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On the diophantine equation $la^x + mb^y = nc^z$

Sivanarayanapandian Subburam*

*Correspondence:
ssrammaths@gmail.com
Gandhi Nagar, Aruppukottai,
Tamil Nadu 626 101, India

Abstract

In this paper, we give an upper bound for the solutions x, y, z of the equation in the title, of magnitude $\left(\log \max\{a, b, c\}\right)^{2+\epsilon}$. This yields an improvement of earlier results of Hu and Le, where the bound is cubic in $\log \max\{a, b, c\}$.

Keyword: Exponential diophantine equation

Mathematics Subject Classification: 11D41, 11D45

1 Introduction

Let a, b and c be coprime integers greater than one and let (x, y, z) be any positive integral solution of the equation $a^x + b^y = c^z$. In 2015, Hu and Le [2] proved that $\max\{x, y, z\} \leq \kappa(\log \max\{a, b, c\})^3$, where $\kappa = 155{,}000$. Recently, the same authors improved the result as $\kappa = 6500$, in [3]. In this paper, we prove the following theorem:

Theorem 1 *For each $0 < \epsilon < 1$, there exists an effectively computable constant $\kappa(\epsilon) > 0$ such that*

$$\max\{x, y, z\} \leq \kappa(\epsilon)(\log \max\{a, b, c\})^{2+\epsilon}.$$

Moreover, if l, m and n are positive integers such that one of $(a \bmod 2, m, n)$, $(b \bmod 2, l, n)$ and $(c \bmod 2, l, m)$ is $(0, 1, 1)$, then all positive integral solutions (x, y, z) of the equation

$$la^x + mb^y = nc^z$$

satisfy the inequality with $\kappa(\epsilon) = \kappa(\epsilon, l, m, n)$, an effectively computable constant depends only on ϵ, l, m and n.

2 Proof of Theorem 1

Though there are better lower bounds in the literature than what Lemma 1 gives, it is sufficient for the present purposes.

Lemma 1 (*Corollary B.*1 *of Shorey-Tijdeman* [4]) *Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be non-zero rational numbers of heights not exceeding $A_1, A_2, \ldots, A_n$, respectively. We assume $A_j \geq 3$ for $1 \leq j \leq n$. Put*

$$\Omega = \prod_{j=1}^{n} \log A_j \text{ and } \Omega' = \prod_{j=1}^{n-1} \log A_j.$$

Then, there exist computable absolute constants κ_0 and κ_1 such that the inequalities

$$0 < \left|\alpha_1^{\beta_1}\alpha_2^{\beta_2}\cdots\alpha_n^{\beta_n} - 1\right| < \exp\left(-(\kappa_0 n)^{\kappa_1 n}\Omega\log\Omega'\log B\right)$$

have no solution in rational integers β_1, β_2, ..., β_n of absolute values not exceeding $B \geq 2$.

For a prime number p and a non-zero integer α, $\mathrm{ord}_p(\alpha)$ is the largest non-negative integer l such that $p^l \mid \alpha$.

Lemma 2 (*Bugeaud* [1]) *Let α_1 and α_2 be non-zero multiplicatively independent integers and let β_1 and β_2 be positive integers. Assume that there exist a positive integer g and a real number E such that*

$$\mathrm{ord}_p\left(\alpha_1^g - 1\right) \geq E > 1/(p-1)$$

and

$$\mathrm{ord}_p\left(\alpha_2^g - 1\right) > 0.$$

Let $A_1 > 1$ and $A_2 > 1$ be real numbers such that

$$\log A_i \geq \max\{\log|\alpha_i|, E\log p\}$$

for $i = 1, 2$. Put

$$\beta = \log\left(\frac{\beta_1}{\log A_2} + \frac{\beta_2}{\log A_1}\right) + \log(E\log p) + 0.4.$$

Consider

$$\Lambda' = \alpha_1^{\beta_1} - \alpha_2^{\beta_2}.$$

Then, we have

$$\mathrm{ord}_p\Lambda' \leq \frac{36.1g}{E^3(\log p)^4}\left(\max\{\beta, 6E\log p, 5\}\right)^2(\log A_1)(\log A_2),$$

if p is odd or if $p = 2$ and $\mathrm{ord}_2(\alpha_2 - 1) \geq 2$.

