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ORIGINAL PAPER

Selling shares to budget-constrained bidders: an experimental study of the proportional auction

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Abstract
We explore the efficiency and revenue of proportional auctions (PA) compared to first price auction (FPA) for budget-constrained bidders. PA auctions have been used in privatization of Russian assets and in cryptocurrency sales, as they can achieve higher efficiency and revenue than FPAs when bidders face severe financial constraints. The experimental results support this in that under a tight budget constraint PA achieved higher revenue and efficiency than FPA, with these results reversed under a looser budget constraint. Detailed patterns of bidding are compared to the theoretical predictions for both PA and FPA.

Keywords Budget-constrained bidders · Proportional auction · First price auction

JEL Classification D44 · D02

1 Introduction

Many auctions sell items that could be sold in parts, for example, shares of a company, mineral rights, and shares of facilities. If bidders are willing and able to buy the entire amount offered, the single-unit first price auction (FPA) in which the highest bidder wins the whole item, allocates the item with maximum efficiency and revenue (Myerson, 1981). However, when bidders' budgets constrain buying the entire item, as shown below, a proportional auction (PA) can be more profitable and

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achieve higher efficiency than the FPA. In PA each bidder submits a single bid, pays the amount bid, and receives a share of the item equal to the amount of the bid divided by the sum of all bids.[12] The present paper experimentally explores revenue and efficiency under FPA compared to PA under both tight and looser budget constraints.

A proportional auction was used as part of Russia's privatization of state-owned enterprises and, more recently, with cryptocurrency sales. When Russia privatized its state-owned enterprises, people were given 10,000 Rupel vouchers which would only enable them to buy very small shares of state enterprises. Under these circumstance economists suggested using the PA auction (Boycko et al, 1994). More recently, PA has been used in cryptocurrency crowd-sales (Boreiko, 2019). For example, in 2017, a blockchain company raised $4.2B by selling cryptocurrency in a PA (Howell et al., 2020).[3]

The main goal of this paper is to conduct an experiment comparing revenue and efficiency of PA and FPA under two different budget constraints in a private values setting: (1) a strong/tight budget constraint where PA is predicted to raise more revenue and be more efficient than FPA and (ii) a looser budget constraint where PA is predicted to be less efficient and raise less revenue than FPA. Experimental outcomes are broadly consistent with the predictions of the theory as the PA achieved higher revenue and efficiency under the tight budget constraint, although PA allocates shares to even the lowest valued bidders. In contrast, under the looser budget constraint FPA achieved higher revenue and efficiency than the PA. Under both FPA and PA bidders tend to bid above the risk-neutral Nash equilibrium. Nevertheless, experimental outcomes were close to predicted outcomes assuming risk neutrality.

The rest of the paper is organized as follows. Section 2 reviews the previous literature focusing on prior studies of budget-constrained bidding. Section 3 introduces the theoretical framework for analyzing the FPA and PA auctions. Section 4 describes the experimental design and hypotheses. The experimental procedures are outlined in Sect. 5, with the experimental results reported in Sect. 6. Section 7 summarizes the outcomes reported.

2 Literature review

There have only been a few experimental studies of auctions that feature budget-constrained bidders. Pitchik and Schotter (1988) study two-stage sequential auctions in which two budget-constrained bidders can each strategically deplete the

[1] The proportional auction is sometimes referred to as the voucher auction (Krishna, 2009) or the proportional-share auction (Dobzinski et al., 2012). Following Brooks and Du (2021), we use proportional auction here.

[2] PA has some similarities to the widely studied Tullock contest (Tullock 1980). But they are quite different. In a Tullock contest, the value of the item is publicly know and the same to everyone, with a winner-take-all outcome.

