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Exact formulas for invariants of Hilbert schemes

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Abstract

A theorem of Göttsche establishes a connection between cohomological invariants of a complex projective surface S and corresponding invariants of the Hilbert scheme of n points on S . This relationship is encoded in certain infinite product q -series which are essentially modular forms. Here we make use of the circle method to arrive at exact formulas for certain specializations of these q -series, yielding convergent series for the signature and Euler characteristic of these Hilbert schemes. We also analyze the asymptotic and distributional properties of the q -series' coefficients.

Keywords: Hilbert schemes, Circle method, Modular forms, Topological invariants, Analytic number theory

Mathematics Subject Classification: 11-XX, 55-XX

1 Introduction and statement of results

K3 surfaces are complex surfaces characterized by their particularly simple Hodge structures and trivial holomorphic tangent bundles. These manifolds are of particular interest to physicists and mathematicians. For physicists, they serve as useful examples of Calabi-Yau manifolds, which are a class of spaces central in string theory, and for mathematicians, they serve as an interesting yet sufficiently simple example in 4-manifold theory and complex differential geometry. These two roles came together in an unexpected way when in [20] Yau and Zaslow conjectured (and later Beauville proved in [4]) that the count of n -nodal curves on a K3 surface is equal to the Euler characteristic $\chi(\text{Hilb}^n(S))$ of Hilbert schemes of n points on a K3 surface S .

Yau and Zaslow's conjecture made use of a previous theorem of Göttsche (see [8, p. 37]) that provides an infinite product encoding the Euler characteristics $\chi(\text{Hilb}^n(S))$ for Hilbert schemes of a K3 surface S . These Euler characteristics can be assembled in the form of the generating function

$$X_S(\tau) := \sum_{n=0}^{\infty} \chi(\text{Hilb}^n(S))q^n = \frac{q}{\Delta(\tau)} = \prod_{n=1}^{\infty} \frac{1}{(1 - q^n)^{24}}, \quad (1.1)$$

where $\Delta(\tau)$ is the modular discriminant and $q := e^{2\pi i\tau}$.

Göttsche stated a more refined infinite product formula concerning Hodge numbers, which are cohomological invariants that can be assembled into Betti numbers and the Euler characteristic. Recently, Manschot and Zapata Rolon [14] studied the asymptotic distribution of linear combinations of these Hodge numbers, which are realized as coeffi-

cients of Laurent polynomials called the χ_y genera for K3 surfaces (see (2.4) for a definition of χ_y). The χ_y genera can also be assembled using the generating function

$$Y_S(y; \tau) := \sum_{m,n} b_S(m; n) y^m q^n := \sum_{n \geq 0} \chi_y(\text{Hilb}^n(S)) q^n. \tag{1.2}$$

Akin to (1.1), Göttsche also provides an infinite product for $Y_S(y; \tau)$ (see Lemma 2.2).

Manschot and Zapata Rolon [14] found for K3 surfaces S that if m is fixed, then as $n \rightarrow \infty$ we have

$$b_S(m; n) \sim \frac{\pi}{3\sqrt{2}} n^{-\frac{29}{4}} \cdot \exp(4\pi\sqrt{n}).$$

Moreover, since the asymptotics for $b_S(m; n)$ do not depend on m , if we define

$$b_S^*(r; n) := \sum_{m \equiv r \pmod{2}} b_S(m; n),$$

it follows that $b_S^*(0; n) \sim b_S^*(1; n)$ as $n \rightarrow \infty$. From a geometric perspective, n corresponds to the length of the Hilbert scheme, and m corresponds to a particular monomial in the χ_y -genus of $\text{Hilb}^n(S)$. Grouping the contributions by the coefficients of these monomials in residue classes mod 2, we obtain an equidistribution in the limit as n goes to infinity.

Göttsche’s formula for Hodge numbers $h^{s,t}(\text{Hilb}^n(S))$ of Hilbert schemes of n points for K3 surfaces is a special case of his more general formula, which states that for any smooth projective complex surface S , we have

$$Z_S(x, y; \tau) := \sum_{n \geq 0} \chi_{\text{Hodge}}(\text{Hilb}^n(S)) q^n = \prod_{n=1}^{\infty} \frac{\prod_{s+t \text{ odd}} (1 - x^{s-1} y^{t-1} q^n)^{h^{s,t}}}{\prod_{s+t \text{ even}} (1 - x^{s-1} y^{t-1} q^n)^{h^{s,t}}}. \tag{1.3}$$

For later use, we define $c_S(s, t; n)$ to be the coefficient of $x^s y^t q^n$ in the power series expansion of $Z_S(x, y; \tau)$.

Remark The *Hodge polynomial* $\chi_{\text{Hodge}}(\text{Hilb}^n(S))$ is a Laurent polynomial in $\mathbb{Z}[x, y, x^{-1}, y^{-1}]$, and we will sometimes write it as $\chi_{\text{Hodge}}(\text{Hilb}^n(S))(x, y)$ to make explicit the Hodge polynomial’s dependence on x and y . If we specialize x and y to ± 1 , then $\chi_{\text{Hodge}}(\text{Hilb}^n(S))(x, y)$ evaluates to different linear combinations of important topological invariants. See Sect. 2 for a more detailed discussion of $\chi_{\text{Hodge}}(\text{Hilb}^n(S))(x, y)$.

We seek exact formulas for sequences assembled from the coefficients of (1.3) for a more general class of smooth projective complex surfaces. We consider the distribution of the coefficients $c_S(s, t; n)$ over residue classes mod 2; namely, if we let

$$c_S^*(r_1, r_2; n) := \sum_{\substack{t \equiv r_1 \pmod{2} \\ s \equiv r_2 \pmod{2}}} c_S(s, t; n), \tag{1.4}$$

we seek to determine the asymptotic properties of the sequences $b_S^*(r; n)$ and $c_S^*(r_1, r_2; n)$. To this end, we consider the asymptotics of the q -series

$$C_S(r_1, r_2; \tau) := \sum_{n \geq 0} c_S^*(r_1, r_2; n) q^n. \tag{1.5}$$

Our main result is the following. See Sect. 2 for a more detailed discussion of the cohomological invariants for which we state exact formulas and asymptotics below.

Theorem 1.1 *Let S be a smooth projective surface. Then we have the following exact formulas:*

(1) If $0 \leq \chi(S) < 24n$, then we have

$$\chi(\text{Hilb}^n(S)) = 2\pi \sum_{j < \frac{\chi(S)}{24}} \sum_{k=1}^{\infty} k^{\chi(S)/2} A_k(-\chi(S), 0, j; n) \chi(\text{Hilb}^j(S)) L^*(0, j, k; n).$$

(2) If $\sigma(S) \leq \chi(S) < 24n$, then we have

$$\begin{aligned} \sigma(\text{Hilb}^n(S)) = & 2\pi \sum_{j < \frac{\chi(S)}{24}} \sum_{\substack{k=2 \\ k \text{ even}}}^{\infty} \frac{A_k(\sigma(S), \Lambda(S), j; n) \sigma(\text{Hilb}^j(S))}{k^{\Lambda'(S)/2}} L^*(0, j, k; n) \\ & + 2\pi \sum_{j < \frac{3\sigma(S) - \chi(S)}{48}} \sum_{\substack{k=1 \\ k \text{ odd}}}^{\infty} \frac{(-1)^n B_k(\sigma(S), \Lambda(S), j; n) a(\Lambda(S), \sigma(S); j)}{2^{\Lambda(S)/2} k^{\Lambda'(S)/2}} L^*(1, j, k; n). \end{aligned}$$

where $\Lambda(S) := -(\sigma(S) + \chi(S))/2$ and $\Lambda'(S) := (\sigma(S) - \chi(S))/2$.

Here $a(\alpha, \beta, j)$, $A_k(\alpha, \beta, j; n)$, $B_k(\alpha, \beta, j; n)$, $L^*(0, j, k; n)$, and $L^*(1, j, k; n)$ are defined in Sects. 3, (2.19), (2.21), (4.4), and (5.4), respectively.

Remark In Theorem 1.1, $A_k(\alpha, \beta, j; n)$ and $B_k(\alpha, \beta, j; n)$ are known as Kloosterman sums, and $L^*(0, j, k; n)$ and $L^*(1, j, k; n)$ are, up to simple multiplicative factors, modified Bessel functions of the first kind.

Theorem 1.1 offers exact formulas as convergent infinite series. These formulas imply the following asymptotics.

Corollary 1.2 *Let S be a smooth projective surface. Then the following are true:*

(1) Suppose $\chi(S) \geq \sigma(S)$.

(a) If $\sigma(S) < 0$, then as $n \rightarrow \infty$, we have

$$\begin{aligned} \sigma(\text{Hilb}^n(S)) \sim & (-1)^n 2^{\frac{7\sigma(S) - 3\chi(S) - 14}{8}} 3^{\frac{\sigma(S) - \chi(S) - 2}{8}} \\ & (\chi(S) - 3\sigma(S))^{\frac{\chi(S) - \sigma(S) + 2}{8}} n^{\frac{\sigma(S) - \chi(S) - 6}{8}} \\ & \cdot \exp\left(\pi \sqrt{\frac{\chi(S) - 3\sigma(S)}{6} n}\right). \end{aligned}$$

(b) If $\sigma(S) > 0$, then as $n \rightarrow \infty$, we have

$$\begin{aligned} \sigma(\text{Hilb}^n(S)) \sim & 2^{\frac{3\sigma(S) - 3\chi(S) - 14}{8}} 3^{\frac{\sigma(S) - \chi(S) - 2}{8}} \\ \chi(S) &^{\frac{\chi(S) - \sigma(S) + 2}{8}} n^{\frac{\sigma(S) - \chi(S) - 6}{8}} \exp\left(\pi \sqrt{\frac{\chi(S)}{6} n}\right). \end{aligned}$$

(c) If $\sigma(S) = 0$ and $\chi(S) \neq 0$, then as $n \rightarrow \infty$, we have

$$\begin{aligned} \sigma(\text{Hilb}^{2n}(S)) \sim & 2^{\frac{-\chi(S) - 3}{2}} 3^{\frac{-\chi(S) - 2}{8}} \\ \chi(S) &^{\frac{2 + \chi(S)}{8}} n^{\frac{-\chi(S) - 6}{8}} \exp\left(\pi \sqrt{\frac{\chi(S)}{3} n}\right). \end{aligned}$$

Moreover, for all n we have

$$\sigma(\text{Hilb}^{2n+1}(S)) = 0.$$

(d) If $\sigma(S) = \chi(S) = 0$, we have for all n ,

$$\sigma(\text{Hilb}^n(S)) = 0.$$

(2) (a) If $\chi(S) > 0$, then, as $n \rightarrow \infty$, we have

$$\chi(\text{Hilb}^n(S)) \sim 2^{\frac{-3\chi(S)-5}{4}} 3^{\frac{-\chi(S)-1}{4}} \chi(S)^{\frac{\chi(S)+1}{4}} n^{\frac{-\chi(S)-3}{4}} \exp\left(\pi\sqrt{\frac{2\chi(S)}{3}}n\right).$$

(b) If $\chi(S) = 0$, then for all n we have

$$\chi(\text{Hilb}^n(S)) = 0.$$

The asymptotics in Corollary 1.2 imply the following asymptotic properties of $b_S^*(r; n)$ and $c_S^*(r_1, r_2; n)$.

