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Mathematics as an Empirical Phenomenon, Subject to Modeling

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Abstract Among the universal attributes of homo sapiens, several have become established as special fields of study—language, art and music, religion, and political economy. But mathematics, another universal attribute of our species, is still modeled separately by logicians, historians, neuroscientists, and others. Could it be integrated into "mathematics studies," a coherent, many-faceted branch of empirical science? Could philosophers facilitate such a unification? Some philosophers of mathematics identify themselves with "positions" on the nature of mathematics. Those "positions" could more productively serve as models of mathematics.

Keywords Mathematics · Modeling · Pluralism · Faulhaber · Pragmatism · Empirical

Modeling is a central feature of contemporary empirical science. There is mathematical modeling, there is computer modeling, and there is statistical modeling, which is half way between. We may recall older models: plaster models of mathematical surfaces, stick-and-ball models of molecules, and the model airplanes that used to be so popular, but now have been promoted into drones.

Today the scholarly or scientific study of any phenomenon, whether physical, biological, or social, implicitly or explicitly uses a model of that phenomenon. A physicist studying heat conduction, for example, may model heat conduction as a fluid flow, or as propagation of kinetic energy of molecules, or as a relativistic or quantum mechanical action. Different models serve different purposes. Setting up a model involves focusing on features of the phenomenon that are compatible with the

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methodology being proposed, and neglecting features that are not compatible with it. A mathematical model in applied science explicitly refrains from attempting to be a complete picture of the phenomenon being modeled.

Mathematical modeling is the modern version of both applied mathematics and theoretical physics. In earlier times, one proposed not a model but a theory. By talking today of a model rather than a theory, one acknowledges that the way one studies the phenomenon is not unique; it could also be studied other ways. One's model need not claim to be unique or final. It merits consideration if it provides an insight that is not better provided by some other model.

It is disorienting to think of mathematics as the thing being modeled, because much of mathematics, starting with elementary arithmetic, already *is* a model of a physical action. Arithmetic, for instance, models the human action of counting.

Philosophy of mathematics, when studying the "positions" of formalism, constructivism, platonism, and so on, is studying models of mathematics, which is in large part a model. It studies second-order models! (Other critical fields like literary and art criticism are also studying models of models.) Being a study of second-order models, philosophy of mathematics constitutes still a higher order of modeling—a third-order model!

At this conference, I will make a few suggestions about the modeling of mathematics.

Empirical Studies of Mathematics

To study any phenomenon, a scholar or scientist must conceptualize it in one way or another. She must focus on some aspects and leave others aside. That is to say, she models it.

Mathematical knowledge and mathematical activity are observable phenomena, already present in the world, already out there, before philosophers, logicians, neuroscientists, or behavioral scientists proceed to study them.

The empirical modeling of social phenomena is a whole industry. Mathematical models, statistical models, and computer models strive to squeeze some understanding out of the big data that is swamping everyone. Mathematical *activity* (in contrast to mathematical *content*) is one of these social phenomena. It is modeled by neuroscience, by logic, by history of mathematics, by psychology of mathematics, anthropology and sociology. These must use verbal modeling for phenomena that are not quantifiable—the familiar psychological and interpersonal variables of daily life, including mathematical life.

Recognizing mathematical behavior and mathematical life as empirical phenomena, we would expect to use various different models, each focusing on a particular aspect of mathematical behavior. Some of these models might be mathematical. For such models, there would be a certain reflexivity or self-reference, since the model then would be part of the phenomenon being modeled.

History, logic, neuroscience, psychology, and other sciences offer different models of mathematics, each focusing on the aspects that are accessible to its method of investigation. Different studies of mathematical life overlap, they have interconnections, but still, each works to its own special standards and criteria. Historians are historians first of all and likewise educators, neuroscientists, and so on. Each special field studying math has its own model of mathematics.

Each of these fields has its particular definition of mathematics. Rival definitions could provoke disagreement, even conflict. Disagreement and conflict are sometimes fruitful or instructive, but often they are unproductive and futile. I hope to convince some members of each profession that his/her viewpoint is not the only one that is permissible. I try to do justice to all, despite the bias from a lifetime as a mathematician.

