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RESEARCH ARTICLE

Characterizations of near-Heyting algebras

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Abstract
A near-Heyting algebra is a join-semilattice with a greatest element such that every principal upset is a Heyting algebra. We will present several characterizations of the concept of near-Heyting algebra. We will show that the class of near-Heyting algebras is a subclass of Hilbert algebras with supremum. We introduce prelinear near-Heyting algebras and present some of their characterizations.

Keywords Near-Heyting algebra · Hilbert algebra · Heyting algebra · Distributive nearlattice

Mathematics Subject Classification 06D75 · 06D20

1 Introduction

It is known that the variety of implication algebras (also known as Tarski algebras) is the algebraic counterpart of the implication fragment of propositional classical logic. Recall that an algebra $\langle A, \to, 1\rangle$ of type $(2, 0)$ is an implication algebra if it satisfies the following identities: $1 \to x = x$, $x \to x = 1$, $x \to (y \to z) = (x \to y) \to (x \to z)$

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and $(x \to y) \to x = (y \to x) \to x$. On the other hand, the class of semi-boolean algebras was introduced by Abbott in [2] as join-semilattices with top element 1 where every principal upset is a Boolean algebra. In [2], Abbott proved that there is a one-to-one correspondence between the class of semi-boolean algebras and the variety of implication algebras. Hence, if $\langle A, \to, 1\rangle$ is an implication algebra, the join $\vee$ can be expressed by means of the implication $\to$ as $x \vee y = (x \to y) \to y$. The meet $\wedge$ is only a partial operation and $x \wedge y$ is defined if and only if the elements x and y have a common lower bound. If $a \in A$ is a common lower bound of x and y, then $x \wedge y$ can be defined as $x \wedge y = (x \to (y \to a)) \to a$, and the complement of x in $[a) = \{x \in A : a \leqslant x\}$ is given by $x \to a$. Therefore, $[a)$ is a Boolean algebra.

It is a natural subject to study join-semilattices where the complement in each principal upset is replaced by the pseudocomplement, that is, join-semilattices with top element 1 where every principal upset is a pseudocomplemented distributive lattice. In [15], the authors named this class of join-semilattices as sectionally pseudocomplemented distributive nearlattices. In [15] it is proved that there is a one-to-one correspondence between the class of sectionally pseudocomplemented distributive nearlattices and a variety of algebras of type $(3, 2, 0)$ satisfying certain identities. It was remarked in [22] that sectionally pseudocomplemented distributive nearlattices can be equivalently defined as join-semilattices with top element 1 where every principal upset is a Heyting algebra. This is why in [22] they decided to name these algebras as near-Heyting algebras.

Since Heyting algebras and Hilbert algebras are closely related, the main aim of this paper is to connect the near-Heyting algebras with Hilbert algebras and obtain several useful characterizations for this class of algebras. We will see several examples showing that near-Heyting algebras arise naturally.

We close this section fixing some notations we use throughout the paper. Our main references for Order and Lattice theory are [16, 23]. Let $\langle P, \leqslant\rangle$ be a poset. A subset $U \subseteq P$ is called an *upset* of P when for all $a, b \in P$, if $a \leqslant b$ and $a \in P$, then $b \in P$. For every $a \in P$, the upset $[a) = \{b \in P : a \leqslant b\}$ is called a *principal upset* of P. We say that P is a *join-semilattice* if there exists the least lower bound (supremum or join) of $\{a, b\}$, for all $a, b \in P$. In a join-semilattice P, for all $a, b \in P$, $a \vee b$ denotes the least lower bound of a and b. In a poset P, for all $a, b \in P$, we write $a \wedge b$ to mean that the greatest upper bound (infimum or meet) of $\{a, b\}$ exists and it is $a \wedge b$.

1.1 Hilbert algebras with supremum

We recall the basics about Hilbert algebras and Hilbert algebras with supremum. Our main references for Hilbert algebras are [17, 27].

Definition 1.1 A *Hilbert algebra* is an algebra $\langle A, \to, 1\rangle$ of type $(2, 0)$ satisfying the following identities:

(H1) $x \to x = 1$,
(H2) $1 \to x = x$,
(H3) $x \to (y \to z) = (x \to y) \to (x \to z)$,
(H4) $(x \to y) \to ((y \to x) \to x) = (y \to x) \to ((x \to y) \to y)$.

In every Hilbert algebra A there can be defined a binary relation $\leqslant$ as follows: $a \leqslant b$ if and only if $a \to b = 1$, for all $a, b \in A$. We present some basic properties of Hilbert algebras needed for what follows.

Lemma 1.2 *Let $\langle A, \to, 1\rangle$ be a Hilbert algebra and $a, b, c \in A$. Then, the following properties hold:*

(H5) $a \to (b \to a) = 1$,
(H6) $[a \to (b \to c)] \to [(a \to b) \to (a \to c)] = 1$,
(H7) *if* $a \to b = 1$ *and* $b \to a = 1$, *then* $a = b$.
(H8) $\leqslant$ *is a partial order on* A *and* 1 *is the greatest element in* $\langle A, \leqslant\rangle$,
(H9) $b \leqslant a \to b$,
(H10) $((a \to b) \to b) \to b = a \to b$,
(H11) *if* $a \leqslant b$, *then* $c \to a \leqslant c \to b$ *and* $b \to c \leqslant a \to c$,
(H12) $a \to (b \to c) = b \to (a \to c)$.

Proposition 1.3 *An algebra $\langle A, \to, 1\rangle$ is a Hilbert algebra if and only if it satisfies conditions* (H5)–(H7).

Definition 1.4 Let A be a Hilbert algebra. A subset $F \subseteq A$ is called an *implicative filter* (also known as *deductive system*) of A if (i) $1 \in F$, and (ii) if $a, a \to b \in F$, then $b \in F$.

Let us denote by $\mathrm{Fi}_{\to}(A)$ the collection of all implicative filters of A. Every implicative filter is an upset of $\langle A, \leqslant\rangle$, and for all $a \in A$, $[a)$ is an implicative filter of A. It is straightforward to check that $\mathrm{Fi}_{\to}(A)$ is an algebraic closure system. For every subset $X \subseteq A$, we denote by $\mathrm{Fig}_{\to}(X)$ the implicative filter of A generated by X. Then, $\langle \mathrm{Fi}_{\to}(A), \cap, \vee, \{1\}, A\rangle$ is a bounded distributive lattice, where $F_1 \vee F_2 = \mathrm{Fig}_{\to}(F_1 \cup F_2)$ for all $F_1, F_2 \in \mathrm{Fi}_{\to}(A)$.

Let A be a Hilbert algebra. A proper implicative filter F of A is said to be *irreducible* when for all $F_1, F_2 \in \mathrm{Fi}_{\to}(A)$, if $F_1 \cap F_2 = F$, then $F_1 = F$ or $F_2 = F$. Let us denote by $\mathrm{X}_{\to}(A)$ the set of all irreducible implicative filters of A.

