Birational classification of toric orbifolds

Johannes Schmitt[1]

Abstract

We give a complete classification of the torus-equivariant birational equivalence classes of smooth proper toric Deligne–Mumford stacks with trivial generic stabilizer in terms of their associated stacky fans.

Keywords Toric orbifolds · Birational classification · Stacky fans

Mathematics Subject Classification 14E05 · 14M25

1 Introduction

An *algebraic orbifold* is a smooth separated irreducible Deligne–Mumford stack of finite type over a field k (assumed to be algebraically closed of characteristic zero) with trivial generic stabilizer. Following [8, 9] we say that such orbifolds $\mathcal{X}, \mathcal{X}'$ are birationally equivalent if there is a third algebraic orbifold $\widehat{\mathcal{X}}$ admitting proper, birational and representable morphisms

$$\mathcal{X} \xleftarrow{f} \widehat{\mathcal{X}} \xrightarrow{g} \mathcal{X}'. \tag{1}$$

In our paper, we consider the torus-equivariant birational classification problem for *smooth proper toric Deligne–Mumford* (DM) *stacks* with generically trivial stabilizer (see below for the precise definition). This problem was previously studied by Levchenko in [10], who considered the case of dimension 2 and obtained some birational invariants. In our paper, we give a full classification in arbitrary dimension.

✉ Johannes Schmitt
johannes.schmitt@math.ethz.ch

1   Departement Mathematik, ETH Zürich, Zürich, Switzerland

2 Main result

To describe our results, we briefly recall the definition and basic properties of smooth toric DM-stacks. These were introduced in [1] and generalize the canonical cover of simplicial toric varieties (cf. [5, Theorem 4.11]). Similarly to the case of normal toric varieties, a smooth toric DM-stack is given by essentially convex-geometric data:

- a finitely generated abelian group N of rank d; we denote the image of the natural map $N \to N_{\mathbb{Q}} = N \otimes_{\mathbb{Z}} \mathbb{Q}$ by $\overline{N}$,
- a simplicial rational polyhedral fan Σ in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$,
- for each ray $\tau \in \Sigma(1)$ a vector $\rho_{\tau} \in \overline{N}$ which generates τ.

Given such a tuple $\boldsymbol{\Sigma} = (N, \Sigma, (\rho_{\tau})_{\tau \in \Sigma(1)})$, called a *stacky fan*, one constructs a smooth Deligne–Mumford stack $\mathcal{X}_{\boldsymbol{\Sigma}}$ (see [1]). The coarse moduli space of $\mathcal{X}_{\boldsymbol{\Sigma}}$ is the toric variety associated to the fan Σ. We have that $N = \overline{N}$ is torsion-free if and only if $\mathcal{X}_{\boldsymbol{\Sigma}}$ is an orbifold, i.e. has generically trivial stabilizer (see [5, Lemma 7.15]). Under this assumption, for $\sigma \in \Sigma$ we denote by $N_{\sigma} \subseteq N$ the sub-lattice spanned by the ρ_{τ} for $\tau \in \sigma(1)$. The index of N_{σ} in $N \cap \mathrm{Span}_{\mathbb{Q}} \sigma$ is the order of the generic stabilizer group on the torus-invariant stratum of $\mathcal{X}_{\boldsymbol{\Sigma}}$ associated to σ. We remark for later use that for a face $\pi \subseteq \sigma$ we have $N_{\pi} = N_{\sigma} \cap \mathrm{Span}_{\mathbb{Q}} \pi$.

