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RESEARCH ARTICLE

On effective ϵ-integrality in orbits of rational maps over function fields and multiplicative dependence

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Abstract
We prove effective bounds for the set of quasi-integral points in orbits of rational maps over function fields under some conditions, generalizing previous work of Hsia and Silverman (Pacific J Math 249(2), 321–342, 2011) for orbits over function fields of characteristic zero. We then use this to prove height bounds for algebraic functions whose orbit under a rational function has multiplicative dependent elements modulo groups of S-units, generalizing recent results over number fields.

Keywords Integral points on orbits · Arithmetic dynamics · Quantitative estimates

Mathematics Subject Classification 37P15 · 11R27

1 Introduction

Let K be a function field of a smooth projective curve over an algebraically closed field of characteristic 0, endowed as usual with a set M_K of absolute values (places) satisfying the product formula, S a finite subset of places of M_K, and $\epsilon > 0$. An element $x \in K$ is said to be *quasi-*(S, ϵ)*-integral* if

$$\sum_{v \in S} \log\left(\max\{|x|_v, 1\}\right) \geqslant \epsilon h([x, 1]),$$

where h is the absolute logarithmic height in $\mathbb{P}^1(K)$ and $[x, 1] \in \mathbb{P}^1(K)$.

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Let $\phi \in K(z)$ be a rational function of degree at least 2. We write $\phi^{(n)}$ for the n-th iterate of ϕ. Let $P \in K$ and let

$$\mathcal{O}_\phi(P) = \{\phi^{(n)}(P) : n \in \mathbb{N}\}$$

denote the forward orbit of P under ϕ. When K is a number field and $\phi^{(2)} = \phi \circ \phi \notin K[z]$, Hsia and Silverman proved [10] that the number of quasi-(S, ϵ)-integral points in the orbit of a point P with infinite orbit is bounded by a constant depending only on ϕ, $\hat{h}_\phi(P)$, ϵ, S, and $[K : \mathbb{Q}]$ (see Sect. 2 for the corresponding definitions). We also note that these results, according to [10, Remark 1], have some applications as to the existence of quantitative estimates for the size of Zsigmondy sets for such orbits and their primitive divisors, as well as to quantitative versions of a dynamical local–global principle in orbits on the projective line. This research was also used to prove finiteness of algebraic numbers having multiplicatively dependent iterated values under rational functions in [4].

In the present paper we study quasi-integrality problems over function fields of characteristic zero (for integrality results over fields of positive characteristic, see [8, 11, 19]). This is presented all over Sect. 2. Making use of such results, this work becomes a place for the study of multiplicative dependence modulo roots of unity of elements in orbits of rational functions over function fields. This generalizes the results of [4] for this context, but now with effective bounds. This is done in Sect. 3. Sometimes one is only able to bound, although effectively, the height of the algebraic functions studied (instead of their cardinality). This is related with the fact that the so-called *Northcott finiteness property* fails over function fields. One can see once again that the ambient of function fields is relevantly different to the number fields' one. For our results, we made use of specific tools from the function fields case. Namely, a recent version of an effective Roth's theorem over function fields due to Wang [21], a deep finiteness gap property for canonical heights of non-isotrivial functions due to Baker [1], and some effective results for the solutions of superellitic equations over function fields in one variable.

2 Effective bounds for quasiintegral points in orbits over function fields

2.1 Canonical heights, distance and dynamics on the projective line

We always assume that K is a fixed function field of a curve over an algebraically closed field $\mathbf{k}$ of characteristic 0 and $K(z)$ is the field of rational functions over K for the rest of the paper. We identify $K \cup \{\infty\} = \mathbb{P}^1(K)$ by fixing an affine coordinate z on $\mathbb{P}^1$, so $\alpha \in K$ is equal to $[\alpha, 1] \in \mathbb{P}^1(K)$, and the point at infinity is $[1, 0]$. In this way, we assume z is the left coordinate for points in $\mathbb{P}^1$, and with respect to this affine coordinate, we identify rational self-maps of $\mathbb{P}^1$ with rational functions in $K(z)$.

If $P = [x_0, \ldots, x_N] \in \mathbb{P}^N(K)$, the naive logarithmic height is given by

$$h(P) = \sum_{v \in M_K} \log(\max_i |x_i|_v),$$

where M_K is the set of places of K satisfying the product formula, and for each $v \in M_K$, $|\cdot|_v$ denotes the corresponding absolute values on K that can be extended to any algebraic closure and respective completions of K, so that h can be well defined on $\overline{K}$. Also, we write K_v for the completion of K with respect to $|\cdot|_v$, and we let $\mathbb{C}_v$ denote the completion of an algebraic closure of K_v. Initially, we also recall that one can define the convergent limit $\hat{h}_f(\alpha) = \lim_{n\to\infty} h(f^{(n)}(\alpha))/d^n$ for any $\alpha \in K$, $f \in K(z)$, called the *canonical height* associated with f. It satisfies $\hat{h}_f(f(\alpha)) = d\hat{h}_f(\alpha)$. When f is not isotrivial, it is a fact that α has infinite orbit (is not *preperiodic*) if and only if $\hat{h}_f(\alpha) > 0$ (see Theorem 2.5).

For each $v \in M_K$, we let ρ_v denote the chordal metric defined on $\mathbb{P}^1(\mathbb{C}_v)$, where we recall that for $[x_1, y_1], [x_2, y_2] \in \mathbb{P}^1(\mathbb{C}_v)$,

$$\rho_v([x_1, y_1], [x_2, y_2]) = \frac{|x_1 y_2 - x_2 y_1|_v}{\max\{|x_1|_v, |y_1|_v\} \max\{|x_2|_v, |y_2|_v\}}.$$

Definition 2.1 The *logarithmic chordal metric function*

$$\lambda_v \colon \mathbb{P}^1(\mathbb{C}_v) \times \mathbb{P}^1(\mathbb{C}_v) \to \mathbb{R} \cup \{\infty\}$$

is defined by

$$\lambda_v([x_1, y_1], [x_2, y_2]) = -\log \rho_v([x_1, y_1], [x_2, y_2]).$$

It is a matter of fact that λ_v is a particular choice of an *arithmetic distance function* as defined by Hsia and Silverman [10] over number fields, which is a local height function $\lambda_{\mathbb{P}^1 \times \mathbb{P}^1, \Delta}$, where Δ is the diagonal of $\mathbb{P}^1 \times \mathbb{P}^1$. The logarithmic chordal metric and the usual metric can relate in the following way.

Lemma 2.2 *Let $v \in M_K$ and let λ_v be the logarithmic chordal metric on $\mathbb{P}^1(\mathbb{C}_v)$. Then for $x, y \in \mathbb{C}_v$ the inequality $\lambda_v(x, y) > \lambda_v(y, \infty)$ implies*

$$\lambda_v(y, \infty) \leqslant \lambda_v(x, y) + \log|x - y|_v \leqslant 2\lambda_v(y, \infty).$$

Proof The proof works in the same way as the proof over number fields appearing in [10, Lemma 3]. □

Now, let $\phi \colon \mathbb{P}^1 \to \mathbb{P}^1$ be a rational map of degree $d \geqslant 2$ defined over K. In this situation we let

$$\phi^{(n)} = \underbrace{\phi \circ \cdots \circ \phi}_{n \text{ times}}$$

with $\phi^{(0)} = \mathrm{Id}$.