Lemma 1.5 ([17]) *Let A be a Hilbert algebra and $F \in \mathrm{Fi}_{\to}(A)$ be proper. Then, F is irreducible if and only if for all $a, b \notin F$, there is $c \notin F$ such that $a, b \leqslant c$.*

Lemma 1.6 ([17]) *Let A be a Hilbert algebra and $F \in \mathrm{Fi}_{\to}(A)$. If $a \notin F$, then there is $P \in \mathrm{X}_{\to}(A)$ such that $F \subseteq P$ and $a \notin P$.*

Corollary 1.7 *Let A be a Hilbert algebra, $a, b \in A$, and $F \in \mathrm{Fi}_{\to}(A)$. Then, $a \to b \notin F$ if and only if there exists $Q \in \mathrm{X}_{\to}(A)$ such that $F \subseteq Q$, $a \in Q$ and $b \notin Q$.*

A Hilbert algebra with supremum is a Hilbert algebra where the associated partial order is a join-semilattice. The class of Hilbert algebras with supremum is a particular class of BCK-algebras with lattice operations studied by Idziak in [26]. Hilbert algebras with supremum were introduced and studied in [12].

Definition 1.8 An algebra $\langle A, \vee, \to, 1\rangle$ of type $(2, 2, 0)$ is called a *Hilbert algebra with supremum*, *HS-algebra* for short, if:

(HS1) $\langle A, \to, 1\rangle$ is a Hilbert algebra,
(HS2) $\langle A, \vee, 1\rangle$ is a join-semilattice with a greatest element 1,
(HS3) $a \to (a \vee b) = 1$,
(HS4) $(a \to b) \to ((a \vee b) \to b) = 1$

Proposition 1.9 *Let* $\langle A, \vee, \to, 1\rangle$ *be an algebra of type* $(2, 2, 0)$. *Then,* $\langle A, \vee, \to, 1\rangle$ *is an HS-algebra if and only if it satisfies* (HS1), (HS2)*, and*

(HS5) *for all* $a, b \in A$, $a \to b = 1$ *if and only if* $a \vee b = b$.

The above proposition tells us that in an HS-algebra A the partial order induced by the join operation $\vee$ and the partial order induced by the implication $\to$ coincide.

Example 1.10 In every join-semilattice $\langle A, \vee, 1\rangle$, it is possible to define the structure of an HS-algebra by defining the implication $\to$ on A by $a \to b = 1$ if $a \leqslant b$, and $a \to b = b$ if $a \nleqslant b$.

Remark 1.11 Let $\langle A, \vee, \to, 1\rangle$ be an HS-algebra and $F \in \mathrm{Fi}_{\to}(A)$ be proper. By Lemma 1.5, F is irreducible if and only if $a \vee b \in F$ implies $a \in F$ or $b \in F$, for all $a, b \in A$.

Proposition 1.12 *Let* $\langle A, \vee, \to, 1\rangle$ *be an HS-algebra. Then, for all* $a, b \in A$*, the following property holds:*

(HS6) $a \vee b \leqslant (a \to b) \to b$.

Remark 1.13 Every implication algebra (see [2]) is an HS-algebra, but there are HS-algebras that are not implication algebras. It is easy to see that the following are equivalent: (i) $\langle A, \vee, \to, 1\rangle$ is an HS-algebra such that $a \vee b = (a \to b) \to b$, for all $a, b \in H$, and (ii) $\langle A, \to, 1\rangle$ is an implication algebra.

1.2 Distributive nearlattices

Now, we recall the basics about distributive nearlattices. Our main reference for distributive nearlattices is [13].

Definition 1.14 A *distributive nearlattice* is a join-semilattice $\langle A, \vee, 1\rangle$ with a greatest element 1 such that for every $a \in A$, the principal upset $[a)$ is a bounded distributive lattice concerning the order induced by $\vee$.

As we can see, distributive nearlattices are a nice generalization of distributive lattices. These algebraic structures were studied by several authors from different points of view: algebraic [3, 7, 8, 10, 11, 14, 15, 20, 24, 25]; topological [9, 21]; and logical [18, 19].

Let $\langle A, \vee, 1\rangle$ be a distributive nearlattice. Let $a \in A$. For every $x, y \in [a)$, $x \wedge_a y$ denotes the meet of $\{x, y\}$ in $[a)$. Notice that if $x, y \in [a) \cap [b)$, then $x \wedge_a y = x \wedge_b y$. Thus, if $x, y \in [a)$, then $x \wedge y$ exists in A, and $x \wedge y = x \wedge_a y$.

In [15], it was proved that there is a one-to-one correspondence between distributive nearlattices and certain algebras of type $(3, 0)$ satisfying some identities, we called them DN-algebras. However, they are different structures. The class of DN-algebras forms a variety, while the class of distributive nearlattices does not. For example, let us consider the distributive nearlattice $\langle \mathbf{2}^3, \vee, 1\rangle$, where $\mathbf{2} = \langle\{0, 1\}, \leqslant\rangle$ is the two-element chain with $0 < 1$ and $\vee$ is defined as usual. It is easy to see that the subalgebra B of $\mathbf{2}^3$ whose elements are the first element, the last element and the dual atoms of $\mathbf{2}^3$ is not a distributive nearlattice.

Definition 1.15 Let $\langle A, \vee, 1\rangle$ be a distributive nearlattice. A subset $F \subseteq A$ is said to be a *filter* when for all $a, b \in A$, (i) $1 \in F$; (ii) if $a \leqslant b$ and $a \in F$, then $b \in F$; and (iii) if $a, b \in F$ and $a \wedge b$ exists in A, then $a \wedge b \in F$.

Let A be a distributive nearlattice. We denote by $\mathrm{Fi}_\wedge(A)$ the collection of all filters of A. It is easy to see that $\mathrm{Fi}_\wedge(A)$ is an algebraic closure system. For every subset $X \subseteq A$, let us denote by $\mathrm{Fig}_\wedge(X)$ the filter of A generated by X. Notice that $\langle \mathrm{Fi}_\wedge(A), \cap, \vee, \{1\}, A\rangle$ is a bounded lattice, where $F \vee G = \mathrm{Fig}_\wedge(F \cup G)$.

Proposition 1.16 *For every distributive nearlattice* $\langle A, \vee, 1\rangle$, $\mathrm{Fi}_\wedge(A)$ *is a distributive lattice.*

A proper filter F of a distributive nearlattice A is said to be *prime* when for all $a, b \in A$, if $a \vee b \in F$, then $a \in F$ or $b \in F$. Let us denote by $\mathrm{X}_\wedge(A)$ the collection of all prime filters of A.

Lemma 1.17 *Let* $\langle A, \vee, 1\rangle$ *be a distributive nearlattice,* $F \in \mathrm{Fi}_\wedge(A)$*, and* $a \in A$*. If* $a \notin F$*, then there is* $P \in \mathrm{X}_\wedge(A)$ *such that* $F \subseteq P$ *and* $a \notin P$.

Lemma 1.18 *Let* $\langle A, \vee, 1\rangle$ *be a distributive nearlattice. Let* $a, b \in A$*. If* $a \nleq b$*, then there is* $P \in \mathrm{X}_\wedge(A)$ *such that* $a \in P$ *and* $b \notin P$.