Corollary 1.3 *Let S be a smooth projective surface. Then the following are true:*

(1) *Suppose that $\chi(S) \geq \sigma(S)$. If $\chi(S) + \sigma(S) > 0$, then as $n \rightarrow \infty$ we have*

$$b_S^*(0; n) \sim b_S^*(1; n).$$

If $\chi(S) + \sigma(S) = 0$, then as $n \rightarrow \infty$ we have

$$b_S^*(1; n) = 0.$$

If $\chi(S) + \sigma(S) < 0$, then as $n \rightarrow \infty$ we have

$$b_S^*(0; n) \sim -b_S^*(1; n).$$

(2) *Suppose that $\chi(S) \geq \sigma(S)$. If $h^{1,0} = 0$, then as $n \rightarrow \infty$ we have*

$$c_S^*(0, 0; n) \sim c_S^*(1, 1; n) \text{ and } c_S^*(0, 1; n) = c_S^*(0, 1; n) = 0.$$

If $h^{1,0} > 0$, then as $n \rightarrow \infty$ we have

$$c_S^*(0, 0; n) \sim c_S^*(1, 1; n) \sim -c_S^*(0, 1; n) = -c_S^*(1, 0; n).$$

Note that when S is a K3 surface, Corollary 1.3 (1) recovers the equidistribution of the $b_S^*(r; n)$ that follows from the work of Manschot and Zapata Rolon in [14].

Remark The Enriques-Kodaira Classification Theorem [3, p. 244] mostly describes the possible Hodge structures of smooth complex surfaces. This theorem lists minimal models for many birational equivalence classes of surfaces, which determine $h^{0,0}$, $h^{1,0}$, $h^{2,0}$, and $\min\{h^{1,1}\}$ for that class. By the blowup construction, every smooth complex projective surface S is birationally equivalent to a smooth complex projective surface S' with $h^{1,1}(S') = h^{1,1}(S) + 1$ and $h^{s,t}(S') = h^{s,t}(S)$ for $(s, t) \neq (1, 1)$. A minimal model of a birational equivalence class is a surface that is not a blowup of any other smooth surface. Since $\chi(S)$ and $\chi(S) - \sigma(S)$ increase linearly with $h^{1,1}(S)$, Theorem 1.1, Corollary 1.2, and Corollary 1.3 apply to all surfaces in each birational equivalence class except those whose Hodge structures fall in a certain finite set. In particular, if the minimal model satisfies the hypotheses of these statements, then all surfaces in that class satisfy them. Excluding surfaces of general type, the only classes of projective surfaces whose minimal models do not satisfy these hypotheses are ruled surfaces of genus $g \geq 2$ (see [3] and [18]).

In Sect. 2, we will define all of the terms above and describe their topological and geometric significance. We will also state important properties of the generating functions of these sequences, which will be crucial in our deduction of the above formulas. In Sect. 3 we will outline our use of the circle method to prove Theorem 6.1, a general result from which Theorem 1.1 and Corollaries 1.2 and 1.3 are derived. Sections 4 and 5 include arguments necessary for this proof. In Sect. 6, these arguments are assembled, and Theorem 6.1, Theorem 1.1, and Corollaries 1.2 and 1.3 are proven. In Sect. 7 we illustrate our results with numerics.

2 Preliminaries

In this section, we present Göttsche’s result and specialize it in terms of weakly holomorphic modular forms. We then provide bounds on certain exponential sums known as Kloosterman sums and bounds on I -Bessel functions. Both Kloosterman sums and Bessel functions will appear in our application of the circle method in Sects. 4 and 5.

A compact complex manifold M has cohomological invariants called the Hodge numbers $h^{s,t} := h^{s,t}(M)$, which are defined as the complex dimensions of the (s, t) -Dolbeault cohomology space $H^{s,t}(M)$ (see [19]). When the context is clear, we will not explicitly indicate dependence of the Hodge numbers $h^{s,t}$ on the manifold M . If M is a Kähler manifold, the Hodge numbers are related to the Betti numbers $b_n(M)$ by the formula

$$b_n(M) = \sum_{s+t=n} h^{s,t}(M) \tag{2.1}$$

(see [19, p. 198]). Moreover, for any manifold M one can construct a manifold $\text{Hilb}^n(M)$ which can be thought of as a smoothed version of the n^{th} symmetric product of M (see [12]). For any smooth projective complex surface S , Göttsche’s formula allows one to compute the Hodge numbers of $\text{Hilb}^n(S)$ for all n from the Hodge numbers of S .

In order to state Göttsche’s result, we first define the Hodge polynomial, which serves as a generating function for the Hodge numbers of M :

$$\chi_{\text{Hodge}}(M)(x, y) := x^{-d/2} y^{-d/2} \sum_{s,t} h^{s,t}(M) (-x)^s (-y)^t, \tag{2.2}$$

where d is the complex dimension of M . We will generally suppress the (x, y) for notational convenience.

On page 37 of [8], Göttsche proved the remarkable fact that one can assemble the Hodge polynomial for $\text{Hilb}^n(S)$ using the Hodge numbers $h^{s,t}(S)$.

Theorem 2.1 (Göttsche) *If S is a smooth projective complex surface, then we have that*

$$Z_S(x, y; \tau) := \sum_{n \geq 0} \chi_{\text{Hodge}}(\text{Hilb}^n(S)) q^n = \prod_{n=1}^{\infty} \frac{\prod_{s+t \text{ odd}} (1 - x^{s-1} y^{t-1} q^n)^{h^{s,t}}}{\prod_{s+t \text{ even}} (1 - x^{s-1} y^{t-1} q^n)^{h^{s,t}}}.$$

Remark Note that for each $n, |s|, |t| \leq n$ in the definition of $\chi_{\text{Hodge}}(\text{Hilb}^n(S))$.

By [2, p. 43], every such smooth projective complex surface S is Kähler, so by Serre duality and Hodge symmetry, the Hodge numbers satisfy the following relations:

$$h^{0,0} = h^{2,2} \quad h^{1,0} = h^{0,1} = h^{1,2} = h^{2,1} \quad h^{2,0} = h^{0,2}.$$

By the additivity of the Hodge numbers, we need only consider the case where M is connected, i.e. $h^{0,0} = 1$. In this case, we obtain:

$$Z_S(x, y; \tau) = \prod_{n=1}^{\infty} \frac{\left((1 - x^{-1}q^n)(1 - xq^n)(1 - y^{-1}q^n)(1 - yq^n) \right)^{h^{1,0}}}{(1 - x^{-1}y^{-1}q^n)(1 - xyq^n) \left((1 - x^{-1}yq^n)(1 - xy^{-1}q^n) \right)^{h^{2,0}} (1 - q^n)^{h^{1,1}}}. \tag{2.3}$$

This polynomial gives rise to other topological invariants upon substituting ± 1 for x and y . For example, the Hirzebruch χ_y -genus of M is the polynomial

$$\chi_y(M) := \sum_{s,t} (-1)^t h^{s,t}(M) y^s. \tag{2.4}$$

We can express this in terms of the Hodge polynomial:

$$\chi_{\text{Hodge}}(\text{Hilb}^n(S))(y, 1) = y^{-n} \sum_{s,t} (-1)^t h^{s,t}(\text{Hilb}^n(S)) (-y)^s = \chi_{-y}(\text{Hilb}^n(S)) y^{-n}.$$

In terms of the Betti numbers, the Euler characteristic $\chi(M)$ is defined as

$$\chi(M) := \sum_n (-1)^n b_n(M). \tag{2.5}$$

Setting $x = 1$ and $y = 1$ in the Hodge polynomial, we see in reference to (2.1) and (2.5) that

$$\chi_{\text{Hodge}}(\text{Hilb}^n(M))(1, 1) = \chi_{-1}(\text{Hilb}^n(M)) = \chi(\text{Hilb}^n(M)). \tag{2.6}$$

On the other hand, setting $x = -1$ and $y = 1$ in the Hodge polynomial gives

$$\chi_{\text{Hodge}}(\text{Hilb}^n(M))(-1, 1) = (-1)^n \chi_1(\text{Hilb}^n(M)) = (-1)^n \sigma(\text{Hilb}^n(S)), \tag{2.7}$$

where the signature $\sigma(M)$ of a d -dimensional complex manifold M is the signature of the intersection pairing on $H^d(M)$ (see [15]). In terms of Hodge numbers of Kähler surfaces S , the signature is given by

$$\sigma(S) = 2h^{2,0} + 2 - h^{1,1}.$$

The discussion above indicates the importance of Göttsche’s infinite product formulas (Theorem 2.1) as a vehicle for studying invariants of complex projective surfaces. This is further illuminated by the following specializations of Theorem 2.1 in terms of these invariants.

Lemma 2.2 *If S is a smooth projective complex surface, then the following are true:*

$$\sum_{n=0}^{\infty} \chi_{-y}(\text{Hilb}^n(S)) y^{-n} q^n = \prod_{n=1}^{\infty} \frac{\left((1 - y^{-1}q^n)(1 - yq^n) \right)^{h^{1,0} - h^{2,0} - 1}}{(1 - q^n)^{h^{1,1} - 2h^{1,0}}}, \tag{2.8}$$

$$\sum_{n=0}^{\infty} \chi(\text{Hilb}^n(S)) q^n = \prod_{n=1}^{\infty} (1 - q^n)^{-\chi(S)}, \tag{2.9}$$

$$\sum_{n=0}^{\infty} (-1)^n \sigma(\text{Hilb}^n(S)) q^n = \prod_{n=1}^{\infty} \frac{(1 - q^n)^{\sigma(S)}}{(1 - q^{2n})^{(\sigma(S) + \chi(S))/2}}. \tag{2.10}$$

Remark We note that (2.9) and (2.10) are alternate expressions for $Z(1, 1; \tau)$ and $Z(1, -1; \tau) = Z(-1, 1; \tau)$, respectively. We can apply the same process to $Z(-1, -1; \tau)$ to obtain

$$Z(-1, -1; \tau) = \prod_{n=1}^{\infty} \frac{(1 - q^n)^{4h^{1,0}}}{(1 - q^{2n})^{\chi(S)+8h^{1,0}}}. \tag{2.11}$$

We now show that these functions can be assembled in linear combinations to give alternate formulas for $B_S(r; \tau)$ and $C_S(r_1, r_2; \tau)$.

Lemma 2.3 *Let S be a smooth projective complex surface. We have*

$$B_S(r, \ell; n) = \frac{1}{2} (Z_S(1, 1; \tau) + (-1)^r Z_S(1, -1; \tau)) \tag{2.12}$$

and

$$C_S(r_1, r_2; \tau) = \frac{1}{4} \sum_{\substack{j_1 \bmod 2 \\ j_2 \bmod 2}} (-1)^{j_2 r_2} (-1)^{j_1 r_1} Z_S((-1)^{j_2}, (-1)^{j_1}; \tau), \tag{2.13}$$

where $C_S(r_1, r_2; \tau)$ is defined by (1.5).

Proof We prove only (2.13), since (2.12) follows from even simpler manipulations. We have

$$\begin{aligned} \sum_{n \geq 0} c_S^*(r_1, r_2; n) q^n &= \frac{1}{4} \sum_{s, t, n} c_S(s, t; n) \sum_{\substack{j_1 \bmod 2 \\ j_2 \bmod 2}} (-1)^{j_2(s+r_2)} (-1)^{j_1(t+r_1)} q^n \\ &= \frac{1}{4} \sum_{\substack{j_1 \bmod 2 \\ j_2 \bmod 2}} (-1)^{j_2 r_2} (-1)^{j_1 r_1} \sum_{s, t, n} c_S(s, t; n) (-1)^{j_2 s} (-1)^{j_1 t} q^n \\ &= \frac{1}{4} \sum_{\substack{j_1 \bmod 2 \\ j_2 \bmod 2}} (-1)^{j_2 r_2} (-1)^{j_1 r_1} Z_S((-1)^{j_2}, (-1)^{j_1}; \tau). \end{aligned}$$

□

2.1 Modularity properties of specializations of Göttsche’s identity

The infinite products (2.9), (2.10), and (2.11) can be written in terms of the Dedekind eta function, the weight 1/2 modular form on $SL_2(\mathbb{Z})$ (with multiplier) that is defined by

$$\eta(\tau) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \tag{2.14}$$

This is done explicitly in the following lemma.