From now on, we only work with toric orbifolds $\mathcal{X}_{\boldsymbol{\Sigma}}$, i.e. smooth toric DM-stacks with generically trivial stabilizer. As for toric varieties, they contain the algebraic torus $T = T_{\mathcal{X}_{\boldsymbol{\Sigma}}} = \mathrm{Spec}(k[N^{\vee}])$ as an open dense substack. Following [10], we say that two such orbifolds $\mathcal{X}$, $\mathcal{X}'$ are T-equivariantly birationally equivalent if there exists a diagram (1) such that $\widehat{\mathcal{X}}$ is an orbifold with an action of T and a dense equivariant embedding of T such that the maps f, g are T-equivariant.[1] It follows from [5, Theorem 7.17] that such an orbifold $\widehat{\mathcal{X}}$ is a toric orbifold in the sense described above, i.e. given by $\widehat{\mathcal{X}} = \mathcal{X}_{\hat{\boldsymbol{\Sigma}}}$ for a stacky fan $\hat{\boldsymbol{\Sigma}} = (N, \widehat{\Sigma}, (\widehat{\rho}_{\tau})_{\tau \in \widehat{\Sigma}(1)})$.

Given two toric DM-stacks $\mathcal{X}_{\boldsymbol{\Sigma}'}$, $\mathcal{X}_{\boldsymbol{\Sigma}}$ containing the same torus T, the identity on T induces a T-equivariant birational map $\mathcal{X}_{\boldsymbol{\Sigma}'} \dashrightarrow \mathcal{X}_{\boldsymbol{\Sigma}}$. The following proposition gives a criterion when this map is a morphism, respectively a representable morphism, in terms of the stacky fans $\boldsymbol{\Sigma}$, $\boldsymbol{\Sigma}'$.

Proposition 2.1 *Let $\boldsymbol{\Sigma} = (N, \Sigma, (\rho_{\tau})_{\tau \in \Sigma(1)})$ and $\boldsymbol{\Sigma}' = (N, \Sigma', (\rho'_{\tau'})_{\tau' \in \Sigma'(1)})$ be two stacky fans with the same underlying group N such that Σ, Σ' are full-dimensional[2] with the same support $|\Sigma| = |\Sigma'|$. Then the birational map $\mathcal{X}_{\boldsymbol{\Sigma}'} \dashrightarrow \mathcal{X}_{\boldsymbol{\Sigma}}$ extending the identity of the torus $T = \mathrm{Spec}(k[N^{\vee}])$ is*

(a) *a morphism if and only if the identity on $N_{\mathbb{R}}$ induces a map of fans $\Sigma' \to \Sigma$ such that for each $\tau' \in \Sigma'(1)$ and any cone $\sigma \in \Sigma$ containing τ', the element $\rho'_{\tau'}$ is an integral linear combination of the ρ_{τ} for $\tau \in \sigma(1)$,*

(b) *a representable morphism if and only if the identity on $N_{\mathbb{R}}$ induces a map of fans $\Sigma' \to \Sigma$ and for any full-dimensional cones $\sigma \in \Sigma(d)$, $\sigma' \in \Sigma'(d)$ with $\sigma' \to \sigma$, we have an equality of sublattices $N_{\sigma} = N_{\sigma'}$.*

---

[1] By composing f, g with the action of suitable points in $T(k)$ on $\mathcal{X}$, $\mathcal{X}'$ we can assume without loss of generality these maps restrict to the identity on T.

[2] Here we say that a rational polyhedral fan in $N_{\mathbb{R}} \cong \mathbb{R}^d$ is full-dimensional if each cone of the fan is a face of a d-dimensional cone inside the fan. This condition is automatic for complete fans.

Proof The fact that the fan-theoretic condition in (a) is sufficient is [1, Remark 4.5], and the converse implication follows from [6, Theorem 3.4]. For part (b) we note that for any cone $\sigma \in \Sigma$, the generic stabilizer group of the associated torus-invariant subset of $\mathcal{X}_\Sigma$ is given by $N \cap \mathrm{Span}_\mathbb{Q} \sigma / N_\sigma$ (as follows from [1, Proposition 4.3]). By part (a), we obtain a morphism if and only if $N_{\sigma'} \subseteq N_\sigma$, and the injectivity of stabilizer groups for the representability of this morphism is then equivalent to the other inclusion $N_{\sigma'} \supseteq N_\sigma$. We use here that it is sufficient to check the condition at the maximal cones since for a face $\pi \subseteq \sigma$ we have $N_\pi = N_\sigma \cap \mathrm{Span}_\mathbb{Q} \pi$.   □