2 Near-Heyting algebras

A *sectionally pseudocomplemented distributive nearlattice* is a distributive nearlattice such that every principal upset is a pseudocomplemented lattice [15]. In every sectionally pseudocomplemented distributive nearlattice $\langle A, \vee, 1\rangle$ is possible to define a binary operation $\to$ as follows: For all $x, y \in A$, $x \to y$ is the pseudocomplemented of $x \vee y$ in $[y)$. In [13, Theorem 5.5.1] it is shown that sectionally pseudocomplemented nearlattices can be defined equivalently as algebras of type (3, 2, 0) satisfying some conditions.

Definition 2.1 ([22]) An algebra $\langle A, \vee, \to, 1\rangle$ of type (2, 2, 0) is said to be a *near-Heyting algebra* if $\langle A, \vee, 1\rangle$ is a distributive nearlattice and the following identities hold:

(NH1) $y \vee (x \to y) = x \to y$,
(NH2) $x \to x = 1$,
(NH3) $1 \to x = x$,
(NH4) $(x \vee z) \wedge_z [((x \vee z) \wedge_z (y \vee z)) \to z] = (x \vee z) \wedge_z (y \to z)$.

Proposition 2.2 (See [13, Theorem 5.5.1]) *If $\langle A, \vee, 1\rangle$ is a sectionally pseudocomplemented distributive nearlattice, then the algebra $\langle A, \vee, \to, 1\rangle$ of type $(2, 2, 0)$ is a near-Heyting algebra, where $x \to y$ is the pseudocomplement of $x \vee y$ in $[y)$, for all $x, y \in A$. Conversely, if $\langle A, \vee, \to, 1\rangle$ is a near-Heyting algebra, then $\langle A, \vee, 1\rangle$ is a sectionally pseudocomplemented distributive nearlattice such that $x \to y$ is the pseudocomplement of $x \vee y$ in $[y)$, for all $x, y \in A$.*

We can notice, from conditions (NH1)–(NH3) of Definition 2.1, that the operation $\to$ behaves like an implication.

Theorem 2.3 *Let $\langle A, \vee, \to, 1\rangle$ be an algebra of type $(2, 2, 0)$. Then, $\langle A, \vee, \to, 1\rangle$ is a near-Heyting algebra if and only if the following conditions hold:* (i) *$\langle A, \vee, 1\rangle$ is a join-semilattice with a greatest element* 1, (ii) *for each $a \in A$, $\langle [a), \wedge_a, \vee, \to, a, 1\rangle$ is a Heyting algebra, and* (iii) *$(x \vee y) \to y = x \to y$, for all $x, y \in A$.*

Proof Let $\langle A, \vee, \to, 1\rangle$ be a near-Heyting algebra. Then, for all $a \in A$, $\langle [a), \wedge_a, \vee, a, {}^{*_a}, 1\rangle$ is a pseudocomplemented distributive lattice, where for each $x \in [a)$, $x^{*_a} = x \to a$. Thus, for all $x, y \in A$,

$$x \to y = (x \vee y)^{*_y} = (x \vee y) \to y.$$

Then, by [4, Theorem IX.2.8] we have that $\langle [a), \wedge_a, \vee, a, \to_a, 1\rangle$ is a Heyting algebra, where

$$x \to_a y = x^{*_{(x \wedge_a y)}} = x \to (x \wedge_a y),$$

for all $x, y \in [a)$. Now, for $x, y \in [a)$, we have

$$\begin{aligned} x \to y &= (x \vee y) \to y = (x \vee y) \to ((x \vee y) \wedge_a y) \\ &= (x \vee y) \to_a y = (x \to_a y) \wedge_a (y \to_a y) = x \to_a y. \end{aligned}$$

Therefore, $\langle [a), \wedge_a, \vee, a, \to, 1\rangle$ is a Heyting algebra, for each $a \in A$.

Assume now that $\langle A, \vee, \to, 1\rangle$ is an algebra satisfying conditions (i)–(iii). Let $a \in A$. Since $\langle [a), \wedge_a, \vee, \to, a, 1\rangle$ is a Heyting algebra, it follows that $\langle [a), \wedge_a, \vee, a, 1\rangle$ is a pseudocomplemented distributive lattice. Moreover, it is clear that $(x \vee a) \to a$ is the pseudocomplement of $x \vee a$ in $[a)$. Hence, $\langle A, \vee, 1\rangle$ is a sectionally pseudocomplemented distributive nearlattice, and by (iii) we have that $x \to y = (x \vee y) \to y$ is the pseudocomplement of $x \vee y$ in $[y)$, for all $x, y \in A$. Therefore, by Proposition 2.2, we obtain that $\langle A, \vee, \to, 1\rangle$ is a near-Heyting algebra. □

Now, if $\langle A, \vee, \to, 1\rangle$ is an algebra of type $(2, 2, 0)$ satisfying only the conditions (i) $\langle A, \vee, 1\rangle$ is a join-semilattice, and (ii) for each $a \in A$, $\langle [a), \wedge_a, \vee, \to, a, 1\rangle$ is a Heyting algebra, we cannot assure that $\langle A, \vee, \to, 1\rangle$ is a near-Heyting algebra, as shown in the following example.

Example 2.4 Consider the join-semilattice $\langle A, \vee, 1\rangle$ depicted in Fig. 1, and the operation $\to$ defined on A as follows: $x \to x = 1$, for all $x \in \{a, b, 1\}$, $1 \to a = a$,

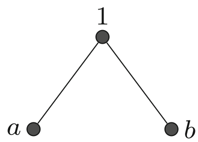

Fig. 1 Corresponding to Example 2.4

$1 \to b = b$, $a \to 1 = 1$, $b \to 1 = 1$, and $a \to b = b \to a = 1$. It is clear that $\langle A, \vee, 1\rangle$ is a distributive nearlattice and $\langle [x), \wedge_x, \vee, \to, x, 1\rangle$ is a Heyting algebra, for each $x \in \{a, b, 1\}$. But $\langle A, \vee, \to, 1\rangle$ is not a near-Heyting algebra because (NH4) is not true for $x = y = a$ and $z = b$. Notice that, in general, the equality $(x \vee y) \to y = x \to y$ is not true.

Lemma 2.5 ([22, Proposition 4.4]) *Let $\langle A, \vee, \to, 1\rangle$ be a near-Heyting algebra. Let $F \in \mathrm{Fi}_\wedge(A)$ and $a, b \in A$. If $a \to b \notin F$, then there exists $P \in \mathrm{X}_\wedge(A)$ such that $F \subseteq P$, $a \in P$ and $b \notin P$.*

Lemma 2.6 ([22, Lemma 5.4]) *Let $\langle A, \vee, \to, 1\rangle$ be a near-Heyting algebra. Let $F \in \mathrm{Fi}_\wedge(A)$ and $a, b \in A$. If $a, a \to b \in F$, then $b \in F$.*

3 Near-Heyting algebras are Hilbert algebras with supremum

In this section we will show that the class of near-Heyting algebras is a subclass of Hilbert algebras with supremum. We also study a weaker class of algebras than near-Heyting.

