



Revisiting the classification of homogeneous 3-Sasakian and quaternionic Kähler manifolds

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Abstract

We provide a new, self-contained proof of the classification of homogeneous 3-Sasakian manifolds, which was originally obtained by Boyer et al. (J Reine Angew Math 455:183–220, [10]). In doing so, we construct an explicit one-to-one correspondence between simply connected homogeneous 3-Sasakian manifolds and simple complex Lie algebras via the theory of root systems. We also discuss why the real projective spaces are the only non-simply connected homogeneous 3-Sasakian manifolds and derive the famous classification of homogeneous positive quaternionic Kähler manifolds due to Alekseevskii (Funct Anal Appl 2(2):106–114, [2]) from our results.

Keywords Sasakian geometry · Quaternionic Kähler manifolds · Homogeneous spaces · Wolf spaces · Root systems

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1 Introduction

3-Sasakian geometry is arguably one of the most important odd-dimensional geometries. It provides a rich source of compact Einstein manifolds, lies “sandwiched”

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between the famous hyperkähler, quaternionic Kähler (qK) and Kähler–Einstein geometries and also has several links to algebraic geometry [9].

The history of the classification of homogeneous manifolds with these interrelated geometries is rather long and complicated, as it involves work from as early as 1961 [6] and as recently as 2020 [11]. In this article, we would like to summarize, revisit and improve upon these results by proving the following

Theorem 1.1 *There is a one-to-one correspondence between simply connected homogeneous 3-Sasakian manifolds and simple complex Lie algebras.*

Given a complex simple Lie algebra \mathfrak{u} , choose a maximal root α of \mathfrak{u} and let \mathfrak{v} denote the direct sum of the subspace $\ker \alpha$ and the root spaces of roots perpendicular to α . Let \mathfrak{g} and \mathfrak{h} be the compact real forms of \mathfrak{u} and \mathfrak{v} , respectively, and write $\mathfrak{k} \cong \mathfrak{sp}(1)$ for the compact real form of the $\mathfrak{sl}(2, \mathbb{C})$ -subalgebra defined by α . Let B denote the Killing form of \mathfrak{g} , set $\mathfrak{g}_1 = (\mathfrak{h} \oplus \mathfrak{k})^{\perp_B}$ and consider the reductive complement $\mathfrak{m} = \mathfrak{h}^{\perp_B} = \mathfrak{k} \oplus \mathfrak{g}_1$. Let G be the simply connected Lie group with Lie algebra \mathfrak{g} and let $H \subset G$ be the connected subgroup with Lie algebra \mathfrak{h} . Define a G -invariant Riemannian metric g on $M = G/H$ by extending the inner product on $T_{eH}M \cong \mathfrak{m}$ given by

$$g|_{\mathfrak{k} \times \mathfrak{k}} = -\frac{1}{4(n+2)} B, \quad g|_{\mathfrak{g}_1 \times \mathfrak{g}_1} = -\frac{1}{8(n+2)} B, \quad g|_{\mathfrak{k} \times \mathfrak{g}_1} = 0.$$

Consider a basis X_1, X_2, X_3 of \mathfrak{k} satisfying the commutator relations $[X_i, X_j] = 2\varepsilon_{ijk}X_k$ and extend $X_i \in \mathfrak{m} \cong T_{eH}M$ to a G -invariant vector field ξ_i on M . Let η_i denote the metric dual of ξ_i and φ_i the G -invariant endomorphism field defined by extending

$$\varphi_i|_{\mathfrak{k}} = \frac{1}{2} \operatorname{ad}_{X_i}, \quad \varphi_i|_{\mathfrak{g}_1} = \operatorname{ad}_{X_i}.$$

Then, $(g, \xi_i, \eta_i, \varphi_i)_{i=1,2,3}$ is a G -invariant 3-Sasakian structure on M .

Conversely, given a simply connected homogeneous 3-Sasakian manifold M , represented as the quotient \tilde{G}/\tilde{H} , where \tilde{G} is a connected Lie group acting effectively on M , then $\tilde{G} = \operatorname{Aut}_0(M)$, the connected component of the 3-Sasakian automorphism group of M , and M is the unique space associated with the complexification of the Lie algebra of \tilde{G} .

Using this characterization, we rediscover the list of homogeneous 3-Sasakian manifolds as given by BOYER, GALICKI and MANN:

Corollary 1.2 *Every homogeneous 3-Sasakian manifold $M = G/H$ (not necessarily simply connected) is isomorphic to one of the following spaces:*

$$\frac{\operatorname{Sp}(n+1)}{\operatorname{Sp}(n)} \cong S^{4n+3}, \quad \frac{\operatorname{Sp}(n+1)}{\operatorname{Sp}(n) \times \mathbb{Z}_2} \cong \mathbb{R}P^{4n+3}, \quad \frac{\operatorname{SU}(m)}{S(U(m-2) \times U(1))},$$

$$\frac{\operatorname{SO}(k)}{\operatorname{SO}(k-4) \times \operatorname{Sp}(1)}, \quad \frac{G_2}{\operatorname{Sp}(1)}, \quad \frac{F_4}{\operatorname{Sp}(3)}, \quad \frac{E_6}{\operatorname{SU}(6)}, \quad \frac{E_7}{\operatorname{Spin}(12)}, \quad \frac{E_8}{E_7}.$$

To avoid redundancy, we need to assume $n \geq 0, m \geq 3$ and $k \geq 7$.

As a consequence, we also arrive at the complete list of homogeneous positive qK manifolds as discovered by ALEKSEEVSKII:

Corollary 1.3 *Every homogeneous positive qK manifold is isometric to one of the spaces*

$$\frac{\mathrm{Sp}(n+1)}{\mathrm{Sp}(n) \times \mathrm{Sp}(1)}, \frac{\mathrm{SU}(m)}{\mathrm{S}(U(m-2) \times U(2))}, \frac{\mathrm{SO}(k)}{\mathrm{SO}(k-4) \times \mathrm{SO}(4)},$$

$$\frac{G_2}{\mathrm{SO}(4)}, \frac{F_4}{\mathrm{Sp}(3)\mathrm{Sp}(1)}, \frac{E_6}{\mathrm{SU}(6)\mathrm{Sp}(1)}, \frac{E_7}{\mathrm{Spin}(12)\mathrm{Sp}(1)}, \frac{E_8}{E_7\mathrm{Sp}(1)},$$

where the Riemannian metric and quaternionic structure are also determined by Theorem 1.1 via the so-called Konishi bundle (see Proposition 2.7 and Sect. 11 for details).

The discussion of Theorem 1.1 and its consequences will be divided into several sections: We begin by recalling basic definitions and features of 3-Sasakian geometry in Sect. 2. We then summarize the history of the classification (Sect. 3) and introduce a certain \mathbb{Z} -grading of semisimple complex Lie algebras based on their root systems (Sect. 4). The first half of the proof of Theorem 1.1 is given in Sect. 5, where we construct homogeneous 3-Sasakian manifolds from simple Lie algebras. The centerpiece of this article is the converse argument in Sect. 6. For reasons that will become apparent during the construction, the special case of the exceptional Lie algebra \mathfrak{g}_2 needs to be relegated to Sect. 7. We complete the proof of Theorem 1.1 by showing that no proper subgroup of the identity component $\mathrm{Aut}_0(M)$ of the automorphism group can act transitively in Sect. 8. In Sect. 9 we compute the isotropy groups described in Corollary 1.2 explicitly for the classical spaces and Lie theoretically via Borel–de Siebenthal theory in the exceptional cases. In Sect. 10, we show that the only non simply connected homogeneous 3-Sasakian manifolds are the real projective spaces $\mathbb{R}P^{4n+3}$, which are the \mathbb{Z}_2 -quotient of the previously described space $S^{4n+3} = \mathrm{Sp}(n+1)/\mathrm{Sp}(n)$. Finally, since our arguments are independent of the classification of homogeneous positive qK manifolds, they allow for an alternative proof of the latter (Sect. 11).

2 Fundamentals of 3-Sasakian geometry

3-Sasakian geometry may be approached from a variety of different starting points, including but not limited to: qK and hyperkähler geometry, Einstein geometry, spin geometry and certain areas of algebraic geometry. For the sake of brevity, we decided to limit the exposition in this article to the necessary minimum. The interested reader is referred to the comprehensive monograph [9, Chapters 6, 8, 13].

Definition 2.1 A Sasakian structure is a tuple $(M^{2n+1}, g, \xi, \eta, \varphi)$, where ξ is a unit length Killing vector field, the one form $\eta = g(\xi, \cdot)$, the endomorphism field $\varphi = -\nabla^g \xi$ and the following curvature condition is satisfied:

$$R(X, \xi)Y = \eta(Y)X - g(X, Y)\xi \quad \text{for all } X, Y \in \mathfrak{X}(M).$$

We refer to the objects ξ, η and φ as the *Reeb vector field*, *contact form* and *almost complex structure* of the Sasakian structure.

Remark 2.2 The definition entails several important identities. Other common definitions of Sasakian manifolds in the literature are usually some selection of these. For $X, Y \in \mathfrak{X}(M)$ we have

$$\varphi^2 = -\text{id} + \eta \otimes \xi, \quad g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0, \\ g(X, \varphi Y) + g(\varphi X, Y) = 0, \quad d\eta(X, Y) = 2g(X, \varphi Y), \quad d\eta(\varphi X, \varphi Y) = d\eta(X, Y).$$

We also remark that the Reeb vector field ξ is characterized uniquely by the properties $\eta(\xi) = 1, d\eta(\xi, \cdot) = 0$ and that the cone over a Sasakian manifold admits a Kähler structure.

Definition 2.3 A *3-Sasakian structure* is a tuple $(M^{4n+3}, g, \xi_i, \eta_i, \varphi_i)_{i=1,2,3}$ such that each $(M, g, \xi_i, \eta_i, \varphi_i)$ is a Sasakian structure and $g(\xi_i, \xi_j) = \delta_{ij}, [\xi_i, \xi_j] = 2\varepsilon_{ijk}\xi_k$, where ε_{ijk} denotes the Levi-Civita symbol and (i, j, k) is a permutation of $(1, 2, 3)$.

Remark 2.4 For any cyclic permutation (i, j, k) of $(1, 2, 3)$ the definition implies the compatibility conditions

$$\varphi_i \circ \varphi_j - \eta_j \otimes \xi_i = \varphi_k, \quad \varphi_i \xi_j = \xi_k, \quad \eta_i \circ \varphi_j = \eta_k.$$

The cone over a 3-Sasakian manifold admits a hyperkähler structure.

Proposition 2.5 ([9, Corollary 13.2.3]) *Every 3-Sasakian manifold M of dimension $4n + 3$ is Einstein with Einstein constant $2(2n + 1)$. Moreover, if M is complete, it is compact with finite fundamental group.*

Definition 2.6 A *3-Sasakian isomorphism* between 3-Sasakian manifolds $(M, g, \xi_i, \eta_i, \varphi_i)$ and $(\tilde{M}, \tilde{g}, \tilde{\xi}_i, \tilde{\eta}_i, \tilde{\varphi}_i)$ is an isometry $\phi: M \rightarrow \tilde{M}$ which satisfies one of the three equivalent conditions $\phi_*\xi_i = \tilde{\xi}_i, \phi^*\tilde{\eta}_i = \eta_i$ or $\phi_* \circ \varphi_i = \tilde{\varphi}_i \circ \phi_*$ for $i = 1, 2, 3$. We will mostly be interested in the case $(\tilde{M}, \tilde{g}, \tilde{\xi}_i, \tilde{\eta}_i, \tilde{\varphi}_i) = (M, g, \xi_i, \eta_i, \varphi_i)$, in which we call ϕ a *3-Sasakian automorphism* of M .

