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ORIGINAL PAPER

A general expenditure system for estimation of consumer demand functions

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Abstract
The class of flexible functional forms for the utility and cost function has been characterized by the pioneering work of Gorman (Some Engel curves. In: Deaton A (ed) Essays in the theory and measurement of consumer behaviour. Cambridge Univ. Press, Cambridge, 1981), known as the Gorman polar form. Despite several decades have elapsed, the economic literature has not found the most general functional form that satisfies Gorman's theorem. This note provides a new general theoretical and parametric formulation of demand functions, labeled general expenditure system (GES), satisfying the Gorman requirement that the Engel curve cannot exceed a polynomial of third degree in expenditure. Estimates show that the GES is a significant generalization of previous popular flexible functions.

Keywords Integrable demand functions · General expenditure system · Gorman polar form

JEL Classification D01 · D11 · C30

1 Introduction

The theory of demand states the theoretical restrictions for a system of equations, which have to be derived from rational consumer behavior. The literature has addressed several issues relative to the utility function specification, such as functional flexibility, functional separability and Engel function curvature. In reality, these have been largely equivalent ways to define and discuss different mathematical

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formulations to represent the fundamental problem in economic theory, namely the rational choice of the optimal vector of consumption quantities, given the vector of prices, the expenditure and the preferences (Barnett and Serletis 2008).

Operationally, there are four equivalent ways to represent consumer choice according to duality theory, namely, utility maximization, cost minimization, minimization of the distance function and derivation of demand functions from the indirect utility function (Deaton and Muellbauer 1980; Blackorby et al. 1978; Chavas and Baggio 2010). In this paper, we adopt the last method, assuming that consumers' preferences can be represented with the indirect utility function and use Roy's identity to derive the Marshallian demand functions.

In this framework, the necessary conditions to parametrize a mathematical function, consistent with the restrictions of the economic theory, are set in the Gorman's theorem (1961, 1981). This theorem states the so-called Gorman polar form in the literature. However, despite several decades have elapsed, the economic literature has not found a unified general functional form that satisfies Gorman's theorem.

The aim of this paper is to propose a new functional form, which is the most general formulation of a demand system satisfying the Gorman theorem. We attempt to provide such formulation, using the minimal requirement stated by Gorman (1981); namely, that the functional form of the system of Engel curves can be a polynomial with a coefficient matrix of at most rank three. In other words, the matrix of Engels curve coefficients cannot have rank. We provide the mathematical function and we estimate a demand system and discuss the results in comparison with previous popular flexible functions.

The paper proceeds as follows. Section 2 provides a brief review of the literature. Section 3 describes the theoretical model. Section 4 provides data and estimation results. Section 5 concludes the paper.

2 Literature review

The literature on the characterization of the demand system for empirical estimation is huge. One of the pioneering contributions dates back to the linear expenditure system (LES), characterized by Klein and Rubin (1947) and estimated by Stone (1954), Pollak and Wales (1969) and others. The LES is characterized by a Cobb–Douglas (CD) utility function, which is derived by the introduction of the committed quantities. From a mathematical point of view, in the LES the Engel curves are homothetic but are not forced to pass through the origin, like in the CD case. Clearly, the LES is linear in expenditure.

A pioneering nonlinear formulation, the Addilog System, was proposed by Houthakker (1960). A more flexible formulation was based on the idea to construct a second order Taylor approximation of an unknown function. This originated the popular Translog, or transcendental-logarithmic, demand system (TL), estimated by Christensen et al. (1975). A more flexible form, the generalized translog (GTL), was introduced and estimated by Pollak and Wales (1980). The idea of a second order function in expenditure was the inspiration for a different function, which has to be quadratic in expenditure. This idea gives rise to the quadratic expenditure system (QES), which has been estimated by Pollak and Wales (1978) and discussed by Howe et al. (1979). The prices independent generalized linearity (PIGL) demand system has been proposed by Muellbauer (1975), who was among the first to address the definition of a function allowing the exact aggregation across consumers. Recall that the exact aggregation entails that the individual demands summation yields a tractable parametric closed form of aggregate demand.

Almost at the same time, the idea of starting from all the desirable properties that a demand system should possess has motivated the parametrization of the almost ideal (AI) demand system, proposed and estimated by Deaton and Muellbauer (1980). A generalized version, the generalized almost ideal (GAI), was proposed and estimated by Bollino (1987). Another generalization into a Box Cox transformation of the AI model was proposed by Matsuda (2006). A generalization of TL and AI was proposed by Lewbel (1987) and subsequently generalized by Bollino (1987) into the generalized almost ideal and translog (GAITL). Banks et al. (1997) proposed another generalization of the AI, including a quadratic term of the log of income, called quadratic almost ideal demand system (QUAIDS).