Definition 3.1 An algebra $\langle A, \vee, \to, 1\rangle$ of type $(2, 2, 0)$ is called a *distributive nearlattice Hilbert algebra*, or *DNH-algebra* for short, if

(DH1) $\langle A, \vee, \to, 1\rangle$ is an HS-algebra, and
(DH2) $\langle A, \vee, 1\rangle$ is a distributive nearlattice.

Thus, a DNH-algebra is a Hilbert algebra with supremum (HS-algebra) where every principal upset $[a)$ is a bounded distributive lattice. For each DNH-algebra $\langle A, \vee, \to, 1\rangle$, we have the collections of filters $\mathrm{Fi}_\wedge(A)$ and prime filters $\mathrm{X}_\wedge(A)$ of the distributive nearlattice $\langle A, \vee, 1\rangle$, and the collections of implicative filters $\mathrm{Fi}_\to(A)$ and irreducible implicative filters $\mathrm{X}_\to(A)$ of the Hilbert algebra $\langle A, \to, 1\rangle$. The reader may want to recall Lemma 1.6, Corollary 1.7, and Lemma 1.18.

Proposition 3.2 *Let $\langle A, \vee, \to, 1\rangle$ be a DNH-algebra. For all $a, b, c, d \in A$, we have*

(DH3) $(a \vee b) \wedge_b (a \to b) \leqslant b$,
(DH4) $c \to (a \wedge b) \leqslant (c \to a) \wedge (c \to b)$*, whenever* $a \wedge b$ *exists,*
(DH5) $a \leqslant b \to c$ *implies* $a \wedge b \leqslant c$*, whenever* $a \wedge b$ *exists.*

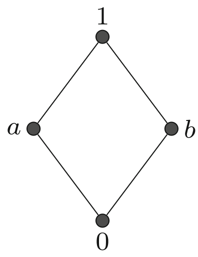

Fig. 2 A DNH-algebra where $X_\wedge(A) \subset X_\rightarrow(A)$

Proof

(DH3) Suppose that $(a \vee b) \wedge_b (a \rightarrow b) \nleqslant b$. Then, by Lemma 1.6 there is $P \in X_\rightarrow(A)$ such that $(a \vee b) \wedge_b (a \rightarrow b) \in P$ and $b \notin P$. Since P is an upset, it follows that $a \vee b, a \rightarrow b \in P$. Now, given that P is irreducible and $b \notin P$, by Remark 1.11 we have $a \in P$. Thus $a, a \rightarrow b \in P$. Then $b \in P$, which is a contradiction. Hence $(a \vee b) \wedge_b (a \rightarrow b) \leqslant b$, for all $a, b \in A$.

(DH4) Assume that $a \wedge b$ exists. Since $a \wedge b \leqslant a$ and $a \wedge b \leqslant b$, it follows by (H11) that $c \rightarrow (a \wedge b) \leqslant c \rightarrow a$ and $c \rightarrow (a \wedge b) \leqslant c \rightarrow b$. Hence $c \rightarrow (a \wedge b) \leqslant (c \rightarrow a) \wedge (c \rightarrow b)$.

(DH5) Assume that $a \wedge b$ exists. Suppose that $a \leqslant b \rightarrow c$ and $a \wedge b \nleqslant c$. Thus, by Lemma 1.6, there exists $P \in X_\rightarrow(A)$ such that $a \wedge b \in P$ and $c \notin P$. Then $a, b \in P$, which implies that $b, b \rightarrow c \in P$. Hence $c \in P$, a contradiction. Therefore, $a \leqslant b \rightarrow c$ implies $a \wedge b \leqslant c$. □

Let $\langle A, \vee, \rightarrow, 1\rangle$ be a DNH-algebra. Notice that for all $a, b \in A$, such that $a \wedge b$ exists, $a \wedge (a \rightarrow b)$ exists. Thus, it follows by (DH5) that $a \wedge (a \rightarrow b) \leqslant b$ because $a \leqslant (a \rightarrow b) \rightarrow b$.

Proposition 3.3 *Let $\langle A, \vee, \rightarrow, 1\rangle$ be a DNH-algebra. Then,* $\mathrm{Fi}_\wedge(A) \subseteq \mathrm{Fi}_\rightarrow(A)$. *In particular,* $X_\wedge(A) \subseteq X_\rightarrow(A)$.

Proof Let $F \in \mathrm{Fi}_\wedge(A)$. Let $a, a \rightarrow b \in F$. Given that F is an upset, we have $a \vee b \in F$. Since $b \leqslant a \vee b$ and $b \leqslant a \rightarrow b$, it follows that $(a \vee b) \wedge (a \rightarrow b)$ exists in A. Then, since $a \vee b, a \rightarrow b \in F$, we have $(a \vee b) \wedge (a \rightarrow b) \in F$. By (DH3), we obtain $b \in F$. Hence $F \in \mathrm{Fi}_\rightarrow(A)$. Now, from the definition of prime filter and by Remark 1.11, it follows that $X_\wedge(A) \subseteq X_\rightarrow(A)$. □

Example 3.4 Consider the join-semilattice $\langle A, \vee, 1\rangle$ depicted in Fig. 2, and the operation $\rightarrow$ defined on A as in Example 1.10. Then, $\langle A, \vee, \rightarrow, 1\rangle$ is a DNH-algebra. It follows that $\mathrm{Fi}_\wedge(A) = \{\{1\}, [a), [b), A\}$, $\mathrm{Fi}_\rightarrow(A) = \{\{1\}, [a), [b), \{a, b, 1\}, A\}$, $X_\wedge(A) = \{[a), [b)\}$ and $X_\rightarrow(A) = \{[a), [b), \{a, b, 1\}\}$. Hence $\mathrm{Fi}_\wedge(A) \subset \mathrm{Fi}_\rightarrow(A)$ and $X_\wedge(A) \subset X_\rightarrow(A)$. Notice also that $a \wedge b \leqslant 0$ but $a \nleqslant b \rightarrow 0 = 0$.

Definition 3.5 An algebra $\langle A, \vee, \to, 1\rangle$ of type $(2, 2, 0)$ is called a *quasi-Heyting algebra* if it is a DNH-algebra, and for all $a, b, c \in A$, satisfies the following condition:

$$a \wedge b \leqslant c \quad \text{implies} \quad a \leqslant b \to c, \tag{R}$$

whenever $a \wedge b$ exists in A.

Remark 3.6 Let $\langle A, \vee, \to, 1\rangle$ be a quasi-Heyting algebra. Then, by (DH5) and condition (R), we obtain that for all $a, b, c \in A$,

$$a \wedge b \leqslant c \quad \text{if and only if} \quad a \leqslant b \to c,$$

whenever $a \wedge b$ exists in A.

Example 3.7 Each Heyting algebra is a quasi-Heyting algebra. Moreover, a quasi-Heyting algebra is a Heyting algebra if and only if it has a least element.

Example 3.8 Implication algebras (also known as Tarski algebras) [1, 2] are also examples of quasi-Heyting algebras.

Proposition 3.9 *Let $\langle A, \vee, \to, 1\rangle$ be a DNH-algebra. Then, the following are equivalent:*

(1) *A is a quasi-Heyting algebra,*
(2) $\mathrm{Fi}_{\wedge}(A) = \mathrm{Fi}_{\to}(A)$,
(3) $\mathrm{X}_{\wedge}(A) = \mathrm{X}_{\to}(A)$.