We denote the group of all such transformations by $\text{Aut}(M)$ and call M a *homogeneous 3-Sasakian manifold* if $\text{Aut}(M)$ acts transitively. Homogeneous 3-Sasakian manifolds are in particular Riemannian homogeneous and thus, complete and compact. The Lie algebra $\text{aut}(M)$ consists of the Killing vector fields $X \in \mathfrak{X}(M)$ such that $\mathcal{L}_X \xi_i = 0, \mathcal{L}_X \eta_i = 0$ and $\mathcal{L}_X \varphi_i = 0$ for $i = 1, 2, 3$.

Proposition 2.7 ([9, Theorem 13.3.13 & Proposition 13.4.5]) *The Reeb vector fields ξ_1, ξ_2, ξ_3 of a 3-Sasakian manifold M generate a 3-dimensional foliation \mathcal{F} , whose space of leaves M/\mathcal{F} is a positive (i.e. its scalar curvature is positive) qK orbifold.*

If M is a homogeneous 3-Sasakian manifold, then $M \rightarrow M/\mathcal{F}$ is a locally trivial Riemannian fibration over a homogeneous positive qK manifold with fiber $\text{Sp}(1)$ or $\text{SO}(3)$.

3 History of the classification

The earliest result concerning our topic was the classification for the related notion of (compact simply connected) homogeneous *complex contact manifolds* (so-called *C-spaces*) by BOOTHBY in 1961 [6], who showed that these are in one-to-one correspondence to simple complex Lie algebras. The much more famous next step was the work of WOLF and ALEKSEEVSKII on qK manifolds in the 1960s. First, WOLF showed in 1961 that there is a one-to-one correspondence between C-spaces and compact simply connected *symmetric* positive qK manifolds [17, Theorem 6.1].

BOOTHBY and WOLF already emphasized the importance of the maximal root in the root system of a simple Lie algebra, which will also play a key role in our construction: WOLF demonstrated that the compact simply connected symmetric positive qK manifolds are precisely of the form $G/N_G(K)$, where G is a compact simple Lie group and $N_G(K)$ denotes the normalizer of the subgroup K corresponding to the compact real form of the subalgebra generated by the root spaces of a maximal root and its negative. These manifolds became known as WOLF spaces. As we will show in this article, the simply connected homogeneous 3-Sasakian manifolds are of the form $G/(C_G(K))_0$, where $(C_G(K))_0$ is the identity component of the centralizer $C_G(K)$ of K in G . In 1968 ALEKSEEVSKII fully classified compact *homogeneous* positive qK manifolds by demonstrating that they are necessarily of the form $G/N_G(K)$ [2, Theorem 1].

By 1994, BOYER, GALICKI and MANN transferred these results to the 3-Sasakian realm [10]. They combined the classification of homogeneous positive qK manifolds with Proposition 2.7 to obtain the following diffeomorphism type classification:

Theorem 3.1 ([10, Theorem C]) *Every homogeneous 3-Sasakian manifold $M = G/H$ (not necessarily simply connected) is precisely one of the following:*

$$\frac{\text{Sp}(n+1)}{\text{Sp}(n)} \cong S^{4n+3}, \quad \frac{\text{Sp}(n+1)}{\text{Sp}(n) \times \mathbb{Z}_2} \cong \mathbb{R}P^{4n+3}, \quad \frac{\text{SU}(m)}{S(U(m-2) \times U(1))},$$

$$\frac{\text{SO}(k)}{\text{SO}(k-4) \times \text{Sp}(1)}, \quad \frac{G_2}{\text{Sp}(1)}, \quad \frac{F_4}{\text{Sp}(3)}, \quad \frac{E_6}{\text{SU}(6)}, \quad \frac{E_7}{\text{Spin}(12)}, \quad \frac{E_8}{E_7}.$$

To avoid redundancy, we need to assume $n \geq 0, m \geq 3$ and $k \geq 7$.

They also provided a more precise description of the 3-Sasakian structures in the four classical cases via *3-Sasakian reduction* [10].

In 1996, BIELAWSKI [5] described the Riemannian structure on these spaces uniformly. Both for his result and for several later discussions, we need to recall the following construction: As was first described systematically by KOBAYASHI and NOMIZU [14], the study of G -invariant geometric objects on a reductive homogeneous space $M = G/H = G/G_p$ can be greatly simplified by instead considering $\text{Ad}(H)$ -invariant algebraic objects on a fixed reductive complement \mathfrak{m} of \mathfrak{h} in \mathfrak{g} . More precisely, the map $\psi: \mathfrak{m} \rightarrow T_pM, X \mapsto \overline{X}_p$ (where \overline{X}_p denotes the fundamental vector field of the left G -action at p) is an isomorphism that allows us to translate between $\text{Ad}(H)$ -invariant tensors on \mathfrak{m} and the restriction of G -invariant tensor fields to T_pM .

While actually working on a more algebro-geometric problem (singularities of nilpotent varieties) and employing very different methods (e.g. Nahm’s differential equation), BIELAWSKI obtained the following

Theorem 3.2 ([5, Theorem 4]) *For every homogeneous 3-Sasakian manifold $M = G/H$ with reductive decomposition $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{h}$, there is a natural decomposition $\mathfrak{m} = \mathfrak{sp}(1) \oplus \mathfrak{m}'$ such that the metric on M corresponds to an inner product on \mathfrak{m} of the form*

$$(X, Y) \mapsto -c B(X_{\mathfrak{sp}(1)}, Y_{\mathfrak{sp}(1)}) - \frac{c}{2} B(X_{\mathfrak{m}'}, Y_{\mathfrak{m}'})$$

where B denotes the Killing form of \mathfrak{g} and $c > 0$ is some constant.

In 2020, the work of DRAPER, ORTEGA and PALOMO gave a new hands-on description of homogeneous 3-Sasakian manifolds [11]. Their study was based on the following

Definition 3.3 ([11, Definition 4.1]) *A 3-Sasakian datum is a pair $(\mathfrak{g}, \mathfrak{h})$ of real Lie algebras such that*

- $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a \mathbb{Z}_2 -graded compact simple Lie algebra whose even part is a sum of two commuting subalgebras,

$$\mathfrak{g}_0 = \mathfrak{sp}(1) \oplus \mathfrak{h};$$

- there exists an $\mathfrak{h}^{\mathbb{C}}$ -module W such that the complexified $\mathfrak{g}_0^{\mathbb{C}}$ -module $\mathfrak{g}_1^{\mathbb{C}}$ is isomorphic to the tensor product of the natural $\mathfrak{sp}(1)^{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C})$ -module \mathbb{C}^2 and W :

$$\mathfrak{g}_1^{\mathbb{C}} \cong \mathbb{C}^2 \otimes W.$$

Their main result is the following

Theorem 3.4 ([11, Theorem 4.2]) *Let $M = G/H$ be a homogeneous space such that H is connected and the Lie algebras $(\mathfrak{g}, \mathfrak{h})$ constitute a 3-Sasakian datum. Consider the reductive complement $\mathfrak{m} := \mathfrak{sp}(1) \oplus \mathfrak{g}_1$ and let $X_1, X_2, X_3 \in \mathfrak{m}$ denote the standard basis of $\mathfrak{sp}(1)$ and ξ_1, ξ_2, ξ_3 the corresponding G -invariant vector fields on M . If g and φ_i are the Riemannian metric and endomorphism fields described in Theorem 1.1 and $\eta_i = g(\xi_i, \cdot)$, then the tuple $(M, g, \xi_i, \eta_i, \varphi_i)_{i=1,2,3}$ constitutes a homogeneous 3-Sasakian structure.*

Furthermore, they conducted a case-by-case study to show that every compact simple Lie algebra admits a 3-Sasakian datum, thus providing a detailed analysis of one homogeneous 3-Sasakian structure (it is, at this point, not clear if there could be more than one such structure on a given space) on each of the diffeomorphism types discovered by BOYER, GALICKI and MANN.

We finish this section by giving an overview of the structure of our proof of Theorem 1.1: In Sect. 5, we first describe a way to construct a simply connected homogeneous

3-Sasakian manifold from a simple complex Lie algebra \mathfrak{u} and a maximal root α of \mathfrak{u} . More precisely, we first utilize the theory of root systems to generate a complexified version $(\mathfrak{u}, \mathfrak{v})$ of a 3-Sasakian datum. We then pass to the compact real forms $(\mathfrak{g}, \mathfrak{h})$ to obtain a “real” 3-Sasakian datum in the sense of Definition 3.3 and apply Theorem 3.4.

For the converse argument in Sects. 6 and 7, we start with a simply connected homogeneous 3-Sasakian manifold $M = G/H$, where G is a compact simply connected Lie group acting almost effectively and transitively on M via 3-Sasakian automorphisms. We prove that the Lie algebra \mathfrak{g} and its complexification $\mathfrak{u} = \mathfrak{g}^{\mathbb{C}}$ are simple and that the 3-Sasakian structure gives rise to a maximal root α of \mathfrak{u} . We can therefore apply the previous construction and then show that this yields the same 3-Sasakian structure that we started with.

Sect. 8 completes the proof of Theorem 1.1 by showing that no subgroup of $\text{Aut}_0(M)$ can act transitively. In particular, this proves that any two homogeneous 3-Sasakian manifolds $M = G/H, M' = G'/H'$ associated with two different simple complex Lie algebras $\mathfrak{g}^{\mathbb{C}} \neq (\mathfrak{g}')^{\mathbb{C}}$ are not isomorphic.

4 Root system preliminaries

Later on, we will need certain basic facts about root systems, which we decided to collect in this section: Let \mathfrak{u} be a (finite-dimensional) semisimple complex Lie algebra. Its Killing form is non-degenerate and thus gives rise to an isomorphism $\mathfrak{u} \rightarrow \mathfrak{u}^*$ and a non-degenerate, symmetric bilinear form $\langle \cdot, \cdot \rangle$ on \mathfrak{u}^* . We fix a Cartan subalgebra $\mathfrak{c} \subset \mathfrak{u}$ and denote the corresponding root system and root spaces by $\Phi \subset \mathfrak{c}^*$ and $\mathfrak{u}_{\alpha} \subset \mathfrak{u}$ for $\alpha \in \mathfrak{c}^*$, respectively.

