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RESEARCH ARTICLE

Bounded reduction for Chevalley groups of types E_6 and E_7

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Dedicated to the memory of the brilliant mathematician Irina Suprunenko

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Abstract

We prove that an element from the Chevalley group of type E_6 or E_7 over a polynomial ring with coefficients in a small-dimensional ring can be reduced to an element of certain proper subsystem subgroup by a bounded number of elementary root elements. The bound is given explicitly. This result is an effective version of the early stabilisation of the corresponding K_1-functor. We also give a part of the proof of similar hypothesis for E_8.

Keywords Chevalley groups · Surjective stability of K_1 · Bounded reduction · Polynomial rings

Mathematics Subject Classification 19B14 · 19B10 · 20G35 · 20G41

1 Introduction

This paper deals with the bounded reduction in exceptional Chevalley groups over polynomial rings.

Chevalley groups over certain rings have bounded width with respect to the elementary generators. For example this holds for Dedekind domains of arithmetic type, see [6, 7, 9, 10, 24, 29, 30, 50, 51]. Results on such *bounded generation* are of great value, for example they are connected to the *congruence subgroup property*, see [26, 31]; to the Margulis–Zimmer conjecture, see [38]; and have applications in studying strong boundedness, see [52–56]. However bounded generation occurs very rarely in the sense that classes of rings for which it is known to hold are pretty narrow. Never-

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theless, for some applications it is enough to have a weaker result such as: bounded length of conjugates of elementary generators (see [47]), bounded length of commutators (see [17, 39, 46]), or bounded generation with respect to a larger set of generators. Bounded reduction is a variation of the last property.

A given Chevalley group G over a given ring is said to have bounded reduction if any element of G can be decomposed as a product of bounded number of elementary generators and one (not necessarily elementary) element from a certain subsystem subgroup. In other words, it means that one can reduce any element to the subsystem subgroup by bounded number of elementary transformations. Without requirement for the number of elementary transformations to be bounded this property is called the surjective stability of the K_1-functor. In papers [15, 32–34, 43, 44] this problem is considered for rings that satisfy certain conditions on stable rank, absolute stable rank, or other similar conditions. Actually, from the proofs of the theorems in these papers one can recover the bound on the required number of elementary transformations, despite the fact that this bound is not stated in papers explicitly. Therefore, these are results on bounded reduction.

However, conditions on stable rank are still very strong. Even though small Jacobson dimension implies small stable rank, rings with large Jacobson dimension usually fail to have small stable rank. In the present paper, we consider another important class of rings. Namely we take a polynomial ring in arbitrary number of variables with coefficients in a small-dimensional ring. Here we use Krull dimension because the techniques require for dimension to behave well with respect to adding an independent variable.

Without the bound on the number of elementary transformations similar result for classical groups is known as early surjective stability of the K_1-functor. For the special linear group this was proved by Suslin in [48]. Similar result for the orthogonal group follows from [49], and for the symplectic group it is proven in [20], see also [14, 21]. Note that if the ring of coefficient is a Dedekind domain or a smooth algebra over a field, then this result for all Chevalley groups follows from the homotopy invariance of the non-stable K_1-functor, see [1, 41, 42].

In the case of special linear and symplectic groups, there are similar results for Laurent polynomial rings, see [22, 23].

In the paper [58], Vaserstein obtained the effective version of the Suslin result, i.e. he proved the bounded reduction for the special linear group over a polynomial ring, and gave this bound explicitly. From this result he deduced that the elementary subgroup of the general linear group over an arbitrary finitely generated commutative ring has Kazhdan's property (T).

In [36], the basic connection between bounded generation and property (T) has been established and used to estimate the Kazhdan Constants for $\mathrm{SL}_n(\mathbb{Z})$. Later the bounds for these constants were improved in [19]. In order to deduce property (T) from the Vaserstein result one needs to refer to [37].

In fact, property (T) for Chevalley groups and groups similar to them has already been studied by other methods, see [11, 12]. However, we believe that the bounded reduction has an independent value, and we aim to study this question for other Chevalley groups. It was noted in the concluding remarks of [58] that the bounded reduction for the symplectic group follows formally from the case of special linear group. In

[16] the bounded reduction for orthogonal groups was established, therefore, closing the problem for classical groups.

In the present paper we deal with exceptional groups. We prove bounded reduction for the groups of types E_6 and E_7; and we give part of the proof for the group E_8.

The main result of the present paper is the following theorem.

Theorem 1.1 *Let C be a commutative Noetherian ring and* $\dim C = D$. *Let* $A = C[x_1, \dots, x_n]$. *Let* $\Delta \leqslant \Phi$ *be one of the following embeddings of root systems:*

(a) $D_5 \leqslant E_6$;
(b) $E_6 \leqslant E_7$.

Assume that

$$D \leqslant \begin{cases} 3 & \text{for } D_5 \leqslant E_6, \\ 4 & \text{for } E_6 \leqslant E_7. \end{cases}$$

For the case $E_6 \leqslant E_7$ assume additionally that C is a Jacobson ring.

Then every element of the group $G(\Phi, A)$ can be reduced to the subgroup $G(\Delta, A)$ by multiplication from the left by N elementary root elements, where

$$N = \begin{cases} 36n^2 + (72D + 80)n + 92 & \text{for } D_5 \leqslant E_6, \\ 52n^2 + (104D + 249)n + 244 & \text{for } E_6 \leqslant E_7. \end{cases}$$

Therefore, this theorem is an extension of [16, 58] to the groups of types E_6 and E_7.

The paper is organised as follows. In Sect. 2, we give all necessary preliminaries and introduce basic notation. In Sect. 3, we recall the notion of an absolute flexible stable rank introduced in [16]. In Sects. 4, 5, 6, and 7 we give the proof of the main result.

2 Preliminaries and notation

2.1 Rings, ideals and dimensions

By a ring we always mean associative and commutative ring with unity.

If R is a ring, then by R^* we denote the set of invertible elements in R. For the elements $r_1, \dots, r_k \in R$, we denote by $\langle r_1, \dots, r_k \rangle$ the ideal in R generated by these elements.

In the present paper we use three different notions of a ring dimension.

- By $\dim R$ we denote Krull dimension of the ring R. That is the supremum of the lengths of all chains of prime ideals.
- By $\operatorname{Jdim} R = \dim \operatorname{Max}(R)$ we denote the dimension of the maximal spectrum $\operatorname{Max}(R)$ of the ring R. It is equal to the supremum of the lengths of all chains of such prime ideals that coincide with its Jacobson radical.

- By BSdim R we denote the Bass–Serre dimension of a ring R. That is the minimal δ such that $\mathrm{Max}(R)$ is a finite union of irreducible Noetherian subspaces of dimension not greater than δ.

Obviously, for a Noetherian ring R we have

$$\mathrm{BSdim}\, R \leqslant \mathrm{Jdim}\, R \leqslant \dim R.$$

The following property of Bass–Serre dimension is well known; see [2, Lemma 4.17].

Lemma 2.1 *Let R be a ring with $\mathrm{BSdim}\, R = d < \infty$. Then it has a finite collection $P_1, \ldots, P_m$ of maximal ideals such that for any $s \in R \setminus \bigcup_i P_i$ we have $\mathrm{BSdim}\, R/(s) < d$. In case where $d = 0$, this means that $s \in R^*$.*

2.2 Chevalley groups

Let Φ be a reduced irreducible root system, let $G(\Phi, -)$, be a simply connected Chevalley–Demazure group scheme over $\mathbb{Z}$ of type Φ (see [8]), and let $T(\Phi, -)$, be a split maximal torus in it. If R is a commutative ring with unit, the group $G(\Phi, R)$ is called the simply connected Chevalley group of type Φ over R.

For a subset X of a group we denote by $\langle X \rangle$ the subgroup generated by X.

To each root $\alpha \in \Phi$ there correspond root unipotent elements $x_\alpha(\xi)$, $\xi \in R$, elementary with respect to T. The group generated by all these elements

$$E(\Phi, R) = \langle x_\alpha(\xi) : \alpha \in \Phi,\ \xi \in R \rangle \leqslant G(\Phi, R)$$

is called the elementary subgroup of $G(\Phi, R)$. For any $N \in \mathbb{N}$ we denote by $E(\Phi, R)^{\leqslant N}$ the subset of $E(\Phi, R)$ consisting of elements that can be expressed as the product of no more than N elementary root elements.

Any inclusion of root systems $\Delta \subseteq \Phi$ induces the homomorphisms $G(\Delta, R) \to G(\Phi, R)$, and $E(\Delta, R) \to E(\Phi, R)$ taking elementary root elements to elementary root elements.

By $U = U(\Phi, R)$ we denote the subgroup of $E(\Phi, R)$ generated by elementary root elements with positive roots, i.e. the unipotent radical of the standard Borel subgroup.

2.3 Basic representations and weight diagrams

Let us fix an order on Φ, and let Φ^+, Φ^- and $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ be the sets of positive, negative, and fundamental roots, respectively. Our numbering of the fundamental roots follows that of [5, 35]. By $\varpi_1, \ldots, \varpi_l$ one denotes the corresponding fundamental weights. Let $W = W(\Phi)$ be the Weyl group of the root system Φ.

Recall that an irreducible representation π of the complex semisimple Lie algebra L is called *basic* (see [28]) if the Weyl group $W = W(\Phi)$ acts transitively on the set $\Lambda^*(\pi)$ of nonzero weights of the representation π. The set of all weights of the representation π we denote by $\Lambda(\pi)$.

In case, where zero is not a weight of the basic representation π, such representation is called *microweight* or *minuscule* representation, and the list of these representations is classically known (see [4]).

Let L be a simple Lie algebra of type Φ. To each complex representation π of the algebra L there corresponds a representation π of the Chevalley group $G = G(\Phi, R)$ on the free R-module $V_\pi = V_\pi(R) = V_\pi(\mathbb{Z}) \otimes_{\mathbb{Z}} R$ (see [28, 45]). If π is faithful, then we can identify G with its image under this representation. Thus, for $g \in G$ and $v \in V$ we write gv for the action of g on v.

Decompose the module $V = V_\pi$ into the direct sum of its weight submodules

$$V = V^0 \oplus \bigoplus_{\lambda \in \Lambda^*(\pi)} V^\lambda.$$

Here all the modules V^λ, $\lambda \in \Lambda^*(\pi)$, are one-dimensional. Matsumoto [28, Lemma 2.3] has shown that there is a special base of weight vectors $e_\lambda \in V^\lambda$, $\lambda \in \Lambda^*(\pi)$, $v^0_\alpha \in V^0$, $\alpha \in \Delta(\pi) = \Pi \cap \Lambda^*(\pi)$ such that the action of root unipotents $x_\alpha(\xi)$, $\alpha \in \Phi$, $\xi \in R$, is described by the following simple formulas:

i. if $\lambda \in \Lambda^*(\pi), \lambda + \alpha \notin \Lambda(\pi)$, then $x_\alpha(\xi) e^\lambda = e^\lambda$,
ii. if $\lambda, \lambda + \alpha \in \Lambda^*(\pi)$, then $x_\alpha(\xi) e^\lambda = e^\lambda \pm \xi e^{\lambda+\alpha}$,
iii. if $\alpha \notin \Lambda^*(\pi)$, then $x_\alpha(\xi) v^0 = v^0$, for any $v^0 \in V^0$,
iv. if $\alpha \in \Lambda^*(\pi)$, then $x_\alpha(\xi) v^{-\alpha} = v^{-\alpha} \pm \xi v^0(\alpha) \pm \xi^2 v^\alpha$, $x_\alpha(\xi) v^0 = v^0 \pm \xi \alpha_*(v^0) v^\alpha$.

Weight diagram of the representation π is a graph whose vertices correspond to the elements of $\Lambda^*(\pi) \sqcup \Delta(\pi)$; and whose edges labeled by the numbers of fundamental roots show the action of the corresponding elementary root elements on the weight basis. These diagrams serve as a great visual aid for calculations in Chevalley groups. The details of how to construct and operate with weight diagrams can be found in [35] (see also [34, 43, 59]).

For a fundamental weight ϖ, we may consider the basic representation with the highest weight ϖ. For simplicity, we call it the representation ϖ.

Recall that in the present paper we study bounded reduction for the following embedding of root systems: $D_5 \leqslant E_6$, $E_6 \leqslant E_7$, and $E_7 \leqslant E_8$. We denote by Φ the bigger system, and by Δ the smaller one. For the rest of the paper we fix the following representation ϖ of the group $G(\Phi, R)$:

(a) $\varpi = \varpi_1$ for $D_5 \leqslant E_6$;
(b) $\varpi = \varpi_7$ for $E_6 \leqslant E_7$;
(c) $\varpi = \varpi_8$ for $E_7 \leqslant E_8$.

By $\lambda_1, \dots, \lambda_{\dim \varpi}$ we denote the weights of this representation with multiplicities, where numbering of weights for (E_6, ϖ_1) and (E_7, ϖ_7) follows that of [15] and [34]. We do not need to fix a numbering for (E_8, ϖ_8), but we agree that the highest weight has number 1, and the lowest weight has number -1.