Proof (1)⇒(2). Assume that $\langle A, \vee, \to, 1\rangle$ is a quasi-Heyting algebra. By Proposition 3.3, we have $\mathrm{Fi}_{\wedge}(A) \subseteq \mathrm{Fi}_{\to}(A)$. Let now $F \in \mathrm{Fi}_{\to}(A)$. We know that F is an upset and $1 \in F$. Let $a, b \in F$ be such that $a \wedge b$ exists in A. By condition (R), we have $a \leqslant b \to (a \wedge b)$. Then, we obtain that $a \wedge b \in F$. Hence $F \in \mathrm{Fi}_{\wedge}(A)$. Therefore, $\mathrm{Fi}_{\to}(A) \subseteq \mathrm{Fi}_{\wedge}(A)$.

(2)⇒(3). It is straightforward from the definition of prime filter and by Remark 1.11.

(3)⇒(1). Assume that $\mathrm{X}_{\wedge}(A) = \mathrm{X}_{\to}(A)$. We only need to prove that condition (R) holds. Let $a, b, c \in A$ be such that $a \wedge b$ exists in A and $a \wedge b \leqslant c$. Suppose that $a \nleqslant b \to c$. Thus, by Lemma 1.6, there is $P \in \mathrm{X}_{\to}(A)$ such that $a \in P$ and $b \to c \notin P$. Then, by Corollary 1.7, there is $Q \in \mathrm{X}_{\to}(A)$ such that $P \subseteq Q$, $b \in Q$, and $c \notin Q$. Since $Q \in \mathrm{X}_{\to}(A) = \mathrm{X}_{\wedge}(A)$, we have that Q is closed under existing finite meets. Thus, because $a, b \in Q$, we obtain $a \wedge b \in Q$. Then $c \in Q$, which is a contradiction. Hence, $a \wedge b \leqslant c$ implies $a \leqslant b \to c$. Then, (R) holds. Therefore, $\langle A, \vee, \to, 1\rangle$ is a quasi-Heyting algebra. □

Theorem 3.10 *Let $\langle A, \vee, \to, 1\rangle$ be a DNH-algebra. The following conditions are equivalent:*

(1) *A is a quasi-Heyting algebra.*
(2) *If $b \wedge c$ exists in A, then $(a \to b) \wedge (a \to c) \leqslant a \to (b \wedge c)$.*

Proof (1)$\Rightarrow$(2). Assume that $\langle A, \vee, \rightarrow, 1\rangle$ is a quasi-Heyting algebra. Let $a, b, c \in A$ be such that $b \wedge c$ exists in A. Since $b \wedge c$ exists, it follows that $b \wedge c \leqslant b \leqslant a \rightarrow b$ and $b \wedge c \leqslant c \leqslant a \rightarrow c$. Thus $(a \rightarrow b) \wedge (a \rightarrow c)$ exists in A. Now suppose, towards a contradiction, that $(a \rightarrow b) \wedge (a \rightarrow c) \nleqslant a \rightarrow (b \wedge c)$. Then, by Lemma 1.6 and Corollary 1.7, there is $P \in \mathrm{X}_{\rightarrow}(A)$ such that $(a \rightarrow b) \wedge (a \rightarrow c) \in P$, $a \in P$, and $b \wedge c \notin P$. Thus $a \rightarrow b, a \rightarrow c \in P$. Since P is an implicative filter, we have $b, c \in P$. Now, by Proposition 3.9, $P \in \mathrm{Fi}_{\rightarrow}(A) = \mathrm{Fi}_{\wedge}(A)$; thus $b \wedge c \in P$, which is a contradiction.

(2)$\Rightarrow$(1). It only remains to verify that condition (R) holds. Let $a, b, c \in A$ and assume that $a \wedge b$ exists and $a \wedge b \leqslant c$. From (H11), we have $b \rightarrow (a \wedge b) \leqslant b \rightarrow c$. By (2), we obtain that $(b \rightarrow a) \wedge (b \rightarrow b) \leqslant b \rightarrow c$. Then $a \leqslant b \rightarrow a \leqslant b \rightarrow c$. Therefore, condition (R) holds. □

Remark 3.11 For every DNH-algebra $\langle A, \vee, \rightarrow, 1\rangle$, by (DH4) we have that condition (2) of Theorem 3.10 is equivalent to

$$a \rightarrow (b \wedge c) = (a \rightarrow b) \wedge (a \rightarrow c)$$

whenever $b \wedge c$ exists in A.

Theorem 3.12 *Let* $\langle A, \vee, \rightarrow, 1\rangle$ *be an algebra of type* $(2, 2, 0)$. *The following are equivalent:*

(1) $\langle A, \vee, \rightarrow, 1\rangle$ *is a quasi-Heyting algebra.*
(2) $\langle A, \vee, \rightarrow, 1\rangle$ *is a near-Heyting algebra.*

Proof (1)$\Rightarrow$(2). Let $\langle A, \vee, \rightarrow, 1\rangle$ be a quasi-Heyting algebra. Let $a \in A$. Notice that $\rightarrow$ is well defined in $[a)$. Indeed, if $x, y \in [a)$, then $a \leqslant y \leqslant x \rightarrow y$. Since $\langle A, \vee, 1\rangle$ is a distributive nearlattice, it follows that $\langle [a), \wedge_a, \vee, a, 1\rangle$ is a bounded distributive lattice. Hence, by Remark 3.6 $\langle [a), \wedge_a, \vee, \rightarrow, a, 1\rangle$ is a Heyting algebra. Let $a, b \in A$. From (HS4) we have that $a \rightarrow b \leqslant (a \vee b) \rightarrow b$. Suppose that $(a \vee b) \rightarrow b \nleqslant a \rightarrow b$. Then by Lemma 1.18 there is $P \in \mathrm{X}_{\wedge}(A)$ such that $(a \vee b) \rightarrow b \in P$ and $a \rightarrow b \notin P$. Hence, by Proposition 3.9 and Corollary 1.7 there exists $Q \in \mathrm{X}_{\rightarrow}(A)$ such that $P \subseteq Q$, $a \in Q$ and $b \notin Q$. Thus $a \vee b \in Q$ and $(a \vee b) \rightarrow b \in Q$, and then $b \in Q$, which is a contradiction. Therefore, by Theorem 2.3 we have that $\langle A, \vee, \rightarrow, 1\rangle$ is a near-Heyting algebra.

(2)$\Rightarrow$(1). Let $\langle A, \vee, \rightarrow, 1\rangle$ be a near-Heyting algebra. By Theorem 2.3 we have that: (i) $\langle A, \vee, 1\rangle$ is a join-semilattice with a greatest element 1, (ii) $\langle [a), \wedge_a, \vee, \rightarrow, a, 1\rangle$ is a Heyting algebra for every $a \in A$, and (iii) $(x \vee y) \rightarrow y = x \rightarrow y$, for all $x, y \in A$. From (ii), for all $x, y, t \in [a)$, we have $x \rightarrow y \in [a)$ and

$$x \wedge_a t \leqslant y \quad \text{if and only if} \quad t \leqslant x \rightarrow y. \tag{3.1}$$

First, let us show that condition (R) is true. Let $a, b, c \in A$ be such that $a \wedge b$ exists in A and $a \wedge b \leqslant c$. Since $a, b, c \in [a \wedge b)$, by (3.1) we obtain $a \leqslant b \rightarrow c$. Therefore, condition (R) holds.