Each root $\alpha \in \Phi$ has an associated coroot $H_{\alpha} \in \mathfrak{c}$ defined as the unique element of $[\mathfrak{u}_{\alpha}, \mathfrak{u}_{-\alpha}]$ satisfying $\alpha(H_{\alpha}) = 2$. Furthermore, $\mathfrak{s}_{\alpha} := \mathfrak{u}_{\alpha} \oplus \mathfrak{u}_{-\alpha} \oplus [\mathfrak{u}_{\alpha}, \mathfrak{u}_{-\alpha}]$ is a subalgebra of \mathfrak{u} which is isomorphic to $\mathfrak{sl}(2, \mathbb{C})$. This isomorphism can be made explicit by choosing an \mathfrak{sl}_2 -triple, i.e. vectors $X_{\alpha} \in \mathfrak{u}_{\alpha}, Y_{\alpha} \in \mathfrak{u}_{-\alpha}$ satisfying the commutation relations

$$[H_{\alpha}, X_{\alpha}] = 2X_{\alpha}, \quad [H_{\alpha}, Y_{\alpha}] = -2Y_{\alpha}, \quad [X_{\alpha}, Y_{\alpha}] = H_{\alpha}. \tag{1}$$

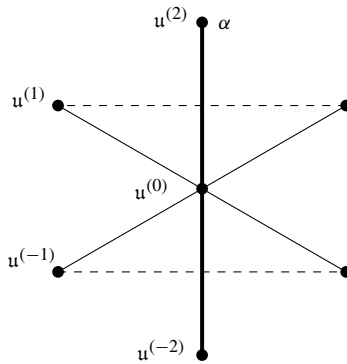
Moreover, it can be shown that for any root $\alpha \in \Phi$ and any linear form $\beta \in \mathfrak{c}^*$:

$$c_{\alpha\beta} := \beta(H_{\alpha}) = \frac{2\langle \beta, \alpha \rangle}{\langle \alpha, \alpha \rangle}.$$

In particular, $c_{\alpha\beta} = 0$ if and only if α and β are perpendicular to each other (with respect to $\langle \cdot, \cdot \rangle$). In case β is also a root, $c_{\alpha\beta}$ is an integer, which we will call the *Cartan number of β with respect to α* . Fixing a root $\alpha \in \Phi$, we can therefore decompose

$$\mathfrak{u} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{u}^{(k)}, \quad \text{where } \mathfrak{u}^{(k)} := \bigoplus_{\substack{\beta \in \mathfrak{c}^* \\ c_{\alpha\beta} = k}} \mathfrak{u}_{\beta}.$$

Since $c_{\alpha\beta}$ is linear in β , this decomposition is in fact a \mathbb{Z} -grading, i.e. $[u^{(k)}, u^{(\ell)}] \subset u^{(k+\ell)}$. We also note that $u^{(k)}$ is precisely the k -eigenspace of $\text{ad}(H_\alpha)$. One can visualize this grading using parallel copies of hyperplanes perpendicular to α , e.g. for the root system A_2 :



The structure of this grading is related to the notion of *maximality* of the root α : Assuming we have chosen a set $\Delta \subset \Phi$ of simple roots, we may introduce a partial order \leq on Φ by stipulating that $\alpha \leq \beta$ if and only if $\beta - \alpha$ is a linear combination of roots in Δ with non-negative coefficients. A root $\alpha \in \Phi$ is called *maximal* if there is a choice of simple roots such that there is no strictly larger root than α with respect to the induced partial order. The following lemma was adapted from [17, Theorem 4.2]:

Lemma 4.1 *For any root $\alpha \in \Phi$, the following statements are equivalent:*

- (i) α is maximal.
- (ii) $|c_{\alpha\beta}| \leq 2$ for all roots $\beta \in \Phi$ and $c_{\alpha\beta} = \pm 2$ if and only if $\beta = \pm\alpha$.

Proof (i) \Rightarrow (ii): It is well known that for $\beta \in \Phi \setminus \{\pm\alpha\}$, the Cartan number is given by $c_{\alpha\beta} = p - q$, where $p, q \in \mathbb{N}_0$ are the greatest non-negative integers such that $\beta + r\alpha \in \Phi$ for every $r \in \{-p, \dots, q\}$ [13, Proposition 2.29]. Suppose there was some $\beta \in \Phi \setminus \{\alpha\}$ such that $c_{\alpha\beta} \geq 2$. Then $p \geq 2$, so that $\beta - \alpha, \beta - 2\alpha \in \Phi$ and their negatives $\alpha - \beta, 2\alpha - \beta \in \Phi$ are roots. In fact, $\alpha - \beta$ has to be a *non-negative* linear combination of simple roots (for some choice of simple roots with respect to which α is maximal), since otherwise $\beta > \alpha$. But then $2\alpha - \beta \geq \alpha$ and maximality of α would imply $2\alpha - \beta = \alpha$, i.e. $\beta = \alpha$. For $\beta \in \Phi \setminus \{-\alpha\}$ such that $c_{\alpha\beta} \leq -2$, we apply this argument to $-\beta$.

(ii) \Rightarrow (i): We may choose a set of simple roots Δ in such a way that $c_{\alpha\beta} \geq 0$ for all $\beta \in \Delta$. This can be achieved by first choosing positive roots using a slight perturbation of the hyperplane perpendicular to α . Let $\beta \in \Phi$ such that $\beta \geq \alpha$, i.e. $\beta - \alpha = \sum_{i=1}^n \lambda_i \alpha_i$, where $\lambda_i \geq 0$ and $\alpha_i \in \Delta$. Then,

$$c_{\alpha\beta} = c_{\alpha\alpha} + c_{\alpha(\beta-\alpha)} = 2 + \sum_{i=1}^n \lambda_i \underbrace{c_{\alpha\alpha_i}}_{\geq 0} \geq 2.$$

Hypothesis (ii) then implies $\beta = \alpha$, so that α is maximal. □

Finally, we remark that in an *irreducible* root system Φ , the maximal root is unique up to the action of the Weyl group: This follows because in an irreducible root system, the maximal root is uniquely determined after choosing simple roots, and any two choices of simple roots can be mapped to each other by the Weyl group.

5 Constructing homogeneous 3-Sasakian manifolds from simple Lie algebras

Our goal in this section is the following construction:

Theorem 5.1 *Let \mathfrak{u} be a simple complex Lie algebra, α a maximal root in its root system, \mathfrak{g} the compact real form of \mathfrak{u} and $\mathfrak{k} \cong \mathfrak{sp}(1)$ the compact real form of the subalgebra $\mathfrak{s}_\alpha = \mathfrak{u}_\alpha \oplus \mathfrak{u}_{-\alpha} \oplus [\mathfrak{u}_\alpha, \mathfrak{u}_{-\alpha}] \cong \mathfrak{sl}(2, \mathbb{C})$. Let G denote the simply connected Lie group with Lie algebra \mathfrak{g} , K the connected subgroup with Lie algebra \mathfrak{k} and $H = (C_G(K))_0$ the identity component of the centralizer $C_G(K)$ of K in G . Then, the simply connected homogeneous space $M = G/H$ admits a homogeneous 3-Sasakian structure whose tensors are given by Theorem 1.1. All possible choices of a maximal root lead to isomorphic 3-Sasakian manifolds.*

Definition 5.2 A *complex 3-Sasakian datum* is a pair $(\mathfrak{u}, \mathfrak{v})$ of complex Lie algebras such that

- $\mathfrak{u} = \mathfrak{u}_{\text{even}} \oplus \mathfrak{u}_{\text{odd}}$ is a \mathbb{Z}_2 -graded simple Lie algebra whose even part is a sum of two commuting subalgebras,

$$\mathfrak{u}_{\text{even}} = \mathfrak{v} \oplus \mathfrak{sl}(2, \mathbb{C});$$

- there exists a \mathfrak{v} -module W such that $\mathfrak{u}_{\text{odd}} \cong \mathbb{C}^2 \otimes W$ as $\mathfrak{u}_{\text{even}}$ -modules.

Remark 5.3 We formulated the above definition in the given way because it allows us to branch off into two cases: Our primary interest in this article will be to consider the *compact* real forms $(\mathfrak{g}, \mathfrak{h})$ of $(\mathfrak{u}, \mathfrak{v})$ which then form a 3-Sasakian datum in the sense of Definition 3.3. On the other hand, one may also look at the real form $(\mathfrak{g}^*, \mathfrak{h})$ of $(\mathfrak{u}, \mathfrak{v})$ given by $\mathfrak{g}^* = \mathfrak{h} \oplus \mathfrak{s}_\alpha \oplus i\mathfrak{g}_1$, to obtain a *generalized 3-Sasakian datum* in the sense of [1]. These give rise to homogeneous negative 3- (α, δ) -Sasakian manifolds by a construction similar to Theorem 3.4, compare [1, Theorem 3.1.1].

Proposition 5.4 *Let \mathfrak{u} be a simple complex Lie algebra and $\alpha \in \Phi$ a maximal root in its root system. Set $\Phi_0 := \{\beta \in \Phi \mid c_{\alpha\beta} = 0\}$ as well as*

$$\mathfrak{v} := \ker \alpha \oplus \bigoplus_{\beta \in \Phi_0} \mathfrak{u}_\beta.$$

Then, $(\mathfrak{u}, \mathfrak{v})$ is a complex 3-Sasakian datum.

Proof Using the \mathbb{Z} -grading from Sect. 4, we let

$$u_{\text{even}} := u^{(-2)} \oplus u^{(0)} \oplus u^{(2)}, \quad u_{\text{odd}} := u^{(-1)} \oplus u^{(1)}.$$

Since $|c_{\alpha\beta}| \leq 2$ for all $\beta \in \Phi$ by Lemma 4.1, we have $u = u_{\text{even}} \oplus u_{\text{odd}}$. Because u_{even} and u_{odd} are comprised of the $u^{(k)}$ with even and odd k respectively, this decomposition is in fact a \mathbb{Z}_2 -grading. We claim that

$$u_{\text{even}} = \mathfrak{s}_\alpha \oplus \mathfrak{v}$$

as a direct sum of Lie algebras, where $\mathfrak{s}_\alpha = u_\alpha \oplus u_{-\alpha} \oplus [u_\alpha, u_{-\alpha}]$. Since $c_{\alpha\beta} = \pm 2$ if and only if $\beta = \pm\alpha$, we have the following vector space decompositions:

$$u_{\text{even}} = u_\alpha \oplus u_{-\alpha} \oplus \mathfrak{c} \oplus \bigoplus_{\beta \in \Phi_0} u_\beta, \quad \mathfrak{c} = [u_\alpha, u_{-\alpha}] \oplus \ker \alpha.$$

In order to show that \mathfrak{v} is indeed a subalgebra of u , note that $[u_\beta, u_\gamma] \subset u_{\beta+\gamma}$ for any $\beta, \gamma \in \Phi_0$. Now if $\beta + \gamma$ is a root, then $\beta + \gamma \in \Phi_0$, so $u_{\beta+\gamma} \subset \mathfrak{v}$. If $\beta + \gamma$ is not a root and not zero, then $u_{\beta+\gamma} = 0 \subset \mathfrak{v}$. If $\beta + \gamma = 0$, then $[u_\beta, u_{-\beta}] = \langle H_\beta \rangle \subset \ker \alpha$ because $\beta \in \Phi_0$. To check that \mathfrak{s}_α and \mathfrak{v} commute, we recall that \mathfrak{v} is a subset of $u^{(0)} = \ker \text{ad}_{H_\alpha}$. For $\beta \in \Phi_0$, we have $[u_{\pm\alpha}, u_\beta] \subset u_{\pm\alpha+\beta} \subset u^{(\pm 2)} = u_{\pm\alpha}$, so $u_{\pm\alpha}$ and u_β commute.

We now verify the second condition from Definition 5.2 for the \mathfrak{v} -module $W := u^{(1)}$. We choose an \mathfrak{sl}_2 -triple $(X_\alpha, Y_\alpha, H_\alpha)$ and identify it (in order) with the three standard matrices

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{C}).$$

This fixes isomorphisms $\mathfrak{s}_\alpha \cong \mathfrak{sl}(2, \mathbb{C})$ and $u_{\text{even}} \cong \mathfrak{v} \oplus \mathfrak{sl}(2, \mathbb{C})$. We consider the following linear map:

$$\begin{aligned} \Psi : u_{\text{odd}} = u^{(-1)} \oplus u^{(1)} &\rightarrow \mathbb{C}^2 \otimes W, \\ X = X^{(-1)} + X^{(1)} &\mapsto (1, 0) \otimes X^{(1)} + (0, 1) \otimes [X_\alpha, X^{(-1)}]. \end{aligned}$$

If $\beta \in \Phi$ is such that $c_{\alpha\beta} = -1$, then $\beta + \alpha$ must be a root and $[u_\alpha, u_\beta] = u_{\alpha+\beta}$. This shows that $\text{ad}_{X_\alpha} : u_\beta \rightarrow u_{\beta+\alpha}$ and, by extension, Ψ are linear isomorphisms. It remains to be shown that Ψ preserves the u_{even} -module structure, where $u_{\text{even}} \cong \mathfrak{v} \oplus \mathfrak{sl}(2, \mathbb{C})$ acts on $\mathbb{C}^2 \otimes W$ via the above fixed isomorphism. We remind the reader of the commutator relations in (1).