For a point $P \in \mathbb{P}^1$, the ϕ-orbit of P is defined as

$$\mathcal{O}_\phi(P) = \{\phi^{(n)}(P) : n \geqslant 0\}.$$

The point P is called *preperiodic for* ϕ if $\mathcal{O}_\phi(P)$ is finite. We set

$$\mathrm{Wander}_K(\phi) = \{P \in \mathbb{P}^1(K) : P \text{ is not preperiodic for } \phi\}.$$

Recall that for $P = [x_0, x_1] \in \mathbb{P}^1(K)$, the height of P is

$$h(P) = \sum_{v \in M_K} \log(\max\{|x_0|_v, |x_1|_v\}).$$

And using the definition of λ_v, we see that

$$h(P) = \sum_{v \in M_K} \lambda_v(P, \infty).$$

For a polynomial $f = \sum a_i z^i$ and an absolute value $v \in M_K$, we define $|f|_v = \max_i\{|a_i|_v\}$ and

$$h(f) = \sum_{v \in M_K} \log|f|_v.$$

Given a rational function $\phi(z) = f(z)/g(z) \in K(z)$ of degree d written in normalized form, let us write $f(z) = \sum_{i \leqslant d} a_i z^i$, $g(z) = \sum_{i \leqslant d} b_i z^i$ with a_d and b_d different from zero, and f and g relatively prime in $K[z]$.

For $v \in M_K$, we set $|\phi|_v = \max\{|f|_v, |g|_v\}$, and then the height of ϕ is defined by

$$h(\phi) := \sum_{v \in M_K} \log|\phi|_v.$$

Proposition 2.3 ([10, Proposition 5(d)]) *Let* ϕ *be a rational function with* $\deg\phi = d \geqslant 2$. *Then for all* $n \geqslant 1$, *we have*

$$h(\phi^{(n)}) \leqslant \left(\frac{d^n - 1}{d - 1}\right) h(\phi) + d^2 \left(\frac{d^{n-1} - 1}{d - 1}\right) \log 8.$$

Lemma 2.4 *For a rational map* $\phi\colon \mathbb{P}^1 \to \mathbb{P}^1$ *of degree* $d \geqslant 2$ *defined over* K *and* $P \in \mathbb{P}^1(\overline{K})$, *it is true that*

(a) $|h(\phi(P)) - dh(P)| \leqslant c_1 h(\phi) + c_2$,
(b) $\hat{h}_\phi(P) = \lim_n h(\phi^{(n)}(P))/d^n$,
(c) $|\hat{h}_\phi(P) - h(P)| \leqslant c_3 h(\phi) + c_4$,

where c_1, c_2, c_3 and c_4 depend only on d.

Proof This is stated in [10, Proposition 6] over number fields. For (a), the same procedure given in [5, Proposition A] works over the referred function fields. For (b) and (c), the proof of the existence of the limit defining the canonical height [18, Theorem 3.20] together with (a) yields the desired. □

Recall that $\phi \in K(z)$ is *isotrivial* if there is a finite extension L/K and a change of coordinates γ defined over L such that $\gamma \circ \phi \circ \gamma^{-1}$ is defined over the field of constants of L.

Also, we say that a set $A \subset \mathbb{P}^1(\overline{K})$ is *isotrivial* if there exists an isomorphism $T : \mathbb{P}^1 \to \mathbb{P}^1$ over $\overline{K}$ such that $T(A) \subset \mathbb{P}^1(\mathbf{k})$, i.e., $T(A)$ is defined over the field of constants of K.

Theorem 2.5 ([1, Theorem 1.6]) *Let $\varphi(z) \in K(z)$ be of degree at least 2, and assume that φ is not isotrivial and $\hat{h}_\varphi$ is the canonical height associated with φ. Then there exists $\varepsilon > 0$ (depending on K and φ) such that the set*

$$\{P \in \mathbb{P}^1(K) : \hat{h}_\varphi(P) \leqslant \varepsilon\}$$

is finite.

2.2 A distance estimate and an effective version of Roth's theorem

We will state three results that will be needed to prove our main theorems. The first one is a result that gives explicit estimates for the dependence on local heights of points and function.

Let us recall that, for a rational function $f(z)$, $P \neq \infty$ and $f(P) \neq \infty$, the *ramification index* of f at P is defined as the order of P as a zero of the rational function $f(z) - f(P)$, i.e.,

$$e_P(f) = \mathrm{ord}_P(f(z) - f(P)).$$

If $P = \infty$, or $f(P) = \infty$, we change coordinates through a linear fractional transformation L, so that $L^{-1}(P) = \beta \neq \infty$, $L^{-1}(f(L(\beta))) \neq \infty$, and define $e_P(f) = e_\beta(L^{-1} \circ f \circ L)$. It will not depend on the choice of L. We say that f is *totally ramified* at P if $e_P(f) = \deg f$. It is also an exercise to show that

$$e_P(g \circ f) = e_P(f) e_{f(P)}(g)$$

for every f, g rational functions and $P \in K \cup \{\infty\}$.

The result is as follows.

Lemma 2.6 *Let $\psi \in K(z)$ be a nontrivial rational function, let $S \subset M_K$ be a finite set of absolute values on K, each extended in some way to $\overline{K}$, and let $A, P \in \mathbb{P}^1(K)$.*

Then

$$\sum_{v\in S}\max_{A'\in\psi^{-1}(A)} e_{A'}(\psi)\lambda_v(P,A') \geqslant \sum_{v\in S}\lambda_v(\psi(P),A) - O(h(A)+h(\psi)+1),$$

where the implied constant depends only on the degree of the map ψ.

Proof This is stated in [10, Proposition 7] over number fields. Its proof uses a higher dimensional version of Lemma 2.4 applied to maps in dimension 1. Lemma 2.4 is enough for our purposes and for that proof, and it also works over function fields. Moreover, the proof uses strong distribution value theorems related with an inverse function theorem in this context due to Silverman therein. There was an error found in these proofs according to [14], which was corrected in [14, Sections 4 and 5]. All these results work over fields equipped with a set of inequivalent absolute values for which the product formula holds. □

We recall that $\beta \in \mathbb{P}^1(\overline{K})$ is an *exceptional point* for ϕ if its backward orbit $\{\gamma \in \mathbb{P}^1(\overline{K}) : \beta \in \mathcal{O}_\phi(\beta)\}$ is finite.

Lemma 2.7 ([10, Lemma 9]) *Fix an integer* $d \geqslant 2$. *Then there exist two positive constants* $\kappa_1 > 0$ *and* $0 < \kappa_2 < 1$ *depending only on* d *such that for all rational functions* $\phi\colon \mathbb{P}^1(K) \to \mathbb{P}^1(K)$, *all points* Q *that are not exceptional for* ϕ, *all integers* $m \geqslant 1$, *and all* $P \in \phi^{-m}(Q)$, *we have that*

$$e_P(\phi^m) \leqslant \kappa_1(\kappa_2 d)^m \quad \textit{for any} \quad m \geqslant 0.$$

Proof The proof of [18, Lemma 3.52] works here, since the Riemann–Hurwitz formula works for this context as well. □

The third result is the following effective version of Roth's theorem over function fields due to Wang.

Lemma 2.8 ([21]) *Let* S *be a finite subset of* M_K. *We assume that each place in* S *is extended to* $\overline{K}$ *in some fashion. Assume that for each* $v \in S$, *we have an element* $\beta_v \in \overline{K}$. *Then, for any* $\mu > 2$, *the elements* $x \in K$ *satisfying*

$$\sum_{v\in S}\log^+|x-\beta_v|_v^{-1} \geqslant \mu h(x)$$

have their heights bounded by an effective constant depending on μ, $|S|$, *the genus of* K, *and the elements* β_v.