It is obvious that $\langle A, \vee, 1\rangle$ is a distributive nearlattice. Thus, (DH2) holds.

Now we need to show that $\langle A, \vee, \rightarrow, 1\rangle$ is an HS-algebra. To this end, we will apply Proposition 1.9. It is clear that condition (HS2) holds. Let $a, b \in A$. Since $a \vee b, b, 1 \in [b)$, by (3.1) we have that $1 \leqslant (a \vee b) \rightarrow b$ if and only if $(a \vee b) \wedge_b 1 \leqslant b$ if and only if $a \vee b = b$. Hence, we obtain $(a \vee b) \rightarrow b = 1$ if and only if $a \vee b = b$. By condition (iii), it follows that $a \rightarrow b = 1$ if and only if $a \vee b = b$. Thus, (HS5) holds true. Now, we will prove that $\langle A, \rightarrow, 1\rangle$ is a Hilbert algebra, i.e., we prove (H5), (H6) and (H7) (see Proposition 1.3). Let $a, b \in A$. Since $a, a \vee b \in [a)$, and $a \wedge_a (a \vee b) \leqslant a$, from (3.1) we have $a \leqslant (a \vee b) \rightarrow a$. Hence, by (iii) we have $a \leqslant b \rightarrow a$, i.e, (H5) holds true. Condition (H7) follows from (HS5). Let $a, b, c \in A$ be such that $a \rightarrow (b \rightarrow c) \nleqslant (a \rightarrow b) \rightarrow (a \rightarrow c)$. From Lemma 1.18 there is $P \in X_\wedge(A)$ such that $a \rightarrow (b \rightarrow c) \in P$ and $(a \rightarrow b) \rightarrow (a \rightarrow c) \notin P$. Now, from Lemma 2.5 there exists $Q \in X_\wedge(A)$ such that $P \subseteq Q$, $a \rightarrow b \in Q$ and $a \rightarrow c \notin Q$. Applying again Lemma 2.5 there exists $Q_1 \in X_\wedge(A)$ such that $Q \subseteq Q_1$, $a \in Q_1$ and $c \notin Q_1$. Since also $a \rightarrow b \in Q \subseteq Q_1$, from Lemma 2.6 we have $b \in Q_1$. Then, from $a \rightarrow (b \rightarrow c) \in P \subseteq Q \subseteq Q_1$, again by Lemma 2.6 we obtain $b \rightarrow c \in Q_1$, and then $c \in Q_1$, which is a contradiction. Hence (H6) holds true. Thus, $\langle A, \vee, \rightarrow, 1\rangle$ is an HS-algebra, and hence (DH1) holds. □

Theorem 3.13 *Let* $\langle A, \vee, \rightarrow, 1\rangle$ *be an algebra of type* $(2, 2, 0)$. *The following are equivalent:*

(1) $\langle A, \vee, \rightarrow, 1\rangle$ *is a quasi-Heyting algebra.*
(2) $\langle A, \vee, \rightarrow, 1\rangle$ *is an HS-algebra such that for each* $a \in A$, $\langle [a), \wedge_a, \vee, \rightarrow, a, 1\rangle$ *is a Heyting algebra.*

Proof (1)⇒(2). If $\langle A, \vee, \rightarrow, 1\rangle$ is a quasi-Heyting algebra, then by Theorem 3.12, A is a near-Heyting algebra. Thus, by Theorem 2.3, we have that $\langle [a), \wedge_a, \vee, \rightarrow, a, 1\rangle$ is Heyting algebra, for all $a \in A$.

(2)⇒(1). It is clear that $\langle A, \vee, 1\rangle$ is a distributive nearlattice. It only remains to verify condition (R). Suppose that $a, b, c \in A$, $a \wedge b$ exists and $a \wedge b \leqslant c$. Since $a, b, c \in [a \wedge b)$, we obtain $a \leqslant b \rightarrow c$ because each upset is a Heyting algebra. Therefore, condition (R) holds true, and thus the proof is complete. □

We present now several examples of near-Heyting algebras showing that these algebraic structures arise naturally.

Example 3.14 Let L be a distributive lattice (not necessarily bounded). Recall that a subset I of L is an *ideal* of L if it is non-empty and for all $a, b \in L$, $a \vee b \in I$ iff $a, b \in I$. Let $\mathrm{Id}(L)$ be the collection of all ideals of L. Then, $\langle \mathrm{Id}(L), \veebar, L\rangle$ is a join-semilattice with top L, where for all $I, J \in \mathrm{Id}(L)$, $I \veebar J = \{a \vee b : a \in I,\ b \in I\}$. Notice that for all $I, J \in \mathrm{Id}(L)$, $I \cap J$ is an ideal of L if and only if $I \cap J \neq \varnothing$. Hence $\langle \mathrm{Id}(L), \veebar, L\rangle$ is a distributive nearlattice. Now for all $I, J \in \mathrm{Id}(L)$, it is defined the operation $\Rightarrow$ as follows: $I \Rightarrow J = \{a \in L : I \cap (a] \subseteq J\}$. It is straightforward show that the algebra $\langle \mathrm{Id}(L), \veebar, \Rightarrow, L\rangle$ satisfies the conditions (H5)–(H7) and (HS5). Hence $\langle \mathrm{Id}(L), \veebar, \Rightarrow, L\rangle$ is a Hilbert algebra with supremum. Moreover it is also easy to check that for all $I, J, K \in \mathrm{Id}(L)$, $I \cap J \subseteq K \iff I \subseteq J \Rightarrow K$. Therefore, $\langle \mathrm{Id}(L), \veebar, \Rightarrow, L\rangle$ is a near-Heyting algebra.

Example 3.15 Let $\langle H, \wedge, \vee, \to, 0, 1\rangle$ be a Heyting algebra (see [4]). Let $H^* = H \setminus \{0\}$. It is clear that $\langle H^*, \vee, \to, 1\rangle$ is a subalgebra of the reduct $\langle H, \vee, \to, 1\rangle$. Thus $\langle H^*, \vee, \to, 1\rangle$ is a Hilbert algebra with supremum. It is known that for all $a \in H$, the principal upset $[a)$ is a Heyting algebra concerning the restrictions of the operations of H (see [4, Theorem IX.2.8]). Hence, for all $a \in H^*$, $\langle [a), \wedge, \vee, \to, a, 1\rangle$ is a Heyting algebra. Therefore, it follows by Theorem 3.13 that $\langle H^*, \vee, \to, 1\rangle$ is a near-Heyting algebra.