If $Z \in \mathfrak{v} \subset u^{(0)}$, then ad_Z preserves the decomposition $u_{\text{odd}} = u^{(-1)} \oplus u^{(1)}$. Since \mathfrak{v} and \mathfrak{s}_α are commuting subalgebras of u , so are their respective adjoint subrepresentations.

tations,

$$\begin{aligned} \Psi([Z, X]) &= (1, 0) \otimes [Z, X^{(1)}] + (0, 1) \otimes [X_\alpha, [Z, X^{(-1)}]] \\ &= (1, 0) \otimes [Z, X^{(1)}] + (0, 1) \otimes [Z, [X_\alpha, X^{(-1)}]] = Z \cdot \Psi(X). \end{aligned}$$

Here, \cdot denotes the adjoint representation of \mathfrak{v} on W , while in the following equations it will signify the standard representation of $\mathfrak{sl}(2, \mathbb{C})$ on \mathbb{C}^2 . Finally, we check the representation of the basis $(X_\alpha, Y_\alpha, H_\alpha)$ of \mathfrak{s}_α :

$$\Psi([X_\alpha, X]) = \Psi([X_\alpha, X^{(-1)}]) = (1, 0) \otimes [X_\alpha, X^{(-1)}] = X_\alpha \cdot \Psi(X).$$

By the Jacobi identity,

$$\begin{aligned} \Psi([Y_\alpha, X]) &= \Psi([Y_\alpha, X^{(1)}]) = (0, 1) \otimes [X_\alpha, [Y_\alpha, X^{(1)}]] \\ &= (0, 1) \otimes \left(\underbrace{[[X^{(1)}, X_\alpha], Y_\alpha]}_{=0} + \underbrace{[[X_\alpha, Y_\alpha], X^{(1)}]}_{=H_\alpha} \right) \\ &= (0, 1) \otimes X^{(1)} = Y_\alpha \cdot \Psi(X). \end{aligned}$$

Ultimately,

$$\begin{aligned} \Psi([H_\alpha, X]) &= \Psi(X^{(1)} - X^{(-1)}) \\ &= (1, 0) \otimes X^{(1)} + (0, -1) \otimes [X_\alpha, X^{(-1)}] = H_\alpha \cdot \Psi(X). \quad \square \end{aligned}$$

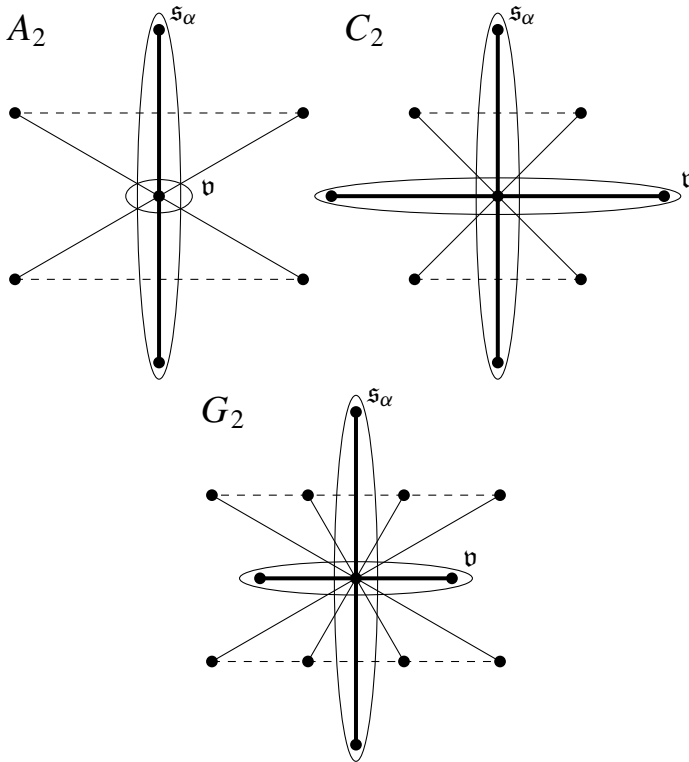
Proof of Theorem 5.1 Starting from a simple complex Lie algebra \mathfrak{u} and a maximal root α , Proposition 5.4 yields a complex 3-Sasakian datum $(\mathfrak{u}, \mathfrak{v})$. As mentioned in Remark 5.3, the compact real forms $(\mathfrak{g}, \mathfrak{h})$ constitute a “real” 3-Sasakian datum in the sense of Definition 3.3 and Theorem 3.4 endows $M = G/H$ with a homogeneous 3-Sasakian structure. Since $\mathfrak{v} = C_{\mathfrak{u}}(\mathfrak{s}_\alpha)$ and thus $\mathfrak{h} = C_{\mathfrak{g}}(\mathfrak{k})$, it follows that $H = (C_G(K))_0$. \square

Example 5.5 Let us illustrate the construction using the special case $\text{rk } \mathfrak{u} = 2$: Here, the only simple Lie algebras are $\mathfrak{sl}(3, \mathbb{C})$, $\mathfrak{sp}(4, \mathbb{C})$ and \mathfrak{g}_2 , corresponding (in order) to the root systems A_2 , C_2 and G_2 . The diagrams on the next page depict the subalgebras \mathfrak{v} and \mathfrak{s}_α from the proposition in these three cases.

The corresponding homogeneous 3-Sasakian manifolds are (in order) the Aloff–Wallach space $W^{1,1} = \text{SU}(3)/S^1$, the 7-sphere $S^7 = \text{Sp}(2)/\text{Sp}(1)$ and the exceptional space $G_2/\text{Sp}(1)$.

We finish this section by showing that the maximal root is in fact an auxiliary choice:

Lemma 5.6 *All possible choices of a maximal root in Proposition 5.4 lead to isomorphic 3-Sasakian manifolds.*



Proof Let \mathfrak{u} be a simple complex Lie algebra, \mathfrak{g} its compact real form, G the corresponding simply connected Lie group and $T \subset G$ a maximal torus. Let $\alpha, \tilde{\alpha}$ denote two maximal roots in the root system Φ of \mathfrak{u} with respect to the Cartan subalgebra given by the complexification of the Lie algebra of T . As mentioned at the end of Sect. 4, the maximal root of Φ is unique up to the action of the Weyl group $W(G) = N_G(T)/T$, so there is a representative $w \in N_G(T)$ such that $\text{Ad}_w^C(H_\alpha) = H_{\tilde{\alpha}}$.

Because the Weyl group acts orthogonally on the root system, $\text{Ad}_w^C : \mathfrak{u} \rightarrow \mathfrak{u}$ maps the \mathbb{Z} -grading $\tilde{\mathfrak{u}}^{(k)}$ with respect to α to the grading $\tilde{\mathfrak{u}}^{(k)}$ with respect to $\tilde{\alpha}$. This implies that $\text{Ad}_w \mathfrak{h} = \tilde{\mathfrak{h}}$, where $\mathfrak{h}, \tilde{\mathfrak{h}} \subset \mathfrak{g}$ are the compact real forms of the subalgebras $\mathfrak{h}, \tilde{\mathfrak{h}} \subset \mathfrak{u}$ considered in Proposition 5.4. Consequently, $wHw^{-1} = \tilde{H}$ for the corresponding connected subgroups $H, \tilde{H} \subset G$ and we have a well-defined diffeomorphism $G/H \rightarrow G/\tilde{H}$, $gH \mapsto wgw^{-1}\tilde{H}$. One easily checks from the definitions in Theorem 1.1 that this map is a 3-Sasakian isomorphism. \square

6 Deconstructing homogeneous 3-Sasakian manifolds

This section is the centerpiece of the article, where we explain a crucial step in the proof of Theorem 1.1, namely:

Theorem 6.1 *Every simply connected homogeneous 3-Sasakian manifold arises from the construction described in Sect. 5.*

From now on, let $(M^{4n+3}, g, \xi_i, \eta_i, \varphi_i)_{i=1,2,3}$ denote a simply connected homogeneous 3-Sasakian manifold and let G be a compact simply connected Lie group acting almost effectively (i.e. the kernel of the action is discrete and hence finite) and transitively on M by 3-Sasakian automorphisms. We will show that the Lie algebra \mathfrak{g} of G and its complexification $\mathfrak{u} = \mathfrak{g}^{\mathbb{C}}$ are simple and describe how the 3-Sasakian structure gives rise to a maximal root α of \mathfrak{u} with respect to a suitably chosen Cartan subalgebra. We can then apply the construction from Sect. 5 and prove that this yields the same 3-Sasakian structure that we started with.

The prototypical example to have in mind is where G is the universal cover of $\text{Aut}_0(M)$, the identity component of the 3-Sasakian automorphism group of M . By Proposition 2.5, M is compact, so by the Myers–Steenrod theorem, the isometry group $\text{Isom}(M)$ of M is a compact Lie group. The subgroup $\text{Aut}(M) \subset \text{Isom}(M)$ of 3-Sasakian automorphisms of M is clearly closed and thus also a compact Lie group. Since M is connected, the identity component $\text{Aut}_0(M)$ still acts transitively. The universal cover of $\text{Aut}_0(M)$ acts almost effectively, transitively and by 3-Sasakian automorphisms. It will follow from the results that we are about to prove that the universal cover of $\text{Aut}_0(M)$ is also compact.

Later on, we will show that, in fact, the effectively acting quotient of any group G satisfying the above assumptions is automatically the full identity component $\text{Aut}_0(M)$ of the automorphism group.

Since G is compact, its Lie algebra \mathfrak{g} is reductive, i.e. decomposes as a direct sum of a semisimple subalgebra and its center $Z(\mathfrak{g})$. We first show that \mathfrak{g} itself is semisimple.

Lemma 6.2 *For $X, Y \in \mathfrak{g}$, the fundamental vector fields satisfy the equation*

$$d\eta_i(\bar{X}, \bar{Y}) = \eta_i([\bar{X}, \bar{Y}]).$$

Notably, evaluating the left-hand side at a point $p \in M$ depends on \bar{X}, \bar{Y} only through their values at p , while the right-hand side a priori depends on the values in a neighborhood of p .