2.3 A bound for the number of quasi-integral points in an orbit

In this section, we show explicit bounds for the number of S-integral points in a given orbit of a wandering point for a dynamical system of rational functions extending previous work by Hsia and Silverman [10].

The next quantitative theorem generalizes [10, Theorem 11] to function fields of characteristic zero. The definitions and strategy of the proof are inspired by their ideas with Diophantine approximation.

Theorem 2.9 *Let $\phi \in K(z)$ be a rational function of respective degree $d \geqslant 2$, and $P \in \mathbb{P}^1(K)$ not preperiodic for ϕ. Fix $A \in \mathbb{P}^1(K)$ which is not an exceptional point of ϕ. For any finite set of places $S \subset M_K$ and any constant $1 \geqslant \epsilon > 0$, define the set of points*

$$\Gamma_{\phi,S}(A, P, \epsilon) := \left\{\phi^{(n)}(P) : \sum_{v\in S} \lambda_v(\phi^n(P), A) \geqslant \epsilon\,\hat{h}_\phi(\phi^n(P))\right\}.$$

(a) *There exist effective constants*

$$\gamma_1 = \gamma_1(\phi, \epsilon, |S|, K, A) \quad \text{and} \quad \gamma_2 = \gamma_2(\phi, \epsilon, |S|, K, A)$$

such that

$$\left\{\phi^{(n)}(P) \in \Gamma_{\phi,S}(A, P, \epsilon) : n > \gamma_1 + \log_d^+\left(\frac{\hat{h}_\phi(A) + h(\phi)}{\hat{h}_\phi(P)}\right)\right\}$$

has bounded height from above by γ_2.

(b) *If ϕ is not isotrivial, then there is an effective constant $\gamma_3(\phi, \epsilon, |S|, K, A)$ that is independent of P such that*

$$\max_P \max\left\{n \geqslant 0 : \phi^{(n)}(P) \in \Gamma_{\phi,S}(A, P, \epsilon)\right\} \leqslant \gamma_3 + \log_d^+\left(\frac{h(\phi)}{\inf_{\hat{h}_\phi(P)>0} \hat{h}_\phi(P)}\right).$$

Proof For simplicity, we write $\Gamma_S(\epsilon)$ instead of $\Gamma_{\Phi,S}(A, P, \epsilon)$. Taking κ_1 and $\kappa_2 < 1$ the constants from Lemma 2.7, we choose $m \geqslant 1$ minimal such that $\kappa_2^m \leqslant \epsilon/5\kappa_1$. Then κ_1, κ_2 and m depend only on d and on ϵ.

If $n \leqslant m$ for all n such that $\phi^{(n)}(P) \in \Gamma_S(\epsilon)$, then

$$\#\,\Gamma_S(\epsilon) \leqslant m \leqslant \frac{\log(5\kappa_1) + \log(\epsilon^{-1})}{\log(\kappa_2^{-1})} + 1,$$

which is in the desired form. If there is an n with $\phi^{(n)}(P) \in \Gamma_S(\epsilon)$ such that $n > m$, we fix n for instance. Then by definition of $\Gamma_S(\epsilon)$ we have

$$\epsilon\hat{h}_\phi(\phi^{(n)}(P)) \leqslant \sum_{v\in S} \lambda_v(\phi^{(n)}(P), A). \tag{2.1}$$

We can write $\phi^{(n)} = \phi^{(m)} \circ \phi^{(n-m)}$ and $\psi = \phi^{(m)}$.

For our chosen m, we denote

$$\mathbf{e}_m := \max_{A'\in\psi^{-1}(A)} e_{A'}(\psi).$$

By Lemma 2.7 and our choice of m, we notice that

$$\mathbf{e}_m \leqslant \kappa_1(\kappa_2)^m \deg\psi \leqslant \epsilon \deg\psi/5.$$

Therefore, Lemma 2.6 yields, for $Q \in \mathbb{P}^1(K)$, that

$$\sum_{v\in S} \lambda_v(\psi(Q), A) - O(h(A) + h(\psi) + 1) \leqslant \mathbf{e}_m \sum_{v\in S} \max_{A'\in\psi^{-1}(A)} \lambda_v(Q, A'). \quad (2.2)$$

Gathering (2.1) and (2.2) with $Q := \phi^{(n-m)}(P)$, we obtain that

$$\epsilon \hat{h}_\phi(\phi^{(n)}(P)) \leqslant \mathbf{e}_m \sum_{v\in S} \max_{A'\in\psi^{-1}(A)} \lambda_v(\phi^{(n-m)}(P), A') + O(h(A) + h(\psi) + 1),$$

where the involved constants depend only on the degree d^m, d and on ϵ.

For each $v \in S$, we choose $A'_v \in \psi^{-1}(A)$ such that

$$\lambda_v(\phi^{(n-m)}(P), A'_v) = \max_{A'\in\psi^{-1}(A)} \lambda_v(\phi^{(n-m)}(P), A'),$$

so that

$$\epsilon \hat{h}_\phi(\phi^n(P)) \leqslant \mathbf{e}_m \sum_{v\in S} \lambda_v(\phi^{(n-m)}(P), A'_v) + O(h(A) + h(\psi) + 1).$$

For instance, we can assume that $z(A') \neq \infty$ for all $A' \in \psi^{-1}(A)$. If this is not the case, we use z for some of the A' and z^{-1} for the others.

Let $S' \subset S$ be the set of places in S defined by

$$S' = \left\{v \in S : \lambda_v(\phi^{(n-m)}(P), A'_v) > \lambda_v(A'_v, \infty)\right\}.$$

Set $S'' := S - S'$. Applying Lemma 2.2 to the places in S', using the definition of S'' and Lemma 2.4 we find, as in [10, p. 337], that

$$\epsilon \hat{h}_\phi(\phi^{(n)}(P)) \leqslant \mathbf{e}_m \sum_{v\in S'} \log\left|z(\phi^{(n-m)}(P)) - z(A'_v)\right|_v^{-1} + \mathbf{e}_m \sum_{v\in S} (2\lambda_v(A'_v, \infty)) + O(h(A) + h(\psi) + 1),$$

and that

$$\sum_{v\in S} \lambda_v(A'_v, \infty) \leqslant \hat{h}_\phi(A) + O(h(\phi) + 1).$$

The constants depend only on m and d. Also, from Proposition 2.3 it follows that $h(\psi) = O(h(\phi) + 1)$.