By $e_1, \dots, e_{\dim \varpi}$ we denote the corresponding weight basis. Thus λ_1 is the highest weight and e_1 is the highest weight vector. For $b \in V_\varpi$ we denote by b_i or b_{λ_i} the corresponding coordinate of b in the basis $e_1, \dots, e_{\dim \varpi}$. We will identify b with the column vector with entries b_i.

By $V_\varpi = V_\varpi(R)$ we denote the underlying module of the representation ϖ. By $\mathrm{Um}_\varpi R$ we denote the set of unimodular vectors in $V_\varpi(R)$, i.e., the set of such vectors $b \in V_\varpi(R)$ that the elements b_i generate the unit ideal in R. By Eq_ϖ we denote the set of equations that determine the orbit of the highest weight vector of the representation ϖ (see [25, 59]). By $\mathrm{Orb}_\varpi R$ we denote the set of vectors from $V_\varpi(R)$ that satisfy the equations from Eq_ϖ. Further set $\mathrm{Um}'_\varpi R = \mathrm{Um}_\varpi R \cap \mathrm{Orb}_\varpi R$; and $\mathrm{Um}''_\varpi R = G(\Phi, R)e_1$.

Let $\Sigma_1 \leqslant \Phi$ be the set roots that have positive coefficient in simple root α_i, $i = 1$ for E_6 and $i = 7$ for E_7, and $i = 8$ for E_8. Therefore, $\Delta \cup \Sigma_1$ is a parabolic set of roots with Δ being the symmetric part, and Σ_1 being the special part. Let U_1 be the unipotent radical of the corresponding parabolic subgroup, and U_1^- be the unipotent radical of the opposite parabolic subgroup.

The following lemmas can be derived from the proof of the Chevalley–Matsumoto decomposition theorem (see [8, 28, 43]).

Lemma 2.2 *Let $b \in \mathrm{Um}'R$ be such that $b_1 = 1$. Then there exists $u \in U_1^-$ such that $b = ue_1$.*

Lemma 2.3 *Let $g \in G(\Phi, R)$ be such that $(ge_1)_1 = 1$. Then*

$$g \in U_1^- \cdot G(\Delta, R) \cdot U_1 = U_1^- U_1 G(\Delta, R).$$

We also need the following lemma.

Lemma 2.4 *If the ring R is semilocal, then* $\mathrm{Um}'R = \mathrm{Um}''R$.

Proof Let $b \in \mathrm{Um}'R$. By Lemma 2.2, it is enough to prove that there exists $g \in G(\Phi, R)$ such that $(gb)_1$ is invertible (we may then make it 1 by a toric element). Let J be the Jacobson radical of the ring R. The reduction map $E(\Phi, R) \to E(\Phi, R/J)$ is surjective; hence it is enough to find $g \in E(\Phi, R/J)$ such that $(g\overline{b})_1$ is invertible in R/J, where $\overline{b}$ is the reduction of b (in fact, we have $G = E$ for both R and R/J). So we may assume that $J = 0$, so R is a product of fields. Moreover, we can look for such g separately for each factor, so we may assume that R is a field. If b has at least one nonzero entry in a position that corresponds to a nonzero weight, then we can take g to be the element of the extended Weyl group that shifts this weight to the highest weight. That concludes the proof for E_6, ϖ_1 and E_7, ϖ_7 because these representations are minuscule. It remains to consider the case for E_8, ϖ_8, where $b \in V^0$. It follows easily from the fact that the lattice E_8 is self-dual that we have $x_{\alpha_i}(1)b \notin V^0$ for at least one simple root α_i; so the problem is reduced to the previous case. □

2.4 Branching tables

From the weight diagram it is immediate to read off the branching of the corresponding representation with respect to a subsystem subgroup. In the case where $\Delta = \langle \Pi \setminus \{\alpha_h\} \rangle$

is the symmetric part of the maximal parabolic subset obtained by dropping the h-th fundamental root the procedure is particularly easy. Then the restriction of π to $G(\Phi, R)$ looks as follows: one has to cut the diagram of π through the bonds with the label h.

Given a representation π of the group $G(\Phi, R)$ and two fundamental roots $\alpha_{h_1}, \alpha_{h_2} \in \Pi$, by "branching table, where vertical lines correspond to cutting through the bonds marked with h_1, and horizontal lines correspond to cutting through the bonds marked with h_2", we mean the table build as follows: at the upper right corner we write the representation π; at the remaining cells of the upper row we write the components of restriction of π to the group $G(\langle\Pi \setminus \{\alpha_{h_1}\}\rangle, -)$; at the remaining cells of the left column we write the components of restriction of π to the group $G(\langle\Pi \setminus \{\alpha_{h_2}\}\rangle, -)$; and in all the remaining cells we write the intersection of the corresponding restrictions. When this intersection is zero we leave the cell blank; and when the intersection or a component is one-dimensional, we denote it by $\circ$, which refers to the node of the weight diagram. The columns of the table, except the left one, are denoted by bold letters, and the rows except the upper one, are denoted by bold numbers.

Here is the example: the branching table for (E_6, ϖ_1), where vertical lines correspond to cutting through the bonds marked with 1, and horizontal lines correspond to cutting through the bonds marked with 6.

		a	**b**	**c**
	E_6, ϖ_1	$\circ$	D_5, ϖ_5	D_5, ϖ_1
1)	D_5, ϖ_1	$\circ$	D_4, ϖ_1	$\circ$
2)	D_5, ϖ_5		D_4, ϖ_4	D_4, ϖ_3
3)	$\circ$			$\circ$

2.5 ASR-condition

Recall that a commutative ring R satisfies the absolute stable rank condition ASR_d if for any row $(b_1, \ldots, b_d)$ with coordinates in R, there exist elements $c_1, \ldots, c_{d-1} \in R$ such that every maximal ideal of R containing the ideal $\langle b_1+c_1b_d, \ldots, b_{d-1}+c_{d-1}b_d\rangle$ contains already the ideal $\langle b_1, \ldots, b_d\rangle$. This notion was introduced in [13] and used in [43, 44] and then in [15, 32–34] to study stability problems.

If we assume that a row $(b_1, \ldots, b_d)$ is unimodular, then the absolute stable rank condition boils down to the usual stable rank condition SR_d (see [3, 57]).

Absolute stable rank satisfies the usual properties, namely for every ideal $I \trianglelefteq R$ the condition ASR_d for R implies ASR_d for the quotient R/I, and if $d \geqslant d'$, then $\mathrm{ASR}_{d'}$ implies ASR_d. Finally, it is well known that if the maximal spectrum of R is a Noetherian space of dimension $\mathrm{Jdim}\, R = d-2$, then both conditions ASR_d and SR_d are satisfied (see [13, 27, 43]).

3 Absolute flexible stable rank

In this section, we recall the definition and the basic properties of *absolute flexible stable rank* introduced in [16]. Here is the definition.

Definition 3.1 A commutative ring A satisfies the *absolute flexible stable rank condition* AFSR_d if for any row $(b_1, \ldots, b_d)$ with coordinates in A, there exists an element $c_1 \in A$ such that for any invertible element $\varepsilon_1 \in A^*$, there exists $c_2 \in A$ such that for any $\varepsilon_2 \in A^*$, ..., there exists $c_{d-1} \in A$ such that for any $\varepsilon_{d-1} \in A^*$, every maximal ideal of A containing the ideal $\langle b_1 + \varepsilon_1 c_1 b_d, \ldots, b_{d-1} + \varepsilon_{d-1} c_{d-1} b_d \rangle$ contains already the ideal $\langle b_1, \ldots, b_d \rangle$.

One can think of it as follows. Two players are playing a game. Player 1 chooses a row $(b_1, \ldots, b_d)$ with coordinates in A. Then they take turns starting with Player 2. Player 2 in his i-th turn chooses an element $c_i \in A$; after that Player 1 in his turn chooses an invertible element $\varepsilon_i \in A^*$. Player 2 wins if after d turns every maximal ideal of A containing the ideal $\langle b_1 + \varepsilon_1 c_1 b_d, \ldots, b_{d-1} + \varepsilon_{d-1} c_{d-1} b_d \rangle$ contains already the ideal $\langle b_1, \ldots, b_d \rangle$. A commutative ring A satisfies the absolute flexible stable rank condition AFSR_d if Player 2 has a winning strategy.

The following lemma shows that the condition AFSR_d holds for small-dimensional rings. That generalises the result of [13].

Lemma 3.2 ([16, Lemma 3.2]) *Let A be a commutative ring. Assume that* $\mathrm{Max}(A)$ *is Noetherian and* $\mathrm{Jdim}\, A \leqslant d - 2$. *Then A satisfies* AFSR_d.

Now the following lemma shows how one can use the AFSR condition.

Lemma 3.3 ([16, Lemma 3.3]) *Let A be a commutative ring, and S be a multiplicative system in A. Assume that the localisation $A[S^{-1}]$ satisfies* AFSR_d. *Then for any row $(b_1, \ldots, b_d)$ with coordinates in $A[S^{-1}]$ and for any $s \in S$, there exist $c_1, \ldots, c_{d-1} \in sA$ such that every maximal ideal of $A[S^{-1}]$ containing the ideal $\langle b_1 + c_1 b_d, \ldots, b_{d-1} + c_{d-1} b_d \rangle$ contains already the ideal $\langle b_1, \ldots, b_d \rangle$.*

4 Reduction of Theorem 1.1 to Propositions 4.2, 4.3 and 4.4

In this section, we divide the proof of Theorem 1.1 into three steps. One of the steps, namely Proposition 4.4, will be formulated and then proved also for the case $E_7 \leqslant E_8$, so that if proofs of Propositions 4.2 and 4.3 are found for this case, it will finish the proof of bounded reduction for Chevalley groups of type E.

Recall that by U_1 we denote the unipotent radical that corresponds to the set $\Sigma_1 \leqslant \Phi$, which is the special part of the parabolic subset of roots $\Delta \cup \Sigma_1$.

Note that

$$|\Sigma_1| = \begin{cases} 16 & \text{for } D_5 \leqslant E_6, \\ 27 & \text{for } E_6 \leqslant E_7. \end{cases}$$

Therefore, Theorem 1.1 follows trivially from the following result and Lemma 2.3.

Theorem 4.1 *Under the condition of Theorem 1.1, for every column $b \in \mathrm{Um}'_{\varpi} A$ there exists a column*

$$b' \in E(\Phi, A)^{\leqslant N} b,$$

where

$$N = \begin{cases} 36n^2 + (72D + 80)n + 60 & \text{for } D_5 \leqslant E_6, \\ 52n^2 + (104D + 249)n + 190 & \text{for } E_6 \leqslant E_7, \end{cases}$$

such that $b'_1 = 1$.

Consider the lexicographic order on the monomials in variables $x_1, \dots, x_n$. That is the order where $x_1^{k_1} \dots x_n^{k_n}$ is bigger than $x_1^{l_1} \dots x_n^{l_n}$ if for some m we have $k_i = l_i$ for $i < m$, and $k_m > l_m$. A polynomial in $A = C[x_1, \dots, x_n]$ is called lexicographically monic if its leading coefficient in lexicographic order is equal to one.

Further we reduce Theorem 4.1 to the following three propositions.

Proposition 4.2 *Under the condition of Theorem 1.1, assuming $n = 0$ (i.e. $A = C$), for every column $b \in \mathrm{Um}'_{\varpi} A$ there exists a column*

$$b' \in E(\Phi, A)^{\leqslant N} b,$$

where

$$N = \begin{cases} 60 & \text{for } D_5 \leqslant E_6, \\ 190 & \text{for } E_6 \leqslant E_7, \end{cases}$$

such that $b'_1 = 1$.

Proposition 4.3 *Let j be a number of a vertex on a weight diagram of the representation ϖ. Under the condition of Theorem 1.1, for every column $b \in \mathrm{Um}'_{\varpi} A$ there exists a column*

$$b' \in E(\Phi, A)^{\leqslant N} b,$$

where

$$N = \begin{cases} 116 & \text{for } D_5 \leqslant E_6, \\ 301 & \text{for } E_6 \leqslant E_7, \end{cases}$$

such that its entry b'_j is lexicographically monic.

We state and prove the third proposition also for the case $E_7 \leqslant E_8$.

Proposition 4.4 *Let B be a commutative ring such that* $\mathrm{BSdim}\, B = d < \infty$. *Let* $A = B[y]$, $b \in \mathrm{Um}'_{\varpi} A$ *be such that its entry b_j is monic, where $j = 24$ for E_6, $j = -1$ for E_7 and E_8. Then*

$$E(\Phi, A)^{\leqslant N} b \cap \mathrm{Um}_{\varpi}(B) \neq \varnothing,$$

where

$$N = \begin{cases} 72d & \text{for } D_5 \leqslant E_6, \\ 104d & \text{for } E_6 \leqslant E_7, \\ 291d & \text{for } E_7 \leqslant E_8. \end{cases}$$

First we need the following lemma.

Lemma 4.5 *Let $f \in C[x_1, \dots, x_n]$ be a lexicographically monic polynomial. Then there exists an invertible change of variables*

$$x_1, \dots, x_n \leftrightarrow y_1, \dots, y_n,$$

such that f becomes monic in y_n.