Lemma 2.4 *Let S be a smooth projective complex surface. We have that*

$$Z_S(1, 1; \tau) = q^{\chi(S)/24} \frac{1}{\eta(\tau)^{\chi(S)}}, \tag{2.15}$$

$$Z_S(-1, 1; \tau) = Z_S(1, -1; \tau) = q^{\chi(S)/24} \frac{\eta(\tau)^{\sigma(S)}}{\eta(2\tau)^{(\sigma(S)+\chi(S))/2}}, \tag{2.16}$$

$$Z_S(-1, -1; \tau) = q^{\chi(S)/24} \frac{\eta(2\tau)^{4h^{1,0}}}{\eta(\tau)^{\chi(S)+8h^{1,0}}}. \tag{2.17}$$

Having expressed $Z_S(\pm 1, \pm 1; \tau)$ as an eta quotient (up to a fractional power of q), we define

$$H_{\alpha,\beta}(q) := q^{\chi(S)/24} \eta(\tau)^\alpha \eta(2\tau)^\beta = Z_S(x, y; \tau), \tag{2.18}$$

where $x, y = \pm 1$ and α and β are determined by x, y , and the Hodge numbers of S . Thus, (up to a fractional power of q), $H_{\alpha,\beta}$ is a modular form on $\Gamma_0(2)$. In our application of the circle method to prove our main results, we use the modularity of $H_{\alpha,\beta}$ to determine the behavior of this function when τ is near the rational number h/k . The modular transformation equations for $H_{\alpha,\beta}$ are given in the following lemma. The q -expansion of $H_{\alpha,\beta}$ gives the behavior of $H_{\alpha,\beta}$ for τ near $i\infty$, and this lemma relates the behavior of $H_{\alpha,\beta}$ near $i\infty$ to its behavior near h/k . We present two transformation equations, corresponding to the two different cusps of $\Gamma_0(2)$. These cusps partition the rationals h/k into two parts, based on the parity of k . The two cusps will make distinct contributions to our final formula.

Lemma 2.5 *Let k be a positive integer, let h satisfy the condition $(h, k) = 1$, and let h' satisfy $hh' \equiv -1 \pmod k$. Further suppose $\text{Re}(z) > 0$. Define*

$$\omega_{\alpha,\beta}(h, k) := \exp(-\pi i(\alpha s(h, k) + \beta s(2h, k))),$$

where $s(h, k)$ is the Dedekind sum

$$s(h, k) := \sum_{r=1}^{k-1} \left(\left(\frac{r}{k} \right) \right) \left(\left(\frac{hr}{k} \right) \right),$$

and $((x))$ is the sawtooth function

$$((x)) := \begin{cases} 0 & x \in \mathbb{Z} \\ x - [x] - \frac{1}{2} & x \notin \mathbb{Z}. \end{cases}$$

(1) *If $2 \mid k$, we have*

$$\begin{aligned} & H_{\alpha,\beta} \left(\exp \left(\frac{2\pi ih}{k} - \frac{2\pi z}{k} \right) \right) \\ &= z^{-\frac{\alpha+\beta}{2}} \omega_{\alpha,\beta}(h, k) \cdot \exp \left((\alpha + 2\beta) \frac{\pi}{12k} \left(z - \frac{1}{z} \right) \right) \\ &\quad \times H_{\alpha,\beta} \left(\exp \left(\frac{2\pi ih'}{k} - \frac{2\pi}{zk} \right) \right). \end{aligned}$$

(2) *If $2 \nmid k$, we have*

$$\begin{aligned} & H_{\alpha,\beta} \left(\exp \left(\frac{2\pi ih}{k} - \frac{2\pi z}{k} \right) \right) \\ &= \frac{z^{-\frac{\alpha+\beta}{2}} \omega_{\alpha,\beta}(h, k)}{2^{\beta/2}} \cdot \exp \left(\frac{\pi}{24k} \left(2(\alpha + 2\beta)z - \frac{2\alpha + \beta}{z} \right) \right) \\ &\quad \times H_{\beta,\alpha} \left(\exp \left(\frac{\pi ih'}{k} - \frac{\pi}{zk} \right) \right). \end{aligned}$$

Proof From [1, p. 96], we know

$$\begin{aligned} & H_{-1,0} \left(\exp \left(\frac{2\pi ih}{k} - \frac{2\pi z}{k} \right) \right) \\ &= z^{\frac{1}{2}} \exp(\pi i s(h, k)) \exp \left(\frac{\pi}{12k} \left(\frac{1}{z} - z \right) \right) H_{-1,0} \left(\exp \left(\frac{2\pi ih'}{k} - \frac{2\pi}{kz} \right) \right), \end{aligned}$$

where $(h, k) = 1$, $\tau = (iz + h)/k$, $hh' \equiv -1 \pmod k$, and $s(h, k)$ is the Dedekind sum defined above. Similarly, [9] shows that if $2 \mid k$, then

$$\begin{aligned} & H_{-1,1} \left(\exp \left(\frac{2\pi ih}{k} - \frac{2\pi z}{k} \right) \right) \\ &= \exp(\pi i(\sigma(h, k))) \exp \left(\frac{\pi}{12k} \left(z - \frac{1}{z} \right) \right) H_{-1,1} \left(\exp \left(\frac{2\pi ih'}{k} - \frac{2\pi}{zk} \right) \right), \end{aligned}$$

where $\sigma(h, k) := s(h, k) - s(2h, k)$, and if $2 \nmid k$

$$\begin{aligned} & H_{-1,1} \left(\exp \left(\frac{2\pi ih}{k} - \frac{2\pi z}{k} \right) \right) \\ &= 2^{-\frac{1}{2}} \exp(\pi i \sigma(h, k)) \exp \left(\frac{\pi}{24k} \left(\frac{1}{z} + 2z \right) \right) H_{-1,1} \left(\exp \left(\frac{\pi ih'}{k} - \frac{\pi}{zk} \right) \right)^{-1}. \end{aligned}$$

To prove the lemma, we note that $H_{\alpha,\beta}(q) = H_{-1,0}(q)^{-(\alpha+\beta)} \cdot H_{-1,1}(q)^\beta$ and $H_{\alpha,0}(q^2) = H_{0,\alpha}(q)$. □

Remark When we apply the circle method to find the coefficients of the q -expansion of $H_{\alpha,\beta}$ in Sects. 3, 4, and 5, we consider $\tau \in \mathbb{H}$ on the horizontal line segment

$$L = \{ \tau \mid \tau = u + N^{-2}i, 0 \leq u \leq 1 \}.$$

As $N \rightarrow \infty$, L approaches the positive real axis directly from above. As discussed in Sect. 3, for the purpose of estimates we partition L into line segments that have midpoints at $h/k + iN^{-2}$ for cusp representatives $0 \leq h/k \leq 1$ and $k \leq N$. When we reparameterize these line segments in terms of the variable z introduced in the statement of Lemma 2.5, they become small vertical line segments just to the right of the origin in the z -plane. Lemma 2.5 allows us to estimate the value of $H_{\alpha,\beta}$ for z on these small vertical line segments.

2.2 Bounds on certain Kloosterman sums

In order to bound the error terms that will result from estimating the contour integral (3.1), we need bounds on Kloosterman sums of the form

$$A_k(\alpha, \beta, j; n) := \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \omega_{\alpha,\beta}(h, k) \exp \left(-\frac{2\pi inh}{k} + \frac{2\pi ih'j}{k} \right) \tag{2.19}$$

$$=: \sum_{\substack{0 \leq h < k \\ (h,k)=1}} A_{h,k}(\alpha, \beta, j; n) \tag{2.20}$$

and

$$B_k(\alpha, \beta, j; n) := \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \omega_{\alpha,\beta}(h, k) \exp\left(-\frac{2\pi inh}{k} + \frac{\pi ih'j}{k}\right) \tag{2.21}$$

$$=: \sum_{\substack{0 \leq h < k \\ (h,k)=1}} B_{h,k}(\alpha, \beta, j; n). \tag{2.22}$$

These sums admit the trivial estimate $O(k)$. However, in the case $\alpha + \beta = 0$, we will need a sharper estimate. In addition, we will occasionally restrict the values of k_i to an interval $N - k < k_i \leq \sigma < N$, which will in turn restrict h' to one or two intervals modulo k . For bounding purposes, it suffices to consider sums of the form

$$\sum'_{\substack{0 \leq h < k \\ (h,k)=1}} A_{h,k}(\alpha, \beta, j; n) \quad \text{and} \quad \sum'_{\substack{0 \leq h < k \\ (h,k)=1}} B_{h,k}(\alpha, \beta, j; n)$$

where the $'$ indicates that h' is restricted to an interval $0 \leq \sigma_1 \leq h' < \sigma_2 \leq k$. Thus we will need the following lemma.

Lemma 2.6 *For α and β fixed, $\alpha + \beta = 0$, the sums*

$$\sum'_{\substack{0 \leq h < k \\ (h,k)=1}} A_{h,k}(\alpha, \beta, j; n) \quad \text{and} \quad \sum'_{\substack{0 \leq h < k \\ (h,k)=1}} B_{h,k}(\alpha, \beta, j; n)$$

are subject to the estimate

$$O(n^{1/3} k^{2/3+\varepsilon})$$

uniformly in σ_1, σ_2 , and j .

Proof The proof is a simple adaptation of those of Theorems 2 and 3 of [9], which follow the proof of Theorem 2 in [13]. Equations (3.5) and (4.11) in [9] together state that if $2 \mid k$, we have

$$\exp(\pi i(s(h, k) - s(2h, k))) = \exp\left(2\pi i\left(\frac{4\phi(uh + vh')}{(k/2, 2)Gk} + r(h, k)\right)\right)$$

where $r(h, k)$ is a rational number that depends on k and $h \pmod 4$, ϕ, G , and v are integers that depend only on k , and u is a polynomial in k . (5.7) states that if $2 \nmid k$, then

$$\exp(\pi i(s(h, k) - s(2h, k))) = \exp\left(2\pi i\left(\frac{\Phi(uh + vh')}{gk} + r(h, k)\right)\right)$$

with r, v and u as before, and Φ and g integers dependent on k . The proof of Lemma 2.6 proceeds as in [9], with u, v , and r replaced with $\beta u, \beta v$ and βr , respectively. \square

2.3 Bounds on I -Bessel functions

As noted in the introduction, an important ingredient in our exact formulas is the modified Bessel function of the first kind, which is also known as the I -Bessel function. Here we recall a number of facts from [7] about I -Bessel functions that will be useful throughout our arguments. We define the I -Bessel function of order ν as

$$I_\nu(z) := \left(\frac{z}{2}\right)^\nu \sum_{k=0}^\infty \frac{\left(\frac{1}{4}z^2\right)^k}{k! \Gamma(\nu + k + 1)}.$$

From this definition it follows that for $0 < x < 1$, we have

$$|I_\nu(x)| < \frac{4}{3} \cdot \left(\frac{x}{2}\right)^\nu. \tag{2.23}$$

We also recall an integral representation of the I -Bessel function which we will use in Sects. 4 and 5: for $\nu \geq 0$, we have

$$I_\nu(z) = \frac{\left(\frac{1}{2}z\right)^{-\nu}}{2\pi i} \int_R t^{\nu-1} \exp\left[t + \frac{z^2}{4t}\right] dt, \tag{2.24}$$

where R is any simple closed contour surrounding the origin.