All the inequalities above together imply that

$$\epsilon \hat{h}_\phi(\phi^n(P)) \leqslant \mathbf{e}_m \sum_{v\in S'} \log\left|z(\phi^{(n-m)}(P)) - z(A'_v)\right|_v^{-1} + O(\hat{h}_\phi(A) + h(\phi) + 1).$$

Let us set some definitions in order to apply Wang's Theorem. We define $\beta_v := A'_v$ and analyze the points $x = \phi^{(n-m)}(P)$ for $\phi^{(n)}(P) \in \Gamma_S(\epsilon)$. Applying Lemma 2.8 for the set of places S' and $\mu = 5/2$, yields that there exist a constant r_1 depending only on K, ϕ, $|S|$ and ϵ such that the set of $\phi^n(P) \in \Gamma_S(\epsilon)$ with $n > m$ can be written as a union

$$\{\phi^n(P) \in \Gamma_S(\epsilon) : n > m\} = T_1 \cup T_2$$

where T_1 has all its elements with height bounded from above by r_1, and

$$T_2 = \left\{\phi^{(n)}(P) \in \Gamma_S(\epsilon) : n > m, \sum_{v\in S'} \log\left|z(\phi^{(n-m)}(P)) - z(A'_v)\right|_v^{-1} \leqslant \frac{5}{2}\, h(\phi^{(n-m)}(P))\right\}.$$

We already have a bound for the height of T_1. We consider the set T_2. Again, using Lemmas 2.4 we derive

$$h(\phi^{(n-m)}(P)) \leqslant \hat{h}_\phi(\phi^{(n-m)}(P)) + c_1 h(\phi) + c_2 = d^{n-m}\hat{h}_\phi(P) + c_1 h(\phi) + c_2.$$

Then, for n with $\phi^{(n)}(P) \in T_2$, using that $\mathbf{e}_m \leqslant \epsilon \deg \psi/5$, we have

$$\begin{aligned}
\epsilon \hat{h}_\phi(\phi^{(n)}(P)) &= \epsilon d^n \hat{h}_\Phi(P) \\
&\leqslant \mathbf{e}_m \sum_{v\in S'} \log\left|z(\phi^{(n-m)}(P)) - z(A'_v)\right|^{-1} + c_9(\hat{h}_\phi(A) + h(\phi) + 1) \\
&\leqslant \left(\epsilon \frac{\deg \psi}{5}\right) \frac{5}{2} (d^{n-m}) \hat{h}_\phi(P) + c_{10}(\hat{h}_\phi(A) + h(\phi) + 1) \\
&= \frac{\epsilon}{2} d^n \hat{h}_\phi(P) + c_{11}(\hat{h}_\phi(A) + h(\phi) + 1).
\end{aligned}$$

Thus,

$$\frac{\epsilon}{2} d^n \hat{h}_\phi(P) \leqslant c_{11}(\hat{h}_\phi(A) + h(\phi) + 1)$$

is equivalent to

$$n \leqslant c_{12} + \log_d^+\left(\frac{\hat{h}_\phi(A) + h(\phi)}{\hat{h}_\phi(P)}\right).$$

We observe that the set $\{z(A'_v) : v \in S'\}$ does not depend on the point P, so the elements in T_1 have height bounded independently of P by the constant r_1. We also note that the quantity

$$\hat{h}_{\phi,K}^{\min} := \inf\left\{\hat{h}_\phi(P) : P \in \mathbb{P}^1(K) \text{ is not preperiodic for } \phi\right\}$$

is strictly positive. This is a consequence of Theorem 2.5. For $\phi^n(P) \in T_1$, we can see from this and Lemma 2.4 that $d^n\hat{h}_\phi(P) = \hat{h}_\phi(\phi^{(n)}(P)) \leqslant r_1 + O(h(\phi)+1)$, and thus

$$n \leqslant \log_d\left(\frac{r_1 + O(h(\phi)+1)}{\hat{h}_{\phi,K}^{\min}}\right)$$

in this case.

Therefore, $\max\{n : \phi^n(P) \in (T_1 \cup T_2)\}$ can be bounded independently of P. □

As a consequence, we recover some results from [11, Theorem 1]. For some results over fields of positive characteristic, see [8], [11, Theorem 2] and [19, Theorem 2].

Corollary 2.10 *Let $S \subset M_K$ be a finite set of places, let R_S be the ring of S-integers of K, and let $d \geqslant 2$. Then, there are effective constants $\gamma_1 = \gamma_1(\phi, |S|, K)$ and $\gamma_2 = \gamma_2(\phi, |S|, K)$ such that for all $\phi \in K(z)$ rational maps of degrees $d \geqslant 2$ with $\phi^{(2)} \notin \overline{K}[z]$, and all $P \in \mathbb{P}^1(K)$ that are not preperiodic for ϕ, the set of S-integers*

$$\left\{\phi^{(n)}(P) : \phi^{(n)}(P) \in R_S \text{ and } n > \gamma_1 + \log_d^+\left(\frac{h(\phi)}{\hat{h}_\phi(P)}\right)\right\}$$

is a set of height bounded from above by γ_2. If ϕ is not isotrivial, then this set is finite and has size effectively bounded in terms of ϕ, $|S|$, K and $\inf_{\hat{h}(f)>0}\hat{h}(f)$.

Proof An element $\alpha \in K$ is in R_S if and only if $|\alpha|_v \leqslant 1$ for all $v \notin S$, or equivalently, if and only if

$$h(\alpha) = \sum_{v\in S}\log\max\{|\alpha|_v, 1\}.$$

Another fact is that

$$\log\max\{|\alpha|_v, 1\} \leqslant \lambda_v(\alpha, \infty).$$

This implies for $\alpha \in R_S$ that $h(\alpha) \leqslant \sum_{v\in S}\lambda_v(\alpha,\infty)$.

Let $n \geqslant 1$ satisfy $z(\phi^{(n)}(P)) \in R_S$. Then

$$h(\phi^{(n)}(P)) \leqslant \sum_{v\in S}\lambda_v(\phi^{(n)}(P), \infty).$$

Lemma 2.4 tells us that

$$h(\phi^{(n)}(P)) \geqslant \hat{h}_\phi(\phi^{(n)}(P)) - c_3h(\phi) - c_4 = \deg(\phi^{(n)})\hat{h}_\phi(P) - c_3h(\phi) - c_4,$$

which implies that

$$d^n \hat{h}_\phi(P) - c_3 h(\phi) - c_4 \leqslant \sum_{v \in S} \lambda_v(\phi^{(n)}(P), \infty).$$

The rest of the proof is divided in two cases: The first one, when

$$d^n \hat{h}_\phi(P) \leqslant 2c_3 h(\phi) + 2c_4.$$

In this case,

$$n \leqslant \log_d^+\left(\frac{2c_3 h(\phi) + 2c_4}{\hat{h}_\phi(P)}\right).$$

In the second case, when $d^n \hat{h}_\phi(P) \geqslant 2c_3 h(\phi) + 2c_4$. Therefore

$$\sum_{v \in S} \lambda_v(\phi^{(n)}(P), \infty) \geqslant \frac{1}{2} d^n \hat{h}_\phi(P) = \frac{1}{2} \hat{h}_\phi(\phi^n(P)).$$

Now Theorem 2.9 with $\epsilon = 1/2$, $A = \infty$ (∞ is not exceptional for ϕ) tells us that either n is at most

$$\gamma_1 + \log_d^+\left(\frac{h(\phi) + \hat{h}_\phi(\infty)}{\hat{h}_\phi(P)}\right),$$

or $\phi^{(n)}(P)$ has height bounded from above by γ_2, where γ_1 and γ_2 are effective constants depending only on K, ϕ and $|S|$. The bounds are of the desired form since $\hat{h}_\phi(\infty) \leqslant h(\infty) + O(1) = 0 + O(1)$. For the claimed finiteness, we note again that if ϕ is not isotrivial then $\hat{h}_\phi(\phi^n(P)) = d^n \hat{h}_\phi(P)$ being bounded by γ_2 together with Theorem 2.5 implies the claimed finiteness with a bound for n which is independent of P, as obtained in the proof of Theorem 2.9 (b). □