Proof The standard formula for the exterior derivative reads

$$d\eta_i(\bar{X}, \bar{Y}) = \bar{X}(\eta_i(\bar{Y})) - \bar{Y}(\eta_i(\bar{X})) - \eta_i([\bar{X}, \bar{Y}]).$$

The Leibniz rule for the Lie derivative implies

$$\bar{X}(\eta_i(\bar{Y})) = \mathcal{L}_{\bar{X}}(\eta_i(\bar{Y})) = (\mathcal{L}_{\bar{X}}\eta_i)(\bar{Y}) + \eta_i(\mathcal{L}_{\bar{X}}\bar{Y}) = \eta_i([\bar{X}, \bar{Y}]),$$

where $\mathcal{L}_{\overline{X}}\eta_i = 0$ because G acts by 3-Sasakian automorphisms. Applying the same reasoning to the second term yields $\overline{Y}(\eta_i(\overline{X})) = -\eta_i([\overline{X}, \overline{Y}])$. \square

Proposition 6.3 *The Lie algebra \mathfrak{g} has trivial center and is therefore semisimple.*

Proof Let $X \in \mathfrak{g}$ be such that $X \neq 0$. Since G acts almost effectively, there is a point $p \in M$ such that $\overline{X}_p \neq 0$ and thus an index $i \in \{1, 2, 3\}$ such that \overline{X}_p is not proportional to $(\xi_i)_p$. We show that there exists some $Y \in \mathfrak{g}$ satisfying $\eta_i([\overline{X}, \overline{Y}]_p) \neq 0$, which implies $[X, Y] \neq 0$: Because G acts transitively, we may choose some $Y \in \mathfrak{g}$ such that $\overline{Y}_p = \varphi_i \overline{X}_p$. From the previous lemma, we have

$$\eta_i([\overline{X}, \overline{Y}]_p) = -d\eta_i(\overline{X}_p, \overline{Y}_p) = -d\eta_i(\overline{X}_p, \varphi_i \overline{X}_p).$$

One of the Sasaki equations in Remark 2.2 reads $\varphi_i^2 \overline{X}_p = -\overline{X}_p + P_i \overline{X}_p$, where P_i denotes the orthogonal projection to the line through $(\xi_i)_p$. Hence,

$$\eta_i([\overline{X}, \overline{Y}]_p) = 2g_p(\overline{X}_p, \overline{X}_p - P_i \overline{X}_p) = 2\|\overline{X}_p - P_i \overline{X}_p\|^2 \neq 0. \quad \square$$

Remark 6.4 The compactness assumption fails for homogeneous negative 3- (α, δ) -Sasakian manifolds. Thus, unlike with the construction in the previous section, a classification cannot be achieved by the method described here. Indeed, in [1] homogenous negative 3- (α, δ) -Sasakian manifolds with a transitive action by a non-semisimple Lie group are constructed.

Since \mathfrak{g} is now both semisimple and the Lie algebra of a compact Lie group, its Killing form B is negative definite. We fix a point $p \in M$ and let $H := G_p$ denote its isotropy group. We write $\theta: G \rightarrow M, g \mapsto g \cdot p$ for the orbit map, which has surjective differential $d\theta_e: T_e G \cong \mathfrak{g} \rightarrow T_p M, X \mapsto \overline{X}_p$. Let $\alpha_i := \theta^* \eta_i$ denote the pullback of the contact form along the orbit map, which we may view — depending on the context — as either a linear form on \mathfrak{g} or as a left-invariant differential one-form on G . In their seminal 1958 article [7], BOOTHBY and WANG exhibited the following results

Lemma 6.5 ([7, Lemmata 2,3,4]) *The one-form α_i is $\text{Ad}(H)$ -invariant, satisfies $\alpha_i(\mathfrak{h}) = 0$ and $d\alpha_i$ has rank $4n + 2$. Furthermore, the Lie algebra of the subgroup $\{g \in G \mid \text{Ad}_g^* \alpha_i = \alpha_i\}$ is given by $\ker d\alpha_i$, contains \mathfrak{h} and has dimension $\dim \mathfrak{h} + 1$.*

We now let $\widetilde{X}_i \in \mathfrak{g}$ denote the Killing dual of α_i , i.e. $B(\widetilde{X}_i, \cdot) = \alpha_i$ and consider $X_i := \widetilde{X}_i/B(\widetilde{X}_i, \widetilde{X}_i)$. Ad-invariance of B implies that $\{g \in G \mid \text{Ad}_g X_i = X_i\}$ and $\{g \in G \mid \text{Ad}_g^* \alpha_i = \alpha_i\}$ coincide, so

$$C_{\mathfrak{g}}(X_i) = \ker d\alpha_i = \mathfrak{h} \oplus \langle X_i \rangle.$$

Proposition 6.6 *The fundamental vector fields \overline{X}_i coincide with the Reeb vector fields ξ_i at the point p and obey the same commutator relations $[X_i, X_j] = 2\varepsilon_{ijk} X_k$, where (i, j, k) is a permutation of $(1, 2, 3)$.*

Proof Clearly, $X_i \in C_{\mathfrak{g}}(X_i) = \ker d\alpha_i$, so that $(\overline{X_i})_p \in \ker(d\eta_i)_p$. Furthermore, we have $1 = \alpha_i(X_i) = (\eta_i)_p(\overline{X_i})_p$. Thus, $(\overline{X_i})_p$ satisfies the uniquely defining equations of the Reeb vector $(\xi_i)_p$. Phrased differently, X_i (viewed as a left-invariant vector field on G) and ξ_i are θ -related. Consequently, the Lie brackets $[X_i, X_j]$ and $[\xi_i, \xi_j] = 2\varepsilon_{ijk}\xi_k$ are also θ -related and in particular, $[\overline{X_i}, \overline{X_j}]_p = 2\varepsilon_{ijk}(\xi_k)_p = 2\varepsilon_{ijk}(\overline{X_k})_p$. Hence, $[X_i, X_j]$ and $2\varepsilon_{ijk}X_k$ could only differ by an element of \mathfrak{h} . But $B(X_k, \mathfrak{h}) = \alpha_k(\mathfrak{h}) = 0$ and $B([X_i, X_j], \mathfrak{h}) = B(X_i, [X_j, \mathfrak{h}]) = 0$, so that also $B([X_i, X_j] - 2\varepsilon_{ijk}X_k, \mathfrak{h}) = 0$. \square

Let \mathfrak{s} be a maximal Abelian subalgebra of \mathfrak{h} . Since $C_{\mathfrak{g}}(X_1) = \mathfrak{h} \oplus \langle X_1 \rangle$, it follows that $\mathfrak{t} := \mathfrak{s} \oplus \langle X_1 \rangle$ is a maximal Abelian subalgebra of \mathfrak{g} . In particular, $\text{rk } G = \text{rk } H + 1$. The Riemannian metric g corresponds to an $\text{Ad}(H)$ -invariant and thus also $\text{ad}(\mathfrak{h})$ -invariant inner product on a reductive complement of our choice. The following lemma states that this inner product is even $\text{ad}(\mathfrak{t})$ -invariant:

Lemma 6.7 *For all $Y, Z \in \mathfrak{g}$, we have*

$$g_p(\overline{[X_i, Y]}_p, \overline{Z}_p) + g_p(\overline{Y}_p, \overline{[X_i, Z]}_p) = 0.$$

Proof Since $\overline{X_i}$ is a Killing vector field (G acts isometrically) that coincides with ξ_i at p , we obtain

$$\begin{aligned} g_p(\overline{[X_i, Y]}_p, \overline{Z}_p) + g_p(\overline{Y}_p, \overline{[X_i, Z]}_p) &= -g_p(\overline{[X_i, Y]}_p, \overline{Z}_p) - g_p(\overline{Y}_p, \overline{[X_i, Z]}_p) \\ &= -(\overline{X_i})_p(g(\overline{Y}, \overline{Z})) = -(\xi_i)_p(g(\overline{Y}, \overline{Z})). \end{aligned}$$

Because the Levi-Civita connection ∇ is metric and torsion free and all G -fundamental fields commute with ξ_i (G acts by 3-Sasakian automorphisms), we have

$$\begin{aligned} (\xi_i)_p(g(\overline{Y}, \overline{Z})) &= g_p(\nabla_{(\xi_i)_p} \overline{Y}, \overline{Z}_p) + g_p(\overline{Y}_p, \nabla_{(\xi_i)_p} \overline{Z}) \\ &= g_p(\nabla_{\overline{Y}_p} \xi_i, \overline{Z}_p) + g_p(\overline{Y}_p, \nabla_{\overline{Z}_p} \xi_i). \end{aligned}$$

Finally, $\nabla \xi_i = -\varphi_i$ and $g(\cdot, \varphi_i \cdot)$ is skew-symmetric. \square

We now move on to the complex picture and let $\mathfrak{u} := \mathfrak{g}^{\mathbb{C}}, \mathfrak{v} := \mathfrak{h}^{\mathbb{C}}, \mathfrak{c} := \mathfrak{t}^{\mathbb{C}}$ and $\alpha := 2i\alpha_1^{\mathbb{C}}|_{\mathfrak{c}}$. Let us consider the vectors $H_\alpha, X_\alpha, Y_\alpha \in \mathfrak{u}$ defined by

$$H_\alpha := \frac{1}{i} X_1, \quad X_\alpha := \frac{1}{2i} (X_2 - iX_3), \quad Y_\alpha := \frac{1}{2i} (X_2 + iX_3),$$

which satisfy the commutation relations

$$[H_\alpha, X_\alpha] = 2X_\alpha, \quad [H_\alpha, Y_\alpha] = -2Y_\alpha, \quad [X_\alpha, Y_\alpha] = H_\alpha.$$

Proposition 6.8 *The linear form α is a root of \mathfrak{u} with respect to \mathfrak{c} , whose root space is given by $\mathfrak{u}_\alpha = \langle X_\alpha \rangle$. Furthermore, $\mathfrak{u}_{-\alpha} = \langle Y_\alpha \rangle$ and H_α is the coroot of α .*

Proof Firstly, $[H_\alpha, X_\alpha] = 2X_\alpha = \alpha(H_\alpha)X_\alpha$. Since X_2, X_3 commute with \mathfrak{h} , the vector X_α commutes with \mathfrak{v} and in particular with $\mathfrak{s}^\mathbb{C}$. Likewise, α_1 vanishes on \mathfrak{h} , so that α vanishes on \mathfrak{v} and in particular on $\mathfrak{s}^\mathbb{C}$. \square

Let $\Phi \subset \mathfrak{c}^*$ denote the root system of \mathfrak{u} with respect to \mathfrak{c} . We consider the \mathbb{Z} -grading of \mathfrak{u} introduced in Sect. 4, viz.

$$\mathfrak{u}^{(k)} := \bigoplus_{\substack{\beta \in \mathfrak{c}^* \\ c_{\alpha\beta} = k}} \mathfrak{u}_\beta.$$

Lemma 6.9 *The 0- and ± 2 -components of the grading are given by $\mathfrak{u}^{(0)} = \mathfrak{v} \oplus \langle H_\alpha \rangle$ and $\mathfrak{u}^{(\pm 2)} = \mathfrak{u}_{\pm\alpha}$, respectively.*

Proof $\mathfrak{u}^{(0)} = \ker \text{ad}_{H_\alpha} = C_{\mathfrak{u}}(H_\alpha) = \mathfrak{v} \oplus \langle H_\alpha \rangle$. Suppose there was a root $\beta \neq \alpha$ such that $c_{\alpha\beta} = 2$. Then $\langle \beta, \alpha \rangle > 0$ and $\beta - \alpha$ was a root satisfying $c_{\alpha(\beta-\alpha)} = 0$. We would need to have $[\mathfrak{u}_\alpha, \mathfrak{u}_{\beta-\alpha}] = \mathfrak{u}_\beta$, but $\mathfrak{u}_{\beta-\alpha} \subset \mathfrak{u}^{(0)} = \mathfrak{v} \oplus \langle H_\alpha \rangle$ and $[\mathfrak{u}_\alpha, \mathfrak{v}] = 0$, $[\mathfrak{u}_\alpha, H_\alpha] = \mathfrak{u}_\alpha$. \square

Proposition 6.10 *The Lie algebras \mathfrak{g} and \mathfrak{u} are simple.*

Proof The semisimple Lie algebra \mathfrak{g} decomposes as a direct sum $\mathfrak{g} = \mathfrak{g}_{(1)} \oplus \dots \oplus \mathfrak{g}_{(m)}$ of simple ideals. Since the Killing form of \mathfrak{g} is negative definite, the same applies to the ideals $\mathfrak{g}_{(i)}$, which thus cannot be the realification of a complex Lie algebra. Therefore, their complexifications $\mathfrak{u}_{(i)} := \mathfrak{g}_{(i)}^\mathbb{C}$ are also simple and yield a similar decomposition $\mathfrak{u} = \mathfrak{u}_{(1)} \oplus \dots \oplus \mathfrak{u}_{(m)}$ into simple ideals [13, Theorem 6.94]. Accordingly, the root system is a disjoint union $\Phi = \Phi_1 \sqcup \dots \sqcup \Phi_m$. We claim that $\mathfrak{g} = \mathfrak{g}_{(i)}$ (and hence $\mathfrak{u} = \mathfrak{u}_{(i)}$), where i is the unique index such that $\alpha \in \Phi_i$.