Using this proposition, we see that fixing Σ, the torus-equivariant morphisms $\mathcal{X}_{\Sigma'} \to \mathcal{X}_\Sigma$ are precisely induced by the choices of a simplicial subdivision $\Sigma' \to \Sigma$ and ray generators $(\rho'_{\tau'})_{\tau' \in \Sigma'(1)}$ such that condition (a) is satisfied. Similarly, the representable morphisms correspond to subdivisions such that (b) is satisfied (in this case the ray generators are uniquely determined by condition (b)).

Theorem 2.2 *Let $\mathcal{X}_\Sigma$ and $\mathcal{X}_{\Sigma'}$ be two proper d-dimensional toric orbifolds together with an identification $T = T_{\mathcal{X}_\Sigma} = T_{\mathcal{X}_{\Sigma'}}$ of their tori. Then $\mathcal{X}_\Sigma$ and $\mathcal{X}_{\Sigma'}$ are T-equivariantly birationally equivalent if and only if for any $\sigma \in \Sigma(d)$ and $\sigma' \in \Sigma'(d)$ such that the interiors of σ and σ' intersect, we have $N_\sigma = N'_\sigma$.*

Proof By definition, we have that $\mathcal{X}_\Sigma$ and $\mathcal{X}_{\Sigma'}$ are T-equivariantly birationally equivalent if and only if we can find a toric orbifold $\widehat{\mathcal{X}}$ (with torus T), together with representable, proper T-equivariant birational morphisms

$$\begin{array}{ccc} & \widehat{\mathcal{X}} & \\ {\scriptstyle f}\swarrow & & \searrow{\scriptstyle g} \\ \mathcal{X}_\Sigma & & \mathcal{X}_{\Sigma'} \end{array} \tag{2}$$

such that f, g restrict to the identity on T. As mentioned before, it follows from [5, Theorem 7.17] that the orbifold $\widehat{\mathcal{X}}$ is given by $\widehat{\mathcal{X}} = \mathcal{X}_{\widehat{\Sigma}}$ for a stacky fan $\widehat{\Sigma} = (N, \widehat{\Sigma}, (\widehat{\rho}_\tau)_{\tau \in \widehat{\Sigma}(1)})$.

All toric orbifolds in diagram (2) are proper, so their fans are full-dimensional with support $N_\mathbb{R}$. By Proposition 2.1, the fan $\widehat{\Sigma}$ is a refinement of Σ, Σ'. For $\sigma \in \Sigma(d)$ and $\sigma' \in \Sigma'(d)$ with overlapping interior, choose a cone $\widehat{\sigma} \in \widehat{\Sigma}(d)$ whose interior maps to $\mathrm{int}(\sigma) \cap \mathrm{int}(\sigma')$. Then the fact that f, g are representable implies $N_\sigma = N_{\widehat{\sigma}} = N_{\sigma'}$ by Proposition 2.1 (b).

Conversely, assume $N_\sigma = N'_\sigma$ for any $\sigma \in \Sigma(d)$ and $\sigma' \in \Sigma'(d)$ such that the interiors of σ and σ' intersect. Then we claim that there exists an iterated stacky star subdivision[3] $\widehat{\Sigma}$ of Σ such that $\widehat{\Sigma}$ is a refinement of Σ'. The analogous statement for non-stacky fans is proven in [3] (see Lemmas in Sections 2.2 and 2.3 and Theorem in Section 2.4 of [3]). The proof is effective, describing a procedure for choosing the sequence of cones to subdivide. Following the same algorithm line-by-line and replacing the primitive generators of rays with the chosen generators ρ_τ gives the desired result

