology theory on $\mathcal{P}_K$. Further, $H_\ell(X)$ has a natural $\Gamma_K = \mathrm{Gal}(K^{\mathrm{sep}}/K)$-action, and we write $H_\ell\colon \mathcal{M}_{K,\mathbb{Q}_\ell} \to \mathrm{Rep}_{\mathbb{Q}_\ell}(\Gamma_K)$ for the enriched ℓ-adic realization functor. The Tate conjecture asserts that H_ℓ is fully faithful when K is a number field.

Example 4.2 Let $K = \mathbb{C}$. For a smooth projective variety X over $\mathbb{C}$ we can form the corresponding complex-analytic manifold X^{an}. Singular cohomology $H_B(X) = H^*_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{Q})$ is a Weil cohomology theory on $\mathcal{P}_\mathbb{C}$. Further, $H_B(X)$ has a $\mathbb{Q}$-Hodge structure, and we write $H_B\colon \mathcal{M}_\mathbb{C} \to \mathbb{Q}\text{-HS}$ for the enriched Betti realization functor. The Hodge conjecture asserts that H_B is fully faithful.

Example 4.3 Let K be a field of characteristic 0. For a smooth projective variety X over K we have the algebraic de Rham complex $\Omega^\bullet_{X/K}$. Algebraic de Rham cohomology $H_{dR}(X) = \mathbb{H}^*(X, \Omega^\bullet_{X/K})$ is a Weil cohomology theory on $\mathcal{P}_K$ (with coefficients in K). Further, $H_{dR}(X)$ has a filtration, and we write $H_{dR}\colon \mathcal{M}_{K,K} \to \mathrm{Fil}_K$ (with Fil_K the category of filtered K-vector spaces) for the corresponding enriched de Rham realization functor.

For a given embedding $\iota\colon \overline{\mathbb{Q}} \to \overline{\mathbb{Q}_\ell}$ let H_ι be the composition $\mathcal{M}_{K,\overline{\mathbb{Q}}} \to \mathcal{M}_{K,\overline{\mathbb{Q}_\ell}} \to \mathrm{Rep}_{\overline{\mathbb{Q}_\ell}}(\Gamma_K)$. The following lemma is taken from [13], with a slight weakening of the hypothesis due to [12], in which it is shown that the Tate conjecture implies the Grothendieck-Serre semisimplicity conjecture.

Lemma 4.4 (Lemma 3.3 [13]) *Assume the Tate and Fontaine–Mazur conjectures, and let K be a number field. If $r_\ell\colon \Gamma_K \to \mathrm{GL}_N(\mathbb{Q}_\ell)$ is an irreducible geometric Galois representation. Then there exists an object M of $\mathcal{M}_{K,\overline{\mathbb{Q}}}$ such that $r_\ell \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}_\ell} \cong H_\iota(M)$.*

5 Proof of main theorem

The first part of the proof follows closely that of [13, Thm 3.1], the main difference being that we have to worry about carrying the bilinear form through the motivic yoga.

Proof of Theorem 1.1 Let $\{\rho_\ell\colon \Gamma_K \to \mathrm{O}(\Lambda \otimes \mathbb{Q}_\ell)\}$ be as in the theorem. Fix an embedding $\iota_0\colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_{\ell_0}}$. Then from Lemma 4.4 (which extends to semisimple geometric representations) we have a motive M in $\mathcal{M}_{K,\overline{\mathbb{Q}}}$ such that $H_{\iota_0}(M) \cong \rho_{\ell_0} \otimes \overline{\mathbb{Q}_{\ell_0}}$. In fact, M has coefficients in some finite extension E of $\mathbb{Q}$ inside $\overline{\mathbb{Q}}$. Let $\rho\colon \mathcal{G}_{K,E} \to \mathrm{GL}_{22,E}$ be the associated motivic Galois representation. Now ι_0 induces some place λ_0 of E. If λ is a finite place of E (say $\lambda \mid \ell$) let ρ_λ be the λ-adic realization of ρ. Then as in [13], for almost all

places v of K there is an equality of the rational numbers

$$\mathrm{tr}(\rho_\lambda(\mathrm{Fr}_v)) = \mathrm{tr}(\rho_{\lambda_0}(\mathrm{Fr}_v)) = \mathrm{tr}(\rho_{\ell_0}(\mathrm{Fr}_v)) = \mathrm{tr}(\rho_\ell(\mathrm{Fr}_v)).$$