Proof Take $K > \deg f$. Set $x_i = y_i + y_n^{K^{n-i}}$, $i = 1, \dots, n-1$, and $x_n = y_n$. □

Now we deduce Theorem 4.1 from Propositions 4.2, 4.3 and 4.4. Take $b \in \mathrm{Um}'_{\varpi} A$. By Proposition 4.3 there exists a column

$$b' \in E(\Phi, A)^{\leqslant N'_1} b,$$

where

$$N'_1 = \begin{cases} 116 & \text{for } D_5 \leqslant E_6, \\ 301 & \text{for } E_6 \leqslant E_7, \end{cases}$$

such that its entry b'_j is lexicographically monic, where j is as in Proposition 4.4. Applying Lemma 4.5, we change variables to $y_1, \dots, y_n$ so that b'_j is now monic in y_n. Now we apply Proposition 4.4 to $B = C[y_1, \dots, y_{n-1}]$. Note that $\mathrm{BSdim}\, B \leqslant \dim B = D + n - 1$. Hence we can obtain a column from

$$E(\Phi, A)^{\leqslant N''_1} b' \cap \mathrm{Um}_{\varpi} B \leqslant E(\Phi, A)^{\leqslant N_1} b \cap \mathrm{Um}_{\varpi} B,$$

where

$$N''_1 = \begin{cases} 72(D + n - 1) & \text{for } D_5 \leqslant E_6, \\ 104(D + n - 1) & \text{for } E_6 \leqslant E_7, \end{cases}$$

and $N_1 = N'_1 + N''_1$.

Repeating this argument n times we can obtain a column from

$$E(\Phi, A)^{\leqslant N_n} b \cap \mathrm{Um}_{\varpi}\, C,$$

where

$$N_n = \begin{cases} 72n(2D+n-1)/2+116n & \text{for } D_5 \leqslant E_6, \\ 104n(2D+n-1)/2+301n & \text{for } E_6 \leqslant E_7. \end{cases}$$

Now Proposition 4.2 implies that there exists

$$b'' \in E(\Phi, A)^{\leqslant N} b,$$

where

$$N = \begin{cases} 72n(2D+n-1)/2+116n+60 & \text{for } D_5 \leqslant E_6, \\ 104n(2D+n-1)/2+301n+190 & \text{for } E_6 \leqslant E_7, \end{cases}$$

such that $b''_1 = 1$.

5 Bounded reduction for low-dimensional rings

In this section, we prove Proposition 4.2. Note that the case E_6 easily follows from the proof of [15, Lemma 2]. Similarly, the proof for the case for the case E_7 can be obtained from the proof of the main theorem in [34]. In order to do so, we must estimate how many elementary root elements it takes to apply [34, Lemma 2]. That proof starts with picking an element $e \in E(E_6, R)$ such that $(ae)_{\lambda_1} \equiv 1 \bmod \mathfrak{u}$ and $(ae)_{\lambda_i} \equiv 0 \bmod \mathfrak{u}$ for $i \neq 1$. Note that it is enough to require $(ae)_{\lambda_1}$ to be invertible modulo $\mathfrak{u}$ and not necessarily congruent to 1. Therefore, the element e can be taken from XU_1 where

$$X = \big\{x_{-\delta_{E_6}}(\xi_1)\, x_{-\delta_{A_5}}(\xi_2)\, x_{-\delta_{D_5(6)}}(\xi_3)\, x_{-\alpha_1}(\xi_4) x_{-\delta_{D_5(1)}}(\xi_5)\, x_{-\alpha_2-\alpha_3-\alpha_4}(\xi_6) \\ x_{-\alpha_2}(\xi_7)\, x_{-\alpha_3}(\xi_8) x_{-\alpha_4}(\xi_9)\, x_{-\alpha_5}(\xi_{10})\, x_{-\alpha_6}(\xi_{11}) : \xi_i \in R\big\},$$

where δ_{E_6} is the maximal root of the system E_6; δ_{A_5} is the maximal root of the system generated by $\alpha_1, \alpha_3, \alpha_4, \alpha_5$ and α_6; $\delta_{D_5(6)}$ is the maximal root of the system generated by $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and α_5; $\delta_{D_5(1)}$ is the maximal root of the system generated by $\alpha_2, \alpha_3, \alpha_4, \alpha_5$ and α_6.

The next step in the proof uses the element $e_1 \in U_1$, hence $ee_1 \in XU_1 \leqslant E^{\leqslant 27}(E_6, R)$. Now it is easy to count that the proof of Lemma 2 takes 67 elementary root elements, and the whole proof of the main theorem in [34] takes 190 elementary root elements.

6 Obtaining a monic polynomial

In this section, we give the proof of Proposition 4.3. First we need some preparation. The following lemma was proved in [16].

Lemma 6.1 ([16, Lemma 5.2]) *Let C be a Noetherian ring, $A = C[x_1, \dots, x_n]$. Let S be a multiplicative system of lexicographically monic polynomials in A. Then we have* $\dim A[S^{-1}] \leqslant \dim C$.

Now we recall a definition from [58].

Definition 6.2 Let A be an associative ring with 1, s be a central element of A, $l \geqslant 2$, $v \in A^{l-1}$ (a column over A), $u \in {}^{l-1}A$ (a row over A). We define an l by l matrix over A by

$$\mu(u, s, v) = \begin{pmatrix} 1_{l-1} + vsu & vs^2 \\ -uvu & 1 - uvs \end{pmatrix}.$$

This matrix is invertible with $\mu(u, s, v)^{-1} = \mu(u, s, -v)$. If $s \in A^*$, then

$$\mu(u, s, v) = \begin{pmatrix} 1_{l-1} & 0 \\ -u/s & 1 \end{pmatrix} \begin{pmatrix} 1_{l-1} & vs^2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1_{l-1} & 0 \\ u/s & 1 \end{pmatrix}.$$

The following lemma was proved in [58].

Lemma 6.3 ([58, Lemma 2.2]) *When $l \geqslant 3$, the matrix $\mu(u, s, v)$ is a product of $7l-3$ elementary transvections in* $\mathrm{GL}(l, R)$.

Now let S be a multiplicative system of lexicographically monic polynomials in A. It follows from Lemmas 6.1 and 3.2 that the ring $A[S^{-1}]$ satisfies AFSR_5 for the case E_6 resp. AFSR_6 for the case E_7, and so does any quotient of $A[S^{-1}]$.

Lemma 6.4 *Let $l \geqslant 3$, let $\mathfrak{I}$ be an ideal in A, and suppose that* $\dim \operatorname{Max} A/\mathfrak{I}[S^{-1}] \leqslant l-2$. *Let $b \in V_{D_l, \varpi_1} A$ be such that it becomes unimodular in $A/\mathfrak{I}[S^{-1}]$. Then there exists a column*

$$b' \in E(D_l, A)^{\leqslant 11l-7} b,$$

such that b'_1 is congruent to a lexicographically monic polynomial modulo $\mathfrak{I}$.

Proof We perform the following steps (Fig. 1).

Step 1. Make the row $(b_2, \dots, b_{-1})$ unimodular in $A/\mathfrak{I}[S^{-1}]$ by $l-1$ elementary elements.

Let $\mathfrak{A} = \mathfrak{I} + \langle b_{-l}, \dots, b_{-1} \rangle \trianglelefteq A[S^{-1}]$. Since $A[S^{-1}]/\mathfrak{A}$ satisfies AFSR_l and the row $(b_1, \dots, b_l)$ is unimodular in $A[S^{-1}]/\mathfrak{A}$, it follows from Lemma 3.3 that there exist $c_2, \dots, c_l \in A$ such that the row $(b_2 + c_2 b_1, \dots, b_l + c_l b_1)$ is unimodular in $A[S^{-1}]/\mathfrak{A}$. Thus by applying the elements $x_{-\alpha_1 - \dots - \alpha_{i-1}}(\pm c_i)$ for $i = 2, \dots, l$, we

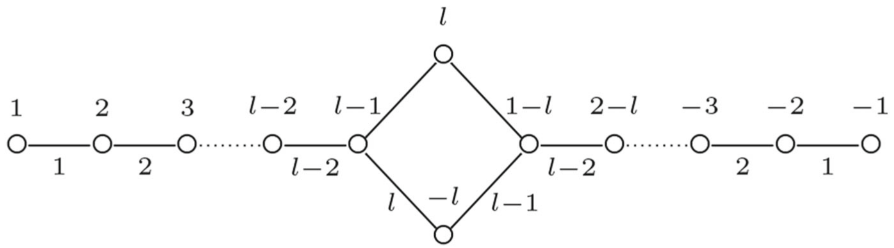

Fig. 1 (D_l, ϖ_1)

make the row $(b_2, \dots, b_l)$ unimodular in $A[S^{-1}]/\mathfrak{A}$ without changing the ideal $\mathfrak{A}$. Thus the row $(b_2, \dots, b_{-1})$ becomes unimodular in $A/\mathfrak{I}[S^{-1}]$.

Step 2. Make the row $(b_1, b_{-l} \dots, b_{-1})$ unimodular in $A/\mathfrak{I}[S^{-1}]$ by $l-1$ elementary elements.

Since the row $(b_2, \dots, b_{-1})$ is unimodular in $A/\mathfrak{I}[S^{-1}]$, it follows that the ideal generated by $\mathfrak{I}$ and $(b_2, \dots, b_{-1})$ in A contains a lexicographically monic polynomial. So for some $f_2, \dots, f_{-1} \in A$, and $f \in \mathfrak{I}$ the polynomial

$$f + \sum_{i=2}^{-1} f_i b_i$$

is lexicographically monic. Multiplying polynomials f and f_i by a large enough power of x_1, we may assume that the polynomial

$$b_1 + f + \sum_{i=2}^{-1} f_i b_i$$

is also lexicographically monic.

Let us now apply the elements $x_{\alpha_1+\dots+\alpha_{i-1}}(f_i)$ for $i = 2, \dots, l$. Then the ideal generated by $\mathfrak{I}$, the new b_1, and old $b_{-l}, \dots, b_{-1}$ contains a lexicographically monic polynomial. However, these elements do not change the ideal generated by $b_{-l}, \dots$, b_{-1}. Hence we actually achieve that the ideal generated by $\mathfrak{I}$, and new $b_1, b_{-l}, \dots, b_{-1}$ contains a lexicographically monic polynomial. Thus the row $(b_1, b_{-l}, \dots, b_{-1})$ becomes unimodular in $A/\mathfrak{I}[S^{-1}]$.

Step 3. Make the row $(b_1, b_{-l} \dots, b_{-2})$ unimodular in $A[S^{-1}]$ by $7l-3$ elementary elements.

Let $\mathfrak{A} = \mathfrak{I} + \langle b_{-l}, \dots, b_{-1}\rangle \trianglelefteq A[S^{-1}]$. Since b_1 is invertible in $A[S^{-1}]/\mathfrak{A}$, it follows that there exist $\xi_2, \dots, \xi_l \in A[S^{-1}]$ such that $b_i - \xi_i b_1 \in \mathfrak{A}$ for $i = 2, \dots, l$. Let s be a common denominator of ξ_i. Set

$$g_1 = \prod_{2 \leqslant i \leqslant l} x_{-\alpha_1-\dots-\alpha_{i-1}}(\pm\xi_i),$$

where signs are such that $(g_1 b)_i = b_i - \xi_i b_1 \in \mathfrak{A}$ for $1 \leqslant i \leqslant l$.

Since $A/\mathfrak{I}[S^{-1}]$ satisfies AFSR_l, it follows from Lemma 3.3 that there exist $c_{-l}, \dots, c_{-2} \in s^2A$ such that every maximal ideal of $A/\mathfrak{I}[S^{-1}]$ containing the ideal $\langle (g_1b)_{-l} + c_{-l}(g_1b)_{-1}, \dots, (g_1b)_{-2} + c_{-2}(g_1b)_{-1}\rangle$ contains already the ideal $\langle (g_1b)_{-l}, \dots, (g_1b)_{-1}\rangle = \mathfrak{A}$. Set

$$g_2 = \prod_{2 \leqslant i \leqslant l} x_{\alpha_1 + \dots + \alpha_{i-1}}(\pm c_{-i}),$$

where signs are such that $(g_2g_1b)_i = (g_1b)_i + c_ib_{-1}$ for $-l \leqslant i \leqslant -2$.

We claim that the elements $(g_2g_1b)_1, (g_2g_1b)_{-l}, \dots, (g_2g_1b)_{-2}$ generate the unit ideal in $A/\mathfrak{I}[S^{-1}]$. Let us prove that.

Assume that some maximal ideal $\mathfrak{M}$ of the ring $A[S^{-1}]$ contains $\mathfrak{I}$ and all the elements $(g_2g_1b)_1, (g_2g_1b)_{-l}, \dots, (g_2g_1b)_{-2}$.

Since applying g_1 does not change the ideal generated by $b_{-l}, \dots, b_{-1}$, by choice of c_i we have $\mathfrak{A} \leqslant \mathfrak{M}$. Hence $(g_1b)_i \in \mathfrak{M}$ for $2 \leqslant i \leqslant l$. Thus $b_1 = (g_2g_1b)_1 + \sum_{2\leqslant i\leqslant 7} \pm c_{-i}(g_1b)_i \in \mathfrak{M}$. However, by the previous step, b_1 and $\mathfrak{A}$ generate a unit ideal. This is a contradiction.