3 Outline of the circle method

In this section, we outline our use of the circle method to find exact formulas for coefficients $a(\alpha, \beta; n)$ of $H_{\alpha, \beta}$ where $\alpha + \beta \leq 0$. This corresponds to those cases where $H_{\alpha, \beta}$ is essentially a modular form of non-positive weight. These cases are special because non-constant holomorphic modular forms do not exist in non-positive weight. We make use of the method of Rademacher, introduced in [17], which improved upon the earlier work of G. H. Hardy and Srinivasa Ramanujan on the partition function. We adapt the implementation of this method along the lines of earlier works by Hagis (see [9, 10], and [11]).

The basic ingredient of the circle method is Cauchy’s integral formula, which we use to obtain

$$a(\alpha, \beta; n) = \frac{1}{2\pi i} \int_C \frac{H_{\alpha, \beta}(q)}{q^{n+1}} dq. \tag{3.1}$$

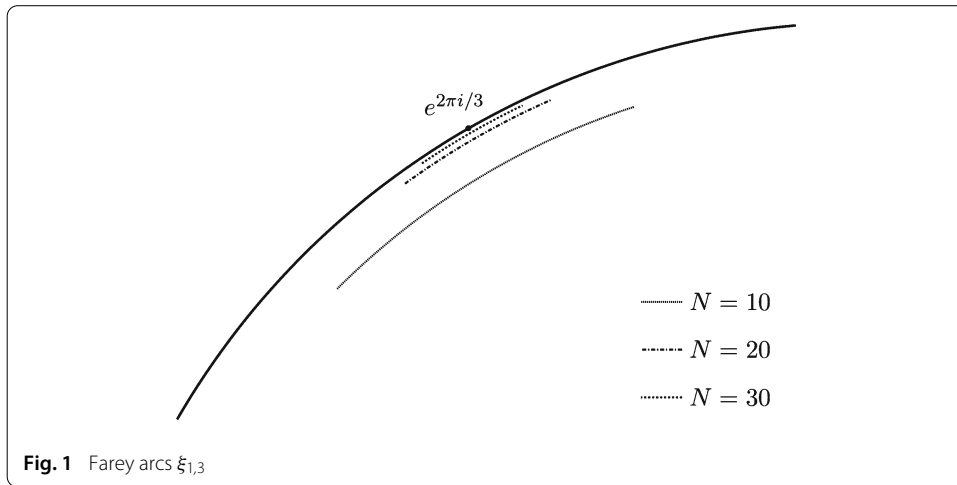
We let C be the circle of radius $e^{-2\pi N^{-2}}$ centered at the origin, where N is an upper bound on the denominator of the cusp representatives under consideration. We emphasize that we choose C to be a circle with this particular radius; there are many paths that could be chosen in applying Cauchy’s integral formula, but the one we choose here is a particular choice following the implementation of the circle method in [9, 10], and [11]. The exponent n will be fixed, and N will later be allowed to approach infinity. As $N \rightarrow \infty$, the main contribution to this integral comes from a dense set of poles on the boundary of the unit circle, where each pole is located at a root of unity $e^{2\pi ih/k}$. The locations of these poles follow from the definition of $H_{\alpha, \beta}$ in (2.18) and the locations of the zeros of $\eta(\tau)$ afforded by the product representation (2.14).

The generating function $H_{\alpha, \beta}$ is, up to a fractional power of q , a modular form with non-positive weight on $\Gamma_0(2)$ in the cases we consider. As shown in Lemma 2.5, we have that $H_{\alpha, \beta}$ has a pole of order $-(\alpha + 2\beta)/24$ at the cusp of $\Gamma_0(2)$ whose representatives have all even denominators and a pole of order $-(2\alpha + \beta)/48$ at the cusp whose representatives have all odd denominators.

Since we have descriptions of $\bar{H}_{\alpha, \beta}$ near all roots of unity, we divide C into *Farey arcs* $\xi_{h,k}$ for $(h, k) = 1$, which decrease in length and approach the point $e^{2\pi ih/k}$ as N increases. Figure 1 shows the Farey arc $\xi_{1,3}$ for $N = 10, 20$, and 30 . By estimating $H_{\alpha, \beta}$ along these Farey arcs, we obtain a convergent series for the coefficients $a(\alpha; \beta, n)$ as $N \rightarrow \infty$. These series are exact formulas which yield Theorem 6.1.

We will need explicit descriptions of these Farey arcs. First, we can rewrite

$$\frac{1}{2\pi i} \int_C \frac{H_{\alpha, \beta}(q)}{q^{n+1}} dq = \sum_{\substack{0 \leq h < k \leq N \\ (h, k) = 1}} \frac{1}{2\pi i} \int_{\xi_{h,k}} \frac{H_{\alpha, \beta}(q)}{q^{n+1}} dq.$$



In the Farey series of order N , we consider the fraction h/k and its two neighbors,

$$\frac{h_1}{k_1} < \frac{h}{k} < \frac{h_2}{k_2}, \tag{3.2}$$

discussed in [16]. Each of h_i and k_i depend on h, k and N as described in [16]. Note that $N - k < k_i \leq N$. On the arc $\xi_{h,k}$, we can introduce the variable θ via the transformation

$$q = \exp \left[-2\pi N^{-2} + 2\pi i \left(\frac{h}{k} + \theta \right) \right] = \exp \left[\frac{2\pi ih}{k} - \frac{2\pi z}{k} \right],$$

where

$$-\vartheta'_{h,k} := -\frac{1}{k(k_1 + k)} \leq \theta \leq \frac{1}{k(k_2 + k)} =: \vartheta''_{h,k}.$$

Setting $z = k(N^{-2} - i\theta)$ on each arc $\xi_{h,k}$, we obtain

$$a(\alpha, \beta; n) = \sum_{\substack{0 \leq h < k \leq N \\ (h,k)=1}} e^{-\frac{2\pi inh}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} H_{\alpha,\beta} \left(e^{\frac{2\pi ih}{k} - \frac{2\pi z}{k}} \right) e^{\frac{2\pi nz}{k}} d\theta.$$

Guided by Lemma 2.5, we break up the sum into even and odd k , writing

$$a(\alpha, \beta; n) = S(0, N; n) + S(1, N; n), \tag{3.3}$$

where

$$S(r, N; n) := \sum_{\substack{0 \leq h < k \leq N \\ (h,k)=1 \\ k \equiv r \pmod{2}}} e^{-\frac{2\pi inh}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} H_{\alpha,\beta} \left(e^{\frac{2\pi ih}{k} - \frac{2\pi z}{k}} \right) e^{\frac{2\pi nz}{k}} d\theta.$$

We henceforth omit dependence on α and β from the names of most variables. Section 2.2 gives estimates on sums of roots of unity known as Kloosterman sums that are needed to bound the error terms. Section 4 extracts behavior from $S(0, N; n)$ that will contribute to the exact formula and bounds the error terms. Throughout the following three sections, n, α , and β are fixed.

4 The even case

4.1 Decomposing $S(0, N; n)$

Here we will extract main term and error term behavior from

$$S(0, N; n) := \sum_{\substack{k=1 \\ k \text{ even}}}^N \sum_{\substack{0 \leq h < k \\ (h,k)=1}} e^{-\frac{2\pi i n h}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} H_{\alpha, \beta} \left(e^{\frac{2\pi i h}{k} - \frac{2\pi z}{k}} \right) e^{\frac{2\pi n z}{k}} d\theta.$$

Towards this goal, we apply the transformation law (1) in Lemma 2.5 for $H_{\alpha, \beta}$ and even k . In the process, we replace the resulting term

$$H_{\alpha, \beta} \left(\exp \left(\frac{2\pi i h'}{k} - \frac{2\pi}{zk} \right) \right) = \sum_{j=0}^{\infty} a(\alpha, \beta; j) \exp \left(\frac{2\pi i h' j}{k} - \frac{2\pi j}{zk} \right)$$

with its series expansion and re-express $z = kw$, yielding

$$\begin{aligned} S(0, N; n) &= \sum_{j=0}^{\infty} \sum_{\substack{k=1 \\ k \text{ even}}}^N \sum_{\substack{0 \leq h < k \\ (h,k)=1}} A_{h,k}(\alpha, \beta, j; n) a(\alpha, \beta; j) k^{-\frac{\alpha+\beta}{2}} \\ &\quad \times \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} w^{-\frac{\alpha+\beta}{2}} \exp \left[w \left((\alpha + 2\beta) \frac{\pi}{12} + 2\pi n \right) \right. \\ &\quad \left. + \frac{1}{w} \left(-(\alpha + 2\beta) \frac{\pi}{12k^2} - \frac{2\pi j}{k^2} \right) \right] d\theta. \end{aligned}$$

The coefficient of $1/w$ inside the above exponential is positive if and only if $j < -(\alpha + 2\beta)/24$. Accordingly, we let

$$S(0, N; n) = Q(0, N; n) + R(0, N; n) \tag{4.1}$$

where $Q(0, N; n)$ is the sum over those $j < -(\alpha + 2\beta)/24$, and $R(0, N; n)$ consists of the remaining terms in $S(0, N; n)$. We will show later that $Q(0, N; n)$ yields mostly main term behavior, whereas $R(0, N; n)$ yields error term behavior. On this note, we observe that if $\alpha + 2\beta \geq 0$, then $Q(0, N; n)$ is in fact an empty sum, which corresponds to the weak growth near $e^{2\pi i h/k}$ for k even discussed at the end of Sect. 2.1. Thus in our analysis of $Q(0, N; n)$ we may assume $\alpha + 2\beta < 0$. Additionally, we will assume $n > -(\alpha + 2\beta)/24$, which guarantees that the coefficient on w inside the exponential is positive.

4.2 Decomposing $Q(0, N; n)$

We now break up $Q(0, N; n)$ into three parts: one which will be extended to a Bessel function in the following subsection, as well as two error terms. Towards this goal, we divide the intervals of integration into three parts according to

$$-\vartheta'_{h,k} = -\frac{1}{k(k+k_1)} \leq -\frac{1}{k(N+k)} < \frac{1}{k(N+k)} \leq \frac{1}{k(k+k_2)} = \vartheta''_{h,k}.$$

Omitting the integrands, the split becomes

$$\begin{aligned} Q(0, N; n) &= \sum_{j < -\frac{\alpha+2\beta}{24}} \sum_{\substack{k=2 \\ k \text{ even}}}^N \sum_{\substack{0 \leq h < k \\ (h,k)=1}} A_{h,k}(\alpha, \beta, j; n) a(\alpha, \beta; j) \\ &\quad \times \left[\int_{-\frac{1}{k(N+k)}}^{\frac{1}{k(N+k)}} + \int_{-\frac{1}{k(k_1+k)}}^{-\frac{1}{k(N+k)}} + \int_{\frac{1}{k(N+k)}}^{\frac{1}{k(k_2+k)}} \right] \\ &=: Q_0(0, N; n) + Q_1(0, N; n) + Q_2(0, N; n). \end{aligned} \tag{4.2}$$

4.3 Extending $Q_0(0, N; n)$

Here we will extend the path of integration of $Q_0(0, N; n)$ to obtain a modified Bessel function. Implementing the variable transformation $w = N^{-2} - i\theta$, we have

$$Q_0(0, N; n) = \sum_{j < -\frac{\alpha+2\beta}{24}} \sum_{\substack{k=2 \\ k \text{ even}}}^N \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{A_{h,k}(\alpha, \beta, j; n)a(\alpha, \beta, j)}{ik^{\frac{\alpha+\beta}{2}}} \cdot \int_{N^{-2}-\frac{i}{k(N+k)}}^{N^{-2}+\frac{i}{k(N+k)}} w^{-\frac{\alpha+\beta}{2}} \times \exp \left[w \left((\alpha + 2\beta) \frac{\pi}{12} + 2\pi n \right) + \frac{1}{w} \left(-(\alpha + 2\beta) \frac{\pi}{12k^2} - \frac{2\pi j}{k^2} \right) \right] dw.$$

We will now extend the path of integration to the rectangle R with vertices

$$\pm N^{-2} \pm \frac{i}{k(N+k)}.$$

Omitting the integrands, we write

$$Q_0(0, N; n) = 2\pi \sum_{j < -\frac{\alpha+2\beta}{24}} \sum_{\substack{k=2 \\ k \text{ even}}}^N \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{A_{h,k}(\alpha, \beta, j; n)a(\alpha, \beta, j)}{k^{\frac{\alpha+\beta}{2}}} \cdot \left\{ \frac{1}{2\pi i} \int_R -\frac{1}{2\pi i} \left[\int_{+,+}^{-,+} + \int_{-,+}^{-,-} + \int_{-,-}^{+,-} \right] \right\}. \tag{4.3}$$

We name $L^*(0, j, k; n) := 1/2\pi i \int_R$, and we name the three remaining integrals $J_1(0, j, k, N; n)$, $J_2(0, j, k, N; n)$, and $J_3(0, j, k, N; n)$, respectively. We will give an exact description of $L^*(0, j, k; n)$ as a Bessel function, and bound each $J_i(0, j, k, N; n)$.