From a mathematical point of view, all of these studies share a common approach, namely, the use of theoretical restrictions as a maintained hypothesis for the construction of a parametric function suitable for empirical estimation. In other words, a parametric representation of the demand functions is postulated to characterize the consumer reaction to given price-expenditure situations. The fundamental restrictions that a demand system must satisfy are three. First, there is additivity, i.e. the sum of expenditures for the desired goods must add up to the total expenditure. Second, there is homogeneity of degree zero in price-expenditure space, i.e. an equiproportional variation of all prices and expenditures must leave the goods choice unchanged. Third, there is symmetry, i.e. the compensated demand elasticity values must be symmetric, reflecting the symmetry of the Slutsky matrix.

In the literature, the development of new functional forms has followed three main operational approaches to specify the number of independent parameters characterizing a demand system. The first is the approach of flexibility of the utility function from which it is possible to derive the systems of demand functions (e.g., Diewert 1974; Christensen et al. 1975; Berndt and Khaled 1979; Appelbaum 1979; Deaton and Muellbauer 1980). The second approach is the functional separability (e.g., Houthakker 1960; Blackorby et al. 1978) which defines boundaries of the consumer choice problem. In this way, it is possible to justify that the consumer choice of sugar is related to the choice of coffee, but it is unrelated to the choice of, say, air conditioning equipment in the house. The third approach consists in defining how expenditure enters the demand equations. This approach leads to the definition of the degree of curvature of the Engel curve (e.g., Muellbauer 1975; Lyssiotou 2012; Howe et al. 1979; Bollino 1987; Lewbel 1987).

3 Theoretical model

The theoretical model is the polynomial demand system due to Gorman (1981):

$$h^i(p,m) = \sum_{k=(1,K)} f^{ik}(p)\, g^{ik}(m), \tag{1}$$

where h^i is the quantity demand for good i (where i = 1,2,…,n), p is a vector of prices, m is income, f^{ik} and g^{ik} are functions of prices and expenditure, respectively. The functional form of (1) is a general polynomial sum of K terms in income. The Gorman theorem states that if the system (1) is descending from a rational utility maximization behavior, then the maximum rank of the matrix of expenditure coefficients is equal to 3 (Russell and Farris 1998). This means that the sum in Eq. (1) cannot exceed three independent terms in expenditure (in other words, K in the sum is at most K = 3) otherwise there is violation of the integrability conditions (additivity, homogeneity and symmetry restrictions) as shown in Lewbel (1990). From an empirical viewpoint, the rank of the demand system establishes the shape of the parametric representation of the Engel curves. For instance, a rank 1 system imposes linear Engel curves, a rank 2 system imposes a second order non-linear shape to Engel curve. Thus, the rank is a property that allows for different degrees of flexibility of the parametrization of Engel curve used to fit the data. The main advantage of a flexible parametrization is that it does not impose a specific functional form. For instance, Pendakur and Sperlich (2010) used Canadian household surveys to analyze the case of the S-shaped curvature of the expenditure share for private transportation, with flatter portions at the bottom and top of the sample distribution. For an interesting discussion about the empirical measurement of the rank (which is beyond the scope of this article), see Gill and Lewbel (1992), Perali (2003) and Menon et al. (2018).

There are special cases of Eq. (1) in the literature.

(i) the LES is rank 1, i.e. linear in expenditure:

$$h^i(p,m) = A^i(p) + B^i(p)\, m. \tag{2}$$

Notice that the first term is known as the committed quantity, while the second term necessarily involves a real price term for each good i. Imposing that the committed quantities are zero yields the CD demand system.

(ii) the PIGL is rank 2:

$$h^i(p,m) = B^i(p)\, m + C^i(p)\, m^\rho, \tag{3}$$

where ρ is a real number ($\rho \neq 1$). The terminology PIGL means prices independent generalized linearity and it is attributed by Muellbauer (1975) to the characteristic of the second income term raised to the power ρ.

(iii) the AI is rank 2:

$$h^i(p,m) = B^i(p)\, m + C^i(p)\, \log(m), \tag{4}$$

and it is characterized by logarithmic Engel curves.

(iv) the QES is rank 3, being quadratic in expenditure:

$$h^i(p,m) = A^i(p) + B^i(p)\,m + C^i(p)\,m^2. \tag{5}$$

Howe et al. (1979) argued that there is no objection in principle to systems "locally quadratic in expenditure", if we confine ourselves to a sub region of all possible price-expenditure situations. These regions would be spanned by the "committed quantities" vector, which is generally defined as the minimum subsistence bundle of the household. Moreover, the collinearity of several total expenditure terms in demand functions is not a problem more serious than the usual price collinearity, provided that suitable parametric restrictions are imposed on the demand functions, as it is done in the translog system.

(v) the QUAIDS is also rank 3, being a quadratic extension of the AI:

$$h^i(p,m) = B^i(p)\,m + C^i(p)\,\log(m) + G^i(p)\left[\log(m)\right]^2. \tag{6}$$

QUAIDS was proposed by Banks et al. (1997) and it has the property that the Engel curves are quadratic in the logarithm of expenditure. It is immediate to derive the AI from (6) as a special case setting $G^i(p)=0$.