Fix $\min\{x, y, z\} \geq 4$. First, we shall assume that c is even. Then a and b are odd. Here, we work modulo 4. Since $a^x + b^y \equiv 0 \pmod 4$ and so it is impossible that both x and y are even, $\wedge' = a^x + b^y$ can be written as $\wedge' = \alpha_1^{\beta_1} - \alpha_2^{\beta_2}$, where α_1, α_2, β_1 and β_2 are integers such that $\alpha_1 \equiv \alpha_2 \equiv 1 \pmod 4$ and $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ is one of $(a, -b, x, y)$, $(b, -a, y, x)$, $(a^2, -b, x/2, y)$ and $(b^2, -a, y/2, x)$. Furthermore, take $p = 2$, $E = 2$, $g = 1$, $A_1 = \max\{4, |\alpha_1|\}$ and $A_2 = \max\{4, |\alpha_2|\}$ in Lemma 2. Since $\gcd(\alpha_1, \alpha_2) = 1$, by Lemma 2, we have

$$\mathrm{ord}_2\left(\alpha_1^{\beta_1} - \alpha_2^{\beta_2}\right) < \kappa_2\left(\log|\alpha_1|\right)\left(\log|\alpha_2|\right)\left(\log\max\{\beta_1, \beta_2\}\right)^2,$$

where κ_2 is a computable absolute constant. Also, it is clear that $z \leq \mathrm{ord}_2(c^z)$. Therefore, we conclude that

$$z \leq 4\kappa_2(\log a)(\log b)\left(\log\max\{x, y\}\right)^2.$$

This provides the bound for $\max\{x, y, z\}$, since $a^x, b^y \leq c^z$.

Next, assume that c is odd. Then a or b is even. Without loss of generality, suppose that a is even. Let $\wedge' = c^z - b^y$. Then, similarly as above, we obtain

$$x \leq 4\kappa_2(\log b)(\log c)\,(\log \max\{y, z\})^2 .$$

Here, we distinguish two subcases. If $c^z \leq a^{2x}$, then the last two inequalities along with the inequality $b^y \leq c^z$ give the required bound. Otherwise, the given equation implies that

$$0 < c^z - b^y \leq c^{z/2} \text{ and so } 0 < 1 - c^{-z}b^y \leq c^{-z/2}.$$

Take $n = 2$, $\alpha_1 = c$, $\alpha_2 = b$, $\beta_1 = -z$, $\beta_2 = y$, $A_1 = \max\{3, c\}$, $A_2 = \max\{3, b\}$ and $B = \max\{y, z\}$ in Lemma 1. Therefore, by the lemma, we write

$$c^{-z/2} \geq 1 - c^{-z}b^y > \exp\{-\kappa_3\,(\log b)\,(\log c)\,(\log\log\min\{b, c\})\log\max\{y, z\}\},$$

where κ_3 is a computable absolute constant. Combine this inequality with $c^z \geq a^{2x}$ and $b^y \leq c^z$ to have the bound.

Next, fix $\min\{x, y, z\} \leq 3$. Then, $a^x \leq c^z$ and $b^y \leq c^z$ provide the required bound, if $\min\{x, y, z\} = z$. Suppose that $\min\{x, y, z\} = x$. Then, we get $c^z - b^y \leq a^3$ and it can be written as that $0 < c^z b^{-y} - 1 \leq a^3 b^{-y}$ if $\max\{y, z\} = y$ or that $0 < 1 - b^y c^{-z} \leq a^3 c^{-z}$ if $\max\{y, z\} = z$. Now, apply Lemma 1 by taking $n = 2$, $\alpha_1 = b$, $\alpha_2 = c$, $A_1 = \max\{3, b\}$, $A_2 = \max\{3, c\}$ and $B = \max\{y, z\}$. Then, we have the bound. Also, similarly as above, we can deal with the case $\min\{x, y, z\} = y$.

For the proof of the second part of the theorem, in the above arguments replace c^z by nc^z if c is even, a^x by la^x if c is odd and a is even, and b^y by mb^y if c is odd and b is even, respectively.

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