Remark 2.11 Theorem 2.9 delivers, in particular, under its conditions, an explicit upper bound for

$$\#\left\{n \geqslant 1 : \frac{1}{\phi^{(n)}(P) - A} \text{ is quasi-}(S, \epsilon)\text{-integral}\right\}.$$

Corollary 2.12 *Under the hypothesis of Theorem 2.9,*

$$\lim_{n \to \infty} \frac{\lambda_v(\phi^{(n)}(P), A)}{d^n} = \lim_{n \to \infty} \frac{\lambda_v(\phi^{(n)}(P), A)}{\hat{h}_\phi(\phi^{(n)}(P))} = 0 \quad \textit{for every} \quad v \in M_K.$$

Proof Applying Theorem 2.9 for the set of places that contains just the place v, we conclude that for every natural n big enough, it will be true that

$$\frac{\lambda_v(\phi^{(n)}(P), A)}{d^n} \leqslant \epsilon \hat{h}_\phi(P).$$

Choosing ϵ sufficiently small, the result is proven. □

3 Multiplicative dependence in orbits over function fields

3.1 *S*-Units, algebraic dynamics, and multiplicative dependence

First we settle some notation. Given three distinct elements $a, b, c \in K$, we write $T_{a,b,c}$ for the unique linear fractional transformation $T(z) = (\alpha z + \beta)/(\delta z + \gamma)$, $(\alpha\gamma - \beta\delta \neq 0)$ such that $T(a) = 0$, $T(b) = \infty$ and $T(c) = 1$.

For any $\varphi \in K(z)$, we define Ω_φ to be the set of elements $f \in K$ such that $T_{a,b,c}(f) \in \mathbf{k}$, for some triple of distinct $a, b, c \in \varphi^{-1}\{\infty\}$, namely

$$\Omega_\varphi := \left\{ f \in K : T_{a,b,c}(f) \in \mathbf{k} \text{ for some distinct } a, b, c \in \varphi^{-1}\{\infty\} \right\}.$$

We also make use of the following notation for wandering points with respect to a rational function $\varphi(z) \in K(z)$.

$$\text{Wander}_K(\varphi) := \left\{ f \in K : f \text{ is not preperiodic under } \varphi \right\}.$$

Here, $S \subset M_K$ is again assumed to be a finite set of places of K, R_S is the respective ring of S-integers in K and R_S^* the corresponding ring of S-units.

We will use the following function field version for the "integer image value" Siegel's theorem.

Lemma 3.1 ([11, Theorem 12 (i), (iv)]) *Let $\varphi(z) \in K(z)$. Suppose that $|\varphi^{-1}\{\infty\}| \geqslant 3$. Then*

$$\{f \in K : \varphi(f) \in R_S\}$$

is a set whose elements f have height bounded from above by a constant $C(\varphi, K, |S|)$. Moreover, the set $\{f \in K : \varphi(f) \in R_S\} \setminus \Omega_\varphi$ is finite, and if $\varphi^{-1}\{\infty\}$ is not isotrivial, then the full set $\{f \in K : \varphi(f) \in R_S\}$ is finite.

The following is a version of [4, Theorem 1.2] for function fields.

Theorem 3.2 *Let $\varphi(z) \in K(z)$ be of degree d at least two.*

(a) *If $|\varphi^{-1}(\{0, \infty\})| \geqslant 3$, then*

$$\{f \in K : \varphi(f) \in R_S^*\}$$

is a set of height bounded from above effectively in terms of φ, $|S|$ *and* K, *and* $\{f \in K : \varphi(f) \in R_S^*\} \setminus \Omega_{\varphi+1/\varphi}$ *is finite.*
If $\varphi^{-1}\{0, \infty\}$ *is not isotrivial, then the full set*

$$\{f \in K : \varphi(f) \in R_S^*\}$$

is finite.

(b) *Suppose that* φ *is not of the form* $\varphi(z) = fz^{\pm d}$. *Let*

$$\mathcal{F}_2(K, \varphi, R_S^*) = \left\{(n, \alpha) \in \mathbb{Z}_{\geqslant 2} \times \mathrm{Wander}_K(\varphi) : \varphi^{(n)}(\alpha) \in R_S^*\right\}.$$

If $(n, \alpha) \in \mathcal{F}_2(K, \varphi, R_S^*)$, *then* α *belongs to a set of height bounded from above effectively in terms of* φ, $|S|$ *and* K. *Also, if* φ *is not isotrivial, then* n *is bounded by an explicit constant depending on* φ, K, $|S|$ *and* $\inf_{\hat{h}_\varphi(f)>0} \hat{h}_\varphi(f)$.

Proof In order to prove (a), we adapt the ideas in the proof of [12, Proposition 1.5 (a)]. Namely, by our assumptions, the function $\psi(z) := \varphi(z) + 1/\varphi(z)$ satisfies the hypothesis of the previous lemma, and since $\psi(\beta) \in R_S$ whenever $\varphi(\beta) \in R_S^*$, it follows that $\{f \in K : \varphi(f) \in R_S^*\}$ is a set of height bounded by $C(\varphi + 1/\varphi, K, |S|)$, $\{f \in K : \varphi(f) \in R_S^*\} \setminus \Omega_\psi$ is finite, and finally, the full set $\{f \in K : \varphi(f) \in R_S^*\}$ will be finite if $\varphi^{-1}\{0, \infty\}$ is not isotrivial.

In order to prove (b), we consider the well-defined map

$$\mathcal{F}_2(K, \varphi, R_S^*) \to \{f \in K : \varphi^{(2)}(f) \in R_S^*\}$$

sending (n, α) to $\varphi^{(n-2)}(\alpha)$. Using the Riemann–Hurwitz formula, which is valid in characteristic zero, the same method as carried out in the proof of [12, Lemma 3.2] can be performed, and such result is also valid in our context, which implies that $|{\varphi^{(2)}}^{-1}(\{0, \infty\})| \geqslant 3$. By (a), this condition implies that the heights of such $\varphi^{(n-2)}(\alpha)$'s are bounded by $C(\varphi^{(2)} + 1/\varphi^{(2)}, K, |S|)$. Hence, we also obtain an effective bound $C_1(\varphi, K, |S|)$ for the canonical heights of such points, and thus also for the heights of the referred α's, due to Lemma 2.4. Looking at a certain $\varphi^{(n-2)}(\alpha)$ among such possibilities, and using by Theorem 2.5 that $\inf_{\hat{h}_\varphi(P)>0} \hat{h}_\varphi(P)$ exists, we have by the same calculations as in the proof of [4, Lemma 2.3] that n is bounded by $2 + \log_d\left(\frac{C_1(\varphi, K, |S|)}{\inf_{\hat{h}_\varphi(P)>0} \hat{h}_\varphi(P)}\right)$. □

The next result generalizes [4, Theorem 1.3] to function fields.

Theorem 3.3 *Let* $r, s \in \mathbb{Z}$ *with* $rs \neq 0$, *and set*

$$\rho = \frac{\log(|s|/|r|)}{\log d} + 1.$$

Let $\phi \in K(z)$ *with degree* $d \geqslant 2$. *Assume that* 0 *is not a periodic point for* ϕ. *Define* $\mathcal{E}_\rho(K, \phi, S, r, s)$ *to be*

$$\left\{(n, k, f, u) \in \mathbb{Z}_{\geqslant \rho} \times \mathbb{Z}_{\geqslant 0} \times \mathrm{Wander}_K(\phi) \times R_S^* : \phi^{(n+k)}(f)^r = u\phi^{(k)}(f)^s\right\}.$$

Let $(n, k, f, u) \in \mathcal{E}_\rho(K, \phi, S, r, s)$. Then f has height bounded from above effectively in terms of $\phi, r, s, |S|$ and K. Moreover, for f out of any fixed set of bounded height from above by δ, we have that n, k and $h(u)$ are effectively bounded in terms of $\phi, r, s, |S|, \delta$ and K.