An effective approach along the line of the Gorman theorem was proposed by Lebwel (1987, 1990) and LaFrance et al. (2006), who presented and labeled other alternative systems of rank 3, with the third term in expenditure being a monomial power, a log or a trigonometric function. Some issues appear unsolved, since in case of some systems of demand functions it is not possible to describe explicitly all the possible utility functions (Lewbel 1987).

In this article we propose a simpler and more compact way to apply the Gorman theorem. We define a new functional form in three terms, labeled general expenditure system (GES):

$$h^i(p,m) = A^i(p) + B^i(p)\,m + C^i(p)\,m^{1+\alpha}, \tag{7}$$

where A^i, B^i and C^i are functions independent of income. The polynomial representation of Eq. (1) can be cast as a Taylor–McLaurin expansion, $h^i(p,m) = \sum_{n=(1,n)} c^i_n (m - f(p))^n$, with suitable restrictions to obtain the most general formulation of a polynomial in three terms in expenditure, given in Eq. (7). Theorem 1 below shows that the GES represents in a unified way all the existing demand functions proposed in the literature, which satisfy the Gorman theorem. In this article we do not consider the rank 4 and 5 special constructions of EASI demand systems (Lewbel and Pendakur 2009) because they do not strictly satisfy the theoretical restrictions of utility maximization problem. The EASI is characterized by "implicit Marshallian demands", whereas the cost function is an affine transformation of the Stone index of deflated log nominal expenditures.

We show a simple general class of functional form for a demand system, which is integrable, i.e. a well-behaved utility function that can be recovered from the utility maximization problem. In other words, we characterize a general polynomial function of income and show that it obeys the Gorman theorem, which is generally used to solve differential equations and recover the underlying utility function from a system of demand functions. A simple way to appreciate this point is to recall that not all the empirical demand functions have this property. For instance, a linear demand system in price and income is not integrable in a utility function (Lau 1976). This motivates the definition of the GES as a general demand system derived from a utility maximization problem or, equivalently, from the indirect utility function.

We preliminarily state the following results in two Lemmas. Lemma 1 shows the properties of a theoretically plausible demand system, which obeys the following properties: adding up, symmetry, homogeneity of degree zero in real prices. Lemma 2 establishes under which conditions a translation procedure maintains the properties of the demand system. Translation means to introduce a new origin of reference for the utility in the good space $\{z_1, z_2, z_N\}$ so that the consumer derives utility from consumption of goods quantities above the minimum level (h^i–z_i). The quantities z_i are known in the literature as committed quantities, as labeled by Klein and Rubin (1947) in the discussion of LES. This clarifies that the LES is a translation of a CD, as CD utility function can be written as: $U = \Sigma\, a_i\, ln(h^i)$ likewise the utility function of the LES can be written as: $U = \Sigma\, a_i\, ln(h^i - z^i)$. Analogously, the GAI is a translation of the AI.

The relevance of Lemma 2 for empirical work is that it does not force all the Engel curves through the origin. Usually, empirical data do not report zero income observations, so that the Engel curve is not observed at the origin, but rather at low levels of income where necessary goods are usually prevailing and luxury goods may be absent in the consumption allocation. For a necessary good, a positive committed quantity implies a positive consumed quantity at zero income level. Conversely, for a luxury good, a negative committed quantity implies a positive consumed quantity beyond a certain level of income.

In the following, for simplicity of notation we have suppressed the argument of the function and use function subscripts to denote partial derivatives of the variable. So, for instance, given the functions $B^i(p)$ and $C^i(p)$, we use B^i and C^i and we denote $\partial B^i(p)/\partial \mathrm{p_j}$ with B^i_j and $\partial C^i(p)/\partial \mathrm{p_j}$ with C^i_j, respectively.

Lemma 1 If Eq. (6) is theoretically plausible, then:

$$\sum p_i C^i = 0 \sum p_i B^i = 1\, A^i = 0, \tag{8}$$

$$C^i_j + \alpha B^j A^i = C^j_i + \alpha B^i A^j, \tag{9}$$

$$B^i_j + B^j B^i = B^j_i + B^i B^j. \tag{10}$$

Proof The budget identity yields $\sum p_j h^j = \sum p_i A^i(p) + \sum p_i B^i(p)m + \sum p_i C^i(p) m^{1+\alpha}$ which establishes (8), because the budget identity implies that $\sum p_j h^j = m$, or that $\sum p_i C^i = 0 \sum p_i B^i = 1 \sum p_i\, A^i = 0$, while non negativity $Ai > 0$ and $\sum p_i\, A^i = 0$ establishes $A^i = 0$.

Slutsky symmetry condition $k_{ij} = k_{ji}$, where $k_{ij} = (1+\alpha) C^i C^j m^{2a+1} + \left[C^i_j + C^j B^i + (1+\alpha) B^j A^i\right] m^{1+a} + \left(B^i_j + B^j B^i\right)$ establishes (9) and (10).

Lemma 2 Translation with committed quantities z_i of a theoretically plausible system of demand functions $h^i(p,m)$ yields the theoretically plausible system $h^{*i}(p,m) = z_i + h^i(p,m^*)$, where $m^* = m - \sum p_j z_j$.