For $j \neq i$, the ideal $\mathfrak{g}_{(j)}$ commutes with $\mathfrak{g}_{(i)} \supset (\mathfrak{u}_\alpha \oplus \mathfrak{u}_{-\alpha}) \cap \mathfrak{g} \ni X_2, X_3$, so $\mathfrak{g}_{(j)} \subset \mathfrak{h} = \mathfrak{g}_p$. Since $\mathfrak{g}_{(j)}$ is an ideal and G is connected, it follows that $\mathfrak{g}_{(j)} = \text{Ad}_g(\mathfrak{g}_{(j)}) \subset \text{Ad}_g(\mathfrak{g}_p) = \mathfrak{g}_{g \cdot p}$ for all $g \in G$. Because the G -action is almost effective, we must have $\mathfrak{g}_{(j)} = 0$. \square

It is well-known that for any root system Φ and any roots $\alpha, \beta \in \Phi$, the Cartan numbers are bounded by $|c_{\alpha\beta}| \leq 3$. Furthermore, the only irreducible case where $|c_{\alpha\beta}| = 3$ occurs is when $\mathfrak{g} = \mathfrak{g}_2$, α is one of the short roots and β is the long root that forms an angle of 150 (210) degrees with α . We relegate the proof that this case cannot actually occur in our situation to the next section.

In all the remaining cases, we have therefore shown that α is a maximal root (cf. Lemma 4.1), so we may carry out the construction from Sect. 5. We now prove that the 3-Sasakian structure obtained this way indeed coincides with the original one we started with. We simplify the analysis by studying the reductive complement $\mathfrak{m} := \mathfrak{h}^\perp_B$.

Lemma 6.11 *The reductive complement \mathfrak{m} decomposes B -orthogonally as*

$$\mathfrak{m} = \langle X_1, X_2, X_3 \rangle \oplus \bigoplus_{\substack{\beta \in \Phi \\ c_{\alpha\beta} = 1}} (\mathfrak{u}_\beta \oplus \mathfrak{u}_{-\beta}) \cap \mathfrak{g} =: \mathfrak{k} \oplus \mathfrak{g}_1.$$

Proof \supset : Clearly, X_1, X_2, X_3 are B -orthogonal to \mathfrak{h} . For $\beta, \gamma \in \mathfrak{c}^*$ with $\beta + \gamma \neq 0$, the subspaces u_β and u_γ are $B^{\mathbb{C}}$ -orthogonal. This implies that for all $\beta \in \Phi$ with $c_{\alpha\beta} = 1$, the subspaces $u_{\pm\beta}$ are also $B^{\mathbb{C}}$ -orthogonal to \mathfrak{h} .

\subset : By Sect. 6.9, both sides of the equation have dimension $4n + 3$. □

We can compare the structure tensors of the two 3-Sasakian structures in question via the isomorphism $\psi : \mathfrak{m} \rightarrow T_pM, X \mapsto \overline{X}_p$. Proposition 6.6 has already shown that the vectors X_i correspond to the Reeb vector fields ξ_i . Looking back at Theorem 1.1, equality of the contact forms is equivalent to the following

Lemma 6.12 $\alpha_i = -B(X_i, \cdot)/4(n + 2)$.

Proof By definition, $\alpha_i = B(X_i, \cdot)/B(X_i, X_i)$. We have

$$\begin{aligned} B(X_1, X_1) &= B^{\mathbb{C}}(iH_\alpha, iH_\alpha) = -B^{\mathbb{C}}(H_\alpha, H_\alpha) = -\text{tr ad}_{H_\alpha}^2 \\ &= -4 \cdot (\dim u^{(2)} + \dim u^{(-2)}) - 1 \cdot (\dim u^{(1)} + \dim u^{(-1)}) = -4(n + 2). \end{aligned}$$

We also have $B(X_2, X_2) = B(X_3, X_3) = -4(n + 2)$, since we could have used the same arguments for a maximal torus of e.g. the form $\mathfrak{s} \oplus \langle X_2 \rangle$. □

Because the contact forms coincide, so do their differentials, which are the fundamental 2-forms. Since the Riemannian metrics are determined by the fundamental 2-forms together with the almost complex structures, it suffices to show that the latter coincide. Let $L_i : \mathfrak{m} \rightarrow \mathfrak{m}$ denote the $\text{Ad}(H)$ -invariant endomorphism of \mathfrak{m} corresponding to the G -invariant endomorphism field φ_i , i.e. $L_i = \psi^{-1} \circ (\varphi_i)_p \circ \psi$. Looking back at Theorem 1.1, the claim reduces to showing that

$$L_i|_{\mathfrak{k}} = \frac{1}{2} \text{ad}_{X_i}, \quad L_i|_{\mathfrak{g}_1} = \text{ad}_{X_i}.$$

The first equation is clear from Proposition 6.6 and the 3-Sasaki equations in Remark 2.4.

Proposition 6.13 *The almost complex structures of the two 3-Sasakian structures in question coincide.*

Proof We first claim that L_1 is not only $\text{ad}(\mathfrak{h})$ - but even $\text{ad}(\mathfrak{t})$ -invariant, i.e. that the endomorphisms L_1 and ad_{X_1} commute on \mathfrak{g}_1 . For all $Y, Z \in \mathfrak{g}_1$, we have

$$\begin{aligned} 2g_p(\overline{Y}_p, \overline{L_1 Z}_p) &= d\eta_1(\overline{Y}_p, \overline{Z}_p) \\ &= d\eta_1(\overline{[X_1, Y]}_p, \overline{[X_1, Z]}_p) \\ &= 2g_p(\overline{[X_1, Y]}_p, \overline{L_1[X_1, Z]}_p) \\ &= -2g_p(\overline{Y}_p, \overline{[X_1, L_1[X_1, Z]}_p). \end{aligned}$$

In the second equation, we used that ad_{X_1} corresponds to an almost complex structure on \mathfrak{g}_1 which is compatible with the common fundamental 2-form $d\eta_1$. The last

equation follows from Lemma 6.7. This shows that $L_1 = -\text{ad}_{X_1} \circ L_1 \circ \text{ad}_{X_1}$ on \mathfrak{g}_1 and consequently, $\text{ad}_{X_1} \circ L_1 = -\text{ad}_{X_1}^2 \circ L_1 \circ \text{ad}_{X_1} = L_1 \circ \text{ad}_{X_1}$.

Let β be a root such that $c_{\alpha\beta} = 1$. Since ad_{H_α} leaves u_β invariant, so does $\text{ad}_{X_1}^\mathbb{C}$. Because L_1 is $\text{ad}(\mathfrak{t})$ -invariant, $L_1^\mathbb{C}$ is $\text{ad}(\mathfrak{c})$ -invariant and thus also leaves u_β invariant. Now, $\text{ad}_{X_1}^\mathbb{C}$ and $L_1^\mathbb{C}$ are \mathbb{C} -linear maps on the one-dimensional subspace u_β which square to $-\text{id}$, so they must be given by multiplication with $\pm i$. Since both endomorphisms commute with complex conjugation, they act on $u_{-\beta} = \overline{u_\beta}$ by multiplication with $\mp i$. Therefore, L_1 and ad_{X_1} coincide on $(u_\beta \oplus u_{-\beta}) \cap \mathfrak{g}$ up to sign. We finish the proof that $L_1 = \text{ad}_{X_1}$ on \mathfrak{g}_1 by observing that for $Y \in \mathfrak{g}_1, Y \neq 0$, Lemma 6.2 implies

$$\begin{aligned} 2g_p(\overline{[X_1, Y]}_p, \overline{L_1 Y}_p) &= d\eta_1(\overline{[X_1, Y]}_p, \overline{Y}_p) \\ &= \eta_1(\overline{[X_1, Y]}_p, \overline{Y}_p) = -\alpha_1(\overline{[X_1, Y]}, \overline{Y}) \\ &= \frac{B(X_1, \overline{[X_1, Y]}, \overline{Y})}{4(n+2)} = -\frac{B([X_1, Y], [X_1, Y])}{4(n+2)} > 0. \end{aligned}$$

Again, we can repeat the arguments for the maximal tori $\mathfrak{s} \oplus \langle X_i \rangle, i = 2, 3$. Even though the root spaces look differently then, the subalgebra \mathfrak{g}_1 is still the same because it can be defined independently of the maximal torus as the B -orthogonal complement of \mathfrak{k} in \mathfrak{m} by virtue of Lemma 6.11. This proves that the almost complex structures in question also coincide for $i = 2, 3$. □

Remark 6.14 In later sections, instead of working with the simply connected, almost effectively acting Lie group G with Lie algebra \mathfrak{g} , we may sometimes turn to a non-simply connected (possibly effectively acting) group \tilde{G} with Lie algebra \mathfrak{g} . For $\mathfrak{g} = \mathfrak{so}(k)$, using $\tilde{G} = \text{SO}(k)$ instead of $G = \text{Spin}(k)$ allows us to describe the corresponding coset space more explicitly via matrices. If we consider a description \tilde{G}/\tilde{H} , then the isotropy group of the \tilde{G} -action on \tilde{G}/\tilde{H} is given by the connected subgroup $H \subset G$ whose Lie algebra coincides with that of \tilde{H} . This follows from the fact that M is simply connected via the long exact sequence of homotopy groups. Hence, \tilde{G}/\tilde{H} and G/H are governed by the same Lie algebraic data and are therefore isomorphic homogeneous 3-Sasakian manifolds.

7 Why the short root of \mathfrak{g}_2 cannot occur

We need to fill the final gap left in the proof of Theorem 6.1 in the previous section:

Proposition 7.1 *Even in the case of a homogeneous 3-Sasakian manifold with automorphism algebra \mathfrak{g}_2 , the root described in Sect. 6 is maximal.*

For the sake of contradiction, let us assume that α was one of the short roots of \mathfrak{g}_2 . Again, we consider the reductive complement $\mathfrak{m} := \mathfrak{h}^{\perp B}$ as well as the maps $\psi: \mathfrak{m} \rightarrow T_p M, X \mapsto \overline{X}_p$ and $L_i := \psi^{-1} \circ (\varphi_i)_p \circ \psi: \mathfrak{m} \rightarrow \mathfrak{m}$. Using the same

arguments as in the proof of Lemma 6.11, we obtain the B -orthogonal decomposition

$$\mathfrak{m} = \langle X_1, X_2, X_3 \rangle \oplus \bigoplus_{\substack{\beta \in \Phi \\ c_{\alpha\beta} \in \{1,3\}}} (\mathfrak{u}_\beta \oplus \mathfrak{u}_{-\beta}) \cap \mathfrak{g}.$$

Under the isomorphism $\psi : \mathfrak{m} \rightarrow T_p M$, this induces a decomposition of the tangent space:

$$T_p M = \langle \xi_1, \xi_2, \xi_3 \rangle \oplus \bigoplus_{\substack{\beta \in \Phi \\ c_{\alpha\beta} \in \{1,3\}}} V_\beta,$$

where $V_\beta := \psi((\mathfrak{u}_\beta \oplus \mathfrak{u}_{-\beta}) \cap \mathfrak{g})$.