[3] PA has also been proposed for selling space on the "cloud" (Teng and Magoulès, 2010; Tsai and Tsai, 2012).

other bidder's budget during the first stage. Kotowski (2011, 2020) theoretically and experimentally studies an FPA in which two bidders have private values and private budget constraints. Ausubel et al. (2017) experimentally studied first and second price auctions in which budget constraints were set endogenously by a financial manager. Kariv et al. (2018) theoretically and experimentally studies a network economy where traders are budget constrained and the price is set by the first-price auction.

The FPA without budget constraints has been widely investigated. The general finding is that the FPA achieves high efficiency and revenue in the absence of a budget constraint, with outcomes reasonably close to predicted levels (Kagel & Levin, 2016). However, subjects tend to consistently bid above the risk neutral Nash equilibrium generating higher revenue than predicted for risk neutral bidders.[4] In contrast, the PA has not been studied experimentally prior to this paper.

3 Theoretical framework

A seller sells a divisible good without a reserve price and n bidders wish to buy the good. All bidders are risk neutral, and each bidder has a private value of the good on sale. Let v_i be the value of the item for bidder i when the bidder receives the whole item. If the bidder receives a share $x \in [0,1]$ of the good, the payoff for the share equals to $v_i\, x$. Values are independently drawn from a distribution $F(v)$. We assume that each bidder has a common budget constrain w, which is the most amount they can spend.

3.1 First price auction

In the first price auction (FPA), each bidder submits a single bid less than or equal to their budget constraint w. The bidder with the highest bid wins the item and pays what she bid. In case of ties, one of the highest bidders is randomly selected as the winner. The FPA has a unique equilibrium when bidders have a common budget constraint (see Milgrom, 2004, for example). For budget-constrained bidders, the bid function has two parts with a cutoff value discontinuously dividing the two. Bidders whose value is lower than the cutoff submit bids according to a typical FPA without a budget constraint, while bidders whose value is greater than the cutoff point submit a bid equal to their budget constraint, w.[5]

[4] While this is consistent with risk aversion, the experimental literature supports a number of alternative explanations (see Kagel and Levin 2016, for a survey of the literature).

[5] Bidders with values above their budget constraint will continue to bid according to the Nash equilibrium bid function as long as the bid is below their budget constraint.

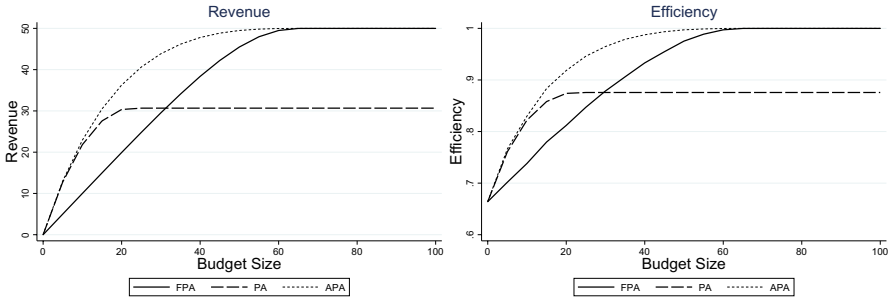

Fig. 1 Efficiency and revenue predictions across different budget constraints

3.2 Proportional auction

In the proportional auction (PA), each bidder submits a single bid up to the budget constraint w and pays what they bid. Then, each bidder receives a fraction of the item equal to their bids divided by the sum of all bids. It is known that the proportional-share auction has a unique equilibrium (Wasser, 2013; Ewerhart, 2014).[6]

It is intractable to get an analytical solution for PA bids because bidders' payoff functions include random variables (other bidders' values) in the denominator. However, equilibrium bids can be computed numerically.[7] The result is that in equilibrium, bids are always less than a bidder's value. Further, bidding above 1/4 $v_i$is dominated as the positive effect of increased earnings is more than offset by the increase in the price for shares won.[8]