Let us look separately at four of the math-studying disciplines and their models.

Logic Among existing models of mathematics, the giant branch of applied logic called formalized mathematics is by far the most prestigious and successful. Being at once a model of mathematics and a branch of mathematics, it has a fascinating self-reflexivity. Its famous achievements are at the height of mathematical depth. Proudly and justifiably, it excludes the psychological, the historical, the personal, the contingent or the transitory aspects of mathematics.

Related but distinct is the recent modeling of mathematical proof in actual code that runs on an actual machine. Such programs come close to guaranteeing that a proof is complete and correct.

Logic sees mathematics as a collection of virtual inscriptions—declarative sentences that could in principle be written down. On the basis of that vision, it offers a model: formal deductions from formal axioms to formal conclusions—formalized mathematics. This vision itself is mathematical. Mathematical logic is a branch of mathematics, and whatever it is saying about mathematics, it is saying about itself—self-reference. Its best results are among the most beautiful in all of mathematics (Godel's incompleteness theorems, Robinson's nonstandard analysis).

This powerful model makes no attempt to resemble what real mathematicians really do. That project is left to others. The logician's view of mathematics can be briefly stated (perhaps over-simplified) as "a branch of applied logic."

The competition between category theory and set theory, for the position of "foundation," can be regarded as a competition within logic, for two alternative logical foundations. Ordinary working mathematicians see them as two alternative models, either of which one may choose, as seems best for any purpose.

The work of *neuroscientists* like Dehaene (1997) is a beginning on the fascinating project of finding how and where mathematical activity takes place on the biophysical level of flesh and blood. Neuroscience models mathematics as an activity of the nervous system. It looks at electrochemical processes in the nervous system of the mathematician. There it seeks to find correlates of her mathematical process. Localization in the brain will become increasingly accurate, as new research technologies are invented. With accurate localization, it may become possible to observe activity in specific brain processes synchronized with conscious mathematical thought. Already, Changeux, in Connes and Changeux (1995) argues forcefully that mathematics is nothing but a brain process.

The neuroscientist's model of mathematics can be summarized (a bit over-simplified) as "a certain kind of activity of the brain, the sense organs and sensory nerves."

History of mathematics is done by mathematicians as well as historians. History models mathematics as a segment of the ongoing story of human culture. Mathematicians are likely to see the past through the eyes of the present, and ask, "Was it important? natural? deep? surprising? elegant?" The historian sees mathematics as a thread in the ever-growing web of human life, intimately interwoven with finance and technology, with war and peace. Today's mathematics is the culmination of all that has happened before now, yet to future viewpoints it will seem like a brief, outmoded stage of the past.

Many *philosophers* have proposed models of mathematics, but without explicitly situating their work in the context of modeling. Lakatos' *Proofs and Refutations* (1976) presents a classroom drama about the Descartes-Euler formula. The problem is to find the correct definition of "polyhedron," to make the Descartes-Euler formula applicable. The successive refinement by examples and counter-examples is implicitly being suggested as a model for mathematical research in general. Of course critics of Lakatos found defects in this model. His neat reconstruction overlooked or omitted inconvenient historical facts. Lakatos argued that his rational reconstruction was more instructive than history itself! This is amusing or outrageous, depending on how seriously you take these matters. It is a clear example of violating the zeroth law of modeling, which is: Never confuse or identify the model with the phenomenon!

Philip Kitcher's *The Nature of Mathematical Knowledge* (1983) sought to explain how mathematics grows, how new mathematical entities are created. He gave five distinct driving forces to account for this. Feferman (1998), in constructing the smallest system of logic that is big enough to support classical mathematics, is also offering us a model of mathematics. Grosholz (2007) in focusing on what she calls "ampliative" moves in mathematical research, is modeling mathematical activity. Cellucci (2006) in arguing that plausible reasoning rather than deductive reasoning is the essential mathematical activity, is also proposing a model of mathematics. In *A Subject With No Object*, Burgess and Rosen (1997) conclude that nominalist reconstructions of mathematics help us better understand mathematics—even though nominalism (they argue) is not very tenable as a philosophical position. This short list reflects my own reading and interests. Many others could be mentioned.