Proof Consider the indirect utility function $F(p,m)$. Applying Roy's identity yields:

$$h^i(p,m) = -F_i(p,m)/F_m(p,m).$$

A transformation $F(p,m) = F(p,m^*)$ yields:

$$\begin{aligned} h^{*i}(p,m) &= -\left[F_i(p,m^*) + F_{m^*}(p,m^*)\,dm^*/dp_i\right] / F_{m^*}(p,m^*)\,dm^*/dm \\ &= -dm^*/dp_i - F_i/F_{m^*} \\ &= z_i + h^i(p,m). \end{aligned}$$

Theorem 1 Any GES of the form (6) can be written as:

$$h^i(p,m) = \left[g^i(p)/(\alpha\, g(p))\right] m + \left[1/\left(\alpha\, g(p)^2\right)\right]\left\{f^i(p) - \left[g^i(p)^i/g(p)\right] f(p)\right\} m^{1+\alpha}, \tag{11}$$

with the indirect utility function given by:

$$F(p,m) = -g(p)/m^\alpha - f(p)/g(p), \tag{12}$$

where $f(p)$ and $g(p)$ are homogeneous of degree α.

Proof There exists n functions k^i such that (11) can be written as $h^i(p,m) = k^i m + C^i(p)\, m^{1+\alpha}$ with $\alpha \neq 0$; this implies $k^i = B^i$.

There exists a function $g(p)$ homogeneous of degree α such that: $k^i = g^i/(\alpha\, g)$.

Define $q^i(p,z) = k^i(p) z$ to get : $q^i_j + q^j q^i z = k^i_j z + k^j k^i z = z\left(k^i_j + k^j k^i\right) = z\left(B^i_j + B^i B^j\right)$ which is symmetric in virtue of (10). Using (9), we get: $\sum p_k\, g_k/a\, g = \sum p_k\, B_k = 1$, proving the homogeneity of $g(p)$: $\sum p_k g_k = a\, g$.

There exists a function $f(p)$ homogeneous of degree a such that: $C_i = (f_i - f\, g_i/g)/(a\, g^2)$, or:

$$f_i = \alpha g^2 C^i + f\, g_i/g.$$

Define: $y^i(p,z) = \alpha\, g^2\, C^i + z\, g_i/g$, yielding:

$$y^i_j + y^j y^i z = a\left(g^2 C^i_j + 2gC^i g^j\right) + z/g^2\left(g_{ij}g - g^i g^j\right) + a\left(g^2 C^j + z\,g_i/(ag)\,g_i/g\right.$$
$$= a\left[g^2 C^i_j + 2g^2 a C^i g_i/(ag)\right] + ag^2 C^j g_i/(ag) + zg_{ij}/g$$
$$= a\,g^2\left[\left(C^i_j + a\,C^i B^j\right) + a\left(C^i B^j + C^j B^i\right)\right] + zg_{ij}/g,$$

which is symmetric in virtue of (10) and direct inspection of the last two terms.

The f(p) is homogeneous using (9): $\sum pk\, fk = a\,g^2 \sum p_k\, A^k + a\,f \sum p_k\, B^k = a\,f.$

The demand system (11) is derived from (12), via Roy's identity.

The operational parameterization of the GES demand functions (using Lemmas 1, 2 and Theorem 1 to characterize the integrability properties, in the spirit of Nocke and Schutz 2017) yields any non-linear form in expenditure:

$$h^i(p,m) = a_i + m^*/p_i[b_i + \left(c_i - b_i\right)/\delta \prod p_{k_k}^{-\alpha c}(m^{*\alpha})], \tag{13}$$

for any real value of α, corresponding to the indirect utility function:

$$F(p,m) = -x(p)/m^{*\alpha} - w(p)/x(p), \tag{14}$$

where $m* = (m - g(p)); g(p) = \sum p_k a_k; x(p) = \delta \prod p_{k_k}^{\alpha b}; w(p) = (1 + \alpha)x(p)^2/\prod p_{k_k}^{\alpha c}$. In Eqs. (13), (14), several restrictions can be applied to obtain many previously known systems. The restriction $\alpha = 1$ yields the QES of Eq. (5), $\alpha = -1$ and $a_i = 0 \forall\ i$, yields the QUAIDS of Eq. (6), $\alpha = -1$, $\delta = (1+\alpha)$ and $a_i = 0 \forall\ i$ yields the AI[1] of Eq. (4), while the restriction $\alpha = 0$, which trivially implies $c_i = b_i \forall\, i$, yields the LES of Eq. (2). In addition, the restriction $a_i = 0 \forall\ i$ yields the PIGL, which belongs to the class of (3). It is also similar to the model of Lafrance (2008) of exactly aggregable demand system.

From an empirical point of view, the GES seems to be a manageable functional form since it adds only one parameter to the QES sharing with this latter the characteristic of being "parsimonious" in the parameters required as the number of commodities increases. GES contains 3n independent parameters. In comparison, among other nonlinear systems, we find that the BTL requires the estimation of $(n^2 + 3n - 2)/2$ independent parameters whereas the GTL—a system of the translog family obtained by introducing "committed quantities" in the BTL—contains $(n^2 + 5n - 2)/2$ parameters.