Figure 1 shows the expected revenue and efficiency across different budget constraints under the parameter values employed in the experiment.[9] For $w \leq 30$, PA achieves higher revenue and efficiency than FPA. This result is reversed for $w > 30$, with FPA achieving higher revenue and efficiency than PA. Also shown are revenue and efficiency for the optimal all pay auction when bidders have a common budget constraint (Laffont & Robert, 1996). At the tight budget constraint employed in the experiment ($w = 20$), both revenue and efficiency are predicted to be substantially higher under APA than PA. At the looser budget constraint employed ($w = 50$), the difference between FPA and APA is relatively small, and substantially higher than predicted under PA. However, APA suffers from the fact that losing bidders still have to pay what they bid, which we conjecture would make the mechanism substantially less attractive to bidders.

[6] Their results easily apply to PA with a budget constraint.

[7] The uniqueness of the equilibrium ensures that the numerical solution will be close enough to the unique theoretical equilibrium. I used a modified version of Wasser (2013)'s MATLAB program to calculate the equilibrium bids. I thank him for kindly sharing his program.

[8] See Appendix A1 for the proof.

[9] Efficiency is defined as $w_{\text{realized}} / w_{\max}$, where S_{realized} is the realized surplus in an auction and $S_{\max}$ is the maximum possible surplus. $w_{\max}$ equals the highest valuation among the bidders and $w_{\text{realized}} = \sum_i \left[x_i v_i \right]$ wher x_i is the share received by bidder i.

Table 1 Equilibrium predictions

		FPA	PA
$w = 20$	Efficiency	81.1%	87.4%
	Revenue	19.9	30.4
$w = 50$	Efficiency	97.5%	87.6%
	Revenue	45.5	31.0

(average values)*

*Predicted values are simulated using 10^6 random draws for values, based on realized bidder values in the experiment. Efficiency is defined as $S_{\text{realized}}/S_{\text{max}}$, where S_{realized} is the realized surplus, and S_{max} is the maximum possible surplus for the treatments

Simulations reporting the effects of changing numbers of bidders going from 2 to 10, with treatment effects similar to those reported on here, are shown in the online appendix. These show that under the tight budget constraint ($w=20$), efficiency and revenue under PA remain slightly below that of the APA throughout. So that it would be a good alternative to APA. In contrast, efficiency of FPA decreases sharply with increased numbers of bidders. Under the looser budget constraint (w = 50), FPA is predicted to achieve higher efficiency and revenue than PA if the number of bidders is small ($n \leq 4$). However, for larger numbers of bidders ($n > 4$), the differences between the two are negligible. Further, with the looser budget constraint, while both revenue and efficiency of APA are greater than under PA and FPA throughout, these differences are relatively small up until $n = 3$. For larger numbers of bidders, however, APA does substantially better on both accounts, with these differences increasing with larger numbers of bidders. The results reported here are limited by the fact that it is not clear what would happen under more general underlying economic conditions, as there is no analytic solution for PA against which to compare outcomes under more general conditions.

4 Experimental design and hypotheses

In each auction, there were three bidders ($n = 3$) with each bidder assigned a private value v independently drawn from a uniform distribution with integer values (0, 1, 2, …, 100).[10] FPA and PA were investigated under two budget constraints: A weak budget constraint $w=50$ and a stronger/tighter budget constraint $w=20$. As a simplification, all bidders had the same budget constraint.[11]

Table 1 shows the equilibrium predictions for efficiency and revenue in FPA compared to PA under the two budget constraints, assuming risk neutrality: On average PA is predicted to have higher revenue and efficiency than FPA under the tight ($w =$

[10] Different values were drawn for each period but the same set of values were used across PA and FPA as well as the different budget constraints.

[11] Without this, outcomes would be dependent on the distribution of budget constraints as well as the distribution of valuations and the corresponding bids.