Analogous to the well-established interaction of history of science and philosophy of science, there has been some fruitful interaction between philosophy of mathematics and history of mathematics. One disappointing example was the great French number theorist Weil (1978), who in his later years took an interest in history, and declared that no two fields have less in common, than philosophy of math and history of math. The philosopher-historian Imre Lakatos, on the other hand, wrote that without philosophy history is lame, and without history, philosophy is blind. Or maybe it is the other way around. Each model is important, none should be ignored.

The collaboration between philosopher Mark Johnson and linguist George Lakoff is exemplary. (*Where mathematics comes from* (2000) by Lakoff and Rafael Nunez, is a major contribution to our understanding of the nature of mathematics.)

There are some eccentric, philosophically oriented *mathematicians.* We try to untangle our own and each others' actions and contributions. We do not always manage to separate the content of mathematics from the activity of mathematics, for to us they are inseparable. We are not offering contributions to philosophy. We are not philosophers, as some philosophers politely inform us. We merely try to report faithfully and accurately what we really do. We are kindly tolerated by our fellow-mathematicians, and are considered "gadflies" by the dominant philosophers.

Byers (2010) introduced ambiguity as an essential aspect of mathematics, and a driving force that leads to the creation of new mathematics.

Several leading mathematicians have written accounts of their own experience in a phenomenological vein; I quote them in *How mathematicians convince each other*, one of the chapters in *Experiencing Mathematics* (Hersh 2014).

My own recent account of mathematicians' proof (Hersh 2014) is another possible model of mathematics. Here it is: A mathematician possesses a mental model of the mathematical entity she works on. This internal mental model is accessible to her direct observation and manipulation. At the same time, it is socially and culturally controlled, to conform to the mathematics community's collective model of the entity in question. The mathematician observes a property of her own internal model of that mathematical entity. Then she must find a recipe, a set of instructions that enables other competent, qualified mathematicians to observe the corresponding property of their corresponding mental model. That recipe is the proof. It establishes that property of the mathematical entity.

This is a verbal, descriptive model. Like any model, it focuses on certain specific features of the situation, and by attending to those features seeks to explain what is going on.

The discussion up to this point has left out of account the far greater part of ongoing mathematical activity—that is, schooling. Teaching and learning. Education.

Teachers and educators will be included in any future comprehensive science of mathematics. They observe a lot and have a lot to say about it. Paul Ernest, in his book *Social constructivism in the philosophy of mathematics* (1997) follows Lakatos (1976) and Wittgenstein, in building his social constructivist model.

Mathematics education has urgent questions to answer. What should be the goals of math education? What methods could be more effective than the present disastrously flawed ones? Mathematics educators carry on research to answer these questions. Their efforts would be greatly facilitated by a well-established overall study of the nature of mathematics.

Why not seek for a unified, distinct scholarly activity of *mathematics studies*: the study of mathematical activity and behavior? Mathematics studies could be established and recognized, in a way comparable to the way that linguistics has established itself, as the study of mathematical behavior, by all possible methods. Institutionally, it would not interfere with or compete with mathematics departments, any more than linguistics departments impinge on or interfere with the long-established departments of English literature, French literature, Russian literature, and so on.

Rather than disdain the aspect of mathematics as an ongoing activity of actual people, philosophers could seek to deepen and unify it. How do different models fit together? How do they fail to fit together? What are their contributions and their shortcomings? What is still missing? This role for philosophy of mathematics would be higher than the one usually assigned to it.

A coherent inclusive study of the nature of mathematics would contribute to our understanding of problem-solving in general. Solving problems is how progress is made in all of science and technology. The synthesizing energy to achieve such a result would be a worthy and inspiring task for philosophy.