If moreover ϕ is not isotrivial, then for any f, we have that n, k and $h(u)$ are effectively bounded in terms of $\phi, K, r, s, |S|$ and $\inf_{\hat{h}_\phi(f)>0} \hat{h}_\phi(f)$.

Remark 3.4 If $r = s = 0$, then $u = 1$ and f can be any element of K. If $r = 0$ and $s \neq 0$ or vice-versa, then due to the multiplicative saturation of R_S^* in $K^*(\gamma^n) \in R_S^*$, $n \neq 0 \Rightarrow \gamma \in R_S^*$, we are reduced to the situation worked out in Theorem 3.2.

Proof Arguing in the same way as in the proof of [4, Theorem 1.3], we can assume that 0 is not an exceptional point for ϕ, i.e., the backwards orbit $\{\gamma : \phi^{(n)}(\gamma) = 0, n \geqslant 0\}$ of 0 under ϕ is infinite.

Suppose first that ϕ is allowed to be isotrivial. We study $(n, k, \alpha) \in \mathbb{Z}_{\geqslant 1} \times \mathbb{Z}_{\geqslant 0} \times \text{Wander}_K(\phi)$ such that

$$\phi^{(n+k)}(\alpha)^r = u\phi^{(k)}(\alpha)^s.$$

Theorem 2.9 (a) applied with $A = 0$ and $\epsilon = 1/3$ gives us effectively computable constants γ_1 and γ_2 such that

$$\left\{\phi^{(n)}(\alpha) \in \Gamma_{\phi,S}(0, \alpha, \epsilon) : n > \gamma_1 + \log_d^+\left(\frac{h(\phi)}{\hat{h}_\phi(\alpha)}\right)\right\} \tag{3.1}$$

has bounded height from above by γ_2.

If $n + k > \gamma_1 + \log_d^+(h(\phi)/\hat{h}_\phi(\alpha))$, then either $\phi^{(n+k)}(\alpha)$ is in the set (3.1) or not. If $\phi^{(n+k)}(\alpha)$ is in the set (3.1), then it has height bounded by γ_2, so that

$$\hat{h}_\phi(\phi^{(n+k)}(\alpha)) = d^{n+k}\hat{h}_\phi(\alpha) = O(\gamma_2 + h(\phi)) \implies \hat{h}_\phi(\alpha) = O(\gamma_2 + h(\phi))$$

and $\hat{h}_\phi(\phi^{(n+k)}(\alpha)) = d^n\hat{h}_\phi(\phi^{(k)}(\alpha)) \Rightarrow \hat{h}_\phi(\phi^{(k)}(\alpha)) = O(\gamma_2 + h(\phi))$, which yields the desired effective bounds for the referred α's, u's, since $\phi^{(n+k)}(\alpha)^r = u\phi^{(k)}(\alpha)^s$. If otherwise $\phi^{(n+k)}(\alpha)$ is not in the set (3.1) even though

$$n + k > \gamma_1 + \log_d^+\left(\frac{h(\phi)}{\hat{h}_\phi(\alpha)}\right),$$

then $\phi^{(n+k)}(\alpha) \notin \Gamma_{\phi,S}(0, \alpha, \epsilon)$ and so

$$\sum_{v\in S} \log^+\big(|\phi^{(n+k)}(\alpha)|_v^{-1}\big) \leqslant \epsilon\hat{h}_\phi(\phi^{(n+k)}(\alpha)). \tag{3.2}$$

In this situation, we can compute

$$\begin{aligned}
h(\phi^{(n+k)}(\alpha)) &= h(\phi^{(n+k)}(\alpha)^{-1}) \\
&= \sum_{v\in S} \log^+ |\phi^{(n+k)}(\alpha)|_v^{-1} + \sum_{v\notin S} \log^+ |\phi^{(n+k)}(\alpha)|_v^{-1} \\
&\leqslant \epsilon \hat{h}_\phi(\phi^{(n+k)}(\alpha)) + \sum_{v\notin S} \log^+ |\phi^{(k)}(\alpha)|_v^{-s/r} \\
&\leqslant \epsilon \hat{h}_\phi(\phi^{(n+k)}(\alpha)) + \left|\frac{s}{r}\right| h(\phi^{(k)}(\alpha)) \\
&\qquad\qquad (\text{using that } h(x) = h(x^{-1}), h(x^d) = dh(x)) \\
&= \epsilon \hat{h}_\phi(\phi^{(n+k)}(\alpha)) + d^{\rho-1} h(\phi^{(k)}(\alpha)) \\
&\leqslant \epsilon \hat{h}_\phi(\phi^{(n+k)}(\alpha)) + d^{\rho-1}\big(\hat{h}_\phi(\phi^{(k)}(\alpha)) + O(h(\phi)+1)\big)
\end{aligned}$$

implying also that

$$(1-\epsilon)d^{n+k}\hat{h}_\phi(\alpha) \leqslant d^{\rho-1+k}\hat{h}_\phi(\alpha) + O(d^{\rho-1}(h(\phi)) + 1)$$

and

$$d^k((1-\epsilon)d^n - d^{\rho-1})\hat{h}_\phi(\alpha) = O(d^{\rho-1}(h(\phi)) + 1).$$

With $\epsilon = 1/3$, the above and $n \geqslant \rho$ yield

$$\begin{gathered}
(1-\epsilon)d^n - d^{\rho-1} \geqslant \frac{4}{3}d^{n-1} - d^{\rho-1} \geqslant \frac{1}{3}d^{n-1} \\
\implies d^{n+k-1}\hat{h}_\phi(\alpha) \leqslant O(d^{\rho-1}(h(\phi)) + 1), \\
\hat{h}_\phi(\alpha) = O(d^{\rho-1}(h(\phi)) + 1),
\end{gathered}$$

which implies as before that $\hat{h}_\phi(\phi^{(n+k)}(\alpha)), \hat{h}_\phi(\phi^{(n)}(\alpha)) \leqslant O(\gamma_2 + h(\phi))$, yielding the desired effective bounds for the referred α's and u's as in the previous situation.

If otherwise $n + k \leqslant \gamma_1 + \log_d^+(h(\phi)/\hat{h}_\phi(\alpha))$, then we start supposing that α is out of a set of bounded height, let us say, that $h(\alpha) \geqslant \delta$ is fixed. In this case, n and k are bounded as claimed and we may assume that they are fixed. Now, we let

$$g(z) = \phi^{(n)}(z)^r/z^s$$

so that we have

$$g(\phi^k(\alpha)) = \phi^{(n+k)}(\alpha)^r/\phi^{(k)}(\alpha)^s \in R_S^*.$$

This says that

$$\phi^{(k)}(\alpha) \in \{f \in K : g(f) \in R_S^*\}.$$

The assumption that 0 is not periodic for ϕ implies that 0 is a pole of g and that $\phi(X) \neq cX^{\pm d}$, which implies that $\phi^{(n)}$ has at least two poles or zeros distinct from 0. Hence g has at least three poles and zeros. Theorem 3.2 (a) now yields that $\phi^{(k)}(\alpha)$ has height effectively bounded as before, and so will be the height of α and $\phi^{(n+k)}(\alpha)$ due to Lemma 2.4 and the effective bound for $n+k$. This implies the claimed effective bound for $h(u)$ as well.