Example 3.16 Let $\langle A, \vee, 1\rangle$ be a join-semilattice with greatest element 1 where every principal upset $[a)$ is a chain. Consider the operation $\to$ given by the partial order of A, that is, $a \to b = 1$ if $a \leqslant b$, and $a \to b = b$ otherwise. Then, it follows by Theorem 2.3 that $\langle A, \vee, \to, 1\rangle$ is a near-Heyting algebra.

Example 3.17 Let Σ the set of all finite binary strings, that is, all finite sequences of zeros and ones; the empty string is included. We order Σ by putting $u \succcurlyeq v$ if and only if $u = v$ or v is a prefix of u (that is, v is a finite initial substring of u). It is straightforward that Σ is a join-semilattice with greatest element (the empty string) concerning the order $\succcurlyeq$. Moreover, for every string $u \in \Sigma$, the principal upset $[u)$ is a (finite) chain. Hence, by the previous example we obtain that Σ is a near-Heyting algebra.

We close this section with a summary of all characterizations of near-Heyting algebra.

Theorem 3.18 *Let $\langle A, \vee, \to, 1\rangle$ be an algebra of type $(2, 2, 0)$. The following are equivalent:*

(1) $\langle A, \vee, \to, 1\rangle$ *is a near-Heyting algebra.*
(2) $\langle A, \vee, 1\rangle$ *is a sectionally pseudocomplemented distributive lattice such that $a \to b$ is the pseudocomplement of $a \vee b$ in $[b)$, for all $a, b, \in A$.*
(3) (i) $\langle A, \vee, 1\rangle$ *is a join-semilattice with a greatest element.*
(ii) *For each $a \in A$, $\langle [a), \wedge_a, \vee, \to, a, 1\rangle$ is a Heyting algebra.*
(iii) $(a \vee b) \to b = a \to b$*, for all $a, b \in A$.*
(4) (DH1) $\langle A, \vee, \to, 1\rangle$ *is an HS-algebra.*
(DH2) $\langle A, \vee, 1\rangle$ *is a distributive nearlattice.*
(R) *$a \wedge b \leqslant c$ implies $a \leqslant b \to c$, for all $a, b, c \in A$ and whenever $a \wedge b$ exists in A.*
(5) (DH1) $\langle A, \vee, \to, 1\rangle$ *is an HS-algebra.*
(DH2) $\langle A, \vee, 1\rangle$ *is a distributive nearlattice.*
(3) $\mathrm{X}_{\wedge}(A) = \mathrm{X}_{\to}(A)$.
(6) (DH1) $\langle A, \vee, \to, 1\rangle$ *is an HS-algebra.*
(DH2) $\langle A, \vee, 1\rangle$ *is a distributive nearlattice.*
(2) $\mathrm{Fi}_{\wedge}(A) = \mathrm{Fi}_{\to}(A)$.
(7) (DH1) $\langle A, \vee, \to, 1\rangle$ *is an HS-algebra.*
(DH2) $\langle A, \vee, 1\rangle$ *is a distributive nearlattice.*
(2) *If $b \wedge c$ exists in A, then $(a \to b) \wedge (a \to c) \leqslant a \to (b \wedge c)$.*
(8) $\langle A, \vee, \to, 1\rangle$ *is an HS-algebra such that for each $a \in A$, $\langle [a), \wedge_a, \vee, \to, a, 1\rangle$ is a Heyting algebra.*

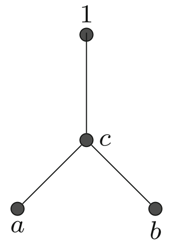

Fig. 3 A non-prelinear near-Heyting algebra

4 Prelinear near-Heyting algebras

In this section, we introduce the concept of prelinear near-Heyting algebra as a natural generalization of prelinear Heyting algebra.

Definition 4.1 Let $\langle A, \vee, \to 1\rangle$ be a near-Heyting algebra. We say that $\langle A, \vee, \to 1\rangle$ is *prelinear* if for all $a, b \in A$, we have

$$(a \to b) \vee (b \to a) = 1.$$

Remark 4.2 If the near-Heyting algebra $\langle A, \vee, \to, 1\rangle$ is prelinear, then the Heyting algebra $[a)$ is prelinear, for all $a \in A$. But the converse is not true. For instance, consider the distributive nearlattice $\langle A, \vee, 1\rangle$ given in Fig. 3. Defining $x \to y = 1$ if $x \leqslant y$, and $x \to y = y$ if $x \nleqslant y$, we obtain that $\langle A, \vee, \to, 1\rangle$ is a DNH-algebra. Then, it is easy to check that $\mathrm{Fi}_{\wedge}(A) = \mathrm{Fi}_{\to}(A)$. Hence $\langle A, \vee, \to, 1\rangle$ is a near-Heyting algebra. For every $x \in A$, $[x)$ is a chain. Thus, $[x)$ is a prelinear Heyting algebra, for all $x \in A$. But $(a \to b) \vee (b \to a) = b \vee a = c \neq 1$. Hence $\langle A, \vee, \to, 1\rangle$ is not prelinear.

Now we will present several characterizations of prelinear near-Heyting algebras. Recall that for every near-Heyting algebra A the lattice filters $\mathrm{Fi}_{\wedge}(A)$ of A coincide with the implicative filters $\mathrm{Fi}_{\to}(A)$ of A, and also $\mathrm{X}_{\wedge}(A) = \mathrm{X}_{\to}(A)$.

Theorem 4.3 *Let $\langle A, \vee, \to 1\rangle$ be a near-Heyting algebra. The following are equivalent:*

(1) $\langle A, \vee, \to 1\rangle$ *is prelinear.*
(2) *For all $P \in \mathrm{X}_{\wedge}(A)$ and all $F \in \mathrm{Fi}_{\wedge}(A) \setminus \{A\}$, if $P \subseteq F$, then F is prime.*
(3) *For all $P \in \mathrm{X}_{\wedge}(A)$, the family $\{F \in \mathrm{Fi}_{\wedge}(A) : P \subseteq F\}$ is a chain.*
(4) *For all $P \in \mathrm{X}_{\wedge}(A)$, the family $\{F \in \mathrm{X}_{\wedge}(A) : P \subseteq F\}$ is a chain.*

Proof (1)$\Rightarrow$(2). Let $P \in \mathrm{X}_{\wedge}(A)$ and $F \in \mathrm{Fi}_{\wedge}(A)\setminus\{A\}$ be such that $P \subseteq F$. Let $a, b \in A$ be such that $a \vee b \in F$. Recall that $(a \vee b) \to b = a \to b$ and $(a \vee b) \to a = b \to a$. Now since $(a \to b) \vee (b \to a) = 1 \in P$ and P is prime, it follows that $a \to b \in P$ or $b \to a \in P$. If $a \to b \in P$, then $(a \vee b) \to b \in P \subseteq F$. As $a \vee b \in F$ and $\mathrm{Fi}_{\wedge}(A) = \mathrm{Fi}_{\to}(A)$, it follows that $b \in F$. Similarly, if $b \to a \in P$, then the obtain that $a \in F$. Hence, F is prime.