About Modeling and the Philosophy of Mathematics

Turning now to the content of mathematics rather than the activity, we are in the realm of present-day philosophy of mathematics.

Philosophers of mathematics seem to be classified by their "positions," as though philosophy of mathematics were mainly choosing a position, and then arguing against other positions. I take Stewart Shapiro's *The Oxford Handbook of Philosophy of Mathematics and Logic* (2005) as a respected representative. "I now present sketches of some main positions in the philosophy of mathematics," he writes.

Six positions appear in the table of contents, and five of them get two chapters, pro and con. Between chapters expounding logicism, intuitionism, naturalism, nominalism, and structuralism, are chapters reconsidering structuralism, nominalism, naturalism, intuitionism, and logicism. "One of these chapters is sympathetic to at least one variation on the view in question, and the other 'reconsiders'." Formalism gets only one chapter, evidently it does not need to be reconsidered.

"A survey of the recent literature shows that there is no consensus on the logical connections betwen the two realist theses or their negations. Each of the four possible positions is articulated and defended by established philosophers of mathematics."

"Taking a position" on the nature of mathematics looks very much like the vice of "essentialism"—claiming that some description of a phenomenon captures what that phenomenon "really is," and then trying to force observations of that phenomenon to fit into that claimed essence. Rival essentialisms can argue for a very long time; there is no way either can force the other to capitulate.

Such is the story of mathematical platonism and mathematical anti-platonism. Balaguer (2001, 2013) has even published a book proving that neither of those two can *ever* be proved or disproved. "He concludes by arguing that it is not simply that we do not currently have any good arguments for or against platonism but that we could never have such an argument." Balaguer's conclusion is correct. It is impossible in principle to *prove or disprove* any model of any phenomenon, for the phenomenon itself is prior to, independent of, our formalization, and cannot be regarded as or reduced to a term in a formal argument.

One natural model for mathematics is as story or narrative. Thomas (2007) suggests such a model. Thinking of mathematical proofs or theories as stories has both obvious merits and defects. Pursuing its merits might have payoffs in research, or in teaching. That would be different from being a fictionalist—taking *the position* that mathematics IS fiction. Thomas (2014) has also suggested litigation and playing a game as models for mathematical activity.

Another natural model for mathematics is as a structure of structures (whether "ante rem" or otherwise). It is easy to see the merits of such a model, and not hard to think of some defects. Pursuing the merits might have a payoff, in benefitting research, or benefitting teaching. This would be a different matter from *being a structuralist*—taking the position that mathematics IS structure.

The model of mathematics as a formal-axiomatic structure is an immense success, settling Hilbert's first and tenth problems, and providing tools for mathematics like nonstandard analysis. It is a branch of mathematics while simultaneously being a model of mathematics, so it possesses a fascinating and bewildering reflexivity. Enjoying these benefits does not require one to be a formalist—to claim that mathematics IS an axiomatic structure in a formal language. Thurston (2006) testifies to the needless confusion and disorientation which that formalist claim causes to beginners in mathematical research.

If a philosopher of mathematics regarded his preferred "position" as a model rather than a theory, he might coexist and interact more easily. Structuralism, intuitionism, naturalism, nominalism/fictionalism and realism/Platonism each has strengths and weaknesses as a model for mathematics. Perhaps the most natural and appealing philosophical tendency for modeling mathematics is phenomenology. The phenomenological investigations of Merleau-Ponty looked at *outer* perception, especially vision. A phenomenological approach to mathematical behavior would try to capture *an inner perception,* the mathematicians' encounter with her own mathematical entity.

If we looked at these theories as models rather than as theories, it would hardly be necessary to argue that each one falls short of capturing all the major properties of mathematics, for no model of any empirical phenomenon can claim to do that. The test for models is whether they are useful or illuminating, not whether they are complete or final.

Different models are both competitive and complementary. Their standing will depend on their benefits in practice. If philosophy of mathematics were seen as modeling rather than as taking positions, it might consider paying attention to mathematics research and mathematics teaching as testing grounds for its models.