By Brauer-Nesbitt, a continuous semisimple Galois representation ρ of Γ_K is determined by $\mathrm{tr}(\rho(\sigma))$ for σ in a dense subset of Γ_K. By Chebotarev we may take the collection Fr_v for v as above and conclude that

$$\rho_\lambda \cong \rho_\ell \otimes_{\mathbb{Q}_\ell} E_\lambda$$

for all λ. Conditions (1) and (3) of our assumption say for a place $\lambda_1 \mid \ell_1$ that $\rho_{\ell_1} \otimes E_{\lambda_1}(1)$ splits as a sum of the trivial representation and an absolutely irreducible representation. By the Tate conjecture, we conclude that $\rho = \mathbf{1}(-1) \oplus \rho'$ (where $\mathbf{1}(-1)$ is the Tate twist of $\mathbf{1}$) with ρ' absolutely irreducible. It follows that each ρ_ℓ is isomorphic to $\mathbb{Q}_\ell(-1) \oplus V_\ell$ with V_ℓ an absolutely irreducible representation of Γ_K.

Lemma 3.4 of [13] shows that ρ' descends to $\mathbb{Q}$, and clearly $\mathbf{1}(-1)$ does, hence ρ descends to $\mathbb{Q}$. We have Γ_K-equivariant pairings $\rho_\ell \otimes \rho_\ell \to \mathbb{Q}_\ell$, and thus there are pairings $\rho_\lambda \otimes_E \rho_\lambda \to E_\lambda$. By the Tate conjecture, there is an isomorphism

$$\left(\mathrm{Sym}^2 \rho_E^\vee\right)^{\mathcal{G}_{K,E}} \otimes_E E_\lambda \cong \left(\mathrm{Sym}^2 \rho_\lambda^\vee\right)^{\Gamma_K}$$

Hence we get a non degenerate $\mathcal{G}_{K,E}$-equivariant pairing $\rho_E \otimes_E \rho_E \to E$. However each local pairing descends to $\mathbb{Q}_\ell$. By Galois descent the map

$$\mathrm{Hom}_{\mathcal{G}_K}(\rho \otimes_{\mathbb{Q}} \rho, \mathbb{Q}) \to \mathrm{Hom}_{\mathcal{G}_{K,E}}(\rho_E \otimes \rho_E, E)^{\Gamma_{\mathbb{Q}}}$$

is an isomorphism, and therefore the pairing on ρ_E descends to a pairing $\rho \otimes_{\mathbb{Q}} \rho \to \mathbb{Q}$.

Now that we have a motivic Galois representation $\rho \colon \mathcal{G}_K \to \mathrm{GL}_{22,\mathbb{Q}}$ whose ℓ-adic realizations are ρ_ℓ, we can do some comparisons. Let $M \in \mathcal{M}_K$ be the corresponding rank 22 motive. Objects of $\mathcal{M}_K$ enjoy the de Rham comparison theorem of p-adic Hodge theory. In particular, for v and ℓ_2 as in condition (4) there are isomorphisms:

$$H_{dR}(M) \otimes_K B_{dR,K_v} \xrightarrow{\sim} H_{\ell_2}(M) \otimes_{\mathbb{Q}_{\ell_2}} B_{dR,K_v}$$

Hence

$$H_{dR}(M) \otimes_K K_v \cong D_{dR,K_v}(H_{\ell_2}(M)) = \left(H_{\ell_2}(M) \otimes_{\mathbb{Q}_{\ell_2}} B_{dR,K_v}\right)^{\Gamma_{K_v}}$$

By assumption (4) and the comparison isomorphism, the Hodge filtration on $H_{dR}(M)$ satisfies

$$\dim_K \mathrm{gr}^i H_{dR}(M) = \begin{cases} 1 & \text{if } i = 0, 2, \\ 20 & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The Betti de-Rham comparison theorem states that

$$H_{dR}(M|_{\mathbb{C}}) \cong H_B(M) \otimes_{\mathbb{Q}} \mathbb{C},$$

so $H_B(M)$ is a Hodge structure of K3 type. It is also a $\mathbb{Q}$-quadratic space, coming from the motivic pairing. We will show that there is an isomorphism of $\mathbb{Q}$-quadratic spaces $H_B(M) \cong \Lambda \otimes \mathbb{Q}$ with Λ the K3 lattice $U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$. First, note that we have comparison isomorphisms $H_B(M) \otimes \mathbb{Q}_\ell \cong H_\ell(M)$ that respect the pairings on both spaces. By assumption $H_\ell(M) \cong \Lambda \otimes \mathbb{Q}_\ell$ as quadratic spaces, so to show that $H_B(M) \cong \Lambda \otimes \mathbb{Q}$

it is enough to show that $H_B(M)$ has the same signature as Λ, which is $(3, 19)$. Now $M = \mathbf{1}(-1) \oplus M'$ with M' absolutely irreducible. Thus we have an orthogonal decomposition of Hodge structures $H_B(M) = \mathbb{Q}(-1) \oplus H_B(M')$ with $H_B(M')$ irreducible.