Lemma 7.2 *The above decomposition of $T_p M$ is g_p -orthogonal.*

Proof If $Y \in (\mathfrak{u}_\beta \oplus \mathfrak{u}_{-\beta}) \cap \mathfrak{g}$, then

$$g_p((\xi_i)_p, \bar{Y}_p) = \eta_i(\bar{Y}_p) = \alpha_i(Y) = B(X_i, Y)/B(X_i, X_i) = 0.$$

Hence, each V_β is g_p -orthogonal to $\langle \xi_1, \xi_2, \xi_3 \rangle$. If β_1, β_2 are roots such that $\beta_1 \neq -\beta_2$, then there exists some $X \in \mathfrak{c}$ such that $\beta_1(X) \neq -\beta_2(X)$. We extend ψ and g_p complex (bi-)linearly, let $Y \in \mathfrak{u}_{\beta_1}, Z \in \mathfrak{u}_{\beta_2}$ and complexify Lemma 6.7 to obtain

$$\begin{aligned} \beta_1(X) g_p(\psi Y, \psi Z) &= g_p(\psi[X, Y], \psi Z) \\ &= -g_p(\psi Y, \psi[X, Z]) = -\beta_2(X) g_p(\psi Y, \psi Z). \end{aligned}$$

Since $\beta_1(X) \neq -\beta_2(X)$, it follows that $\psi \mathfrak{u}_{\beta_1}$ and $\psi \mathfrak{u}_{\beta_2}$ are g_p -orthogonal. This implies that for $\beta \neq \pm\gamma$, the subspaces V_β and V_γ are g_p -orthogonal. \square

Lemma 7.3 *For all $Y, Z \in \mathfrak{g}$, we have*

$$g_p(\bar{Y}_p, \overline{L_i Z}_p) = 0 \iff B(X_i, [Y, Z]) = 0.$$

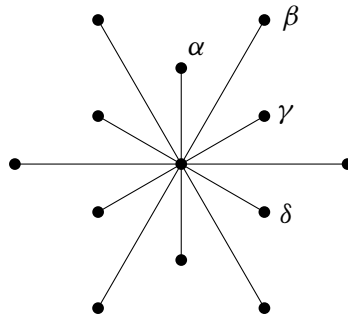
Proof By virtue of Lemma 6.2,

$$\begin{aligned} 2g_p(\bar{Y}_p, \overline{L_i Z}_p) &= d\eta_i(\bar{Y}_p, \overline{Z}_p) = \eta_i([\bar{Y}, \overline{Z}]_p) \\ &= -\alpha_i([Y, Z]) = -\frac{B(X_i, [Y, Z])}{B(X_i, X_i)}. \end{aligned} \quad \square$$

Lemma 7.4 *For any root $\beta \in \Phi$, we have $\varphi_2 V_\beta \subset V_{\beta+\alpha} \oplus V_{\beta-\alpha}$.*

Proof Let $\gamma \in \Phi$ be such that $\gamma \neq \sigma\alpha + \tau\beta$ for all $\sigma, \tau \in \{\pm 1\}$. Then, $\sigma\beta + \tau\gamma \notin \{\pm\alpha\}$ for all $\sigma, \tau \in \{\pm 1\}$. Consequently, the subspace $[\mathfrak{u}_\beta \oplus \mathfrak{u}_{-\beta}, \mathfrak{u}_\gamma \oplus \mathfrak{u}_{-\gamma}]$ is $B^{\mathbb{C}}$ -orthogonal to $\mathfrak{u}_\alpha \oplus \mathfrak{u}_{-\alpha} \ni X_\alpha, Y_\alpha$ and thus also to $X_2 = i(X_\alpha + Y_\alpha)$ (see the equations above Proposition 6.8). The previous lemma now implies that $\varphi_2 V_\beta$ is g_p -orthogonal to V_γ . The claim then follows from Lemma 7.2. \square

Proof of Proposition 7.1 Let us label some of the roots of \mathfrak{g}_2 according to the following diagram:



Lemma 7.4 implies that $\varphi_2 V_\beta \subset V_\gamma$. Since φ_2 is injective on the horizontal space, we in fact have $\varphi_2 V_\beta = V_\gamma$. Another application of Lemma 7.4 yields $\varphi_2 V_\gamma \subset V_\beta \oplus V_\delta$. If we can show that there exists some $Y \in V_\gamma$ such that $\varphi_2 Y$ has a non-trivial V_δ -component, then we arrive at a contradiction to the fact that $\varphi_2^2 = -\text{id}$ on V_β .

Let Z^* denote the complex conjugate of a vector $Z \in \mathfrak{u}$. We can choose $X_\gamma \in \mathfrak{u}_\gamma$, $X_\delta \in \mathfrak{u}_\delta$ in such a way that

$$[X_\gamma, X_\delta^*] = X_\alpha = \frac{1}{2i} (X_2 - iX_3).$$

We note that

$$X_\alpha^* = -\frac{1}{2i} (X_2 + iX_3) = -Y_\alpha, \quad i(X_\alpha - X_\alpha^*) = X_2,$$

and

$$B([X_\gamma + X_\gamma^*, i(X_\delta^* - X_\delta)], X_2) = B(i(X_\alpha - X_\alpha^*), X_2) = B(X_2, X_2) \neq 0.$$

Lemma 7.3 finally implies that $\varphi_2 Y$ has a V_δ -component for $Y := \overline{(X_\gamma + X_\gamma^*)}_p$. \square

8 Why no proper subgroup of $\text{Aut}_0(M)$ acts transitively

We have shown that any simply connected homogeneous 3-Sasakian manifold M is obtained from a complex 3-Sasakian datum as explained in Sect. 5. Hence, M can be written as G/H , where G is a simply connected compact simple Lie group. As G is simple, we pass to the effectively acting finite quotient \tilde{G} which is then a subgroup of $\text{Aut}_0(M)$. We can now write $M = \tilde{G}/\tilde{H}$. Below we show that $\mathfrak{g} = \text{Lie } \tilde{G} = \mathfrak{aut}(M)$. This concludes Theorem 1.1 as M determines its 3-Sasakian datum.

Proposition 8.1 *If a subgroup $\tilde{G} \subset \text{Aut}_0(M)$ acts transitively on a simply connected homogeneous 3-Sasakian manifold M , then $\tilde{G} = \text{Aut}_0(M)$.*

Proof We show that the Lie algebra \mathfrak{g} of \tilde{G} is tied to purely geometric data of M . Recall the setup from Theorem 1.1: We have a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{m} = \mathfrak{h}^{\perp B}$ and $\mathfrak{m} = \mathfrak{k} \oplus \mathfrak{g}_1$, where \mathfrak{k} corresponds to the Reeb vector fields ξ_i and $\mathfrak{g}_1 = (\mathfrak{h} \oplus \mathfrak{k})^{\perp B}$. Note that we have the commutator relations (cf. Proposition 5.4)

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, [\mathfrak{h}, \mathfrak{k}] = 0, [\mathfrak{h}, \mathfrak{g}_1] \subset \mathfrak{g}_1, [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, [\mathfrak{k}, \mathfrak{g}_1] \subset \mathfrak{g}_1, [\mathfrak{g}_1, \mathfrak{g}_1] \subset \mathfrak{g}_0 = \mathfrak{h} \oplus \mathfrak{k}.$$

Consider the subspace $[\mathfrak{m}, \mathfrak{m}] + \mathfrak{m} \subset \mathfrak{g}$. Using the commutator relations we find that this is an ideal in \mathfrak{g} and thus (Proposition 6.10) already \mathfrak{g} itself. Hence, the knowledge of \mathfrak{m} embedded in the Lie algebra of Killing vector fields $\text{isom}(M)$ on M via $X \mapsto \bar{X}$ alone determines $\mathfrak{g} \subset \text{isom}(M)$.

We now characterize \mathfrak{m} as the subset of Killing fields whose covariant derivatives satisfy a certain behavior at $o = e\tilde{H}$. By analogy, recall that in a symmetric space, the analogue of \mathfrak{m} can be characterized as the Killing fields whose covariant derivative vanishes at o . Let ∇ be the Levi-Civita connection on M , and $\alpha : \mathfrak{m} \times \mathfrak{m} \rightarrow \mathfrak{m}$ the associated Nomizu operator defined by

$$\overline{\alpha(X, Y)}_o = \nabla_{\bar{X}_o} \bar{Y} - \overline{[X, Y]}_o.$$

It satisfies

$$\alpha(X, Y) = \begin{cases} 0, & X \in \mathfrak{k} \text{ and } Y \in \mathfrak{g}_1, \\ \frac{1}{2}[X, Y]_{\mathfrak{m}}, & X, Y \in \mathfrak{k} \text{ or } X, Y \in \mathfrak{g}_1, \\ [X, Y]_{\mathfrak{m}}, & X \in \mathfrak{g}_1 \text{ and } Y \in \mathfrak{k}, \end{cases}$$

see [11, Theorem 4.2]. Thus, by definition of the Nomizu operator, we have

$$\nabla_{\bar{X}_o} \bar{Y} = \overline{\alpha(X, Y)}_o + \overline{[X, Y]}_o,$$

which means that

$$\nabla_{\bar{X}_o} \bar{Y} = \begin{cases} \frac{3}{2}\overline{[X, Y]}_o = -3 \sum \eta_i(\bar{Y}_o)\varphi_i \bar{X}_o, & X \in \mathfrak{k}, \\ 2\overline{[X, Y]}_o = -2 \sum \eta_i(\bar{Y}_o)\varphi_i \bar{X}_o, & X \in \mathfrak{g}_1 \end{cases}$$

for $Y \in \mathfrak{k}$ and

$$\nabla_{\bar{X}_o} \bar{Y} = \begin{cases} \overline{[X, Y]}_o = \sum_{i=1}^3 \eta_i(\bar{X}_o)\varphi_i(\bar{Y}_o), & X \in \mathfrak{k}, \\ \frac{3}{2}\overline{[X, Y]}_{\mathfrak{k}} = -\frac{3}{2} \sum_{i=1}^3 d\eta_i(\bar{X}_o, \bar{Y}_o)\xi_i, & X \in \mathfrak{g}_1 \end{cases}$$

for $Y \in \mathfrak{g}_1$, where we used Lemma 6.2 in the last equation. Hence, the fundamental vector field of $Y \in \mathfrak{m} = \mathfrak{k} \oplus \mathfrak{g}_1$ satisfies

$$\begin{aligned} \nabla_v \bar{Y} = & -3 \sum_{i,j=1}^3 \eta_i(v) \eta_j(\bar{Y}_o) \varphi_i \xi_j - 2 \sum_{j=1}^3 \eta_j(\bar{Y}_o) \varphi_j (v_{\mathcal{H}}) \\ & + \sum_{i=1}^3 \eta_i(v) \varphi_i(\bar{Y}_o) \mathcal{H} - \frac{3}{2} \sum_{i=1}^3 d\eta_i(v, \bar{Y}_o) \xi_i \end{aligned} \tag{2}$$

for all $v \in T_oM$ and where $v_{\mathcal{H}}$ denotes the projection of v to $\mathcal{H} = \bigcap \ker \eta_i$. Note that $(\nabla \bar{Y})_o$ depends only on the value $\bar{Y}_o \in T_oM$. We now consider the maps

$$\mathfrak{m} \rightarrow \{Y \text{ Killing field on } M \mid Y \text{ satisfies (2) for all } v \in T_oM\} \rightarrow T_oM,$$

where the first map is $Y \mapsto \bar{Y} \in \text{isom}(M)$ and the second is evaluation at o . The evaluation map is injective, as for Killing fields Y_1, Y_2 in the middle space with $(Y_1)_o = (Y_2)_o$, by Property (2) also $(\nabla Y_1)_o = (\nabla Y_2)_o$, which implies that $Y_1 = Y_2$. Since $\mathfrak{m} \cong T_oM$ both maps are isomorphisms. Thus,

$$\mathfrak{m} = \{Y \text{ Killing field on } M \mid Y \text{ satisfies (2) for all } v \in T_oM\} \subset \text{isom}(M).$$

Therefore we have shown that every connected Lie group \tilde{G} with Lie algebra \mathfrak{g} acting effectively and transitively on M has the same Lie algebra, namely $\mathfrak{g} = \text{aut}(M)$. The corresponding connected subgroup of $\text{Aut}(M)$ is then $\tilde{G} = \text{Aut}_0(M)$. \square

9 Determining the isotropy

Having proven Theorem 1.1, we now derive the precise list given in Corollary 1.2. By Theorems 5.1 and 6.1 any simply-connected homogeneous 3-Sasakian manifold can be written in the form G/H , where G is a simply-connected simple Lie group and $H = (C_G(K))_0$, where $K \subset G$ is the connected subgroup with Lie algebra $\mathfrak{k} \cong \mathfrak{sp}(1)$ determined by a maximal root. In this section, we will determine the isotropy groups H , thereby proving Corollary 1.2 in the simply connected case. The classical cases are dealt with in the following

Proposition 9.1 *For $G = \text{Sp}(n+1)$, $\text{SU}(m)$ and $\tilde{G} = \text{SO}(k)$, the isotropy groups are given by $H = \text{Sp}(n)$, $S(U(m-2) \times U(1))$ and $\tilde{H} = \text{SO}(k-4) \times \text{Sp}(1)$, respectively.*

Proof We use the explicit description of the root systems of the compact groups provided in [16, Chapter 11].