Can we imagine these rival schools settling for the status of alternative models, each dealing with its own part of the phenomenon of interest, each aspiring to offer some insight and understanding? The structuralist, platonist, and nominalist could accept that in the content of mathematics, even more than in heat conduction or electric currents, no single model is complete. Progress would be facilitated by encouraging each in his own contribution, noticing how different models overlap and connect, and proposing when a new model may be needed. A modeling paradigm would substitute competition for conflict. One philosophical modeler would allow the other modeler his or her model. By their fruits would they be judged.

Frege expelled psychologism and historicism from respectable philosophy of mathematics. Nevertheless, it is undeniable that mathematics is a historical entity, and that mathematical work or activity are mental work and activity. Its history and its psychology are essential features of mathematics. We cannot hope to understand mathematical activity while forbidding attention to the mathematician's mind.

As ideologies, historicism or psychologism are one-sided and incomplete, as was logicisms' reduction of mathematics to logic. We value and admire logic without succumbing to logicism. We can see the need for the history of mathematics and the psychology of mathematics, without committing ourselves to historicism or psychologism.

The argument between fictionalists, platonists, and structuralists seems to suppose that some such theory could be or should be the actual truth. But mathematics is too complex, varied and elaborate to be encompassed in any model. An all-inclusive model would be like the map in the famous story by Borges—perfect and inclusive because it was identical to the territory it was mapping.

Formalists, logicists, constructivists, and so on can each try to provide understanding without discrediting each other, any more than the continuum model of fluids contradicts or interferes with the kinetic model.

Some Elementary Number Theory

Since nothing could be more tedious than 20 pages of theorizing about mathematics without a drop of actual mathematics, I end with an example from the student magazine *Eureka* (2013) which also appeared in the *College Mathematics Journal* (2012). It is an amusing, instructive little sample of mathematicians' proof, and a possible test case for different models of mathematics.

A high-school exercise is to find a formula for the sum of the first n cubes. You quickly sum

$$1 + 8 + 27 + 64 + 125\ldots$$

and find the successive sums

$$1, 9, 36, 100, 225\ldots$$

You immediately notice that these are the squares of

$$1, 3, 6, 10, 15$$

which are the sums of the first n integers for

$$n = 1, 2, 3, 4 \text{ and } 5.$$

If we denote the sum of the pth powers of the integers, from the first up to the nth, as the polynomial $S_p(n)$, which always has degree $p + 1$, then our discovery about the sum of cubes is very compact:

$$S_3(n) = [S_1(n)]^2$$

What is the reason for this surprising relationship? Is it just a coincidence?

A simple trick will explain the mystery. We will see that the sums of odd powers—the first, third, fifth, or seventh powers, and so on—are always *polynomials in the sum of the first n integers*. If you like, you could call this a "theorem."

I will give you instructions. To start, just make a table of the sums of pth powers of the integers, with

$p = 0$ in the first row,

$p = 1$ in the second row,

$p = 2$ in the third row,

$p = 3$ in the fourth row,

Instead of starting each row at the left side of the page, start in the middle of the page, like this:

0	1	2	2	4	5...
0	1	3	6	10	15...
0	1	5	14	30	55...
0	1	9	36	100	225...

Now notice that nothing prevents you from extending these rows *to the left*—by successive *subtractions* of powers of integers, instead of adding! In the odd rows, subtracting negative values, you obtain positive entries. Here is what you get:

−5	−4	−3	−2	−1	0	0	1	2	3	4	5
15	10	6	3	1	0	0	1	3	6	10	15
−55	−30	−14	−5	−1	0	0	1	5	14	30	55
225	100	36	9	1	0	0	1	9	36	100	225

The double appearance of 0 in each row results from the fact that in the successive subtractions, a subtraction of 0 occurs between the subtractions of 1 to the pth power and (-1) to the pth power.

Notice the symmetry between the right and left half of each row. The symmetry of the first and third row is opposite to the symmetry of the second and fourth. These two opposite kinds of symmetry are called "odd" and "even," respectively.