We compute the possible signatures on $H_B(M')$. First, there is a smooth projective X such that $M' \hookrightarrow h^k(X)(j)$ for some integers k, j where $h^k(X)(j)$ is the motive whose realization is $H^k(X)(j)$. The pairing on M' is up to a scalar multiple the same as the intersection pairing coming from a polarization L on X because $\dim_{\mathbb{Q}}(\mathrm{Sym}^2 \rho'^{\vee})^{\mathcal{G}_K} = 1$. Since M' has weight 2, we know that k is even and $k - 2j = 2$. There is a decomposition of motives [1, Prop 5.2.5.1]

$$h^k(X) = \bigoplus_{r \leq k} L^r h_{\mathrm{prim}}^{k-2r}(X)(-r)$$

so $M' \subset L^r h_{\mathrm{prim}}^{k-2r}(X)(j-r)$ for some r and thus $H_B^{1,1}(M') \subset L^r H_{\mathrm{prim}}^{j+1-r,j+1-r}(X)$. The intersection pairing on this subspace is definite by the Hodge index Theorem [18, Thm 6.32]. By Lemma 2.2 the only possible signatures of $H_B(M)$ are $(3, 19), (7, 15), (11, 11), (15, 7)$ and $(19, 3)$ but since the form is definite on the 19-dimensional subspace $H^{1,1}(M') \subset H_B(M) \otimes \mathbb{C}$, the signature must be $(3, 19)$ or $(19, 3)$. The quadratic form restricted to $H_B(\mathbf{1}(-1))$ is determined by an element α of $\mathbb{Q}^{\times}/(\mathbb{Q}^{\times})^2$. Assumption (1) assures that the image of $H_B(\mathbf{1}(-1))$ under the comparison isomorphism

$$H_B(\mathbf{1}(-1)) \otimes \mathbb{Q}_{\ell} \cong H_{\ell}(\mathbf{1}(-1))$$

maps to the line spanned by the vector $(e + df) \otimes 1$ in $\Lambda \otimes \mathbb{Q}_{\ell}$ (using the notation in Theorem 1.1). The fact that $(e+df)^2 = 2d$ shows that the image of α in $\mathbb{Q}_{\ell}^{\times}/(\mathbb{Q}_{\ell}^{\times})^2$ is $2d$ for almost all ℓ. Thus by the Grunwald–Wang theorem, we have that $\alpha = 2d$. Consequently, there is an isomorphism of quadratic spaces $H_B(\mathbf{1}(-1)) \cong \mathbb{Q}(2d)$ where the bilinear form on $\mathbb{Q}(2d)$ is given by $(a, b) = 2dab$. In particular, the pairing on $H_B(\mathbf{1}(-1))$ is positive definite. We conclude that the signature on $H_B(M)$ is $(3, 19)$ which completes the proof that $H_B(M) \cong \Lambda \otimes \mathbb{Q}$ as quadratic spaces. Let $i \colon \Lambda \hookrightarrow H_B(M)$ be an embedding and write $\Lambda(M)$ for the image of Λ under this embedding.

Now $\Lambda(M)$ has an induced $\mathbb{Z}$-Hodge structure from that on $H_B(M)$. By surjectivity of the period map (Theorem 3.1) we know that there is a K3 surface X over $\mathbb{C}$ with a Hodge isometry $H^2(X, \mathbb{Z}) \cong \Lambda \subset H_B(M)$. The Hodge conjecture implies that $M|_{\mathbb{C}} \cong h^2(X)$ and further that this isomorphism respects the pairings on each motive. For each $\sigma \in \mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$ we have

$$h^2(X)^{\sigma} \cong M|_{\mathbb{C}}^{\sigma} = M|_{\mathbb{C}} \cong h^2(X),$$