20) budget constraint. In contrast, under the looser budget constraint ($w = 50$), average revenue and efficiency are predicted to be higher under FPA than PA. Although it is well-known that bidders are likely to bid above the risk neutral Nash in FPAs, the extent of overbidding tends to be minimized in auctions with three bidders (Cox et al, 1988; Dyer et al, 1989), the number of bidders employed here. Numerical analysis of the impact of risk aversion on bidding in PA shows that low value bidders will bid lower than the risk neutral Nash equilibrium, but the differences are quite small.

5 Experimental procedures

Six sessions were run, three for FPA and three for PA with 15 to 21 subjects in each session.[12] Subjects participated in ten auctions under $w = 20$, followed by ten auctions with $w = 50$.[13]In each period, subjects were assigned integer values randomly drawn from [0, 100] with subjects randomly rematched into different groups in each auction. In FPA, each bidder could submit an integer bid up to the budget constraint *w*. The highest bidder won the item and paid the amount bid. In the case of ties, one of highest bidders was randomly selected to be the winner. Similarly, in PA, bidders could submit integer bids up to the budget constraint. After all bids were submitted, each bidder paid their bid and received a fraction of the item equal to their bid divided by the sum of all bids. Subjects were provided with an on-screen calculator for PA sessions where they could enter potential bids for their opponents along with their bid, to easily calculate their expected payoff for different bids.[14]

Subjects participated in three practice auctions bidding against two computerized bidders whose values were the same for all sessions. This was to ensure that everyone had the same experience in the dry runs.[15] The change in the budget constraint was announced prior to the start of period 11. Subjects were paid the sum of their earnings across all periods.

In each auction, subjects were given 1 min (1.5 min for the first three paid auctions) to make decisions. Subjects were provided with starting capital balances of 100 experimental currency units (ECUs) with earnings added to this and losses subtracted from it. Bidders were told they could lose money, but no restrictions were imposed on bids to prevent this. They were also told that if their cash balance went to zero or negative, they would be bankrupt and no longer able to bid. This never happened.

Following each auction, feedback was provided in the form of a table reporting values, bids, allocations, and earnings of all bidders. All values and earnings

[12] FPA sessions had 21 subjects in each session, while PA sessions had 18,21,15 subjects in the three sessions, respectively. All subjects participated in a single experimental session.

[13] One limitation to the present experiment is that there are no sessions with budget constraints moving from the looser to the tighter budget constraint.

[14] This was provided for PA because of the more complicated nature of determining potential payoffs.

[15] Computer bids were equilibrium bids with noise, with the noise added so as not to suggest equilibrium bids.

Table 2 Average predicted and realized efficiency and revenue

	Budget	Predicted		Realized	
		FPA	PA	FPA	PA
Efficiency	$w = 20$	81.1% (0.8%)	87.4% (0.4%)	81.2% (1.7%)	87.0% (0.6%)
	$w = 50$	97.5% (0.3%)	87.6% (0.4%)	96.2% (0.6%)	90.2% (0.6%)
Revenue	$w = 20$	19.9 (0.07)	30.4 (0.92)	19.8 (0.10)	29.7 (0.91)
	$w = 50$	45.5 (0.65)	31.0 (0.95)	46.6 (0.55)	37.8 (1.56)

*Standard errors of the mean are in parentheses

*Predicted outcomes based on the random draws in the experiment

were denominated in ECUs. Final earnings were converted into dollars at the rate of 20ECUs = \$1. Earnings averaged \$19.82 per subject in FPA and \$19.50 in PA, with sessions lasting 1.5 h on average.

The experiment was conducted in the Ohio State University Experimental Economics Laboratory between Mar 2019 and Sep 2019. Most of the subjects who participated in the experiment were undergraduate students drawn from all disciplines, and all were recruited through ORSEE (Greiner, 2015). Each subject participated in no more than one experimental session. The experiment was computerized and programmed using z-Tree (Fischbacher, 2007).