Since applying g_1^{-1} does not change the ideal generated by elements $b_1, b_{-l}, \dots$, b_{-2}, we obtain that the elements $(g_1^{-1}g_2g_1b)_i$, where $i = 1, -l, \dots, -2$, generate the unit ideal in $A[S^{-1}]$.

It remains to notice that the element $g_1^{-1}g_2g_1$ is the image of the matrix $\mu(u, s, v)$ for certain u and v under the embedding $G(A_{l-1}, A) \to G(D_l, A)$ as a subsystem subgroup. Therefore, by Lemma 6.3, $g_1^{-1}g_2g_1 \in E(D_l, A)^{\leqslant 7l-3}$.

Step 4. Make the row $(b_{-l} \dots, b_{-2})$ unimodular in $A/\mathfrak{I}[S^{-1}]$ by $l-1$ elementary elements.

Since $A/\mathfrak{I}[S^{-1}]$ satisfies AFSR_l and the row $(b_1, b_{-l} \dots, b_{-2})$ is unimodular in $A/\mathfrak{I}[S^{-1}]$, it follows from Lemma 3.3 that there exist $c_{-l}, \dots, c_{-2} \in A$ such that the row $(b_{-l} + c_{-l}b_1, \dots, b_{-2} + c_{-2}b_1)$ is unimodular in $A/\mathfrak{I}[S^{-1}]$. Thus by applying the elements $x_{\alpha_2+\dots+\alpha_{i-1}-\delta}(\pm c_{-i})$ for $i = 2, \dots, l$, we make the row $(b_{-l} \dots, b_{-2})$ unimodular in $A/\mathfrak{I}[S^{-1}]$.

Step 5. Make b_1 congruent to a lexicographically monic polynomial modulo $\mathfrak{I}$ by $l-1$ elementary elements.

Since the row $(b_{-l} \dots, b_{-2})$ is unimodular in $A/\mathfrak{I}[S^{-1}]$, it follows that the ideal generated by $\mathfrak{I}$ and $(b_{-l}, \dots, b_{-2})$ in A contains a lexicographically monic polynomial. So for some $f_1, f_{-l}, \dots, f_{-3} \in A$ and $f \in \mathfrak{I}$, the polynomial

$$f + \sum_{-l\leqslant i\leqslant -2} f_ib_i$$

is lexicographically monic. Multiplying polynomials f_i and f by a large enough power of x_1, we may assume that the polynomial

$$b_{-2} + f + \sum_{-l\leqslant i\leqslant -2} f_ib_i$$

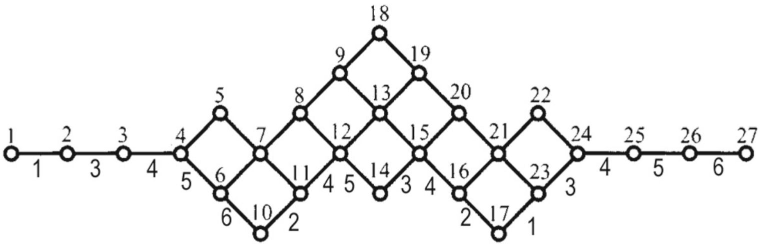

Fig. 2 (E_6, ϖ_1)

is also lexicographically monic.

Now applying the elements $x_{\delta-\alpha_2-\dots-\alpha_{i-1}}(\pm f_{-i})$ for $i = 2, \dots, l$, we achieve that b_1 is congruent to a lexicographically monic polynomial modulo $\mathfrak{I}$. □

Remark 6.5 One can notice that the proof above repeats the proof of [16, Proposition 4.2], which on its turn basically repeats the proof of stability theorem for K_1-functor given in [43].

Now we prove Proposition 4.3 for the case (E_6, ϖ_1).

Proof Consider the branching table for (E_6, ϖ_1), where vertical lines correspond to cutting through the bonds marked with 1, and horizontal lines correspond to cutting through the bonds marked with 6.

	E_6, ϖ_1	**a** ∘	**b** D_5, ϖ_5	**c** D_5, ϖ_1
1)	D_5, ϖ_1	∘	D_4, ϖ_1	∘
2)	D_5, ϖ_5		D_4, ϖ_4	D_4, ϖ_3
3)	∘			∘

Take $b \in \mathrm{Um}'_{(E_6,\varpi_1)} A$. We need to obtain a lexicographically monic polynomial by 116 elementary elements. Since the Weyl group acts transitively on weights, it does not matter in which position to obtain a lexicographically monic polynomial. Let us make it with b_1. We perform the following steps (Fig. 2).

Step 1. Make the row that consists of elements in all the cells except **a1** unimodular in $A[S^{-1}]$ by four elementary elements.

Let $\mathfrak{A} \trianglelefteq A[S^{-1}]$ be the ideal generated by all the elements b_i except for $b_1, \dots, b_4, b_6$. Since $A[S^{-1}]/\mathfrak{A}$ satisfies AFSR_5, and the row $(b_1, \dots, b_4, b_6)$ is unimodular in $A[S^{-1}]/\mathfrak{A}$, it follows from Lemma 3.3 that there exist $c_2, c_3, c_4, c_6 \in A$ such that the row $(b_2 + c_2b_1, \dots, b_6 + c_6b_1)$ is unimodular in $A[S^{-1}]/\mathfrak{A}$. Thus by applying the elements $x_{\alpha_1}(\pm c_2), \dots, x_{\alpha_1+\alpha_3+\alpha_4+\alpha_5}(\pm c_6)$, we make the row (b_2, b_3, b_4, b_6) unimodular in $A[S^{-1}]/\mathfrak{A}$ without changing the ideal $\mathfrak{A}$. Thus the row that consists of elements in all the cells except **a1** becomes unimodular in $A[S^{-1}]$.

Step 2. Make the row that consists of elements in columns **a** and **c** unimodular in $A[S^{-1}]$ by 16 elementary elements.

Since the row $(b_2, \dots, b_{27})$ is unimodular in $A[S^{-1}]$, it follows that the ideal generated by $(b_2, \dots, b_{27})$ in A contains a lexicographically monic polynomial. So for some $f_2, \dots, f_{27} \in A$, the polynomial

$$\sum_{i=2}^{27} f_i b_i$$

is lexicographically monic. Multiplying polynomials f_i by a large enough power of x_1, we may assume that the polynomial

$$b_1 + \sum_{i=2}^{27} f_i b_i$$

is also lexicographically monic.

Let us now apply the elements $x_{\lambda_1-\lambda_i}(\pm f_i)$ for all λ_i from the column **b**. Then the ideal generated by the new b_1 and old b_λ, where λ is from the column **c**, contains a lexicographically monic polynomial. However, these elements do not change the ideal generated by b_λ, where λ is from the column **c**. Hence we actually achieve that the ideal generated by new b_1, and b_λ, where λ is from the column **c** contains a lexicographically monic polynomial. Thus the row of elements in columns **a** and **c** becomes unimodular in $A[S^{-1}]$.

Step 3. Make the row that consists of elements in cells **a1** and **c1** unimodular in $A[S^{-1}]$ by 48 elementary elements.

Apply Lemma 6.4 to the column **c** and the ideal generated by b_1.

Step 4. Make the element b_1 lexicographically monic by 48 elementary elements.

Apply Lemma 6.4 to the row **1** and the zero ideal. □

Remark 6.6 One can notice that the proof above basically repeats the proof of stability theorem for K_1-functor given in [15].

Before we prove Proposition 4.3 for (E_7, ϖ_7), we need one more lemma.

Lemma 6.7 *Under the condition of Theorem 1.1 in case $E_6 \leqslant E_7$. Let $\mathfrak{I}$ be an ideal in A. Let $b \in \mathrm{Orb}_{E_6,\varpi_1} A/\mathfrak{I}$ be such that it becomes unimodular in $A/\mathfrak{I}[S^{-1}]$. Then there exists a column vector*

$$b' \in E(E_6, A)^{\leqslant 91} b,$$

such that the row (b'_1, b'_{18}) is unimodular in $A/\mathfrak{I}[S^{-1}]$.

Proof Let us choose in each irreducible component of $\mathrm{Max}(A/\mathfrak{I}[S^{-1}])$ a maximal ideal $\mathfrak{u}_i$, $i \in I$. Next denote by $\widetilde{\mathfrak{u}_i}$ the preimage of $\mathfrak{u}_i$ in A.

For each i choose a maximal ideal $\mathfrak{v}_i \in \mathrm{Max}\, A$ such that it contains $\widetilde{\mathfrak{u}_i}$, and b is unimodular in $A/\mathfrak{v}_i$. Let us show that we can do it. The column b is unimodular in $A/\mathfrak{I}[S^{-1}]$; hence the ideal in A generated by $\mathfrak{I}$ and entries of b contains a lexicographically monic polynomial f. Clearly, the ideal $\widetilde{u_i}$ is prime and $f \notin \widetilde{\mathfrak{u}_i}$. Since C is a

Jacobson ring and A is finitely generated over C, it follows that A is a Jacobson ring; hence there exists $\mathfrak{v}_i \in \operatorname{Max} A$ such that $\widetilde{\mathfrak{u}_i} \leqslant \mathfrak{v}_i$ and $f \notin \mathfrak{v}_i$. Then b is unimodular in $A/\mathfrak{v}_i$.

Set $I_1 = \{i \in I : \mathfrak{v}_i = \widetilde{\mathfrak{u}_i}\}$, and $I_2 = \{i \in I : \mathfrak{v}_i \neq \widetilde{\mathfrak{u}_i}\}$, so $I = I_1 \sqcup I_2$.

Now we perform the following steps.

Step 1. Achive that $b_1 \notin \bigcup_{i \in I} \mathfrak{v}_i$ by 11 elementary elements.

In other words, we should make b_1 invertible in $A/\big(\bigcap_{i \in I} \mathfrak{v}_i\big)$. The ring $A/\big(\bigcap_{i \in I} \mathfrak{v}_i\big)$ is semilocal, so we can do it in the same way we did in Sect. 5.

Step 2. Without changing b_1, make the row $(b_2, \dots, b_{27})$ is unimodular in $A/\mathfrak{I}[S^{-1}]$, and achive that elements $b_5, b_7, b_8, b_9, b_{11}, \dots, b_{27}$ belong to $\bigcap_{i \in I_1} \mathfrak{v}_i$ by element from U_1, i.e. by 16 elementary elements.

It follows from Lemma 2.2 that for some $b' = u_1 b$, where $u_1 \in U_1$, we have $b'_\lambda \in \bigcap_{i \in I_1} \mathfrak{v}_i$ for all $\lambda \neq \lambda_1$. Now let $\mathfrak{A} \trianglelefteq A[S^{-1}]$ be the ideal generated by $\mathfrak{I}$ and the elements $b'_5, b'_7, b'_8, b'_9, b'_{11}, \dots, b'_{27}$ except for $b'_1, \dots, b'_4, b'_6, b'_{10}$. Then $\mathfrak{A} \leqslant \bigcap_{i \in I_1} \mathfrak{v}_i$. Since $A[S^{-1}]/\mathfrak{A}$ satisfies AFSR_6, and the row $(b'_1, \dots, b'_4, b'_6, b'_{10})$ is unimodular in $A[S^{-1}]/\mathfrak{A}$, it follows from Lemma 3.3 that there exist $c_2, c_3, c_4, c_6, c_{10} \in A$ such that the row $(b'_2 + c_2 b'_1, \dots, b'_{10} + c_{10} b'_1)$ is unimodular in $A[S^{-1}]/\mathfrak{A}$. Thus by applying the elements $x_{\alpha_1}(\pm c_2), \dots, x_{\alpha_1+\alpha_3+\dots+\alpha_6}(\pm c_{10})$, we make the row $(b'_2, b'_3, b'_4, b'_6, b'_{10})$ unimodular in $A[S^{-1}]/\mathfrak{A}$ without changing the ideal $\mathfrak{A}$. Thus the row $(b'_2, \dots, b'_{27})$ becomes unimodular in $A/\mathfrak{I}[S^{-1}]$.

The composition of u_1 with the elements as above is then the required element of U_1.

Step 3. Preserving the fact that the image of b_1 in $A/\mathfrak{I}[S^{-1}]$ does not belong to $\bigcup_{i \in I} \mathfrak{u}_i$, make the row that consists of elements in columns **a** and **c** (we use the same branching table as in the proof above) unimodular in $A/\mathfrak{I}[S^{-1}]$ by 16 elementary elements.

Since the row $(b_2, \dots, b_{27})$ is unimodular in $A/\mathfrak{I}[S^{-1}]$, it follows that the ideal generated by $\mathfrak{I}$ and elements $b_2, \dots, b_{27}$ in A contains a lexicographically monic polynomial. So for some $f_2, \dots, f_{27} \in A$ and $f \in I$, the polynomial

$$f + \sum_{k=2}^{27} f_k b_k$$

is lexicographically monic. Clearly for any $i \in I_2$ the ideal $\mathfrak{v}_i$ contains some $h_i \in S$. Multiplying polynomials f and f_k by $\prod_{i \in I_2} h_i$ and then by a large enough power of x_1, we may assume that, firstly all the f_k belong to $\bigcap_{i \in I_2} \mathfrak{v}_i$, and secondly, that the polynomial

$$b_1 + f + \sum_{i=2}^{27} f_i b_i$$

is lexicographically monic.