where $M|_{\mathbb{C}}$ is the image of M under base change $\mathcal{M}_K \to \mathcal{M}_{\mathbb{C}}$ and $h^2(X)^{\sigma}, M|_{\mathbb{C}}^{\sigma}$ are the σ-conjugates of $h^2(X)$ and $M|_{\mathbb{C}}$. Since these isomorphisms respect the pairing, upon applying Betti realization we see that X is isogenous to each conjugate X^{σ}, hence by Corollary 6.4, X admits a model over $\overline{\mathbb{Q}}$. We denote this model by $X_{\overline{\mathbb{Q}}}$ and write $X_{\mathbb{C}}$ for the complex K3 surface above. The map

$$\mathrm{Hom}_{\mathcal{M}_{\overline{\mathbb{Q}}}}(h^2(X_{\overline{\mathbb{Q}}}), M|_{\overline{\mathbb{Q}}}) \to \mathrm{Hom}_{\mathcal{M}_{\mathbb{C}}}(h^2(X_{\mathbb{C}}), M|_{\mathbb{C}})$$

is an isomorphism, and hence there is an isomorphism $\alpha \colon h^2(X_{\overline{\mathbb{Q}}}) \to M|_{\overline{\mathbb{Q}}}$ in $\mathcal{M}_{\overline{\mathbb{Q}}}$. Let L be a finite extension of K such that both X and α are defined over L. Hence we get an isomorphism $\alpha \colon h^2(X) \to M|_L$ in $\mathcal{M}_L$ which yields isomorphisms

$$H_{\ell}(\alpha) \colon H^2(X_{\overline{L}}, \mathbb{Q}_{\ell}) \xrightarrow{\sim} H_{\ell}(M) = \rho|_{\Gamma_L} \otimes \mathbb{Q}_{\ell}$$

<div align="right">□</div>

6 Descent

In this section we prove that complex K3 surfaces that are isogenous to all of their $\overline{\mathbb{Q}}$-conjugates admit models over number fields. This will follow from a general spreading out argument.

Proposition 6.1 *Let L be an extension of a characteristic zero field K and E and F subextensions of L. Suppose that*

(1) *K is algebraically closed in one of E or F.*
(2) *E and F are algebraically disjoint over K.*

Then E and F are linearly disjoint over K.

Proof Let $\{x_i\}_{i \in I}$ be a transcendence basis for E over K and define $E_0 = K(\{x_i\}_{i \in I})$. Likewise let $\{y_j\}_{j \in J}$ be a transcendence basis for F over K and define $F_0 = K(\{y_j\}_{j \in J})$.

First, notice that E_0 and F_0 are linearly disjoint over K. This follows from the fact that they are algebraically disjoint over K, and E_0 and F_0 are purely transcendental extensions of K see by [3, Ch. V, §14, Prop. 14]. It also follows from the same proposition that E and F_0 are linearly disjoint over K, as are E_0 and F. The theorem holds as long as EF_0 and E_0F are linearly disjoint over E_0F_0, by [3, Ch. V, §14, Prop. 8]. In what follows, we will assume that K is algebraically closed in E.

We will show that $E \cap F = K$. We already know that $E_0 \cap F_0 = K$ as they are linearly disjoint over K. Any element $\alpha \in E \cap F$ is contained in a finite extension of E_0 and F_0 so we may assume for now that E is finite over E_0 and F is finite over F_0. We claim that $\mathrm{Tr}_{E/E_0}(\alpha) \in K$. Indeed since $\alpha \in EF_0$ and $\alpha \in E_0F$ we have the following equalities

$$
\begin{aligned}
[EF : E_0F]\mathrm{Tr}_{E_0F/E_0F_0}(\alpha) &= \mathrm{Tr}_{E_0F/E_0F_0}(\mathrm{Tr}_{EF/E_0F}(\alpha)) \\
&= \mathrm{Tr}_{EF/E_0F_0}(\alpha) \\
&= \mathrm{Tr}_{EF_0/E_0F_0}(\mathrm{Tr}_{EF/EF_0}(\alpha)) \\
&= [EF : EF_0]\mathrm{Tr}_{EF_0/E_0F_0}(\alpha)
\end{aligned}
$$