Suppose now that ϕ is not isotrivial. This allows us to apply Theorem 2.9 (b) with $A = 0$ and $\epsilon = 1/3$, yielding an effective constant

$$\gamma_4 := \gamma_3(\phi, \epsilon, |S|, K) + \log_d^+\left(\frac{h(\phi)}{\inf_{\hat{h}_\phi(P)>0} \hat{h}_\phi(P)}\right)$$

such that

$$\max\left\{m \geqslant 0 : \sum_{v\in S} \log^+(|\phi^{(m)}(\alpha)|_v^{-1}) \geqslant \epsilon \hat{h}_\phi(\phi^{(m)}(\alpha))\right\} \leqslant \gamma_4.$$

We use this constant γ_4 to consider $(n, k, \alpha) \in \mathbb{Z}_{\geqslant 1} \times \mathbb{Z}_{\geqslant 0} \times \mathrm{Wander}_K(\phi)$ such that

$$|\phi^{(n+k)}(\alpha)|_v^r = |\phi^{(k)}(\alpha)|_v^s \quad \text{for all} \quad v \in M_K \setminus S.$$

If $n+k < \gamma_4$, then we will already have the desired bound for n and k. Otherwise, we have $n+k \geqslant \gamma_4$.

In this case Theorem 2.9 tells us that (n, k, α) satisfies inequality (3.2) again. We then proceed similarly as in the computations after inequality (3.2) to obtain

$$d^{n+k-1}\hat{h}_\phi(\alpha) \leqslant O(d^{\rho-1}(h(\phi)) + 1) \quad \text{and} \quad \hat{h}_\phi(\alpha) = O(d^{\rho-1}(h(\phi)) + 1),$$

and hence

$$n, k \leqslant \gamma_4 + \log_d^+\left(\frac{h(\phi)}{\inf_{\hat{h}_\phi(P)>0} \hat{h}_\phi(P)}\right)$$

for an effective constant $\gamma_4(K, |S|, \phi, \rho)$, which is as the bound claimed for n, k in the isotrivial case again. Whether $n+k < \gamma_4$ or $n+k \geqslant \gamma_4$, the two inequalities above together with $\phi^{(n+k)}(\alpha)^r = u\phi^{(k)}(\alpha)^s$ yield the desired effective height bound for the referred u's when ϕ is not isotrivial, completing the proof. □

As pointed out in the beginning of the [4, Subsection 1.4], we remark that results like Theorem 3.3 would follow from sufficiently strong *Dynamical Zsigmondy Primitive Divisor Theorems*. In the present situation of function fields, such a general dynamical primitive divisor theorem is proven in the work of Bridy and Tucker [6].

3.2 Polynomial dynamics and multiplicative dependence over function fields in one variable

In this subsection, we deal with polynomials and restrict the S-integers to the set of roots of unity. In this setting, one can obtain independence of the exponents r, s from Theorem 3.3.

The result stated below is analogous to a Schinzel–Tijdeman result, now over the fields treated here.

Lemma 3.5 ([7]) *Let $f \in K[X]$ be a polynomial of degree n with at least two distinct zeros in some algebraic closure of K. Then the equation*

$$f(x) = y^m \quad in \quad x, y \in K, \quad m \in \mathbb{Z}_{>0}, \quad y \notin \mathbf{k},$$

implies

$$m \leqslant B(h(f), K),$$

where $B = B(h(f), K)$ is an effective constant.

The next result generalizes [4, Theorem 1.7] to function fields in one variable, when R_S^* is replaced by the set of roots of unity $\mathbb{U}_K$ in K.

Theorem 3.6 *Let $\phi \in K[z]$ be a polynomial of degree $d \geqslant 3$ such that ϕ and $\phi^{(2)}$ have no multiple roots, and that 0 is not a periodic point of ϕ. Then*

(a)

$$\mathcal{M} := \left\{ f \in K : \begin{array}{l} (\phi^{(m)}(f))^r = u(\phi^{(n)}(f))^s, u \in \mathbb{U}_K \\ m > n > 0, (r, s) \in \mathbb{Z}^2 \setminus \{(0,0)\} \end{array} \right\}$$

is a set of height effectively bounded from above by bounds depending on ϕ and K.

(b) *If $f \in \mathcal{M}$ is such that $(\phi^{(m)}(f))^r = u(\phi^{(n)}(f))^s$ with $rs \neq 0, m > n > 0$ and ϕ is not isotrivial, then m and n are effectively bounded from above by bounds depending on ϕ, K, and $\inf_{\hat{h}_\phi(P)>0} \hat{h}_\phi(P)$.*

Proof In order to use our previous results and their language, we make $S := \varnothing$. In this way, we have that $\mathbb{U}_K \subset R_S^*$. Let $\alpha \in \text{Wander}_\phi(K)$ be such that there exist non-negative integers $m > n > 0$, integers r and s, and $u \in \mathbb{U}_K \subset R_S^*$ such that

$$(\phi^{(m)}(\alpha))^r = u(\phi^n(\alpha))^s. \tag{3.3}$$

If $r = 0$ or $s = 0$, then the result follows from Theorem 3.2. Thus, we may assume that $rs \neq 0$. Also, due to the saturation of $\mathbb{U}_K$ and R_S^*, we may assume that $\gcd(r, s) = 1$. We enlarge S by adding to it the places for which ϕ has bad reduction, namely, $v \in M_K$ such that if $f(z) = c_0 + c_1 z + \cdots + c_d z^d$, then either $v(c_i) < 0$ for some i or $v(c_d) > 0$. In this way we obtain a new finite set of places S_ϕ. Moreover, proving the results for the larger S_ϕ gives results that are stronger than the original statements. One can also see that

$$S_{\phi^{(k)}} \subset S_\phi \quad \text{for all } k \geqslant 1.$$

Since $rs \neq 0$, replacing r, s by $-r, -s$ if necessary, we may assume that $r > 0$. We now divide the proof in Cases A and B, depending on whether α lies in R_{S_ϕ} or not.

Case A: $\alpha \in R_{S_\phi}$**.**

By definition one can check that $\phi^{(k)}(\alpha) \in R_{S_\phi}$ for every $k \geqslant 0$, so that

$$v(\phi^{(k)}(\alpha)) \geqslant 0 \quad \text{for all } k \geqslant 0, \quad v \notin S_\phi.$$

We now divide this case in some subcases.

Case A.1: $r > 0, s < 0$**.**

Here, equation (3.3) becomes

$$(\phi^{(m)}(\alpha))^r(\phi^{(n)}(\alpha))^t = u \quad \text{with } t = -s > 0.$$

As $u \in R^*_{S_\phi}$, it follows that

$$rv(\phi^{(m)}(\alpha)) + tv(\phi^{(n)}(\alpha)) = 0 \quad \text{for all } v \notin S_\phi.$$

Since $r, t > 0$, this implies that

$$v(\phi^{(m)}(\alpha)) = v(\phi^{(n)}(\alpha)) = 0 \quad \text{for all } v \notin S_\phi,$$

and hence that $\phi^{(m)}(\alpha) \in R^*_S$. The desired conclusions for (a) and (b) follow directly from Theorem 3.2.