---

[3] This is a variant of the usual star subdivision of a fan along a cone which takes into account the chosen ray generators ρ_τ, see [4] for the definition.

for stacky fans. We claim that taking $\widehat{\mathcal{X}} = \mathcal{X}_{\widehat{\boldsymbol{\Sigma}}}$, we obtain a diagram (2) of proper representable morphisms as desired. Indeed, for the conditions of Proposition 2.1 (b), we already checked that that the underlying fan $\widehat{\Sigma}$ of $\widehat{\boldsymbol{\Sigma}}$ refines Σ, Σ'. On the other hand, the map f induced by iterated stacky star subdivision is representable, so for a cone $\widehat{\sigma} \in \widehat{\Sigma}(d)$ mapping to $\sigma \in \Sigma(d)$ and $\sigma' \in \Sigma'(d)$, we have $N_{\widehat{\sigma}} = N_{\sigma}$. But then $\text{int}(\sigma) \cap \text{int}(\sigma') \supseteq \text{int}(\widehat{\sigma}) \neq \varnothing$, so by assumption $N_{\sigma'} = N_{\sigma} = N_{\widehat{\sigma}}$ as desired. $\square$

As a consequence of the above result, we can give a complete classification of the birational equivalence classes of proper toric orbifolds. To state it, we introduce the following notion (which we did not find in this precise shape in the literature).

Definition 2.3 A *conical polyhedral partition* of $N_{\mathbb{R}}$ is a finite collection $(C_i)_{i \in I}$ of non-empty subsets $\varnothing \neq C_i \subseteq N_{\mathbb{R}}$ such that

- each C_i is a finite union of full-dimensional rational polyhedral cones,
- the union of the C_i is all of $N_{\mathbb{R}}$,
- the interiors of the C_i are pairwise disjoint.

We claim that for such a conical polyhedral partition, one can in fact find a fan Σ_0 with support $N_{\mathbb{R}}$ such that each of the C_i is a union of cones in $\Sigma_0(d)$. Indeed, one way to obtain Σ_0 is to subdivide $N_{\mathbb{R}}$ at each defining hyperplane of each of the full-dimensional rational polyhedral cones used to cover the sets C_i. Conversely, given a complete fan Σ_0 and a partition I of $\Sigma(d)$ into non-empty subsets, the sets $C_i = \bigcup_{\sigma \in I} \sigma$ form a conical polyhedral partition.

Definition 2.4 A *sublattice coloring* of N is a conical polyhedral partition $(C_{N'})_{N' \in \mathcal{N}}$ indexed by a finite set $\mathcal{N}$ of finite-index sublattices $N' \subseteq N$ such that for any N', $N'' \in \mathcal{N}$, we have $C_{N'} \cap C_{N''} \cap N' = C_{N'} \cap C_{N''} \cap N''$.

Given a stacky fan $\boldsymbol{\Sigma}$ with an underlying fan Σ which is complete, we define

$$\mathcal{N} = \mathcal{N}(\boldsymbol{\Sigma}) = \{N_{\sigma} : \sigma \in \Sigma(d)\}$$

as the set of lattices associated to maximal cones of Σ, and given such an $N' \in \mathcal{N}$ we denote by

$$C_{N'} = \bigcup_{\substack{\sigma \in \Sigma(d) \\ N_{\sigma} = N'}} \sigma$$

the associated union of maximal cones σ with lattice $N_{\sigma} = N'$. We claim that $(C_{N'})_{N' \in \mathcal{N}}$ is a sublattice coloring of N. Indeed, by the remark below Definition 2.3 we see that $(C_{N'})_{N'}$ is a conical polyhedral partition of $N_{\mathbb{R}}$. Thus the only non-trivial property to check is $C_{N'} \cap C_{N''} \cap N' = C_{N'} \cap C_{N''} \cap N''$. It follows from the fact that $C_{N'} \cap C_{N''}$ is a union of cones $\pi \in \Sigma$ combined with the previous observation that for π a face of $\sigma \in \Sigma$ we have $N_{\pi} = N_{\sigma} \cap \text{Span}_{\mathbb{Q}} \pi$.