In addition, the GES parametrization has committed quantities, like QES, GAI and LES. Conversely, CD, AI, PIGL and QAIDS do not have committed quantities.

[1] Recall that $\log(m) = \left(m^{1+\alpha}\right)/(1 + \alpha)\, for\, \alpha \rightarrow -1.$

Table 1 Estimation and tests of model

Model estimation	Log L values and LR test = 2($\log L_0 - \log L_1$)
Systems without committed quantities	
CD	LogL = – 49,369
AI	LogL = – 46,867
	Test of AI vs CD: LR = (49,369–46,249) × 2 = 6240; df = 2
Systems with committed quantities	
LES	LogL = – 39,844
	Test of LES vs CD: LR = (49,369–39,844) × 2 = 19,050; df = 216
GAI	LogL = – 39,719
	Test of GAI vs AI: LR = (46,249–39,719) × 2 = 13,060; df = 216
QES	LogL = – 39,753
	Test of QES vs LES LR = (39,844–39,753) × 2 = 182; df = 2
GES	LogL = – 39,523
	Test of GES vs GAI LR = (39,719–39,523) × 2 = 392; df = 1
	Test of GES v QES LR = (39,753–39,523) × 2 = 460; df = 1
	Test of GES vs LES LR = (39,844–39,523) × 2 = 642; df = 3

LogL value of log of likelihood, *LR* likelihood ratio test (Chi-square), *df* degrees of freedom of the Chi-square test

4 Data and estimation

We use data on household consumption for 109 countries in the world, aggregated in a two-good bundle, the composite good (h^1) and the energy services (h^2), with capital stock constraint, taken from Atalla et al. (2018). Inclusion of capital stock avoids the risk of empirical bias (Deaton and Muellbauer 1981) and takes account of the energy services impact (Schaffrin and Reiblin 2015).

We have parametrized the effect of the capital stock as an additional term to the demand equations of the form: $k_i * K$, where K is a measure of the capital stock and k_i are demand specific parameters. This data consists of the aggregate household expenditure for the goods and the respective prices for a group of 109 countries for the period 1978–2012, for a total of 1891 observations, representing more than 95% of world population. The panel is unbalanced, because data periods differ across countries. Atalla et al. (2018) estimated a model of aggregate residential and commercial energy demand using a GAI parametrization. He discussed also the non-linear and globally stationary characteristics of the data. Table 2 shows the results of the Kapetanios et al. (2003) non-linear test, which detects non-stationarity hypothesis against non-linear but globally stationary

Table 2 Detailed parameter estimates

	CD	AI	LES	GAI	QES	GES
b_1	0.939**	0.472**	0.983**	0.811**	0.953**	0.994**
a_1			242,644	−239,104	265,699	−249,375
a_2			2067**	778**	2084**	2124**
k_1		0.911E−04**	−0.641E−04**	−0.234E−04**	−0.467E−05**	−0.134E−04**
k_2		0.107E−03**	0.222E−04**	0.199E−04**	0.392E−04**	0.129E−04**
c_1		0.0388**		0.0209**	0.8835**	1.14748**
d_1		0.0346**		0.012**		
a					1	−3**
δ					0.148E−07**	1**
Rsq Eq. 1	0.989		0.999	0.999	0.999	0.999
Rsq Eq. 2	0.706		0.995	0.997	0.994	0.997

**Significant at 1%; Rsq refer to equations of good 1 and 2, respectively

ESTAR (Exponential Smooth Transition Autoregressive) model. According to the tests, the demand variables exhibit a stationary but non-linear behavior. This is the main motivation for the choice of this data set. We want to apply a general functional form allowing non-linear Engel curves and we apply it to data that show relevant non-linearity.

The estimation of several functional forms, from the most restrictive to the general, CD, LES, AI, GAI, QES and GES, with a non-linear FIML procedure, is conducted on the 1891 observations (Table 1). Notice that we focused on systems with committed quantities and we report only CD and AI for comparison with their respective generalizations. All structural parameters are significant (Table 2). We do not report the country specific committed quantity parameters. The zero restriction on the capital stock parameters is rejected. In the GES, α has been estimated with a grid search in the range $\{-4, +4\}$, obtaining the maximum of the likelihood function at $\alpha = -2$, with the scale parameter $\delta = 1$.

The estimated systems are nested, in the sense that imposing appropriate restrictions on the parameters (13) yields more restrictive forms (see also Lafrance et al. 2006). Below, some examples as follows:

The restriction $\alpha = 1$ yields the parametrization of the QES:

$$h^i(p,m) = a_i + m^*/p_i\left[b_i + \left(c_i - b_i\right)/\delta \prod p_{k_k}^{-ac}(m^*)\right]. \tag{15}$$

Imposing $\alpha = -1$ and $\delta = (1 + \alpha)$ and trivially $d_i = c_i = b_i \forall i$ yields the parametrization of the GAI:

$$h^i(p,m) = a_i + m^*/p_i\left[b_i + \sum c_j \ln\left(p_j\right) + d_i m^*/P*\right], \tag{16}$$

where $P* = \sum w_j\, ln(p_j)$ *and* w_j are the budget shares.