4.4 Expressing $L^*(0, j, k; n)$ as a Bessel function

In this subsection, we express $L^*(0, j, k; n)$ in terms of a modified Bessel function of the first kind.

In the remaining subsections, we will bound each of the error terms that we have encountered, demonstrating that the only significant contribution from $S(0, N; n)$ comes from $L^*(0, j, k; n)$. To $L^*(0, j, k; n)$ we apply the variable transformation

$$u = w \left((\alpha + 2\beta) \frac{\pi}{12} + 2\pi n \right),$$

which gives

$$L^*(0, j, k; n) = \left[(\alpha + 2\beta) \frac{\pi}{12} + 2\pi n \right]^{\frac{\alpha+\beta}{2}-1} \frac{1}{2\pi i} \int_{R'} u^{-\left(\frac{\alpha+\beta}{2}\right)} \times \exp \left[u + \frac{1}{u} \left\{ \left[-(\alpha + 2\beta) \frac{\pi}{12k^2} - \frac{2\pi j}{k^2} \right] \left[(\alpha + 2\beta) \frac{\pi}{12} + 2\pi n \right] \right\} \right] du.$$

It follows from the integral representation of the I -Bessel function (2.24) that

$$L^*(0, j, k; n) = \left[-(\alpha + 2\beta) \frac{\pi}{12k^2} - \frac{2\pi j}{k^2} \right]^{\frac{1}{2}-\frac{\alpha+\beta}{4}} \left[(\alpha + 2\beta) \frac{\pi}{12} + 2\pi n \right]^{-\frac{1}{2}+\frac{\alpha+\beta}{4}} I_\nu(s_0(j, k)) \tag{4.4}$$

where

$$\nu := 1 - \frac{\alpha + \beta}{2}, \quad s_0(j, k) := 2\sqrt{\left[-(\alpha + 2\beta) \frac{\pi}{12k^2} - \frac{2\pi j}{k^2} \right] \left[(\alpha + 2\beta) \frac{\pi}{12} + 2\pi n \right]}.$$

4.5 Bounding $R(0, N; n)$

In this section we will bound $R(0, N; n)$, which we recall is given by

$$\begin{aligned}
 R(0, N; n) &= \sum_{j \geq -\frac{\alpha+2\beta}{24}} \sum_{\substack{k=1 \\ k \text{ even}}}^N \sum_{\substack{0 \leq h < k \\ (h,k)=1}} A_{h,k}(\alpha, \beta, j; n) a(\alpha, \beta; j) \\
 &\quad \times \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} (kw)^{-\frac{\alpha+\beta}{2}} \exp \left[w \left((\alpha + 2\beta) \frac{\pi}{12} + 2\pi n \right) \right. \\
 &\quad \left. + \frac{1}{w} \left(-(\alpha + 2\beta) \frac{\pi}{12k^2} - \frac{2\pi j}{k^2} \right) \right] d\theta.
 \end{aligned}$$

We will show that $R(0, N; n) = O(N^{-\delta})$ for some $\delta > 0$. Setting

$$E := E(w, \alpha, \beta, n, k, j) := \exp \left[w \left((\alpha + 2\beta) \frac{\pi}{12} + 2\pi n \right) + \frac{1}{w} \left(-(\alpha + 2\beta) \frac{\pi}{12k^2} - \frac{2\pi j}{k^2} \right) \right]$$

and splitting the integral into three parts as in (4.2), we obtain

$$\begin{aligned}
 R(0, N; n) &= \sum_{j \geq -\frac{\alpha+2\beta}{24}} \sum_{\substack{k=1 \\ k \text{ even}}}^N \sum_{\substack{0 \leq h < k \\ (h,k)=1}} A_{h,k}(\alpha, \beta, j; n) a(\alpha, \beta; j) \\
 &\quad \times \left[\int_{-\frac{1}{k(N+k)}}^{\frac{1}{k(N+k)}} (kw)^{-\frac{\alpha+\beta}{2}} E d\theta + \sum_{\ell=k_1+k}^{N+k-1} \int_{-\frac{1}{k\ell}}^{-\frac{1}{k(\ell+1)}} (kw)^{-\frac{\alpha+\beta}{2}} E d\theta \right. \\
 &\quad \left. + \sum_{\ell=k_2+k}^{N+k-1} \int_{\frac{1}{k\ell}}^{\frac{1}{k(\ell+1)}} (kw)^{-\frac{\alpha+\beta}{2}} E d\theta \right] \\
 &= \sum_{j \geq -\frac{\alpha+2\beta}{24}} a(\alpha, \beta; j) \sum_{\substack{k=1 \\ k \text{ even}}}^N \int_{-\frac{1}{k(N+k)}}^{\frac{1}{k(N+k)}} (kw)^{-\frac{\alpha+\beta}{2}} E d\theta \sum_{\substack{0 \leq h < k \\ (h,k)=1}} A_{h,k}(\alpha, \beta, j; n) \tag{4.5}
 \end{aligned}$$

$$\begin{aligned}
 &+ \sum_{j \geq -\frac{\alpha+2\beta}{24}} a(\alpha, \beta; j) \sum_{\substack{k=1 \\ k \text{ even}}}^N \sum_{\substack{0 \leq h < k \\ (h,k)=1}} A_{h,k}(\alpha, \beta, j; n) \sum_{\ell=k_1+k}^{N+k-1} \int_{-\frac{1}{k\ell}}^{-\frac{1}{k(\ell+1)}} (kw)^{-\frac{\alpha+\beta}{2}} E d\theta \\
 &\tag{4.6}
 \end{aligned}$$

$$\begin{aligned}
 &+ \sum_{j \geq -\frac{\alpha+2\beta}{24}} a(\alpha, \beta; j) \sum_{\substack{k=1 \\ k \text{ even}}}^N \sum_{\substack{0 \leq h < k \\ (h,k)=1}} A_{h,k}(\alpha, \beta, j; n) \sum_{\ell=k_2+k}^{N+k-1} \int_{\frac{1}{k\ell}}^{\frac{1}{k(\ell+1)}} (kw)^{-\frac{\alpha+\beta}{2}} E d\theta. \\
 &\tag{4.7}
 \end{aligned}$$

Our goal is to bound the expressions (4.5), (4.6), and (4.7). We begin by bounding $|E|$. As a reminder, we have $w = N^{-2} - i\theta$ and $|\theta| \leq 1/Nk$, so that $\text{Re}(1/w) \geq k^2/2$ and $\text{Re}(w) = N^{-2}$. Therefore for $j \geq -(\alpha + 2\beta)/24$,

$$|E| \leq \exp \left[N^{-2} \left((\alpha + 2\beta) \frac{\pi}{12} + 2\pi n \right) - (\alpha + 2\beta) \frac{\pi}{24} - \pi j \right].$$

Thus for $N > 0$, we have

$$\exp \left[N^{-2} \left((\alpha + 2\beta) \frac{\pi}{12} + 2\pi n \right) - (\alpha + 2\beta) \frac{\pi}{24} \right] = O(1). \tag{4.8}$$

Noting that the series $\sum_{j=0}^{\infty} a(\alpha, \beta; j)e^{-\pi j}$ defining $H_{\alpha, \beta}(i/2)$ converges absolutely, we have that for fixed n , (4.5) admits the estimate

$$\begin{aligned} & O \left(\sum_{\substack{k=1 \\ k \text{ even}}}^N \int_{-\frac{1}{k(N+k)}}^{\frac{1}{k(N+k)}} (kw)^{-\frac{\alpha+\beta}{2}} d\theta \sum_{\substack{0 \leq h < k \\ (h,k)=1}} A_{h,k}(\alpha, \beta, j; n) \right) \\ &= O \left(\sum_{\substack{k=1 \\ k \text{ even}}}^N \frac{N^{\frac{\alpha+\beta}{2}-1}}{k} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} A_{h,k}(\alpha, \beta, j; n) \right). \end{aligned}$$

If $\alpha + \beta < 0$, we can use the trivial Kloosterman sum bound $O(k)$ to show that

$$O \left(\sum_{\substack{k=1 \\ k \text{ even}}}^N \frac{N^{\frac{\alpha+\beta}{2}-1}}{k} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} A_{h,k}(\alpha, \beta, j; n) \right) = O \left(N^{\frac{\alpha+\beta}{2}} \right). \tag{4.9}$$

If $\alpha + \beta = 0$, we can make use of Lemma 2.6 to show that for fixed n ,

$$O \left(\sum_{\substack{k=1 \\ k \text{ even}}}^N \frac{N^{\frac{\alpha+\beta}{2}-1}}{k} \sum_{\substack{0 \leq h < k \\ (h,k)=1}} A_{h,k}(\alpha, \beta, j; n) \right) = O \left(\sum_{\substack{k=1 \\ k \text{ even}}}^N \frac{k^{-1/3+\varepsilon}}{N} \right) = O(N^{-1/3+\varepsilon}). \tag{4.10}$$

The summands (4.6) and (4.7) are handled in a similar way. The only important difference is that we must first switch the order of summation as in [16, p. 507] to obtain

$$\sum_{\substack{j \geq -\frac{\alpha+2\beta}{24} \\ k \text{ even}}}^{\infty} a(\alpha, \beta; j) \sum_{\substack{k=1 \\ k \text{ even}}}^N \sum_{\ell=N+1}^{N+k-1} \int_{-\frac{1}{k\ell}}^{-\frac{1}{k(\ell+1)}} (kw)^{-\frac{\alpha+\beta}{2}} E \sum_{\substack{0 \leq h < k \\ (h,k)=1 \\ N-k < k_2 \leq \ell-k}} A_{h,k}(\alpha, \beta, j; n) d\theta. \tag{4.11}$$

The desired bound is now obtained via the same methods used to bound (4.5). As described in Sect. 2.2, Lemma 2.6 is equipped to handle the incomplete Kloosterman sum in (4.11). By (4.9), (4.10), and corresponding statements for (4.6) and (4.7), we obtain in all cases

$$R(0, N; n) = O(N^{-\delta}) \tag{4.12}$$

for some $\delta > 0$.