Let us now apply the elements $x_{\lambda_1 - \lambda_i}(\pm f_i)$ for all λ_i from the column **b**. Clearly we preserve the fact that $b_1 \notin \bigcup_{i \in I_2} \mathfrak{v}_i$; hence we preserve the fact that the image of b_1 in $A/\mathfrak{I}[S^{-1}]$ does not belong to $\bigcup_{i \in I_2} \mathfrak{u}_i$. Further the ideal generated by $\mathfrak{I}$, the new b_1, and old b_λ, where λ is from the column **c**, contains a lexicographically monic polynomial. However, these elements do not change the ideal generated by b_λ, where λ is from the column **c**. Hence we actually achieve that the ideal generated by new b_1, and b_λ, where λ is from the column **c** contains a lexicographically monic polynomial. Thus the row of elements in columns **a** and **c** becomes unimodular in $A[S^{-1}]$. In addition, the ideal generated by b_λ, where λ is from the column **c** is contained in $\bigcap_{i\in I_1} \mathfrak{v}_i = \bigcap_{i\in I_1} \widetilde{\mathfrak{u}_i}$, because we make it so at Step 2, and it was not changed. Hence the image of new b_1 in $A/\mathfrak{I}[S^{-1}]$ does not belong to $\bigcup_{i\in I_1} \mathfrak{u}_i$.

Step 4. Make the row (b_1, b_{18}) unimodular in $A/\mathfrak{I}[S^{-1}]$ by 48 elementary elements.

Since the image of b_1 in $A/\mathfrak{I}[S^{-1}]$ does not belong to $\bigcup_{i\in I} \mathfrak{u}_i$, it follows that $\dim \operatorname{Max} A/(\mathfrak{I} + \langle b_1\rangle)[S^{-1}] \leqslant \dim \operatorname{Max} A/\mathfrak{I}[S^{-1}] - 1 \leqslant 3$. Therefore, we can apply Lemma 6.4 to the column **c** and the ideal $\mathfrak{I} + \langle b_1\rangle$. □

Remark 6.8 One can notice that the proof above basically repeats the proof of [34, Lemma 2].

Remark 6.9 The lemma above is the only place, where we use the assumption that C is a Jacobson ring. It is easy to see that this assumption can be lifted, if we assume that $\dim C \leqslant 3$.

Now we prove Proposition 4.3 for the case (E_7, ϖ_7).

Proof Consider the branching table for (E_7, ϖ_7), where vertical lines correspond to cutting through the bonds marked with 1, and horizontal lines correspond to cutting through the bonds marked with 7.

		a	**b**	**c**
	E_7, ϖ_7	D_6, ϖ_1	D_6, ϖ_6	D_6, ϖ_1
1)	∘	∘		
2)	E_6, ϖ_6	D_5, ϖ_1	D_5, ϖ_5	∘
3)	E_6, ϖ_1	∘	D_5, ϖ_4	D_5, ϖ_1
4)	∘			∘

Take $b \in \mathrm{Um}'_{(E_7,\varpi_7)} A$. We need to obtain a lexicographically monic polynomial by 301 elementary elements. Since the Weyl group acts transitively on weights, it does not matter in which position to obtain a lexicographically monic polynomial. Let us make it with b_1. We perform the following steps (Fig. 3).

Step 1. Make the row that consists of elements in all the cells except **a1** unimodular in $A[S^{-1}]$ by five elementary elements.

Let $\mathfrak{A} \trianglelefteq A[S^{-1}]$ be the ideal generated by all the elements b_i except for $b_1, \dots, b_5$, b_7. Since $A[S^{-1}]/\mathfrak{A}$ satisfies AFSR_6, and the row $(b_1, \dots, b_5, b_7)$ is unimodular in $A[S^{-1}]/\mathfrak{A}$, it follows from Lemma 3.3 that there exist $c_2, \dots, c_5, c_7 \in A$ such that the row $(b_2 + c_2 b_1, \dots, b_7 + c_7 b_1)$ is unimodular in $A[S^{-1}]/\mathfrak{A}$. Thus by applying the elements $x_{\alpha_7}(\pm c_2), \dots, x_{\alpha_7+\dots+\alpha_3}(\pm c_7)$, we make the row $(b_2, \dots, b_5, b_7)$ unimodular

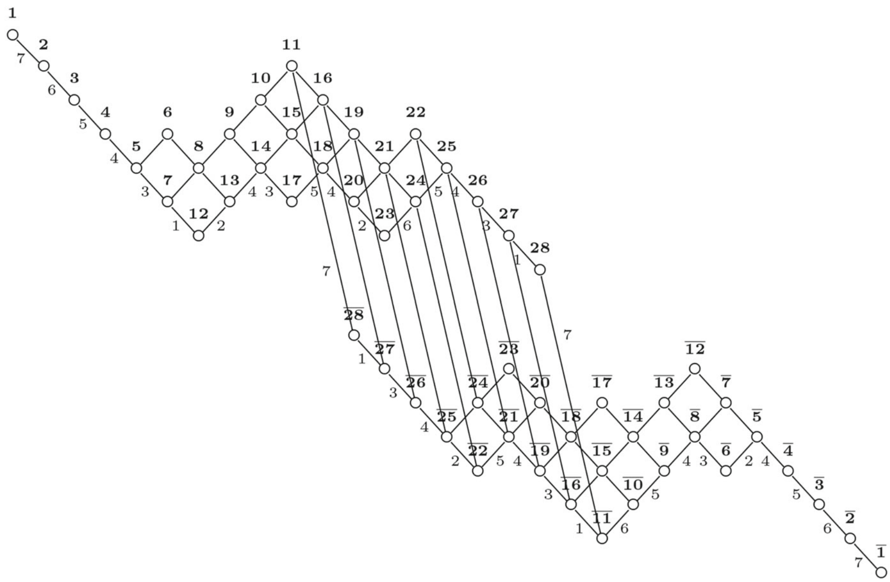

Fig. 3 (E_7, ϖ_7)

in $A[S^{-1}]/\mathfrak{A}$ without changing the ideal $\mathfrak{A}$. Thus the row that consists of elements in all the cells except **a1** becomes unimodular in $A[S^{-1}]$.

Step 2. Make the row that consists of elements in the rows **1**, **3**, and **4** unimodular in $A[S^{-1}]$ by 27 elementary elements.

Since the row $(b_2, \dots, b_{-1})$ is unimodular in $A[S^{-1}]$, it follows that the ideal generated by $(b_2, \dots, b_{-1})$ in A contains a lexicographically monic polynomial. So for some $f_2, \dots, f_{-1} \in A$, the polynomial

$$\sum_{i=2}^{-1} f_i b_i$$

is lexicographically monic. Multiplying polynomials f_i by a large enough power of x_1, we may assume that the polynomial

$$b_1 + \sum_{i=2}^{-1} f_i b_i$$

is also lexicographically monic.

Let us now apply the elements $x_{\lambda_1 - \lambda_i}(\pm f_i)$ for all λ_i from the row **2**. Then the ideal generated by the new b_1 and old b_λ, where λ is from the rows **3** and **4**, contains a lexicographically monic polynomial. However, these elements do not change the ideal generated by b_λ, where λ is from the rows **3** and **4**. Hence we actually achieve that the ideal generated by new b_1, and b_λ, where λ is from the rows **3** and **4** contains a lexicographically monic polynomial. Thus the row of elements in the rows **1**, **3**, and **4** becomes unimodular in $A[S^{-1}]$.

Step 3. Make the row that consists of elements in the cells **a1**, **b3**, **c3**, and **c4** unimodular in $A[S^{-1}]$ by five elementary elements.

Let $\mathfrak{A} \trianglelefteq A[S^{-1}]$ be the ideal generated by all the elements b_λ for λ in rows **1**, **3**, and **4**, except for $b_{-28}, \ldots, b_{-23}$. Since $A[S^{-1}]/\mathfrak{A}$ satisfies AFSR_6, and the row $(b_{-28}, \ldots, b_{-23})$ is unimodular in $A[S^{-1}]/\mathfrak{A}$, it follows from Lemma 3.3 that there exist $c_{-27}, \ldots, c_{-23} \in A$ such that the row $(b_{-27} + c_{-27}b_{-28}, \ldots, b_{-23} + c_{-23}b_{-28})$ is unimodular in $A[S^{-1}]/\mathfrak{A}$. Thus by applying the elements $x_{-\alpha_1}(\pm c_{-27}), \ldots,$ $x_{-\alpha_1-\alpha_3-\cdots-\alpha_6}(\pm c_{-23})$, we make the row $(b_{-27}, \ldots, b_{-23})$ unimodular in $A[S^{-1}]/\mathfrak{A}$ without changing the ideal $\mathfrak{A}$. Thus the row that consists of elements in the cells **a1**, **b3**, **c3**, and **c4** becomes unimodular in $A[S^{-1}]$.

Step 4. Make the row that consists of elements in the cells **a1**, **a3**, **c3**, and **c4** unimodular in $A[S^{-1}]$ by 16 elementary elements.

Let Γ be the set of weights in cells **a1**, **b3**, **c3**, and **c4**. Since the row that consists of elements $\{b_\lambda : \lambda \in \Gamma\}$ is unimodular in $A[S^{-1}]$, it follows that the ideal generated by $\{b_\lambda : \lambda \in \Gamma\}$ in A contains a lexicographically monic polynomial. So for some $f_\lambda \in A$, where $\lambda \in \Gamma$ the polynomial

$$\sum_{\lambda\in\Gamma} f_\lambda b_\lambda$$

is lexicographically monic. Multiplying polynomials f_λ by a large enough power of x_1, we may assume that the polynomial

$$b_{-28} + \sum_{\lambda\in\Gamma} f_\lambda b_\lambda$$

is also lexicographically monic.

Let us now apply the elements $x_{\lambda_{-28}-\lambda}(\pm f_\lambda)$ for all λ_i from the cell **b3**. Then the ideal generated by the new b_{-28} and old b_λ, where λ is from the cells **a1**, **c3**, and **c4**, contains a lexicographically monic polynomial. However, these elements do not change the ideal generated by b_λ, where λ is from the cells **a1**, **c3**, and **c4**. Hence we actually achieve that the ideal generated by new b_1, and b_λ, where λ is from the cells **a1**, **c3**, and **c4**, contains a lexicographically monic polynomial. Thus the row of elements in the cells **a1**, **a3**, **c3**, and **c4** becomes unimodular in $A[S^{-1}]$.

Step 5. Make the row that consists of elements in the cells **a1**, **a2**, **a3**, and **c2** unimodular in $A[S^{-1}]$ by 59 elementary elements.

Apply Lemma 6.4 to the column **c** and the ideal generated by elements from the column **a**.

Step 6. Make the row that consists of elements in the cells **a1**, **a2**, and **c2** unimodular in $A[S^{-1}]$ by 39 elementary elements.

Let $\mathfrak{A} \trianglelefteq A[S^{-1}]$ be the ideal generated by elements from column **a**. Since b_{28} is invertible in $A[S^{-1}]/\mathfrak{A}$, it follows that there exist $\xi_{-7},\ldots,\xi_{-11} \in A[S^{-1}]$ such that

$b_i - \xi_i b_{28} \in \mathfrak{A}$ for $i = -7, \dots, -11$. Let s be a common denominator of ξ_i. Set

$$g_1 = \prod_{-11 \leqslant i \leqslant -7} x_{\lambda_i - \lambda_{28}}(\pm \xi_i),$$

where signs are such that $(g_1 b)_i = b_i - \xi_i b_{28} \in \mathfrak{A}$ for $-11 \leqslant i \leqslant -7$.

Since $A[S^{-1}]$ satisfies AFSR_6, it follows from Lemma 3.3 that there exist $c_7, \dots, c_{11} \in s^2 A$ such that every maximal ideal of $A[S^{-1}]$ containing the ideal $\langle (g_1 b)_7 + c_7 (g_1 b)_{-28}, \dots, (g_1 b)_{11} + c_{11}(g_1 b)_{-28} \rangle$ contains already the ideal $\langle (g_1 b)_7, \dots, (g_1 b)_{11}, (g_1 b)_{-28} \rangle = \mathfrak{A}$. Set

$$g_2 = \prod_{7 \leqslant i \leqslant 11} x_{\lambda_i - \lambda_{-28}}(\pm c_i),$$

where signs are such that $(g_2 g_1 b)_i = (g_1 b)_i + c_i b_{-28}$ for $7 \leqslant i \leqslant 11$.

We claim that the elements $(g_2 g_1 b)_\lambda$, where λ is in the cells **a1**, **a2**, and **c2**, generate the unit ideal in $A[S^{-1}]$. Let us prove that.

Assume that some maximal ideal $\mathfrak{M}$ of the ring $A[S^{-1}]$ contains all the elements $(g_2 g_1 b)_\lambda$, where λ is in the cells **a1**, **a2**, and **c2**.