Case A.2: $r > 0, s \geqslant 2$**.**

Since $\gcd(r, s) = 1$, we can choose a and b with $ar + bs = 1$ so that (3.3) becomes

$$\phi^{(m)}(\alpha) = u^a\big((\phi^{(n)}(\alpha))^a(\phi^{(m)}(\alpha))^b\big)^s.$$

Since $\phi^{(m)}(\alpha) \in R_{S_\phi}$ and $u \in \mathbb{U}_K$, we have that $(\phi^{(n)}(\alpha))^a(\phi^{(m)}(\alpha))^b \in R_{S_\phi}$. If $(\phi^{(n)}(\alpha))^a(\phi^{(m)}(\alpha))^b \in R^*_{S_\phi}$, then $\phi^{(m)}(\alpha) \in R^*_{S_\phi}$ and we have the desired results for (a) and (b) by Theorem 3.2. If $(\phi^{(n)}(\alpha))^a(\phi^{(m)}(\alpha))^b \notin R^*_{S_\phi}$, then we also have that $(\phi^{(n)}(\alpha))^a(\phi^{(m)}(\alpha))^b \notin \mathbf{k}$. Writing

$$\phi^{(m)}(\alpha) = \phi(\phi^{(m-1)}(\alpha)),$$

we use Lemma 3.5 to conclude that the exponent $s \geqslant 2$ is effectively bounded in terms of ϕ and K (as the logarithmic height of any root of unity is zero).

If we first assume that $s \geqslant 3$, then we can apply [2, Proposition 4.6] to effectively bound the heights of the referred $\phi^{(m-1)}(\alpha)$'s, and then $h(\alpha)$ as desired, as well as m in case ϕ is not isotrivial, after using the procedure used in the proof of Theorem 3.2 (b) via Lemma 2.4 and Theorem 2.5. This proves (a) and (b) for this subcase.

If on the other hand $s = 2$, we can apply [2, Proposition 4.7] in case $d \geqslant 3$, and obtain the conclusions of (a) and (b) similarly. If $s = d = 2$, then $\phi^{(m)}(\alpha) = \phi^{(2)}(\phi^{(m-2)}(\alpha))$, with $m \geqslant 2$, we can use again [2, Proposition 4.7] to conclude (a) and (b) similarly.

Case A.3: $r \geqslant 2, s = 1$**.**

If $n \geqslant 2$, then the same conclusions from Case A.1 as above hold (replacing m by n and r by s). It is enough then to consider the case $n = 1$, so that (3.3) becomes

$$\phi(\alpha) = u^{-1}(\phi^{(m)}(\alpha))^r.$$

If $r \geqslant 3$, then we can apply [2, Proposition 4.6] again to conclude as in Case A.2. If otherwise $r = 2$, we can apply Theorem 3.3 to conclude that (a) and (b) are true in this situation.

Case A.4: $r = 1, s = 1$**.**

This case is covered by Theorem 3.3.

Case B: $\alpha \notin R_{S_\phi}$**.**

Choosing $v \in M_K \setminus S_\phi$ such that $v(\alpha) < 0$, we have also that $v \notin S_{\phi^{(k)}}$ for all $k \geqslant 1$. In this case, we have that $v(\phi^{(k)}(\alpha)) = d^k v(\alpha)$ for all $k \geqslant 0$ and hence (3.3) implies

$$rd^m v(\alpha) = sd^n v(\alpha).$$

Therefore, since $\gcd(r, s) = 1$, we have that $r = 1$ and $s = d^{m-n}$. Writing $m = n+k$, (3.3) becomes

$$\phi^{(n+k)}(\alpha) = u(\phi^{(n)}(\alpha))^{d^k}.$$

If $k = 1$, then $\phi^{(n+1)}(\alpha) = u(\phi^{(n)}(\alpha))^d$, and we can use Theorem 3.3 to conclude that (a) and (b) are valid in this case. Otherwise $k \geqslant 2$ and α satisfies $\phi^{(n+k)}(\alpha) = u(\phi^{(n)}(\alpha))^{d^k}$, $d \geqslant 3$, so we consider the curve

$$\frac{\phi^{(2)}}{u}(X) = Y^{d^k}.$$

Notice that $\phi^{(n+k-2)}(\alpha)$ is a solution in X for the curve, and then $h(\phi^{(n+k-2)}(\alpha))$ as well as $\hat{h}_\phi(\phi^{(n+k-2)}(\alpha))$ are bounded only in terms of ϕ and K due to [13, Chapter VIII, Theorem 16]. This implies effective bounds for $h(\alpha)$ in terms of ϕ and K, and effective bounds for n, k in terms of ϕ, K and $\inf_{\hat{h}(f)>0} \hat{h}(f)$ in case ϕ is not isotrivial again, as we wanted to show. □

3.3 Iterates as zeros of split polynomials

Here we would like to point that [4, Theorem 1.10] is also true over function fields of characteristic 0, making use of Theorem 2.5. We start recalling the following definition.

Definition 3.7 A *multilinear polynomial* with split variables is a vector of polynomials

$$F(\mathbf{T}_1, \ldots, \mathbf{T}_k) = \sum_{i=1}^{r} c_i \prod_{j \in J_i} \mathbf{T}_j \in K[\mathbf{T}_1, \ldots, \mathbf{T}_k]$$

for some disjoint partition $J_1 \cup \cdots \cup J_r = \{1, \ldots, s\}$ and $c_i \in K^*$, $i = 1, \ldots, r$.

Theorem 3.8 *Let K be a function field of a smooth projective curve over an algebraically closed field of characteristic 0. Let $F(\boldsymbol{T}_1, \ldots, \boldsymbol{T}_k) \in K[\boldsymbol{T}_1, \ldots, \boldsymbol{T}_k]$ be a multilinear polynomial with split variables and let $\phi \in K(z)$ be a non-isotrivial rational function degree $d \geqslant 2$.*

The set of $\alpha \in \overline{K}$ not preperiodic for ϕ, for which there exists a k-tuple of distinct non-negative integers $(n_1, \ldots, n_k)$ satisfying

$$F(\phi^{(n_1)}(\alpha), \ldots, \phi^{(n_k)}(\alpha)) = 0$$

is a set of bounded height. If $d \geqslant 3$, such heights are bounded effectively in terms of ϕ, F and K.

Moreover, there are only finitely many k-tuples of integers $n_1 > n_2 > \cdots > n_k$ satisfying $F(\phi^{(n_1)}(\alpha), \ldots, \phi^{(n_k)}(\alpha)) = 0$, and there is a bound for such integers that depends only on ϕ, F, K and $\inf_{\hat{h}_\phi(P)>0} \hat{h}_\phi(P)$ and is independent of α.

Proof The proof follows the number field situation [4, Theorem 1.10] almost ipsis literis, except that we have to point out that the quantity $C_2(K, f)$ in that proof is replaced here by $\inf_{\hat{h}_\phi(P)>0} \hat{h}_\phi(P)$, which we know to exist due to Theorem 2.5. □

Remark 3.9 It would be interesting and one should be able to extend some of the present results to arithmetic function fields and finitely generated fields by using a very recent extension of Roth's theorem to such context (see [20]), which makes use of a new construction of global height functions associated with polarizations, due to Moriwaki. Over finitely generated fields, one can define a complete set of places for which a more general integral product formula holds and also define local heights that can be integrated over such places to obtain the global heights (see [20, Section 3]). In order to extend Theorem 2.9 (a) using similar ideas used here, one would have to prove the distribution relations and/or the inverse function theorem of [14] with such new language from [20].

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