4.6 Bounding $Q_1(0, N; n)$ and $Q_2(0, N; n)$

In this subsection, we will bound those segments of the Farey arcs that do not contribute to the Bessel integral. We will explicitly bound just $Q_2(0, N; n)$ because similar arithmetic yields the same bound for $Q_1(0, N; n)$. First, as in (4.6) and (4.7) we split our path of integration into many smaller intervals, obtaining

$$\begin{aligned} Q_2(0, N; n) &= \sum_{\substack{j < -\frac{\alpha+2\beta}{24} \\ k \text{ even}}} \sum_{k=1}^N \sum_{\substack{0 \leq h < k \\ (h,k)=1}} A_{h,k}(\alpha, \beta, j; n) a(\alpha, \beta; j) k^{-\frac{\alpha+\beta}{2}} \\ &\quad \times \sum_{l=k_2+k}^{N+k-1} \int_{\frac{1}{k(\ell+1)}}^{\frac{1}{k\ell}} w^{-\frac{\alpha+\beta}{2}} E d\theta. \end{aligned}$$

Now, as in 4.11, we switch the order of summation, which yields

$$Q_2(0, N; n) = \sum_{j < -\frac{\alpha+2\beta}{24}} a(\alpha, \beta; j) \sum_{\substack{k=1 \\ k \text{ even}}}^N k^{-\frac{\alpha+\beta}{2}} \sum_{l=N+1}^{N+k-1} \int_{\frac{1}{k(l+1)}}^{\frac{1}{kl}} w^{-\frac{\alpha+\beta}{2}} E d\theta$$

$$\times \sum_{\substack{0 \leq h < k \\ (h,k)=1 \\ N-k < k_2 \leq l-k}} A_{h,k}(\alpha, \beta, j; n).$$

We have, similarly to (4.8), that $E = O(1)$.

Since $|w| \leq (2/(N^2k^2))^{1/2}$, we have

$$\sum_{l=N+1}^{N+k-1} \int_{\frac{1}{k(l+1)}}^{\frac{1}{kl}} w^{-\frac{\alpha+\beta}{2}} E d\theta = O\left((Nk)^{\frac{\alpha+\beta}{2}-1}\right).$$

Here and throughout, the constant implied by the big Oh notation depends at most on α, β, n . The following steps are analogous to the final steps of Sect. 4.5, where the cases $\alpha + \beta < 0$ and $\alpha + \beta = 0$ are considered separately. We conclude that

$$Q_1(0, N; n), Q_2(0, N; n) = O(N^{-\delta}) \tag{4.13}$$

for some $\delta > 0$.

4.7 Bounding $J_1(0, j, k, N; n), J_2(0, j, k, N; n), J_3(0, j, k, N; n)$

As a reminder, we have

$$J_1(0, j, k, N; n) := \int_{N^{-2} + \frac{i}{k(N+k)}}^{-N^{-2} + \frac{i}{k(N+k)}} w^{-\frac{\alpha+\beta}{2}} E dw,$$

$$J_3(0, j, k, N; n) := \int_{-N^{-2} - \frac{i}{k(N+k)}}^{+N^{-2} - \frac{i}{k(N+k)}} w^{-\frac{\alpha+\beta}{2}} E dw,$$

and

$$J_2(0, j, k, N; n) := \int_{-N^{-2} + \frac{i}{k(N+k)}}^{-N^{-2} - \frac{i}{k(N+k)}} w^{-\frac{\alpha+\beta}{2}} E dw.$$

We first consider J_1 and J_3 . Here one easily finds $\text{Re}(w) \leq N^{-2}$ and $\text{Re}(1/w) \leq 4k^2$, and from this obtains $E = O(1)$. Since the interval length is $2N^{-2}$ and $|w| \leq \sqrt{2}/kN$, we have that J_1 and J_3 are $O\left((kN)^{(\alpha+\beta)/2-1}\right)$.

For J_2 , we need only note that $\text{Re}(w)$ and $\text{Re}(1/w)$ are negative to show that $E = O(1)$. Since the length of the interval is $2/N(k + N)$, we see that J_2 is $O\left((kN)^{(\alpha+\beta)/2-1}\right)$. We proceed as in Sects. 4.5 and 4.6 to show that

$$\sum_{\substack{k=2 \\ k \text{ even}}}^N A_k(\alpha, \beta, j; n) \{J_1 + J_2 + J_3\} = O(N^{-\delta}) \tag{4.14}$$

for some $\delta > 0$, as desired.

Recalling our decomposition of $Q_0(0, N; n)$ in (4.3) and the bound given by (4.14) above, we can now estimate

$$Q_0(0, N; n) = 2\pi \sum_{j < -\frac{\alpha+2\beta}{24}} \sum_{\substack{k=2 \\ k \text{ even}}}^N \frac{A_k(\alpha, \beta, j; n)a(\alpha, \beta, j)}{k^{\frac{\alpha+\beta}{2}}} L^*(0, j, k; n) + O(N^{-\delta}). \quad (4.15)$$

5 The odd case

The odd case follows a manner very similar to the even case, and the discrepancies are all consequences of the differences between the transformation formulas (1) and (2) in Lemma 2.5 for even and odd k .

5.1 Decomposing $S(1, N; n)$

Here we will extract main term and error term behavior from

$$S(1, N; n) = \sum_{\substack{0 \leq h < k \leq N \\ (h,k)=1 \\ k \text{ odd}}} e^{-\frac{2\pi ih}{k}} \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} H_{\alpha,\beta} \left(e^{\frac{2\pi ih}{k} - \frac{2\pi z}{k}} \right) e^{\frac{2\pi nz}{k}} d\theta.$$

We apply the transformation law (2) in Lemma 2.5 for $H_{\alpha,\beta}$ and odd k , obtaining

$$\begin{aligned} S(1, N; n) &= \sum_{j=0}^{\infty} \sum_{\substack{k=1 \\ k \text{ odd}}}^N \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{B_{h,k}(\alpha, \beta, j; n)a(\beta, \alpha; j)}{2^{\beta/2} k^{\frac{\alpha+\beta}{2}}} \\ &\times \int_{-\vartheta'_{h,k}}^{\vartheta''_{h,k}} w^{-\frac{\alpha+\beta}{2}} \exp \left[w \left((\alpha + 2\beta) \frac{\pi}{12} + 2\pi n \right) \right. \\ &\left. + \frac{1}{w} \left(-(2\alpha + \beta) \frac{\pi}{24k^2} - \frac{\pi j}{k^2} \right) \right] d\theta \end{aligned}$$

as in Sect. 4.1.

In this case the coefficient on $1/w$ inside the above exponential is positive if and only if $j < -(2\alpha + \beta)/24$. Thus we stratify $S(1, N; n)$ via

$$S(1, N; n) = Q(1, N; n) + R(1, N; n), \quad (5.1)$$

where $Q(1, N; n)$ is the sum over those $j < -(2\alpha + \beta)/24$. As in the even case, $Q(0, N; n)$ will yield mostly main term behavior, whereas $R(0, N; n)$ will yield error term behavior. We continue to assume that $n > -(\alpha + 2\beta)/24$.

5.2 Decomposing $Q(1, N; n)$

We will now break up $Q(1, N; n)$ into three parts, which play the same roles as in Sect. 4.2. Omitting the integrands, we have

$$\begin{aligned} Q(1, N; n) &= \sum_{j < -\frac{2\alpha+\beta}{24}} \sum_{\substack{k=1 \\ k \text{ odd}}}^N \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{B_{h,k}(\alpha, \beta, j; n)a(\beta, \alpha; j)}{2^{\beta/2} k^{\frac{\alpha+\beta}{2}}} \\ &\times \left[\int_{-\frac{1}{k(N+k)}}^{\frac{1}{k(N+k)}} + \int_{-\frac{1}{k(k_1+k)}}^{-\frac{1}{k(N+k)}} + \int_{\frac{1}{k(N+k)}}^{\frac{1}{k(k_2+k)}} \right] \\ &=: Q_0(1, N; n) + Q_1(1, N; n) + Q_2(1, N; n). \end{aligned} \quad (5.2)$$

5.3 Extending $Q_0(1, N; n)$

In this subsection, we will extend the path of integration of $Q_0(1, N; n)$ to obtain a modified Bessel function. Via the variable transformation $w = N^{-2} - i\theta$, we can write

$$\begin{aligned}
 Q_0(1, N; n) &= \sum_{j < -\frac{2\alpha+\beta}{24}} \sum_{\substack{k=1 \\ k \text{ odd}}}^N \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{B_{h,k}(\alpha, \beta, j; n) a(\beta, \alpha; j)}{i 2^{\beta/2} k^{\frac{\alpha+\beta}{2}}} \\
 &\times \int_{N^{-2} - \frac{i}{k(N+k)}}^{N^{-2} + \frac{i}{k(N+k)}} w^{-\frac{\alpha+\beta}{2}} \exp \left[w \left((\alpha + 2\beta) \frac{\pi}{12} + 2\pi n \right) \right. \\
 &\left. + \frac{1}{w} \left(-(2\alpha + \beta) \frac{\pi}{24k^2} - \frac{\pi j}{k^2} \right) \right] dw.
 \end{aligned}$$

Considering the rectangle R described in Section 4.3, we write

$$\begin{aligned}
 Q_0(1, N; n) &= 2\pi \sum_{j < -\frac{2\alpha+\beta}{24}} \sum_{\substack{k=1 \\ k \text{ odd}}}^N \sum_{\substack{0 \leq h < k \\ (h,k)=1}} \frac{B_{h,k}(\alpha, \beta, j; n) a(\beta, \alpha; j)}{2^{\beta/2} k^{\frac{\alpha+\beta}{2}}} \\
 &\times \left\{ \frac{1}{2\pi i} \int_R -\frac{1}{i} \left[\int_{+,+}^{-,+} + \int_{-,+}^{-,-} + \int_{-,-}^{+,-} \right] \right\}. \tag{5.3}
 \end{aligned}$$

Omitting the integrands, we write $L^*(1, j, k; n) := 1/2\pi i \int_R$, and we name the three remaining integrals $J_1(1, j, k, N; n)$, $J_2(1, j, k, N; n)$, and $J_3(1, j, k, N; n)$, respectively.

5.4 Expressing $L^*(1, j, k; n)$ as a Bessel function

Now, we will express $L^*(1, j, k; n)$ exactly as a modified Bessel function of the first kind. To $L^*(1, j, k; n)$ we apply the same variable transformation as in Sect. 4.4, which gives

$$\begin{aligned}
 L^*(1, j, k; n) &= \left[-(2\alpha + \beta) \frac{\pi}{24k^2} - \frac{\pi j}{k^2} \right]^{\frac{1}{2} - \frac{\alpha+\beta}{4}} \\
 &\times \left[(\alpha + 2\beta) \frac{\pi}{12} + 2\pi n \right]^{-\frac{1}{2} + \frac{\alpha+\beta}{4}} I_\nu(s_1(j, k)), \tag{5.4}
 \end{aligned}$$

where

$$s_1(j, k) := 2 \sqrt{\left[-(2\alpha + \beta) \frac{\pi}{24k^2} - \frac{\pi j}{k^2} \right] \left[(\alpha + 2\beta) \frac{\pi}{12} + 2\pi n \right]}$$

and ν is the same as in Sect. 4.4.