Since applying g_1 does not change the ideal generated by elements from column **a**, by choice of c_i we have $\mathfrak{A} \leqslant \mathfrak{M}$. Hence $(g_1 b)_i \in \mathfrak{M}$ for $-11 \leqslant i \leqslant -7$. Thus $b_{28} = (g_2 g_1 b)_{28} + \sum_{7 \leqslant i \leqslant 11} \pm c_i (g_1 b)_{-i} \in \mathfrak{M}$. However, by the previous step, b_{28} and $\mathfrak{A}$ generate a unit ideal. This is a contradiction.

Since applying g_1^{-1} does not change the ideal generated by elements from the cells **a1**, **a2**, and **c2**, we obtain that the elements $(g_1^{-1} g_2 g_1 b)_\lambda$, where λ is in the cells **a1**, **a2**, and **c2**, generate the unit ideal in $A[S^{-1}]$.

It remains to notice that the element $g_1^{-1} g_2 g_1$ is the image of the matrix $\mu(u, s, v)$ for certain u and v under the embedding $G(A_5, A) \to G(E_7, A)$ as a subsystem subgroup. Therefore, by Lemma 6.3, $g_1^{-1} g_2 g_1 \in E(E_7, R)^{\leqslant 39}$.

Step 7. Make the row that consists of elements in the cells **a1**, and **a2** unimodular in $A[S^{-1}]$ by 91 elementary elements.

It follows from the proof of [34, Lemma 1] that the elements in the row **2**, taken modulo the ideal $\langle b_1 \rangle \trianglelefteq A$, form an element of $\mathrm{Orb}_{E_6, \varpi_6} A/\langle b_1 \rangle$. Therefore, we can apply Lemma 6.7 to the row **2** and the ideal $\langle b_1 \rangle$.

Step 8. Make the element b_1 lexicographically monic by 59 elementary elements.

Apply Lemma 6.4 to the column **a** and the zero ideal. □

7 Eliminating a variable

In this section, we give the proof of Proposition 4.4. First we need some preparation.

For $\Phi = E_6, E_7$ or E_8, let $\Sigma_2 \leqslant \Phi$ be the set roots that have positive coefficient in simple root α_2, and Δ_2 be the set roots that have zero coefficient in α_2. Therefore, $\Delta_2 \cup \Sigma_2$ is a parabolic set of roots with Δ_2 being the symmetric part, and Σ_2 being the special part. Let U_2 be the unipotent radical of the corresponding parabolic subgroup, and U_2^- be the unipotent radical of the opposite parabolic subgroup.

Note that

$$|\Sigma_2| = \begin{cases} 21 & \text{for } E_6, \\ 42 & \text{for } E_7, \\ 92 & \text{for } E_8. \end{cases}$$

Further let Λ be the set of weights of the representation ϖ. Denote by $\Lambda_i \leqslant \Lambda$ the subset of such weights λ that in the decomposition of $\lambda_1 - \lambda$ in simple roots the coefficient in α_2 is equal to i. Therefore,

$$\Lambda = \bigcup_{i=0}^{i_{\max}} \Lambda_i,$$

where

$$i_{\max} = \begin{cases} 2 & \text{for } E_6, \\ 3 & \text{for } E_7, \\ 6 & \text{for } E_8. \end{cases}$$

For an ideal $I \trianglelefteq R$, set

$$U_2^-(I) = \big\langle x_\gamma(\xi) : \gamma \text{ has negative coefficient in } \alpha_2, \xi \in I \big\rangle.$$

Lemma 7.1 *Let R be a commutative ring. Let $0 \leqslant r \leqslant i_{\max} - 1$. Let $b, b' \in \mathrm{Orb}_\varpi R$ be such that for all $0 \leqslant i \leqslant r$ and for all $\lambda \in \Lambda_i$ we have $b_\lambda = b'_\lambda$. Let $I = \langle b_\lambda - b'_\lambda : \lambda \in \Lambda \rangle \trianglelefteq R$. Suppose that the elements $\{b_\lambda : \lambda \in \Lambda_0\}$ generate the unit ideal. Let $\gamma_1, \dots, \gamma_p \in \Phi$ be all the roots with coefficient in α_2 being equal to $-(r+1)$. Then there exists an element $u = x_{\gamma_1}(\xi_1) \dots x_{\gamma_p}(\xi_p)$, where all the ξ_j are in I, such that for all $0 \leqslant i \leqslant r+1$ and for all $\lambda \in \Lambda_i$ we have $(ub)_\lambda = b'_\lambda$.*

Remark 7.2 In particular, this means that if $r+1$ is bigger than the maximal coefficient in α_2, then the assumptions on b and b' imply that $b = b'$.

Proof Note that such an element u does not change the elements b_λ for $\lambda \in \bigcup_{i \leqslant r} \Lambda_i$. Therefore, we must ensure the equalities $(ub)_\lambda = b'_\lambda$ only for $\lambda \in \Lambda_{r+1}$. It is easy to see that these equalities are linear equations in ξ_j. We must seek ξ_j in form $\xi_j = \sum_\lambda \zeta_{j,\lambda}(b_\lambda - b'_\lambda)$. Therefore, we have a system of linear equations in $\zeta_{j,\lambda}$. For a system of linear equations over a ring, existence of a solution is a local property, see, for example, [18, Proposition 1]. Hence it is enough to consider the case where R is a local ring.

Note that the system Δ_2 has type $A_{|\Lambda_0|-1}$ and the summand of the representation ϖ that corresponds to Λ_0 is the vector representation of $G(\Delta_2, -)$. Hence in the local case, since the elements $\{b_\lambda : \lambda \in \Lambda_0\}$ generate the unit ideal, there exists an element $g \in G(\Delta_2, R) \leqslant G(\Phi, R)$ such that $(gb)_{\lambda_1} = 1$ and $(gb)_\lambda = 0$ for all $\lambda \in \Lambda_0 \setminus \{\lambda_1\}$. The same equalities hold for gb'. It follows by Lemma 2.2 that $gb = u_1 e_1$ and $gb' = u_1' e_1$ for some $u_1, u_1' \in U_1^-$. Since $(gb)_\lambda = (gb')_\lambda = 0$ for all $\lambda \in \Lambda_0 \setminus \{\lambda_1\}$, it follows that $u_1, u_1' \in U_2^-$. We denote by $U_2^{-(j)}$ the subgroup of U_2^- generated by root subgroups for all roots that have coefficient in α_2 less than or equal to $-(j+1)$. Similarly, we define the subgroup $U_2^{-(j)}(I)$. Since $g \in G(\Delta_2, R)$, the assumptions on b and b' imply that $(gb)_\lambda = (gb')_\lambda$ for all $\lambda \in \bigcup_{i \leqslant r} \Lambda_i$; hence we have $u_1 \equiv u_1' \mod U_2^{-(r)}$. Moreover, it is easy to see that the ideal generated by elements $(gb)_\lambda - (gb')_\lambda$ for all $\lambda \in \Lambda$ is equal to I. Therefore, $u_1 \equiv u_1' \mod U_2^{-(r)}(I)$. Set $\widetilde{u} = u_1' u_1^{-1} \in U_2^{-(r)}(I)$. Then we have

$$b' = g^{-1} g b' = g^{-1} u_1' e_1 = g^{-1} \widetilde{u} u_1 e_1 = g^{-1} \widetilde{u} g b.$$

Since $g \in G(\Delta_2, R)$, it follows that $g^{-1} \widetilde{u} g \in U_2^{-(r)}(I)$. Therefore, $g^{-1} \widetilde{u} g = u \widehat{u}$, where $u = x_{\gamma_1}(\xi_1) \dots x_{\gamma_p}(\xi_p)$, where $\xi_j \in I$, and $\widehat{u} \in U_2^{-(r+1)}$. Then we have $(ub)_\lambda = (g^{-1} \widetilde{u} g b)_\lambda = b'_\lambda$ for $\lambda \in \Lambda_{r+1}$. □

Lemma 7.3 *Let R be a commutative ring. Let $b, b' \in \mathrm{Orb}_\varpi R$ be such that for all $\lambda \in \Lambda_0$ we have $b_\lambda = b'_\lambda$. Let $I = \langle b_\lambda - b'_\lambda : \lambda \in \Lambda \rangle \trianglelefteq R$. Suppose that the elements $\{b_\lambda : \lambda \in \Lambda_0\}$ generate the unit ideal. Then there exists an element $u \in U_2^-(I)$ such that $ub = b'$.*

Proof Follows from Lemma 7.1 by induction. □

Lemma 7.4 *Let R be a commutative Noetherian ring, $s \in R$. Then there exists $k \in \mathbb{N}$ such that for any $b, b' \in \mathrm{Orb}_\varpi R$ that satisfy the following conditions:*

- *for all $\lambda \in \Lambda_0$ we have $b_\lambda = b'_\lambda$,*
- *$s \in \langle b_\lambda : \lambda \in \Lambda_0 \rangle \trianglelefteq R$,*
- *$b_\lambda - b'_\lambda$ is divisible by s^k for all $\lambda \in \Lambda$,*

there exists an element $u \in U_2^-$ such that $ub = b'$.

Proof The annihilators of the elements s^i, $i \in \mathbb{N}$, form an ascending chain

$$\mathrm{Ann}\, s \leqslant \mathrm{Ann}\, s^2 \leqslant \cdots.$$

Since the ring R is Noetherian, it follows that for some $l \in \mathbb{N}$, we have $\mathrm{Ann}\, s^{l+m} = \mathrm{Ann}\, s^l$ for any $m \in \mathbb{N}$.

Now consider the ring $\mathbb{Z}[\{\widetilde{b}_\lambda : \lambda \in \Lambda\}][\{\widetilde{b'}_\lambda : \lambda \in \Lambda\}][\{\widetilde{a}_\lambda : \lambda \in \Lambda_0\}]$ of polynomials over $\mathbb{Z}$ in $2|\Lambda| + |\Lambda_0|$ variables. Set

$$\widetilde{R} = \mathbb{Z}[\{\widetilde{b} : \lambda \in \Lambda\}][\{\widetilde{b'} : \lambda \in \Lambda\}][\{\widetilde{a} : \lambda \in \Lambda_0\}]/\mathfrak{I},$$

where the ideal $\mathfrak{I}$ is generated by the following elements: equations form Eq_ϖ for the column $\widetilde{b}$, equations form Eq_ϖ for the column $\widetilde{b'}$, elements $\widetilde{b}_\lambda - \widetilde{b'}_\lambda$ for all $\lambda \in \Lambda_0$.

Then set

$$\tilde{s} = \sum_{\lambda\in\Lambda_0} \widetilde{a}_\lambda \widetilde{b}_\lambda.$$

It follows by Lemma 7.3 that over the ring $\widetilde{R}[\tilde{s}^{-1}]$ there exists an element $\widetilde{u} \in U_2^-$ such that $\widetilde{u}\widetilde{b} = \widetilde{b}'$ in $\widetilde{R}[\tilde{s}^{-1}]$. Let $\widetilde{u} = x_{\gamma_1}(\widetilde{\xi_1}) \dots x_{\gamma_q}(\widetilde{\xi_q})$, where γ_j are roots that have negative coefficient in α_2. Moreover, we can choose $\widetilde{\xi_j}$ to be in the ideal generated by $\widetilde{b}_\lambda - \widetilde{b'}_\lambda$ in $\widetilde{R}[\tilde{s}^{-1}]$. Let $\widetilde{k} \in \mathbb{N}$ be such that for all i elements $\tilde{s}^{\tilde{k}}\widetilde{\xi_j}$ belong to the ideal generated by $\widetilde{b}_\lambda - \widetilde{b'}_\lambda$ in $\widetilde{R}$, i.e.

$$\widetilde{\xi_j} = \tilde{s}^{-\tilde{k}} \sum_{\lambda\in\Lambda} \widetilde{\zeta}_{j,\lambda}(\widetilde{b}_\lambda - \widetilde{b'}_\lambda), \quad \widetilde{\zeta}_{j,\lambda} \in \widetilde{R}.$$

We claim that we can take $k = \widetilde{k} + l$. Let $b, b' \in \operatorname{Orb}_\varpi R$ satisfy the conditions. Then there exists a ring homomorphism $\varphi\colon \widetilde{R} \to R$ that maps $\widetilde{b}$ to b, $\widetilde{b'}$ to b', and $\tilde{s}$ to s. Set $\zeta_{j,\lambda} = \varphi(\widetilde{\zeta}_{j,\lambda})$. Let $b_\lambda - b'_\lambda = s^k c_\lambda$ for all $\lambda \in \Lambda$. We claim that we can take $u = x_{\gamma_1}(\xi_1) \dots x_{\gamma_q}(\xi_q)$, where

$$\xi_j = s^l \sum_{\lambda\in\Lambda} \zeta_{j,\lambda} c_\lambda.$$

It is easy to see that the homomorphism $\widetilde{R}[\tilde{s}^{-1}] \to R[s^{-1}]$ induced by φ sends $\widetilde{u}$ to u. Therefore, $ub = b'$ over $R[s^{-1}]$, i.e. for any $\lambda \in \Lambda$ we have $(ub)_\lambda - b'_\lambda \in \operatorname{Ann} s^m$ for some m. On the other hand, since all the ξ_j are divisible by s^l, it follows that $(ub)_\lambda - b_\lambda$ are divisible by s^l; hence $(ub)_\lambda - b'_\lambda$ are divisible by s^l. Let $(ub)_\lambda - b'_\lambda = s^l\theta_\lambda$. Then $s^{m+l}\theta_\lambda = s^m((ub)_\lambda - b'_\lambda) = 0$, i.e. $\theta_\lambda \in \operatorname{Ann} s^{m+l} = \operatorname{Ann} s^l$. Therefore, $(ub)_\lambda - b'_\lambda = s^l\theta_\lambda = 0$, i.e. $ub = b'$. □

Set $\Lambda'_0 = \Lambda_0 \setminus \{\nu\}$, where ν is the lowest weight in Λ_0.