Corollary 2.5 *The T-equivariant birational equivalence classes of proper toric orbifolds $\mathcal{X}_{\boldsymbol{\Sigma}}$ with torus $T = \text{Spec}(k[N^{\vee}])$ are in bijection to sublattice colorings of N by sending the class $[\mathcal{X}_{\boldsymbol{\Sigma}}]$ to the sublattice coloring $(C_{N'})_{N' \in \mathcal{N}(\boldsymbol{\Sigma})}$ above.*

Proof Theorem 2.2 implies that two proper toric orbifolds are birationally equivalent if and only if they have the same associated sublattice coloring. This shows that their birational equivalence classes inject into the set of sublattice colorings.

Conversely, given such a sublattice coloring $(C_{N'})_{N' \in \mathcal{N}}$ choose a fan Σ_0 such that each $C_{N'}$ is a union of maximal cones of Σ_0. By performing a barycentric subdivision, we can assume without loss of generality that Σ_0 is simplicial. For each cone $\sigma \in \Sigma_0$ choose $N' \in \mathcal{N}$ such that $\sigma \subseteq C_{N'}$ and define $N_\sigma = \mathrm{Span}(\sigma) \cap N'$. By the definition of sublattice coloring, this sublattice $N_\sigma \subseteq N$ is independent of the choice of the set $C_{N'}$ containing σ. We would like to construct a stacky fan Σ_0' whose underlying fan Σ_0' refines Σ_0 and such that the lattices on the cones of Σ_0' are the restrictions of the lattices N_σ.

To obtain Σ_0', observe that associated to each cone σ of Σ_0 we have two lattices: the lattice N_σ above, and its sublattice N_σ' spanned by the primitive generators $\rho_\tau \in \tau \cap N_\tau \cong \mathbb{N}$ of the rays $\tau \in \sigma(1)$. We define the multiplicity $\mathrm{mult}(\sigma)$ as the index of N_σ' in N_σ. If some of the multiplicities are strictly greater than 1, we can perform iterated star subdivision on Σ_0 to reduce their multiplicities, as explained in [2, Section 8.2] (adapting the procedure explained there to use the lattices N_σ for operations on the cone σ). After finitely many steps we arrive at a simplicial refinement Σ_0' of Σ_0 with multiplicity 1 on each of the cones, giving the desired stacky fan $\Sigma_0' = (N, \Sigma_0', (\rho_{\tau'})_{\tau' \in \Sigma_0'(1)})$. □

In particular, the above result easily shows that fixing the torus T there are infinitely many T-equivariant birational equivalence classes of proper toric orbifolds $\mathcal{X}$ with torus T, even when fixing a bound $b \geqslant 2$ on the size of the stabilizer groups of points in $\mathcal{X}$. This is in stark contrast to the classical case of toric varieties, which are all birationally equivalent.[4]

Remark 2.6 The proofs of Theorem 2.2 and Corollary 2.5 featured many combinatorial and convex geometric operations (like the multiplicity reduction above). A more conceptual and geometric approach is possible via the theory of possibly singular toric DM-stacks and their associated *KM fans* in the sense of [7]. Here a (lattice) KM fan is a triple $F = (N, \Sigma, (N_\sigma)_{\sigma \in \Sigma})$ of a rational polyhedral fan Σ in the lattice N together with choices of sublattices $N_\sigma \subseteq N$ associated to its cones which are finite index in $\mathrm{Span}(\sigma) \cap N$ and compatible under face inclusions. Associated to F, the paper [7] defines a toric, but possibly singular DM-stack $\mathcal{X}(F)$. Any stacky fan Σ defines a KM fan F by taking N_σ to be the lattice spanned by the chosen ray generators ρ_τ of σ as before. The constructed toric DM-stacks $\mathcal{X}(\Sigma)$ and $\mathcal{X}(F)$ agree and for a KM fan F, we have that $\mathcal{X}(F)$ is smooth if and only if F comes from a stacky fan.