5.5 Bounding the odd case error terms

$R(1, N; n)$, $Q_\mu(1, N; n)$, and $J_\nu(1, j, k, N; n)$ for $\mu = 1, 2$ and $\nu = 1, 2, 3$ are bounded in a manner which is nearly identical to the corresponding arguments in the even case. The main differences between these expressions and their even case counterparts are the coefficient $a(\beta, \alpha; j)$ from the expansion of $H_{\beta,\alpha}$ and the coefficient

$$-(2\alpha + \beta) \frac{\pi}{24k^2} - \frac{\pi j}{k^2}$$

of $1/w$ in the exponential. Note that in Sects. 4.5, 4.6, and 4.7, the terms $-(\alpha + 2\beta)/12$ in the coefficient of $1/w$, which correspond to $-(2\alpha + \beta)/24$ in the odd case, may as well have been arbitrary negative real numbers that are fixed in this discussion. The same holds for -2π as the coefficient of j , except when we used the absolute convergence of the series

defining $H_{\alpha,\beta}(i/2)$ in Sect. 4.5. In the odd case, we need to use the absolute convergence of the series defining $H_{\beta,\alpha}(i/4)$ to obtain the same result. Running the same arguments with these minor adjustments, we obtain

$$R(1, N; n) = O(N^{-\delta}), \tag{5.5}$$

$$Q_\mu(1, N; n) = O(N^{-\delta}), \tag{5.6}$$

and

$$\sum_{\substack{0 \leq h < k \\ (h,k)=1}} B_{h,k}(\alpha, \beta, j; n) \left(\sum_v J_v(1, j, k, N; n) \right) = O(N^{-\delta}) \tag{5.7}$$

for some $\delta > 0$, where $\mu = 1, 2$ and $\nu = 1, 2, 3$.

Recalling our decomposition of $Q_0(1, N; n)$ in (5.3) and the bounds given by (5.7) above, we conclude that

$$Q_0(1, N; n) = 2\pi \sum_{j < -\frac{2\alpha+\beta}{24}} \sum_{\substack{k=1 \\ k \text{ odd}}}^N \frac{B_k(\alpha, \beta, j; n)a(\beta, \alpha, j)}{2^{\beta/2}k^{\frac{\alpha+\beta}{2}}} L^*(1, j, k; n) + O(N^{-\delta}). \tag{5.8}$$

6 Proof of Theorem 1.1 and Corollaries 1.2 and 1.3

We now apply our work from Sects. 3, 2.2, and 4 to obtain exact formulas for $a(\alpha, \beta; n)$, and afterwards we will extract asymptotic behavior from those formulas. Finally, we will express these exact and asymptotic formulas in terms of our topological invariants to prove Theorem 1.1 and Corollary 1.2.

Theorem 6.1 *Let $\alpha + \beta \leq 0$, and assume $n > -(\alpha + 2\beta)/24$. Then*

$$a(\alpha, \beta; n) = 2\pi \sum_{j < -\frac{\alpha+2\beta}{24}} \sum_{\substack{k=2 \\ k \text{ even}}}^\infty \frac{A_k(\alpha, \beta, j; n)a(\alpha, \beta, j)}{k^{\frac{\alpha+\beta}{2}}} L^*(0, j, k; n) \tag{6.1}$$

$$+ 2\pi \sum_{j < -\frac{2\alpha+\beta}{24}} \sum_{\substack{k=1 \\ k \text{ odd}}}^\infty \frac{B_k(\alpha, \beta, j; n)a(\beta, \alpha, j)}{2^{\beta/2}k^{\frac{\alpha+\beta}{2}}} L^*(1, j, k; n). \tag{6.2}$$

Proof Making use of the decompositions (4.1) and (4.2), as well as the estimates (4.12), (4.13), and (4.15), we have

$$S(0, N; n) = 2\pi \sum_{j < -\frac{\alpha+2\beta}{24}} \sum_{\substack{k=2 \\ k \text{ even}}}^N \frac{A_k(\alpha, \beta, j; n)a(\alpha, \beta, j)}{k^{\frac{\alpha+\beta}{2}}} L^*(0, j, k; n) + O(N^{-\delta}).$$

Similarly, by the decompositions (5.1) and (5.2), as well as the estimates (5.5), (5.6), and (5.8), we have

$$S(1, N; n) = 2\pi \sum_{j < -\frac{2\alpha+\beta}{24}} \sum_{\substack{k=1 \\ k \text{ odd}}}^N \frac{B_k(\alpha, \beta, j; n)a(\beta, \alpha, j)}{2^{\beta/2}k^{\frac{\alpha+\beta}{2}}} L^*(1, j, k; n) + O(N^{-\delta}).$$

But by the parity split in (3.3), these estimates imply that

$$\begin{aligned}
 a(\alpha, \beta; n) = & 2\pi \sum_{j < -\frac{\alpha+2\beta}{24}} \sum_{\substack{k=2 \\ k \text{ even}}}^N \frac{A_k(\alpha, \beta, j; n)a(\alpha, \beta, j)}{k^{\frac{\alpha+\beta}{2}}} L^*(0, j, k; n) \\
 & + 2\pi \sum_{j < -\frac{2\alpha+\beta}{24}} \sum_{\substack{k=1 \\ k \text{ odd}}}^N \frac{B_k(\alpha, \beta, j; n)a(\beta, \alpha; j)}{2^{\beta/2}k^{\frac{\alpha+\beta}{2}}} L^*(1, j, k; n) \\
 & + O(N^{-\delta})
 \end{aligned}$$

for some $\delta > 0$. Keeping n fixed and letting $N \rightarrow \infty$, we obtain (6.1). □

Remark In [5], Bringmann and Ono obtained exact formulas for the coefficients of harmonic Maass forms of non-positive weight using the theory of Maass-Poincaré series. Our results can therefore also be obtained using their work.

In the following corollary, we make use of Theorem 6.1 to obtain approximations for $a(\alpha, \beta; n)$ for large n . We will see that the main contribution for large n comes from one of the first two summands in k .

Corollary 6.2 *Suppose that $\alpha + \beta \leq 0$.*

(1) *If $\alpha = 0$ and $\beta \neq 0$, then as $n \rightarrow \infty$ we have*

$$a(0, \beta; 2n) \sim 2^{\frac{3\beta-5}{4}} 3^{\frac{\beta-1}{4}} (-\beta)^{\frac{1-\beta}{4}} n^{\frac{\beta-3}{4}} \exp\left(\pi\sqrt{\frac{-2\beta}{3}n}\right).$$

(2) *If $\alpha < 0$, then as $n \rightarrow \infty$ we have*

$$a(\alpha, \beta; n) \sim 2^{\frac{2\alpha+\beta-3}{2}} 3^{\frac{\alpha+\beta-1}{4}} (-(2\alpha + \beta))^{\frac{-\alpha-\beta+1}{4}} n^{\frac{\alpha+\beta-3}{4}} \exp\left(\pi\sqrt{\frac{-(2\alpha + \beta)}{3}n}\right).$$

(3) *If $\alpha > 0$, then as $n \rightarrow \infty$ we have*

$$a(\alpha, \beta; n) \sim (-1)^n 2^{\frac{3\alpha+3\beta-7}{4}} 3^{\frac{\alpha+\beta-1}{4}} (-(\alpha + 2\beta))^{\frac{1-\alpha-\beta}{4}} n^{\frac{\alpha+\beta-3}{4}} \exp\left(\pi\sqrt{\frac{-(\alpha + 2\beta)}{6}n}\right).$$

Proof Consider the case $\alpha < 0$. Let

$$a(\alpha, \beta; n) =: 2\pi \left(M^-(\alpha, \beta; n) + E^-(\alpha, \beta; n) \right),$$

where we let

$$M^-(\alpha, \beta; n) := \frac{B_1(\alpha, \beta, 0; n)a(\beta, \alpha; 0)}{2^{\frac{\beta}{2}}} L^*(1, 0, 1; n)$$

and

$$E^-(\alpha, \beta; n) := 2\pi \left(\sum_{j=0}^{\lfloor \frac{\alpha+2\beta}{24} \rfloor} \sum_{\substack{k=2 \\ k \equiv 0 \pmod{2}}}^{\infty} f_A(\alpha, \beta, j, k; n) + \sum_{j=0}^{\lfloor \frac{2\alpha+\beta}{24} \rfloor} \sum_{\substack{k=3 \\ k \equiv 1 \pmod{2}}}^{\infty} f_B(\alpha, \beta, j, k; n) \right), \tag{6.3}$$

where

$$f_A(\alpha, \beta, j, k; n) := \frac{A_k(\alpha, \beta, j; n)a(\alpha, \beta, j)}{k^{\frac{\alpha+\beta}{2}}}L^*(0, j, k; n)$$

and

$$f_B(\alpha, \beta, j, k; n) := \frac{B_k(\alpha, \beta, j; n)a(\beta, \alpha, j)}{2^\beta k^{\frac{\alpha+\beta}{2}}}L^*(1, j, k; n).$$

To prove Corollary 6.2 for $\alpha < 0$, it suffices to show that $E^-(\alpha, \beta; n) = o(L^*(1, 0, 1; n))$ and then to carry out the necessary simplification of $2\pi M^-(\alpha, \beta; n)$.

Using the bound (2.23), the monotonicity of the I -Bessel functions, and the trivial bound on the Kloosterman sums, it follows from (6.3) that

$$\begin{aligned} E^-(\alpha, \beta; n) &\leq \left(\frac{-(\alpha + 2\beta)}{\alpha + 2\beta + 24n}\right)^{\frac{\nu}{2}} \\ &\quad \times \sum_{j=0}^{\lfloor \frac{-\alpha+2\beta}{24} \rfloor} |a(\alpha, \beta, j)| \left(K_A(\alpha, \beta, j; n) + \lceil s_0(j, 1) \rceil I_\nu(s_0(0, 2)) \right) \\ &\quad + \left(\frac{-(2\alpha + \beta)}{\alpha + 2\beta + 48\pi n}\right)^{\frac{\nu}{2}} \\ &\quad \times \sum_{j=0}^{\lfloor \frac{-2\alpha+\beta}{24} \rfloor} \frac{|a(\beta, \alpha; j)|}{2^{\beta/2}} \left(K_B(\alpha, \beta, j; n) + \lceil s_1(j, 1) \rceil I_\nu(s_1(0, 3)) \right), \end{aligned}$$

where

$$K_A(\alpha, \beta, j; n) := \frac{4}{3} \sum_{\substack{k > \lceil s_0(j, 1) \rceil \\ k \equiv 0 \pmod{2}}} \frac{A_k(\alpha, \beta, j; n)}{k^{\nu+1}} \left(\frac{s_0(0, 1)}{2}\right)^\nu,$$

and

$$K_B(\alpha, \beta, j; n) := \frac{4}{3} \sum_{\substack{k > \lceil s_1(j, 1) \rceil \\ k \equiv 1 \pmod{2}}} \frac{B_k(\alpha, \beta, j; n)}{k^{\nu+1}} \left(\frac{s_1(0, 1)}{2}\right)^\nu.$$

If $\alpha + \beta < 0$, we use the trivial bound on the Kloosterman sums to obtain

$$K_A(\alpha, \beta, j; n) \leq \frac{4}{3} \left(\frac{s_0(0, 1)}{2}\right)^\nu \frac{s_0(j, 1)^{1-\nu}}{1-\nu}$$

and

$$K_B(\alpha, \beta, j; n) \leq \frac{4}{3} \left(\frac{s_1(0, 1)}{2}\right)^\nu \frac{s_0(j, 1)^{1-\nu}}{1-\nu},$$

and if $\alpha + \beta = 0$, we use Lemma 2.6 to obtain

$$K_A(\alpha, \beta, j; n) = O(n^{\frac{1}{3}} s_0(j, 1)^{-\frac{1}{3}+\epsilon})$$

and

$$K_B(\alpha, \beta, j; n) = O(n^{\frac{1}{3}} s_1(j, 1)^{-\frac{1}{3}+\epsilon}).$$

Therefore, since $E^-(\alpha, \beta; n)$ is bounded by a finite sum of rational functions and I -Bessel functions, it follows from

$$I_\nu(x) \sim \frac{e^x}{\sqrt{2\pi x}}$$

that $E^-(\alpha, \beta; n) = o(L^*(1, 0, 1; n))$, and that the statement of Corollary 6.2 follows from a simplification of $2\pi M(\alpha, \beta; n)$.