Lemma 7.5 *Let B be a commutative Noetherian ring, $A = B[y]$. Let $b = b(y) \in \operatorname{Um}' A$, and $s \in B \cup \langle b(y)_\lambda : \lambda \in \Lambda'_0\rangle$. Then there exists $m \in \mathbb{N}$ such that*

$$b(y + s^m z) \in E(\Phi, A[z])^{\leqslant N} b(y),$$

where

$$N = \begin{cases} 65 & \textit{for } D_5 \leqslant E_6, \\ 94 & \textit{for } E_6 \leqslant E_7, \\ 152 & \textit{for } E_7 \leqslant E_8. \end{cases}$$

Proof Take k from Lemma 7.4 (for $R = A[z]$) and set $m = k+2$. Let $b = \begin{pmatrix} b^0 \\ b^1 \end{pmatrix}$, where b_0 is a column with entries b_λ for $\lambda \in \Lambda_0$ and b_1 is a column with entries b_λ for $\lambda \notin \Lambda_0$. Recall that the system Δ_2 has type $A_{|\Lambda_0|-1}$ and the summand of the representation ϖ

that corresponds to Λ_0 is the vector representation of $G(\Delta_2, -)$. Therefore, it follows from [58, Corollary 2.4] that there exists an element $g(z) \in E(\Delta_2, A[z])^{\leqslant 8|\Lambda_0|-4}$ such that $g(z)\,b^0(y) = b^0(y+s^2z)$. Moreover, it follows from the proof that g is congruent to the identity element modulo z, see the proof of [16, Lemma 6.5] for details.

Therefore, for some $\widetilde{b}^1 \in A[z]^{|\Lambda\setminus\Lambda_0|}$ we have

$$b' = \begin{pmatrix} b^0(y+s^m z) \\ \widetilde{b}^1 \end{pmatrix} = g(s^k z)\,b \in E(\Phi, A[z])^{\leqslant 8|\Lambda_0|-4} b.$$

In addition, $\widetilde{b}^1$ is congruent to $b^1(y)$, and hence to $b^1(y+s^m z)$, modulo s^k. Now applying Lemma 7.4 to vectors b' and $b(y+s^m z)$, we obtain that

$$b(y+s^m z) \in E(\Phi, A[z])^{\leqslant|\Sigma_2|} b' \subseteq E(\Phi, A[z])^{\leqslant 8|\Lambda_0|-4+|\Sigma_2|} b.$$

Here we used that $s \in B$, so the shift of the variable does not change the fact that $s \in \langle b_\lambda : \lambda \in \Lambda_0\rangle$. □

Lemma 7.6 *There is an element $w \in W(E_8)$ such that $w(\alpha_2) = \alpha_8$, $w(\alpha_4) = \alpha_7$, $w(\alpha_5) = \alpha_6$, and $w(-\delta_{A_8}) = \delta$, where δ_{A_8} is the maximal root of the subsystem generated by $\alpha_1, \alpha_3, \ldots, \alpha_8, \delta$.*

Proof Let δ_{A_7} be the maximal root of the subsystem generated by $\alpha_1, \alpha_3, \ldots, \alpha_8$. Let us show that there is $w' \in W(E_8)$ such that $w(\alpha_2) = \alpha_8$, $w(\alpha_4) = \alpha_7$, $w(\alpha_5) = \alpha_6$, and $w(-\delta_{A_7}) = \delta$. Note that all the roots in the condition belong to the subsystem of type D_8 generated by $\alpha_2, \alpha_3, \ldots, \alpha_8, \delta$. We can realise this D_8 in the Euclidean space with the orthonormal basis $e_1, \ldots, e_8$ so that $\delta = e_1 - e_2, \alpha_8 = e_2 - e_3, \alpha_7 = e_3 - e_4, \alpha_6 = e_4 - e_5, \alpha_5 = e_5 - e_6, \alpha_4 = e_6 - e_7, \alpha_2 = e_7 - e_8, \alpha_3 = e_7 + e_8, \delta_{A_7} = e_1 + e_8$. An element form $W(D_8)$ can perform any permutation of e_i and in addition replace any even number of e_i with $-e_i$. So we can take $w' \in W(D_8)$ to be the element such that $w'(e_1) = e_1$, $w'(e_2) = e_8$, $w'(e_3) = e_7$, $w(e_4) = e_6$, $w(e_5) = -e_5$, $w(e_6) = -e_4$, $w(e_7) = -e_3$, $w(e_8) = -e_2$.

Now we can take $w = w' w_{\alpha_1}$, where w_{α_1} is the reflection with respect to α_1. □

Lemma 7.7 *Let B be a commutative ring, $P_1, \ldots, P_m$ be distinct maximal ideals in B, $A = B[y]$, $b = b(y) \in \mathrm{Um}'_{\varpi} A$ such that b_j is monic, where $j = 24$ for E_6, $j = -1$ for E_7 and E_8. Then there exists a column vector*

$$b^{(1)} \in E(\Phi, A)^{\leqslant N} b$$

such that $b^{(1)}_j$ is monic and

$$\big(\langle b^{(1)}_\lambda : \lambda \in \Lambda'_0\rangle \cap B\big) \setminus \bigcup_{i=1}^m P_i \neq \varnothing,$$

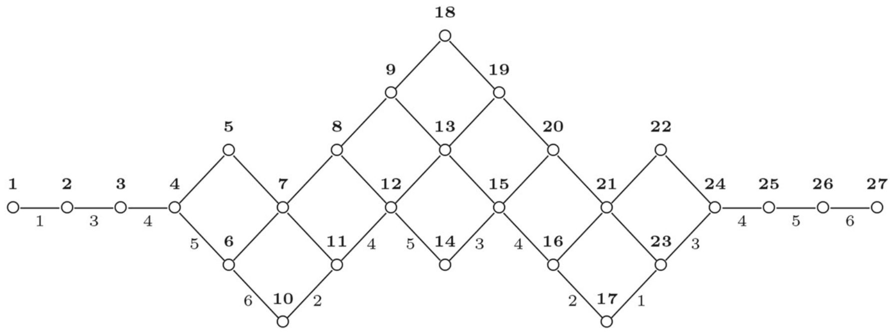

Fig. 4 (E_6, ϖ_1)

where

$$N = \begin{cases} 7 & \text{for } D_5 \leqslant E_6, \\ 10 & \text{for } E_6 \leqslant E_7, \\ 139 & \text{for } E_7 \leqslant E_8. \end{cases}$$

Proof Set

$$R = B/\left(\bigcap_{i=1}^{m} P_i\right) = \prod_{i=1}^{m} B/P_i.$$

First we show that the last condition on $b^{(1)}$ holds if $b_1^{(1)}$ is monic and the elements $\{b_\lambda^{(1)} : \lambda \in \Lambda_0'\}$ generate the unit ideal in $R[y]$.

Let $c_\lambda \in A$, where $\lambda \in \Lambda_0'$, be such that $\sum_{\lambda\in\Lambda_0'} c_\lambda b_\lambda^{(1)} \equiv 1 \bmod P_i$ for every i.

Set $f = \sum_{\lambda\in\Lambda_0'\setminus\{\lambda_1\}} c_\lambda b_\lambda^{(1)}$. Then $b_1^{(1)}$ and f are coprime in $B/P_i[y]$ for every i.

Since $b_1^{(1)}$ is monic, it follows that the resultant $\operatorname{res}(b_1^{(1)}, f)$ modulo P_i is equal to the resultant of $b^{(1)}$ taken modulo P_i and f taken modulo P_i (even if f modulo P_i has smaller degree).

Therefore, we have

$$\operatorname{res}(b_1^{(1)}, f) \in \left(\langle b_\lambda^{(1)} : \lambda \in \Lambda_0'\rangle \cap B\right) \setminus \bigcup_{i=1}^{m} P_i.$$

Thus it remains to prove that a given column $b \in \mathrm{Um}'_\varpi A$, with b_j being monic, can be transformed by N elementary elements so that b_1 becomes monic, b_j remains monic, and the new elements $\{b_\lambda : \lambda \in \Lambda_0'\}$ generate the unit ideal in $R[y]$.

Proof for (E_6, ϖ_1). Here we perform the following steps (Fig. 4).

Step 1. Make the polynomial b_3 monic and the row $(b_1, \dots, b_{16}, b_{18}, \dots b_{22}, b_{24})$ unimodular in $R[y]$ by the element $x_\delta(\xi)$.

Since R is a product of fields and b_{24} is monic, it follows that the ring $R[y]/\langle b_{24}\rangle$ is semilocal; hence it is easy to see that there exists $\widetilde{\xi}$ such that the row $((x_\delta(\widetilde{\xi})b)_1, \dots, (x_\delta(\widetilde{\xi})b)_{16}, (x_\delta(\widetilde{\xi})b)_{18}, \dots (x_\delta(\widetilde{\xi})b)_{22}, (x_\delta(\widetilde{\xi})b)_{24})$ is unimodular in $R[y]$. Therefore, if we take

$$\xi = \widetilde{\xi} + y^K b_{24}$$

for some $K \in \mathbb{N}$, then we guarantee that the row $(b_1, \dots, b_{16}, b_{18}, \dots b_{22}, b_{24})$ becomes unimodular in $R[y]$. It remains to notice that if K is large enough, then we also make b_3 monic.

Step 2. Make the polynomial b_2 monic and the row $(b_1, \dots b_{21}, b_{23})$ unimodular in $R[y]$ by the element $x_{\alpha_3}(\xi)$.

This is done similarly to Step 1.

Step 3. Make the polynomial b_1 monic and the row $(b_1, \dots b_{17})$ unimodular in $R[y]$ by the element $x_{\alpha_1}(\xi)$.

This is done similarly to Step 1.

Step 4. Make the row $(b_1, b_2, b_3, b_4, b_6)$ unimodular in $R[y]$ by the element $x_{\alpha_2}(\xi_4)x_{\delta_{D_4}}(\xi_3)x_{\alpha_6}(\xi_2)x_{\delta_{D_5}}(\xi_1)$, where δ_{D_5} is the maximal root of the subsystem generated by $\alpha_2, \dots, \alpha_6$, and δ_{D_4} is the maximal root of the subsystem generated by $\alpha_2, \dots, \alpha_5$.

Existence of such $\xi_1, \dots, \xi_4$ follows easily from the fact that $R[y]/\langle b_1\rangle$ is semilocal.

Note that neither of steps change b_{24}; hence it remains monic. Also Step 4 does not change b_1; hence it remains monic after being made so in Step 3.

Proof for (E_7, ϖ_7). Consider the branching table for (E_7, ϖ_7), where vertical lines correspond to cutting through the bonds marked with 1, and horizontal lines correspond to cutting through the bonds marked with 7.

		a	**b**	**c**
	E_7, ϖ_7	D_6, ϖ_1	D_6, ϖ_6	D_6, ϖ_1
1)	$\circ$	$\circ$		
2)	E_6, ϖ_6	D_5, ϖ_1	D_5, ϖ_5	$\circ$
3)	E_6, ϖ_1	$\circ$	D_5, ϖ_4	D_5, ϖ_1
4)	$\circ$			$\circ$

Now we perform the following steps, which are similar to those for E_6.

Step 1. Make the polynomial in the cell **a3** monic and the row that consists of elements in the cells **a1**, **a2**, **a3**, **b2**, **b3**, and **c4** unimodular in $R[y]$ by the element $x_\delta(\xi)$.

Step 2. Make the polynomial b_2 (highest weight in the cell **a2**) monic and the row that consists of elements in the cells **a1**, **a2**, **a3**, **b2**, **c2**, the upper half of the cell **b3** with respect to cutting through the bonds marked with 6, and the element that correspond to the highest weight in the cell **c3** unimodular in $R[y]$ by the element $x_{\delta_{D_6}}(\xi)$, where δ_{D_6} is the maximal root of the subsystem generated by $\alpha_2, \dots, \alpha_7$.

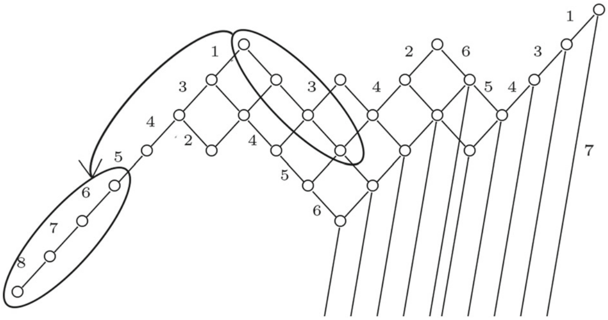

Fig. 5 Part of (E_8, ϖ_8) and action of w

Step 3. Make the polynomial b_1 monic and the row that consists of elements in the cells **a1**, **a2**, **b2**, and **c2** unimodular in $R[y]$ by the element $x_{\alpha_7}(\xi)$.