With this formalism in mind, we can sketch a geometric interpretation of the proofs presented above. Indeed, given complete stacky fans Σ, Σ' with compatible lattices on their maximal cones as in Theorem 2.2, choose any common refinement Σ'' of their underlying fans. Then the cones σ'' of Σ'' carry natural sublattices $N_{\sigma''}$ of $N \cap \mathrm{Span}(\sigma)$ induced from the associated lattices of their coarsenings in Σ, Σ' (which are compatible by assumption). Denote by $F'' = (N, \Sigma'', (N_{\sigma''})_{\sigma''})$ the associated KM fan. By functoriality, the associated toric DM-stack $\mathcal{X}(F'')$ admits toric maps

---

to $\mathcal{X}_{\mathbf{\Sigma}}, \mathcal{X}_{\mathbf{\Sigma'}}$ which are representable by [7, Theorem 3.11.2]. Composing these with a toric resolution of singularities $\widehat{\mathcal{X}} \to \mathcal{X}(F'')$ gives the diagram (2) showing that $\mathcal{X}_{\mathbf{\Sigma}}, \mathcal{X}_{\mathbf{\Sigma'}}$ are T-equivariantly birationally equivalent.

Similarly, in Corollary 2.5 given a sublattice coloring $(C_{N'})_{N'}$ choose a fan Σ_0 such that each $C_{N'}$ is a union of maximal cones of Σ_0. Given a cone $\sigma \in \Sigma_0$ contained in some $C_{N'}$, we obtain the lattice $N_\sigma = N' \cap \mathrm{Span}(\sigma)$ (which is independent of the choice of $C_{N'}$ containing σ by the definition of sublattice colorings). Again we obtain a KM fan $F = (N, \Sigma_0, (N_\sigma)_\sigma)$, and we can choose any toric resolution $\widehat{\mathcal{X}}$ of $\mathcal{X}(F)$, whose birational equivalence class then induces the given sublattice coloring $(C_{N'})_{N'}$.

References

1. Borisov, L.A., Chen, L., Smith, G.G.: The orbifold Chow ring of toric Deligne-Mumford stacks. J. Amer. Math. Soc. **18**(1), 193–215 (2005)
2. Danilov, V.I.: The geometry of toric varieties. Uspekhi Mat. Nauk **33**(2), 85–134 (1978) (in Russian)
3. De Concini, C., Procesi, C.: Complete symmetric varieties. II. Intersection theory. In: Hotta, R. (ed.) Algebraic Groups and Related Topics. Advanced Studies in Pure Mathematics, vol. 6, pp. 481–513. North-Holland, Amsterdam (1985)
4. Edidin, D., More, Y.: Partial desingularizations of good moduli spaces of Artin toric stacks. Michigan Math. J. **61**(3), 451–474 (2012)
5. Fantechi, B., Mann, E., Nironi, F.: Smooth toric Deligne–Mumford stacks. J. Reine Angew. Math. **648**, 201–244 (2010)
6. Geraschenko, A., Satriano, M.: Toric stacks I: the theory of stacky fans. Trans. Amer. Math. Soc. **367**(2), 1033–1071 (2015)
7. Gillam, W.D., Molcho, S.: A Theory of Stacky Fans (2015). arXiv:1512.07586
8. Kresch, A., Tschinkel, Yu.: Birational types of algebraic orbifolds. Sb. Math. **212**(3), 319–331 (2021)
9. Kresch, A., Tschinkel, Yu.: Birational geometry of Deligne–Mumford stacks (2023). arXiv:2312.14061
10. Levchenko, D.: Birational invariants of toric orbifold surfaces. Ann. Univ. Ferrara Sez. VII Sci. Mat. **69**(2), 463–471 (2023)