(That is because the graphs of the odd and even power functions have those two opposite kinds of symmetry. The even powers 2, 4, and so on, have the same values in the negative direction as in the positive direction. For degree 2, the graph is the familiar parabola of $y = x^2$, with axis of symmetry on the y-axis. The fourth power, sixth power, and so on have more complicated graphs, but they all are symmetric with respect to the vertical axis. The graphs of the odd powers, on the other hand (the first, third, fifth and so on), are symmetric in the opposite way, taking negative values in the negative direction (in the "third quadrant") and symmetric with respect to a point, the origin of coordinates.)

The two opposite symmetries in your little table suggest that the sum functions of the integers raised to even powers are odd polynomials, and the sums of odd powers are even polynomials.

Continuing to the left is done by *subtracting* $(-n)^p$. For the odd powers p, this is *negative*, so the result is *adding* n^p. That is the same as what you would do to continue *to the right*, adding the pth power of the next integer. Therefore, the observed symmetry for odd powers will continue for all n, and for every odd p, not just the $p = 1$ and $p = 3$ that we can read off our little table.

But surprise! The center of symmetry is not at

$$n = 0$$

but halfway between 0 and -1! Therefore, as the table shows, for odd p the polynomial $S_p(n)$ satisfies the shifted symmetry identity

$$S_p(-n) = S_p(n-1).$$

Therefore, for odd p, the squares, fourth powers and higher terms of $S_p(n)$ are even powers of $(n + ½)$. A sum of those *even* powers is the same thing as a sum of *all* powers of $(n + 1/2)^2$, which would be called "a polynomial in $(n + 1/2)^2$." To complete our proof, we need only show that

$$(n + 1/2)^2 = 2\,S_1 + 1/4.$$

Now $S_1(n)$ is very familiar, everybody knows that it is equal to

$$n(n+1)/2.$$

(There is a much-repeated anecdote about how this was discovered by the famous Gauss when he was a little boy in school.)

So then, multiplying out,

$$2\,S_1 = n^2 + n.$$

We do a little high-school algebra:

$$(n + 1/2)^2 = n^2 + n + ¼ = 2\,S_1 + 1/4,$$

so for odd p we do have S_p as a polynomial in S_1, as claimed.

I leave it to any energetic reader to work out $S_5(n)$ as a polynomial in $S_1(n)$. Since S_5 has degree 6, and S_1 is quadratic, S_5 will be cubic as a polynomial in S_1. There are only three coefficients to be calculated!

This little proof in elementary number theory *never even needed to state an axiom or hypothesis.* The rules of arithmetic and polynomial algebra did not need to be made explicit, any more than the rules of first-order logic. Without an axiom or a hypothesis or a premise, where was the logic?

Given an interesting question, we dove right into the mathematics, and swam through it to reach the answer. We started out, you and I, each possessing our own internal model of mathematical tables, of the integers, and of polynomials in one variable. These models match, they are congruent. In particular, we agree that an odd power of a negative number is negative, and that subtracting a negative number results in adding a positive number.

I noticed that continuing the table to the left led to interesting insights. So I gave you instructions that would lead you to those insights. You followed them, and became convinced. My list of instructions is the proof!

One could elaborate this example into formalized logic. But, what for? More useful would be making it a test for competing models of mathematics (formerly "positions."). How would the structuralist account for it? The nominalist, the constructivist, the platonist, the intuitionist? Which account is more illuminating? Which is more credible? How do they fit together? Are any of them incompatible with each other?

You may wonder, "Am I serious, asking a philosopher to take up modeling, instead of arguing for his chosen position against opposing positions?"

Yes. I am serious. The philosopher will then be more ready to collaborate with historians and cognitive scientists. The prospect for an integrated field of mathematics studies will improve.

However, such a turn is not likely to be made by many. If philosophy is all about "taking a position" and arguing against other positions, a switch from position-taking to modeling might bring a loss of standing among philosophers.

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