The restriction $\alpha = 0$, which trivially implies $c_i = b_i = 0 \forall i$, yields the parametrization of the LES

Table 3 Estimated price and income elasticities—sample averages

	ELP11	ELP22	ELY1	ELY2	ELK1	ELK2
LES	− 0.980	− 0.014	1.296	0.256	− 0.002	0.006
QES	− 0.955	− 0.034	1.257	0.609	− 0.001	0.002
GAI	− 0.986	− 0.109	1.318	0.013	− 0.037	0.214
GES	− 0.981	− 0.027	1.301	0.010	− 0.007	0.094

All values are significant at 1%

ELP11 own price elasticity of good h^1; ELP2 = own price elasticity of good h^2, *ELY1* income elasticity of good h^1, *ELY2* income elasticity of good h^2, *ELK1* capital stock elasticity of good h^1, *ELK2* capital stock elasticity of good h^2

$$h^i(p,m) = a_i + m^*/p_i b_i. \tag{17}$$

The log likelihood and the related LR tests (Table 1) show that the GES provides a better fit than the GAI, AI, QES, LES, CD, as the LR test rejects the restrictions of these latter systems. All parameters are significant. All the systems with committed quantities show significant estimations. Precisely, the test of GES against QES and against GAI yields LR = 460 and LR = 392 with 1 degree of freedom, respectively. In addition, the tests of GAI against AI and of LES against CD show the significance of the committed quantities. The rejection of LES suggests the existence of non-linearity. Given that both GAI and QES are nested in the parametric form of GES, these tests confirm that GES is a significant generalization of previous known systems.

We estimated the sample average income and price elasticities (Table 3). We report the estimated quantities (Fig. 1) and the estimated Engel curves for the selected functional forms and the countries (Fig. 2).

Note that in the GES, the estimated price elasticity of the composite good (h^1) is close to unity, while the energy good (h^2) is price inelastic, with the absolute value lowest for LES and highest for GAI. The values for GES, around to 0.03 is consistent with the literature. In addition, the income elasticity of h^1 is greater than one, in the range 1.25–1.32, while income elasticity of energy is small and significantly lower than 1. This is consistent with the negative worldwide trend of energy intensity, which has declined by around 1.1% per year. The elasticity to capital stock is generally negative for the composite good and positive for the energy good, which are plausible values.

Note that the estimations are quite accurate. The world sample actual and the estimated values are reported in the top graphs of Fig. 1, along with the pattern for emerging (lower income) and developed countries (higher income). Focusing on energy (h^2), note that there are some appreciable differences at higher income levels between the LES and the other non-linear systems. This is also evident for the Engel curves, showing a non-linear pattern for h^2 in some countries (Fig. 2). It is interesting to note that the composite good h^1 is a luxury for Russia and China. Concerning the energy good there are some differences:

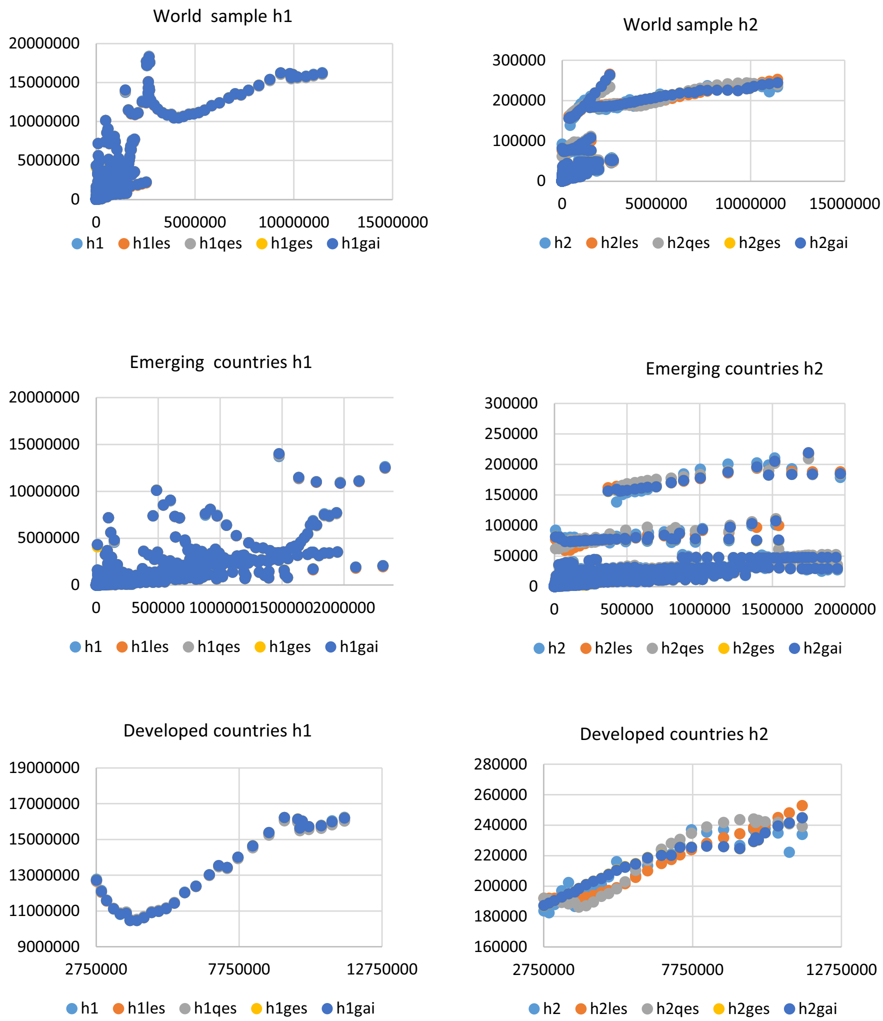

Fig. 1 Estimated quantities for goods h^1 and h^2 for the sample and selected groups of countries. h1 and h2 = actual values; suffix: les, qes, ges, gai = estimated values of the systems

energy is generally a necessary good, except in the case of QES for Germany, UK and US. In general, the GES estimated curves represent accurately the Engel curve pattern.

A further comparison between the linear and non-linear systems is shown in Fig. 3, which presents the ratio of the estimated Engel curves of GES QES and GAI to the Engel curves of LES. The horizontal axis is ordered by country income level. Note that the most appreciable difference from the linear case occurs for the lower income countries. In particular, in the case of GES, the ratio to LES is slightly above 1 for energy in the middle and in the high-income

Fig. 2 Estimated Engel curves for goods h^1 and h^2 for selected countries. h1les = h1 estimated values of LES; h2les = h2 estimated values of LES; h1qes = h1 estimated values of QES; h2qes = h2 estimated values of QES; h1ges = h1 estimated values of GES; h2ges = h2 estimated values of GES; h1gai = h1 estimated values of GAI; h2gai = h2 estimated values of GAI

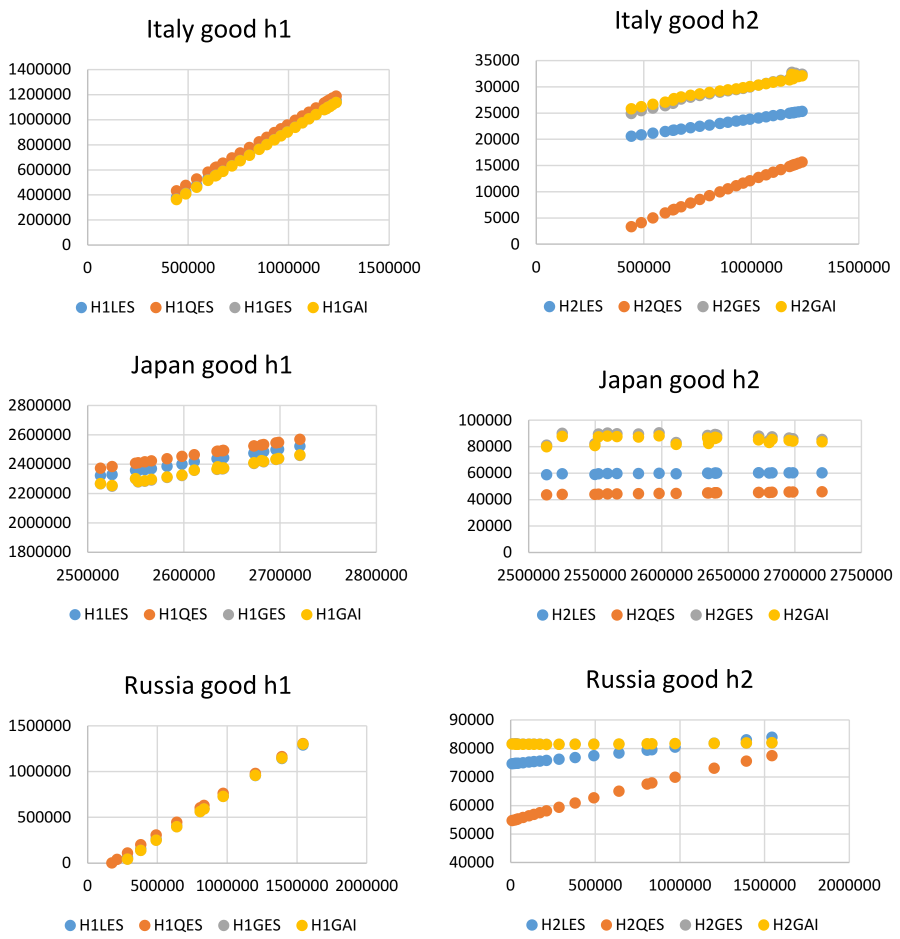

Fig. 2 (continued)

levels. The differences are relatively more appreciable at the high income levels, as shown in the bottom graph of Fig. 3.