Step 4. Make the row $(b_1, b_2, b_3, b_4, b_5, b_7)$ unimodular in $R[y]$ by the element $x_{\alpha_2}(\xi_7)\, x_{\alpha_2+\alpha_3+\alpha_4}(\xi_6)\, x_{\delta_{D_5(1)}}(\xi_5)\, x_{\alpha_1}(\xi_4) x_{\delta_{D_5(6)}}(\xi_3)\, x_{\delta_{A_5}}(\xi_2)\, x_{\delta_{E_6}}(\xi_1)$, where δ_{E_6} is the maximal root of the system generated by $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ and α_6; δ_{A_5} is the maximal root of the system generated by $\alpha_1, \alpha_3, \alpha_4, \alpha_5$ and α_6; $\delta_{D_5(6)}$ is the maximal root of the system generated by $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and α_5; $\delta_{D_5(1)}$ is the maximal root of the system generated by $\alpha_2, \alpha_3, \alpha_4, \alpha_5$ and α_6.

Proof for (E_8, ϖ_8). Here we perform the following steps.

Step 0. Make the row (b_1, b_{-1}) unimodular in $R[y]$ by the element $u \in U$.

Since $R[y]/\langle b_{-1}\rangle$ is semilocal, by Lemma 2.4 there exists $g \in G(E_8, R[y]/\langle b_{-1}\rangle)$ such that $gb = e_1$ in $R[y]/\langle b_{-1}\rangle$. By [40, Theorem 1.1], we have $g = hu_1vu$, where $h \in T$, $u, u_1 \in U$, and $v \in U^-$. Therefore, we have $ub = v^{-1}u_1^{-1}h^{-1}e_1$ in $R[y]/\langle b_{-1}\rangle$; hence $(ub)_1$ is invertible in $R[y]/\langle b_{-1}\rangle$. Clearly u can be lifted to the element of $U(\Phi, B[y])$. Note that $(ub)_{-1} = b_{-1}$; hence the row $((ub)_1, (ub)_{-1})$ is unimodular in $R[y]$.

Now consider the subsystem $D_8 \leqslant E_8$ generated by $\alpha_2, \alpha_3, \dots, \alpha_8, \delta$. If we restrict our representation to the group $G(D_8, -)$ one of the summands will be the representation (D_8, ϖ_8). Take $w \in W(E_8)$ from Lemma 7.6. If we move our subsystem D_8 with element w, then the highest weight of the representation (D_8, ϖ_8) becomes the highest weight of the entire (E_8, ϖ_8). In addition, three weights next to it become weights from Λ_0' (Fig. 5). It is clear that lowest weight of (D_8, ϖ_8) becomes the lowest weight of (E_8, ϖ_8).

Consider the weight diagram for (D_8, ϖ_8). If we cut it through the bonds marked with 2 (here marks refer to the numbering of simple root in D_8 as shown in Fig. 6), then we obtain the union of diagrams (D_6, ϖ_6), $(D_6, \varpi_5) \otimes (A_1, \varpi_1)$, and (D_6, ϖ_6).

Diagram for (D_6, ϖ_5) differs from the diagram for (D_6, ϖ_6) by swaping two labels; so essentially we have four copies of diagram (D_6, ϖ_6). We give number 1 to the one containing the highest weight, number 2 to the upper half of the component $(D_6, \varpi_5) \otimes (A_1, \varpi_1)$, number 3 to its lower half, and number 4 to the one containing the lowest weight. Now we give to every vertex of the diagram (D_8, ϖ_8) the number

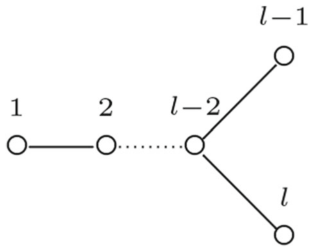

Fig. 6 Numbering of simple roots in D_l

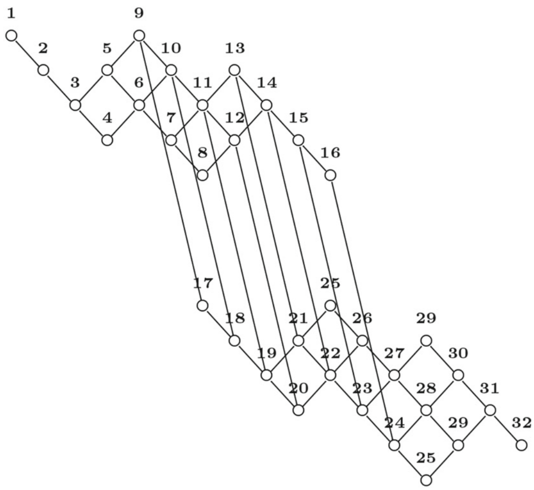

Fig. 7 (D_6, ϖ_6)

of the form i/j, where i is the number of weight in (D_6, ϖ_6) according to Fig. 7, and j is the number of the copy of (D_6, ϖ_6).

Now it remains to prove the following statement. For any column vector $b = b(y) \in V_{(D_8,\varpi_8)}A$ such that it becomes unimodular in $R[y]$ and that $b_{32/4}$ is monic, there exists a column vector

$$b^{(1)} \in E(\Phi, A)^{\leqslant 19} b$$

such that $b^{(1)}_{32/4}$ and $b^{(1)}_{1/1}$ are monic and the row $(b^{(1)}_{1/1}, b^{(1)}_{2/1}, b^{(1)}_{3/1}, b^{(1)}_{5/1})$ is unimodular in $R[y]$.

We prove this statement similarly to how we proved it for (E_6, ϖ_1) and (E_7, ϖ_7). Here we perform the following steps (numbering of roots is as in Fig. 6).

Step 1. Make the polynomial $b_{32/1}$ monic and simultaneously make the row that consists of elements $\{b_{i/j} : 1 \leqslant i \leqslant 32, 1 \leqslant j \leqslant 3\} \cup \{b_{32/4}\}$ unimodular in $R[y]$ by the element $x_{\delta_{D_8}}(\xi)$.

Step 2. Make the polynomial $b_{8/1}$ monic and simultaneously make the row that consists of elements $\{b_{i/j} : 1 \leqslant i \leqslant 24, 1 \leqslant j \leqslant 3\} \cup \{b_{32/1}, b_{8/4}\}$ unimodular in $R[y]$ by the element $x_{\delta_{D_6}}(\xi)$, where δ_{D_6} is the maximal root of the subsystem generated by $\alpha_3, \dots \alpha_8$.

Step 3. Make the row of elements $\{b_{i/j} : 1 \leqslant i \leqslant 16, 1 \leqslant j \leqslant 3\} \cup \{b_{32/1}, b_{8/4}\}$ unimodular in $R[y]$ by the element $x_{\alpha_3}(\xi)$ (the polynomial $b_{8/1}$ remains the same).

Step 4. Make the polynomial $b_{2/1}$ monic and simultaneously make the row that consists of elements $\{b_{i/j} : 1 \leqslant i \leqslant 14, 1 \leqslant j \leqslant 3\} \cup \{b_{26/1}, b_{2/4}\}$ unimodular in $R[y]$ by the element $x_{\delta_{D_4}}(\xi)$, where δ_{D_6} is the maximal root of the subsystem generated by $\alpha_5, \dots, \alpha_8$.

Step 5. Make the row of elements $\{b_{i/j} : 1 \leqslant i \leqslant 12, 1 \leqslant j \leqslant 3\} \cup \{b_{26/1}, b_{2/4}\}$ unimodular in $R[y]$ by the element $x_{\alpha_5}(\xi)$ (the polynomial $b_{2/1}$ remains the same).

Step 6. Make the row of elements $\{b_{i/j} : 1 \leqslant i \leqslant 8, 1 \leqslant j \leqslant 3\} \cup \{b_{22/1}, b_{2/4}\}$ unimodular in $R[y]$ by the element $x_{\alpha_4}(\xi)$ (the polynomial $b_{2/1}$ remains the same).

Step 7. Make the row of elements $\{b_{i/1} : 1 \leqslant i \leqslant 8\} \cup \{b_{i,j} : 1 \leqslant i \leqslant 7, 2 \leqslant j \leqslant 3\} \cup \{b_{22/1}, b_{2/4}\}$ unimodular in $R[y]$ by the element $x_{\alpha_7}(\xi)$ (the polynomial $b_{2/1}$ remains the same).

Step 8. Make the row of elements $\{b_{i/1} : 1 \leqslant i \leqslant 8\} \cup \{b_{i,j} : 1 \leqslant i \leqslant 6, 2 \leqslant j \leqslant 3\} \cup \{b_{22/1}, b_{2/4}\}$ unimodular in $R[y]$ by the element $x_{\alpha_6+\alpha_8}(\xi)$ (the polynomial $b_{2/1}$ remains the same).

Step 9. Make the polynomial $b_{1/1}$ monic and simultaneously make the row that consists of elements $\{b_{i/1} : 1 \leqslant i \leqslant 7\} \cup \{b_{i,j} : i \in \{1, 2, 3, 5\}, 2 \leqslant j \leqslant 3\} \cup \{b_{21/1}, b_{1/4}\}$ unimodular in $R[y]$ by the element $x_{\alpha_8}(\xi)$.

Step 10. Make the row $(b^{(1)}_{1/1}, b^{(1)}_{2/1}, b^{(1)}_{3/1}, b^{(1)}_{5/1})$ unimodular in $R[y]$ by the element $x_{\alpha_7}(\xi_{10})\, x_{\alpha_6}(\xi_9)\, x_{\alpha_4}(\xi_8)\, x_{\alpha_5}(\xi_7)\, x_{\alpha_3}(\xi_6)\, x_{\alpha_4}(\xi_5)\, x_{\alpha_2}(\xi_4)\, x_{\alpha_3}(\xi_3)\, x_{\alpha_1}(\xi_2)\, x_{\alpha_2}(\xi_1)$. □

Now we are ready to prove Proposition 4.4. For simplicity, we will write

$$v \xrightarrow{N} w$$

instead of

$$w \in E(\Phi, R)^{\leqslant N} v,$$

where v and w are columns in V_ϖ.

Set

$$N_1 = \begin{cases} 65 & \text{for } D_5 \leqslant E_6, \\ 94 & \text{for } E_6 \leqslant E_7, \\ 152 & \text{for } E_7 \leqslant E_8, \end{cases}$$

and

$$N_2 = \begin{cases} 7 & \text{for } D_5 \leqslant E_6, \\ 10 & \text{for } E_6 \leqslant E_7, \\ 139 & \text{for } E_7 \leqslant E_8. \end{cases}$$

Applying Lemmas 2.1 and 7.7 d times, we obtain elements $s_1, \ldots, s_d \in B$ and columns $b = b^{(0)}, b^{(1)}, \ldots, b^{(d)} \in \mathrm{Um}'_{\varpi} A$ such that, firstly,

$$b^{(i)} \xrightarrow{N_2} b^{(i+1)}, \quad i = 0, \ldots, d-1,$$

secondly, $s_i \in \langle b_\lambda^{(i)} : \lambda \in \Lambda'_0 \rangle$ for $i = 1, \ldots, d$, and thirdly, $\mathrm{BSdim}\, B/(s_1, \ldots, s_{i+1}) < \mathrm{BSdim}\, B/(s_1, \ldots, s_i)$ for $i = 0, \ldots, d-1$. In particular, the elements $s_1, \ldots, s_d$ generate the unit ideal.

By Lemma 7.5 we have

$$b^{(i)}(y) \xrightarrow{N_1} b^{(i)}(y + s_i^{m_i} z)$$

in $A[z]$.

Therefore, we have the following chain of transformations in $A[z_1, \ldots, z_d]$:

$$\begin{aligned} b = b^{(0)}(y) \to b^{(1)}(y) \to b^{(1)}(y + s_1^{m_1} z_1) \to b^{(2)}(y + s_1^{m_1} z_1) \to \cdots \\ \to b^{(d)}\big(y + s_1^{m_1} z_1 + \cdots + s_{d-1}^{m_{d-1}} z_{d-1}\big) \\ \to b^{(d)}\big(y + s_1^{m_1} z_1 + \cdots + s_d^{m_d} z_d\big). \end{aligned}$$

Thus we have

$$b(y) \xrightarrow{d(N_1+N_2)} b^{(d)}\big(y + s_1^{m_1} z_1 + \cdots + s_d^{m_d} z_d\big).$$

Since the elements $s_1, \ldots, s_d$ generate the unit ideal, it follows that so do the elements $s_1^{m_1}, \ldots, s_d^{m_d}$. Specializing the indeterminates z_i to elements in yB, we make $y + s_1^{m_1} z_1 + \cdots + s_d^{m_d} z_d$ equal to zero; this concludes the proof of Proposition 4.4.

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