If $\alpha > 0$, the proof of Corollary 6.2 follows *mutatis mutandis* if we let

$$a(\alpha, \beta; n) := 2\pi \left(M^+(\alpha, \beta; n) + E^+(\alpha, \beta; n) \right),$$

where we let

$$M^+(\alpha, \beta; n) := \frac{A_2(\alpha, \beta, 0; n)a(\beta, \alpha; 0)}{2^{\frac{\alpha+\beta}{2}}} L^*(0, 0, 2; n)$$

and

$$E^+(\alpha, \beta; n) := 2\pi \left(\sum_{j=0}^{\lfloor \frac{\alpha+2\beta}{24} \rfloor} \sum_{\substack{k=4 \\ k \equiv 0 \pmod{2}}}^{\infty} f_A(\alpha, \beta, j, k; n) + \sum_{j=0}^{\lfloor \frac{2\alpha+\beta}{24} \rfloor} \sum_{\substack{k=1 \\ k \equiv 1 \pmod{2}}}^{\infty} f_B(\alpha, \beta, j, k; n) \right).$$

Finally, we observe that $a(0, \beta; 2n) = a(\beta, 0, n)$, and so making the necessary substitutions for the case $\alpha < 0$ yields the case $\alpha = 0$. □

We are now in a position to prove Theorem 1.1, our exact formulas for $\chi(\text{Hilb}^n(S))$ and $\sigma(\text{Hilb}^n(S))$, and Corollary 1.2, our asymptotic formulas for $\chi(\text{Hilb}^n(S))$, $\sigma(\text{Hilb}^n(S))$, $b^*(r, 2; n)$ and $c^*(r_1, r_2; n)$. At this point, the work is a simple application of Theorem 6.1 and Corollary 6.2.

Proof of Theorem 1.1 The proof follows from Lemma 2.2 and Theorem 6.1. We note only that in the derivation of (1.1) the sum over odd k is actually a sum over even j , since $a(0, -\chi(S); j) = 0$ for odd j . Noting this and replacing j with $2j$, we obtain (1.1). □

Proof of Corollary 1.2 This follows from Corollary 6.2 and (2.9) and (2.10) in Lemma 2.2. □

Proof of Corollary 1.3 The coefficients of our functions $Z_S(x, y; \tau)$, where $x, y = \pm 1$, are of the form $a(\alpha, \beta; n)$ for some α and β determined by (2.11) and Lemma 2.2. For convenience we define

$$a_S(x, y; n) := a(\alpha, \beta; n).$$

By Corollary 6.2, each of these coefficient sequences is asymptotic to a function of the form $c_1 n^{c_2} \exp(\sqrt{Gn})$, where c_1, c_2 , and G only depend x, y , and S . We will let $G_S(x, y)$ denote this value G . Note that $G_S(x, y)$ essentially determines the growth of the sequence $a_S(x, y; n)$ in the sense that if $G_S(x_1, y_1) < G_S(x_2, y_2)$, then $a_S(x_1, y_1; n) = o(a_S(x_2, y_2; n))$.

The proof of (1) now follows easily from Corollary 6.2, (2.9), (2.10), and (2.11). As a reminder, (2.12) tells us

$$B_S(r, 2; \tau) = \frac{1}{2} (Z(1, 1; \tau) + (-1)^r Z(1, -1; \tau)).$$

One makes use of Corollary 6.2 to check that $G_S(1, 1) > G_S(1, -1)$ if and only if $\sigma(S) + \chi(S) > 0$ and $G_S(1, 1) < G_S(1, -1)$ if and only if $\sigma(S) + \chi(S) < 0$. Note that $G_S(1, 1) = G_S(1, -1)$ whenever $\chi(S) = -\sigma(S)$, and in this case $Z_S(1, 1; \tau) = Z_S(1, -1; \tau)$, so that $b_S^*(1; n) = 0$ for all n . Also, if $\chi(S) \leq 0$, $(\chi(S), \sigma(S)) \neq (0, 0)$, we have from Theorem 15.1 in [6] that $a_S(1, 1; n) = o(a_S(1, -1; n))$. Since this requires $\sigma(S) + \chi(S) \leq 0$, we can remove the hypothesis that $\chi(S) > 0$.

To prove (2), we first recall that (2.13) gives

$$C_S(r_1, r_2; \tau) = \frac{1}{4} \sum_{\substack{j_1 \bmod 2 \\ j_2 \bmod 2}} (-1)^{j_2 r_2} (-1)^{j_1 r_1} Z_S((-1)^{j_2}, (-1)^{j_1}; \tau).$$

We note that by (2.11), the hypotheses of Corollary 6.2 are always satisfied for $Z(-1, -1; \tau)$, since

$$4h^{1,0} - (\chi(S) - 8h^{1,0}) = -(2h^{0,0} + 2h^{2,0} + h^{1,0}) \leq 0$$

for all S . Note also that for $\chi(S) \geq 0$, we have

$$G_S(-1, -1) - G_S(1, 1) = \frac{2\chi(S) + 12h^{1,0}}{3} - \frac{2\chi(S)}{3} = 4h^{1,0} \geq 0.$$

If $h^{1,0} = 0$, we have $Z_S(1, 1; \tau) = Z_S(-1, -1; \tau)$. If $\chi(S) \leq 0$, we have $a_S(1, 1; n) = o(a_S(-1, -1; n))$ by Theorem 15.1 in [6]. Since $\chi(S) \leq 0$ requires $h^{1,0} > 0$, we have

$$a_S(1, 1; n) + (-1)^{r_1+r_2} a_S(-1, -1; n) \sim K a_S(-1, -1; n),$$

where

$$K := \begin{cases} 2 & h^{1,0} = 0 \text{ and } r_1 + r_2 \equiv 0 \pmod{2} \\ 0 & h^{1,0} = 0 \text{ and } r_1 + r_2 \equiv 1 \pmod{2} \\ 1 & h^{1,0} > 0 \text{ and } r_1 + r_2 \equiv 0 \pmod{2} \\ -1 & h^{1,0} > 0 \text{ and } r_1 + r_2 \equiv 1 \pmod{2}. \end{cases}$$

We now see that if $G_S(-1, -1) > G_S(1, -1)$, then $c_S^*(r_1, r_2; n) \sim (K/4)a_S(-1, -1; n)$, which gives the desired result. To conclude the proof, one simply checks that this inequality holds for both $\sigma(S) \geq 0$ and $\sigma(S) < 0$. □

7 Examples

Here we illustrate Theorem 1.1 and Corollaries 1.2 and 1.3 with examples of numerical computations.

Example 7.1 To demonstrate Theorem 1.1, we focus on S , a 2-dimensional torus blown up at one point, a Hirzebruch surface $\Sigma_0 \cong \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{P}^2$. We have $\chi(S) = 1, \chi(\mathbb{P}^2) = 3, \sigma(\mathbb{P}^2) = 1, \chi(\Sigma_0) = 4$, and $\sigma(\Sigma_0) = 0$, so that we can consider the functions

$$\begin{aligned} \sum_{n=0}^{\infty} \chi(\text{Hilb}^n(S))q^n &= H_{-1,0}(q) = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + 11q^6 + \dots \\ \sum_{n=0}^{\infty} (-1)^n \sigma(\text{Hilb}^n(\Sigma_0))q^n &= H_{0,-2}(q) = 1 + 0q + 2q^2 + 0q^3 + 5q^4 + 0q^5 + 10q^6 + \dots \\ \sum_{n=0}^{\infty} (-1)^n \sigma(\text{Hilb}^n(\mathbb{P}^2))q^n &= H_{1,-2}(q) = 1 - q + q^2 - 2q^3 + 3q^4 - 4q^5 + 5q^6 + \dots \end{aligned}$$

Table 1 lists $a_2(-1, 0; n), a_2(0, -2; n)$, and $a_2(1, -2; n)$ for small values of n , where $a_N(\alpha, \beta; n)$ is the approximation obtained from Theorem 1.1 by summing over $1 \leq k \leq N$. The rows correspond to the series above in order.

As Corollary 1.2 indicates, the quality of these approximations improves as $n \rightarrow \infty$. Moreover, choosing larger values of N gives better approximations. Table 2 gives approximations when $N = 75$.

Table 1 Approximate values in Theorem 1.1, $N = 2$

n	1	2	3	4	5	6
$a_2(-1, 0; n)$	1.0029...	2.0808...	2.9340...	5.0296...	7.0278...	10.9325...
$a_2(0, -2; n)$	0	2.1281...	0	4.8883...	0	10.1650...
$a_2(1, -2; n)$	-0.8747...	1.3314...	-1.9544...	2.7902...	-3.8958...	5.3410...

Table 2 Approximate values in Theorem 1.1, $N = 75$

n	1	2	3	4	5	6
$a_{75}(-1, 0; n)$	0.9999..	2.0005...	2.9999...	4.9999...	6.9999...	10.9999...
$a_{75}(0, -2; n)$	0.0001...	1.9999...	-0.0002...	5.0001...	-0.0000...	9.9999...
$a_{75}(1, -2; n)$	-1.0004...	1.0003...	-2.0000...	3.0005...	-3.9994...	5.0003...

Table 3 Comparative asymptotic properties of $b_{S'}^*(r, 2; n)$

n	5	10	15	20	25
$\Theta_{S'}^0(n)$	0.5714...	0.5054...	0.4977...	0.4993...	0.5000...
$\Theta_{S'}^1(n)$	-0.4285...	-0.4946...	-0.5023...	-0.5006...	-0.5000...

Table 4 Comparative asymptotic properties of $c_{S'}^*(r_1, r_2; n)$

n	5	10	15	20	25
$\Theta_{S'}^{0,0}(n)$	0.2505...	0.2500...	0.2500...	0.2500...	0.2500...
$\Theta_{S'}^{0,1}(n)$	-0.2499...	-0.2499...	-0.2500...	-0.2500...	-0.2500...
$\Theta_{S'}^{1,0}(n)$	-0.2499...	-0.2499...	-0.2500...	-0.2500...	-0.2500...
$\Theta_{S'}^{1,1}(n)$	0.2495...	0.2499...	0.2499...	0.2499...	0.2499...

Example 7.2 For an illustration of Corollary 1.3, let S' be $C_2 \times \mathbb{P}^1$ blown up at two points, where C_2 is a curve of genus 2. In Tables 3, and 4 we take

$$\Theta_{S'}^r(n) := \frac{b^*(r, 2; n)}{\sum_{r \bmod 2} |b^*(r, 2; n)|} \quad \text{and} \quad \Theta_{S'}^{r_1, r_2}(n) := \frac{c^*(r_1, r_2; n)}{\sum_{\substack{r_1 \bmod 2 \\ r_2 \bmod 2}} |c^*(r_1, r_2; n)|}.$$

Corollary 1.3 (1) states that $\Theta_{S'}^0(n) \rightarrow 1/2$ and $\Theta_{S'}^1(n) \rightarrow 1/2$ as $n \rightarrow \infty$, which is illustrated by Table 3. Corollary 1.3 (2) states that $\Theta_{S'}^{0,0}(n), \Theta_{S'}^{1,1}(n) \rightarrow 1/4$ and $\Theta_{S'}^{0,1}(n), \Theta_{S'}^{1,0}(n) \rightarrow -1/4$, which is illustrated by Table 4.

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