(2)$\Rightarrow$(3). Let $P \in X_{\wedge}(A)$. Let $F, G \in Fi_{\wedge}(A)$ be such that $P \subseteq F \cap G$. Suppose $F \nsubseteq G$ and $G \nsubseteq F$, that is, there is $a \in F \backslash G$ and there is $b \in G \backslash F$. Consider the filter $Q = Fig_{\wedge}(P \cup \{a \vee b\})$. We show that $a, b \notin Q$. Suppose that $a \in Q$. Notice that $Q = Fig_{\wedge}(P, a \vee b) = Fig_{\rightarrow}(P, a \vee b) = \{x \in A : (a \vee b) \rightarrow x \in P\}$ (see [17, p. 18]). Then $b \rightarrow a = (a \vee b) \rightarrow a \in P$. Thus, $b \in G$ and $b \rightarrow a \in G$. Then $a \in G$, a contradiction. Similarly if $b \in Q$. Thus $Q \neq A$, and since $P \subseteq Q$, it follows by hypothesis that Q is prime. This is a contradiction because $a \vee b \in Q$ and $a, b \notin Q$. Therefore, $F \subseteq G$ or $G \subseteq F$.

(3)$\Rightarrow$(4). It is immediate.

(4)$\Rightarrow$(1). Suppose there exist $a, b \in A$ such that $(a \rightarrow b) \vee (b \rightarrow a) < 1$. Then there exists $P \in X_{\rightarrow}(A)$ such that $(a \rightarrow b) \vee (b \rightarrow a) \notin P$. Thus, $a \rightarrow b \notin P$ and $b \rightarrow a \notin P$. Since $a \rightarrow b \notin P$, then there exists $Q_1 \in X_{\rightarrow}(A)$ such that $P \subseteq Q_1$, $a \in Q_1$ and $b \notin Q_1$. Similarly, since $b \rightarrow a \notin P$, then there exists $Q_2 \in X_{\rightarrow}(A)$ such that $P \subseteq Q_2$, $b \in Q_2$ and $a \notin Q_2$. As $X_{\rightarrow}(A) = X_{\wedge}(A)$ and $Q_1, Q_2 \in \{F \in X_{\wedge}(A) : P \subseteq F\}$ is a chain, then $Q_1 \subseteq Q_2$ or $Q_2 \subseteq Q_1$. If $Q_1 \subseteq Q_2$, then $a \in Q_2$ which is a contradiction. If $Q_2 \subseteq Q_1$, then $b \in Q_1$ and again we have a contradiction. Hence, $\langle A, \vee, \rightarrow 1\rangle$ is prelinear. □

Theorem 4.4 *Let $\langle A, \vee, \rightarrow 1\rangle$ be a near-Heyting algebra. The following are equivalent:*

(1) $\langle A, \vee, \rightarrow 1\rangle$ *is prelinear.*
(2) $x \vee y = ((x \rightarrow y) \rightarrow y) \wedge_{x \vee y} ((y \rightarrow x) \rightarrow x)$.
(3) $x \rightarrow (y \vee z) = (x \rightarrow y) \vee (x \rightarrow z)$.

Proof (1)$\Rightarrow$(2). By (HS6) we have $x \vee y \leqslant (x \rightarrow y) \rightarrow y$ and $y \vee x \leqslant (y \rightarrow x) \rightarrow x$. So,

$$x \vee y \leqslant ((x \rightarrow y) \rightarrow y) \wedge_{x \vee y} ((y \rightarrow x) \rightarrow x)$$

We see the other inequality. Let $a, b, c \in A$ be such that $a \leqslant c$ and $b \leqslant c$. Take

$$d = ((a \rightarrow b) \rightarrow b) \wedge_{a \vee b} ((b \rightarrow a) \rightarrow a).$$

Since $a \leqslant c$, it follows that $d \rightarrow a \leqslant d \rightarrow c$. As $d \leqslant (b \rightarrow a) \rightarrow a$, then by (H10) we have

$$b \rightarrow a = ((b \rightarrow a) \rightarrow a) \rightarrow a \leqslant d \rightarrow a.$$

Thus $b \rightarrow a \leqslant d \rightarrow c$. Analogously, $a \rightarrow b \leqslant d \rightarrow c$. Then

$$1 = (a \rightarrow b) \vee (b \rightarrow a) \leqslant d \rightarrow c$$

and $d \rightarrow c = 1$, i.e., $d \leqslant c$. We conclude that for all $a, b \in A$,

$$a \vee b = ((a \rightarrow b) \rightarrow b) \wedge_{a \vee b} ((b \rightarrow a) \rightarrow a).$$

(2) ⇒ (3). Let $a, b, c \in A$. By hypothesis and by Remark 3.11, we have

$$\begin{aligned} a \to (b \vee a) &= a \to [((b \to c) \to c) \wedge_{b \vee c} ((c \to b) \to b)] \\ &= [a \to ((b \to c) \to c)] \wedge_{b \vee c} [a \to ((c \to b) \to b)] \\ &\overset{\text{(H3)}}{=} [(a \to (b \to c)) \to (a \to c)] \wedge_{b \vee c} [(a \to (c \to b)) \to (a \to b)] \\ &\overset{\text{(H3)}}{=} [((a \to b) \to (a \to c)) \to (a \to c)] \wedge_{b \vee c} [((a \to c) \to (a \to b)) \to (a \to b)] \\ &= (a \to b) \vee (a \to c). \end{aligned}$$

Therefore, for all $a, b, c \in A$ we have $a \to (b \vee c) = (a \to b) \vee (a \to c)$.

(3) ⇒ (1). Let $a, b \in A$. Then by (NH2), by hypothesis and (iii) of Theorem 2.3 we have

$$1 = (a \vee b) \to (b \vee a) = [(a \vee b) \to a] \vee [(a \vee b) \to b] = (b \to a) \vee (a \to b).$$

Thus $(a \to b) \vee (b \to a) = 1$. Hence, the near-Heyting algebra $\langle A, \vee, \to 1\rangle$ is prelinear. □

5 Future work

The main contribution of the present article was to prove several characterizations of what we call near-Heyting algebras. We believe these may be useful in future investigations about the class of near-Heyting algebras. We show the connections between the concept of near-Heyting algebra and Hilbert algebra and Heyting algebra. Indeed, we show that every near-Heyting algebra is a Hilbert algebra with supremum, and for every element a in a near-Heynting algebra A, $[a)$ is a Heyting algebra.

Taking into account that for every near-Heyting algebra A, we have $\mathrm{Fi}_{\wedge}(A) = \mathrm{Fi}_{\to}(A)$, we believe that it would be possible to develop a topological duality for the algebraic category of near-Heyting algebras following the techniques in [9, 12]. This path is a Stone-like approach. On the other hand, we believe it may be developed a Priestley/Esakia-style duality for the near-Heyting algebras. This could be achieved by a direct approach, taking the collection $\{\varphi(a) : a \in A\} \cup \{\varphi(b)^c : b \in A\}$ as a subbasis for a topology on $\mathrm{Fi}_{\wedge}(A) = \mathrm{Fi}_{\to}(A)$, for each near-Heyting algebra A. An alternative path to obtain a Priestley/Esakia-style duality could be as follows: first, try to get the "free Heyting extension" of every near-Heyting algebra, and then follows the approach given in [5, 6].

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