Notice that $\mathrm{Tr}_{EF_0/E_0F_0}(\alpha) = \mathrm{Tr}_{E/E_0}(\alpha)$ as $EF_0 \cong E \otimes_{E_0} E_0F_0$ (as the two fields are linearly disjoint over E_0 by the first remarks, and E is algebraic over E_0) and trace is invariant under extension of scalars. Therefore

$$
\mathrm{Tr}_{E/E_0}(\alpha) = \mathrm{Tr}_{EF_0/E_0F_0}(\alpha) = \frac{[EF : E_0F]}{[EF : EF_0]}\mathrm{Tr}_{E_0F/E_0F_0}(\alpha) = \frac{[EF : E_0F]}{[EF : EF_0]}\mathrm{Tr}_{F/F_0}(\alpha) \in F_0
$$

and consequently $\mathrm{Tr}_{E/E_0}(\alpha) \in E_0 \cap F_0 = K$. However, by the same reasoning $\mathrm{Tr}_{E/E_0}(\alpha^n) \in K$ for $n = 0, 1, 2, \ldots, [E : E_0]$. From Newton's identities, we see that the minimal polynomial of α over E_0 has coefficients in K, and therefore α is algebraic over K. But from the assumption that K is algebraically closed in E we see that $\alpha \in K$, as required.

Finally, to show that EF_0 and E_0F are linearly disjoint over E_0F_0 we may enlarge E and F so that they are normal over E_0 and F_0 respectively. Then EF_0 and E_0F are normal over E_0F_0 and hence it is enough to show that $EF_0 \cap E_0F = E_0F_0$. Suppose $x \in EF_0 \cap E_0F$ and write $x = ef_0 = e_0f$ for $e_0 \in E_0, e \in E, f_0 \in F_0$ and $f \in F$. Then $ee_0^{-1} = ff_0^{-1} \in E \cap F = K$. Therefore $e = (ee_0^{-1})e_0 \in KE_0 = E_0$ and therefore $x \in E_0F_0$ as required. □

Corollary 6.2 *Let Ω be an extension of a field k with Ω and k algebraically closed of characteristic zero. Suppose that Ω has uncountable transcendence degree over k. Then*

for any subextension E of Ω that is countably generated over k, there is an element $\sigma \in \mathrm{Aut}(\Omega/k)$ such that E and $\sigma(E)$ are linearly disjoint over k.

Proof Let $x_1, x_2, \ldots$ be a transcendence basis of E over k, and let $y_1, y_2, \ldots$ be any elements of Ω such that the collection $\{x_1, x_2, \ldots, y_1, y_2, \ldots\}$ are algebraically independent over k. Let $\sigma \in \mathrm{Aut}(\Omega/k)$ be any element satisfying $\sigma(x_i) = y_i$. By construction the fields E and $\sigma(E)$ are algebraically disjoint over k. As k is algebraically closed, the previous theorem shows E and $\sigma(E)$ are linearly disjoint over k. $\qquad\qquad\square$

Lemma 6.3 *Let Ω be an extension of a field k with Ω and k algebraically closed of characteristic zero, and Ω of uncountable transcendence degree over k. Let X be a variety over Ω whose conjugates X^σ for $\sigma \in \mathrm{Aut}(\Omega/k)$ are contained in a countable set. Then X admits a model over k.*

Proof We can choose $\sigma_1, \sigma_2 \ldots \in \mathrm{Aut}(\Omega/k)$ so that every conjugate of X is isomorphic to X^{σ_i} for some i. For each i, X^{σ_i} is defined over a finitely generated field extension K_i over k. Let $K \subset \Omega$ be the composite in Ω of $K_1, K_2 \ldots$ so that $X^{\sigma_1}, X^{\sigma_2}, \ldots$ (hence any conjugate of X) admit models over K. Note that K countably generated over k. Let $\tau \in \mathrm{Aut}(\Omega/k)$ be any automorphism with $\tau(K)$ and K linearly disjoint over k which exists by Corollary 6.2. Suppose X_0 is a model of X over K_i for some i. As X^τ is isomorphic (over Ω) to X^{σ_j} for some j, we know that X_0^τ is a model of X^{σ_j} over $\tau(K_i)$. Thus X^{σ_j} admits a model over K_j and $\tau(K_i)$. These are finitely generated and linearly disjoint over k, so X^{σ_j} admits a model over k, (this can be seen from the proof of [5, Thm. 1]). Hence X admits a model over k. $\square$

Corollary 6.4 *If X is a complex K3 surface and X is isogenous to X^σ for all $\sigma \in \mathrm{Aut}(\mathbb{C}/\overline{\mathbb{Q}})$ then X admits a model over $\overline{\mathbb{Q}}$.*

Proof By assumption $X^\sigma \in \mathcal{S}(X)$ where $\mathcal{S}(X)$ is the isogeny class of X. From Lemma 3.2 $\mathcal{S}(X)$ is countable and by the previous lemma it follows that X has a model over $\overline{\mathbb{Q}}$. $\qquad\square$

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