6 Experimental results

6.1 Efficiency and revenue

Table 2 shows predicted and realized efficiency and revenue under the two budget constraints. Realized efficiencies were broadly consistent with the theoretical predictions: PA achieved significantly higher efficiency (87.7%) than FPA (81.2%) under $w = 20$ for all three sessions (Mann–Whitney test, $p < 0.05$ using session averages as the unit of observation), and FPA achieved significantly higher efficiency (96.2%) than PA (90.2%) under $w = 50$ for all three sessions ($p < 0.05$ using session averages as the unit of observation). Realized efficiencies for FPA and PA were also quite close to the theoretical predictions—all within 3 percentage points of the predicted efficiencies.[16]

Average realized revenue was broadly consistent with the theoretical predictions as well: Under $w = 20$, PA raised significantly more revenue on average (29.7 ECUs) than FPA (19.8 ECUs) across all three sessions (Mann–Whitney test, $p < 0.05$ using

[16] There were only minor, and nonsystematic differences between early and later auctions, so these are not reported.

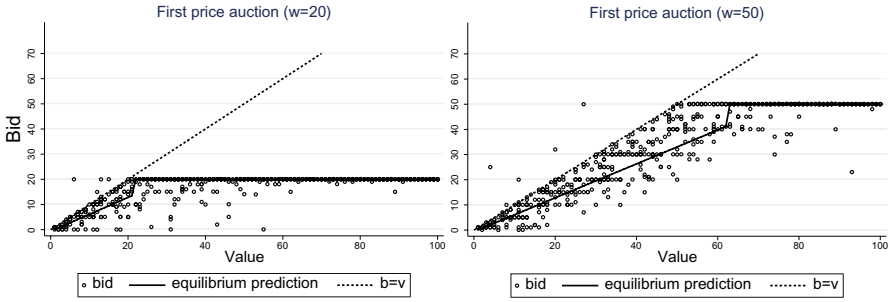

Fig. 2 Bid plots in the first price auction

session averages as the unit of observation). While under $w = 50$, FPA raised significantly more revenue on average (46.6 ECUs) than PA (37.8 ECUs) across all three sessions (Mann–Whitney test, $p < 0.05$ using session averages as the unit of observation). Realized revenues were, for the most part, quite close to the theoretical predictions, although PA achieved noticeably higher revenue (37.8) than predicted (30.4) under $w = 50$. As shown below, this resulted from high-value bidders bidding well above 1/4 v_i benchmark in PA with $w = 50$. Finally, note that average revenue under PA with $w = 20$ was greater than the maximum revenue possible (20) under FPA.

Result 1: PA achieved significantly higher efficiency and revenue than FPA under thetight($w = 20$) budget constraint, while FPA had higher revenue and efficiency under the weaker budget constraint ($w = 50$). Realized outcomes were quite close to equilibrium predictions, except that PA achieved noticeably higher revenue than predicted under $w = 50$, as reported on below, high bidders consistently bid above the 1/4 v_i benchmark. These results are consistent with the idea that with severely budget-constrained bidders, sellers can be better off using a PA as opposed to an FPA auction, with higher average efficiency under PA as well.

6.2 Bidding behavior

6.2.1 First price auction

Figure 2 shows submitted bids in FPA sessions under $w = 20$ and $w = 50$. The solid lines are equilibrium bid functions, with the dashed lines representing bids equal to value. Under $w = 20$, budget-constrained bidders were expected to pool at their budget constraint, which they did 87.3% of the time. Bidders with values below 22 were expected to bid two-thirds of their value (the risk neutral Nash equilibrium bid) but as is typical of FPAs (Kagel & Levin, 2016), 44.2% of these bids were more than 1 ECU above this benchmark.[17]

[17] 18.8% bid below the RNNE, with 37% within 1 ECU of the RNNE benchmark.