5 Conclusions

This paper has proposed a new demand system that is the most general formulation of the Gorman polar form. We have named this function general expenditure system GES.

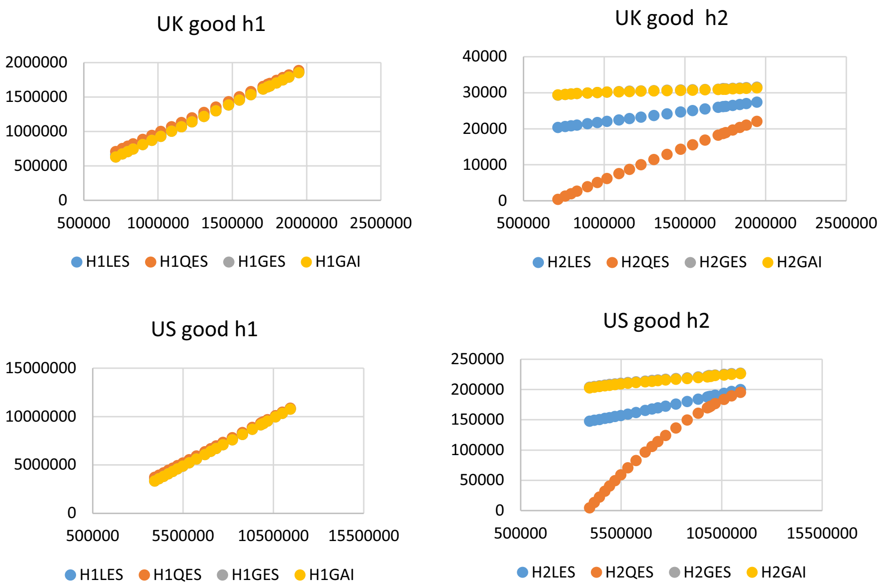

Fig. 2 (continued)

The parametrization of GES is a statistically significant generalization of previously known demand systems of rank 2 and 3, in accordance with Gorman's definition, such as the quadratic expenditure system, and the almost ideal demand system.

The empirical relevance of these results is that we find a more flexible functional form to fit non-linear Engel curve behavior, which is crucial in assessing the different responses to income for consumers with different income levels. We find that the most appreciable differences from the linear case occurs for the lower income

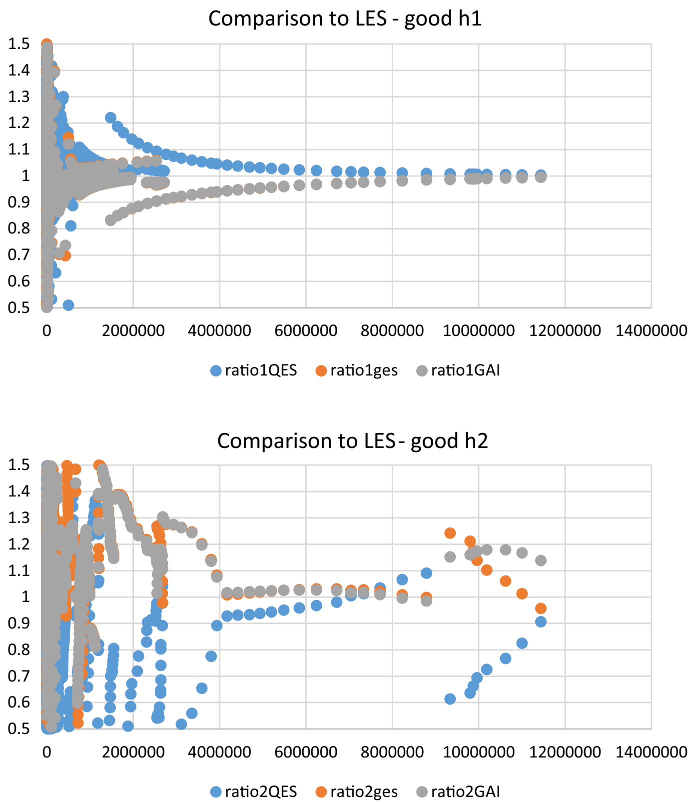

Fig. 3 Comparison of Engel curves of goods h^1 and h^2 of GAI GES QES with respect to LES. ratio1QES = ratio of estimated values of h1 Engel curve QES to LES; ratio1GES = ratio of estimated values of h1 Engel curve GES to LES; ratio1GAI = ratio of estimated values of h1 Engel curve GAI to LES; ratio2QES = ratio of estimated values of h2 Engel curve QES to LES; ratio2GES = ratio of estimated values of h2 Engel curve GES to LES; ratio2GAI = ratio of estimated values of h2 Engel curve GAI to LES

countries. This is relevant for a policy making perspective, because it shows that, if income effects are non-linear, the use of linear Engel curves could bias the estimated welfare. As an example, consider a tax on energy or a poverty support program. A linear Engel curve tends to result in overestimation of welfare loss for low-income consumers on the other hand in underestimation of welfare loss for high-income consumers. Hence, the new flexible GES is proved to be a relevant tool for the analysis and provides a more accurate empirical basis for policy-making.

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