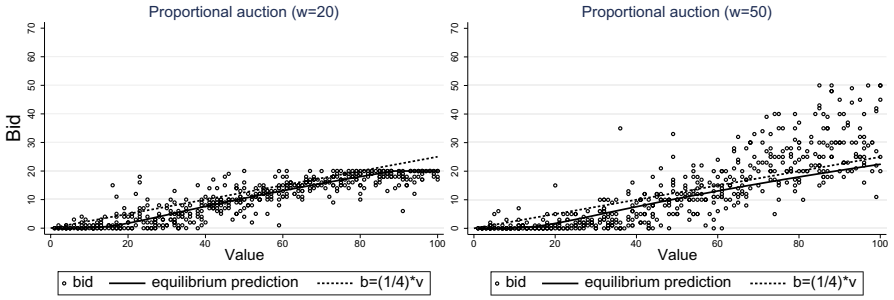

Fig. 3 Bid plots in the proportional auction

Under $w = 50$, budget-constrained bidders pooled at the budget constraint 88.6% of the time, with 11.4% bidding below it. 56% of unconstrained bids were more than 1ECU above the RNNE benchmark, as is typical. Not surprisingly, 49.3% of bidders with values between the cutoff point (63) and the budget constraint pooled at the budget constraint, rather than below it, as predicted under the RNNE, as the budget constraint may have served as a focal point.[18]

6.3 Proportional auction

Figure 3 plots PA bids under the two budget constraints. The solid lines are equilibrium bids, with the dashed lines at 1/4 v_i, the cutoff point for undominated bids. Under $w = 20$, 55.9% of all bids were within 1ECU of the theoretical prediction, with 24.8% (19.3%) bidding below (above) equilibrium. The net effect was that realized revenue and efficacy were quite close to predicted levels (see Table 2).

Under $w = 50$, realized bids were more scattered relative to predicted bids than with $w = 20$. Bidders with values below 40 bid close to predicted levels: 57.4% of all bids were within 1ECU of equilibrium, with 29% (13.5%) bidding below (above) equilibrium. However, bidders with values greater than 40 showed substantial bidding above equilibrium (58.3%), with 53.1% bidding above the 1/4 v_i benchmark for dominated bids. The latter resulted in lower profits than had they bid according to the 1/4 v_i benchmark. But these were mainly opportunity costs as opposed to negative earnings, as the frequency of negative earnings was not much higher than the benchmark (13.3% versus 9.9%).

Result 2: Bidding in FPAs was largely consistent with equilibrium predictions under both budget constraints, except for the tendency to bid above the RNNE, widely reported in FPAs without budget constraints. Bidding in PA was close to the equilibrium prediction under $w = 20$. However, with $w = 50$, a substantial number of high value bidders submitted dominated bids. This can be accounted for by the fact that

[18] Bidding more than one's value was exceedingly rare: 0.63% under $w = 20$ and 0.79% under $w = 50$. Bids above the budget constraint were not allowed.

it is not obvious that bidding above the 1/4 v_i benchmark is dominated, while this did not result in a much higher frequency of negative earnings than predicted, these dominated bids resulted in higher revenue (lower earnings) than predicted under PA.

7 Summary and conclusions

This paper experimentally investigates proportional auctions (PA) compared to first price auctions (FPA) when bidders' budgets constrain bidding for the entire item. The results show that as predicted, under a strong/tight budget constraint, revenue and efficiency would be higher under PA than FPA. The results also show that, as predicted, with a significantly looser budget constraint, revenue and efficiency were higher under FPA than PA. These results provide some, albeit limited, support for using the PA format in cases where items are divisible, and bidders face a strong budget constraint.

The current study can be extended in several ways. The most obvious way would be to explore the impact of heterogeneous budget constraints and increased numbers of bidders between the two auction formats. In addition, it would be worthwhile to investigate PA compared to uniform price auctions, the alternative format commonly used when bidders demand, or can only afford, shares of the auctioned item.

Supplementary Information The online version contains supplementary material available at https://doi.org/10.1007/s40881-022-00119-x.

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