$G = \mathrm{Sp}(n + 1)$: We may choose the maximal root α such that $\mathfrak{k} = \{\mathrm{diag}(0_n, \mathfrak{sp}(1))\}$ (by letting $\alpha = \gamma_{n+1}$ in TAPP’s notation). Accordingly,

$$\begin{aligned} K &= \{\mathrm{diag}(I_n, \mathrm{Sp}(1))\}, \\ C_G(K) &= \{\mathrm{diag}(\mathrm{Sp}(n), \pm 1)\}, \\ H &= (C_G(K))_0 = \{\mathrm{diag}(\mathrm{Sp}(n), 1)\}. \end{aligned}$$

$G = \mathrm{SU}(m)$: We may choose α such that $\mathfrak{k} = \{\mathrm{diag}(0_{m-2}, \mathfrak{su}(2))\}$ (by letting $\alpha = \alpha_{m-1,m}$ in TAPP’s notation). Accordingly,

$$\begin{aligned} K &= \{\mathrm{diag}(I_{m-2}, \mathrm{SU}(2))\}, \\ C_G(K) = H &= \{\mathrm{diag}(\mathrm{SU}(m - 2), zI_2) \mid z \in U(1)\} \cap \mathrm{SU}(m). \end{aligned}$$

$\tilde{G} = \mathrm{SO}(k)$: We recall that there are two embeddings $\mathrm{Sp}(1)^+, \mathrm{Sp}(1)^- \subset \mathrm{SO}(4)$, depending on whether $\mathrm{Sp}(1)$ is viewed as acting on $\mathbb{H} \cong \mathbb{R}^4$ by multiplication from the left or right, respectively. We may choose α such that $\mathfrak{k} = \{\mathrm{diag}(0_{k-4}, \mathfrak{sp}^-(1))\}$ (by letting $\alpha = \alpha_{[k/2]-1, [k/2]}$ in TAPP’s notation). Accordingly,

$$\begin{aligned} \tilde{K} &= \{\mathrm{diag}(I_{k-4}, \mathrm{Sp}^-(1))\}, \\ C_{\tilde{G}}(\tilde{K}) = \tilde{H} &= \{\mathrm{diag}(\mathrm{SO}(k - 4), \mathrm{Sp}^+(1))\}. \quad \square \end{aligned}$$

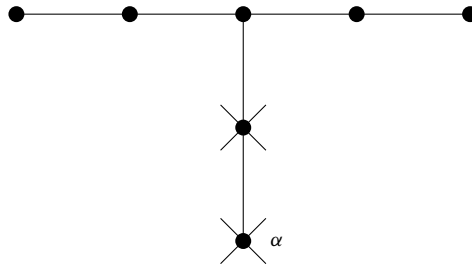
We now present a different method based on Borel–de Siebenthal theory, which allows us to first understand the isotopy algebra \mathfrak{h} in the exceptional cases:

Using the same notation as before, we let $\mathfrak{s} \subset \mathfrak{h}$ be a maximal Abelian subalgebra and consider the maximal Abelian subalgebra $\mathfrak{t} := \mathfrak{s} \oplus \langle X_1 \rangle$ of \mathfrak{g} . Let α denote the maximal root that vanishes on $\mathfrak{s}^{\mathbb{C}}$. We fix a set of positive roots of \mathfrak{g} using a slight perturbation of the hyperplane perpendicular to the maximal root α . By intersecting this hyperplane with \mathfrak{s} , we also obtain a notion of positive root for \mathfrak{h} . By the very definition of root spaces, as \mathfrak{h} commutes with X_1 , any root of \mathfrak{h} becomes, by extending it by 0 on X_1 , a root of \mathfrak{g} .

Proposition 9.2 *The simple \mathfrak{h} -roots are precisely those simple \mathfrak{g} -roots perpendicular to α .*

Proof By our notions of positivity, any \mathfrak{h} -simple root is also \mathfrak{g} -simple: If an \mathfrak{h} -root is the sum of two positive \mathfrak{g} -roots, both of them have to lie in the hyperplane perpendicular to α . Conversely, recall that by Proposition 5.4 the roots of \mathfrak{h} are exactly those roots β perpendicular to the maximal root α . □

We can thus determine the isotropy type of H by deleting the nodes in the Dynkin diagram of G corresponding to simple roots that are not perpendicular to α . For each simple G , these were determined by BOREL and DE SIEBENTHAL in [8]: In the table on p. 219 they draw the Dynkin diagrams for every simple \mathfrak{g} , extended by the lowest root (denoted P in their notation). In order to find the isomorphism type of H one therefore only needs to erase this lowest root, as well as all roots connected to it. As an example, consider the Dynkin diagram of E_6 :



Deleting α as well as the unique simple root connected to it results in the Dynkin diagram of $SU(6)$: The homogeneous 3-Sasakian manifold corresponding to E_6 is $E_6/SU(6)$.

Remark 9.3 If one removes only the nodes in the Dynkin diagram of G that are connected to α , and not α itself, one obtains the Dynkin diagram of the normalizer $N_G(K)$, which then yield the Wolf spaces $G/N_G(K)$. Note that by the list in [8], in all cases except $G = SU(n)$ the maximal root α is connected to only one other node, which means that in these cases the groups H and $N_G(K)$ are semisimple, whereas in the case $G = SU(n)$ the groups H and $N_G(K)$ have a one-dimensional center. Furthermore, in the cases except $SU(n)$, the normalizer $N_G(K)$ is a maximal subgroup of maximal rank: the types of such groups are exactly those that were classified by BOREL and DE SIEBENTHAL in [8]: Given a simple compact Lie group G , one adds the lowest root to the Dynkin diagram and removes one other simple root from it.

Going through the list in [8], one obtains the Lie algebras of the isotropy groups of the homogeneous spaces occurring in Corollary 1.2. As we determined the isotropy groups in the classical cases above, in order to finish the proof of this corollary in the simply connected case, we only need to argue that in the exceptional cases the isotropy groups are simply connected. ISHITOYA and TODA showed in [12, Corollary 2.2] that in the cases $G = G_2, F_4, E_6, E_7, E_8$, we have $\pi_2(G/N_G(K)) = \mathbb{Z}_2$, which is, because G is simply connected, equivalent to $\pi_1(N_G(K)) = \mathbb{Z}_2$. (See also [8, Remarque II, p. 220] for how to compute the fundamental group of a maximal subgroup of G of maximal rank.) Moreover, by [12, Theorem 2.1] the normalizer $N_G(K)$ is of the form $N_G(K) = (H \times Sp(1))/\mathbb{Z}_2$, which then implies that H is simply connected.

10 Why $\mathbb{R}P^{4n+3}$ is the only non-simply connected homogeneous 3-Sasakian manifold

Having treated the simply connected case of Corollary 1.2, our goal is now to prove the following

Theorem 10.1 *The only homogeneous 3-Sasakian manifolds which are not simply connected are the real projective spaces $\mathbb{R}P^{4n+3}$.*

Let $M = G/\overline{H}$ be a homogeneous 3-Sasakian manifold (not necessarily simply connected), where G is a simply connected compact Lie group and \overline{H} is possibly

disconnected. The universal cover of G/\overline{H} is given by G/H , where H denotes the identity component of \overline{H} , and the homogeneous 3-Sasakian structure lifts to the simply connected space G/H . As shown in Sect. 6, the automorphism group G has to be simple. The vectors $X_i \in \mathfrak{g}$ from Sect. 6 span a subalgebra $\mathfrak{k} := \langle X_1, X_2, X_3 \rangle \cong \mathfrak{sp}(1)$ and we let $K \subset G$ denote the corresponding connected subgroup.

Since H is the identity component of \overline{H} , we have $\overline{H} \subset N_G(H)$. Furthermore, the 3-Sasakian structure descends from G/H to G/\overline{H} , so $\overline{H} \subset C_G(K)$. Conversely, any subgroup $\overline{H} \subset N_G(H) \cap C_G(K)$ containing H allows us to define a 3-Sasakian structure on G/\overline{H} . Summarizing, the non-simply connected quotients of a given simply connected homogeneous 3-Sasakian manifold G/H are classified by the subgroups of the group

$$(N_G(H) \cap C_G(K))/H.$$

Thus, it suffices to show that this quotient is \mathbb{Z}_2 for $G = \text{Sp}(n+1)$ and trivial otherwise.

Lemma 10.2 *The numerator $N_G(H) \cap C_G(K)$ is the subgroup generated by $H \cup Z(K)$.*

Proof Clearly, $H \cup Z(K) \subset N_G(H) \cap C_G(K)$. The vector $X_1 \in \mathfrak{g}$ is the infinitesimal generator of a circle subgroup $S_1 \subset K$. Since $C_{\mathfrak{g}}(X_1) = \mathfrak{h} \oplus \langle X_1 \rangle$ and the centralizer of any torus (not necessarily maximal) in a compact connected Lie group is always connected, it follows that $C_G(S_1)$ is the subgroup generated by $H \cup S_1$. Consequently, any $g \in C_G(K) \subset C_G(S_1)$ can be represented as $g = hg_1$ for some $h \in H, g_1 \in S_1$. Since $H \subset C_G(K)$, we have $g_1 = h^{-1}g \in C_G(K) \cap K = Z(K)$. \square

Proposition 10.3 *The quotient $(N_G(H) \cap C_G(K))/H$ is \mathbb{Z}_2 for $G = \text{Sp}(n+1)$ and trivial otherwise.*

Proof By the previous lemma, it suffices to check if $Z(K)$ is contained in H .

$G = \text{Sp}(n+1)$: We have already seen in Sect. 9 that the center $Z(K) = \{\text{diag}(I_n, \pm 1)\}$ is **not** contained in $H = \{\text{diag}(\text{Sp}(n), 1)\}$.

$G = \text{SU}(m)$: We have also shown that in this case $Z(K) = \{\text{diag}(I_{m-2}, \pm I_2)\}$ is contained in $H = S(U(m-2) \times U(1))$.

$G = \text{Spin}(k)$: We have seen that for $\tilde{G} = \text{SO}(k)$, the center $Z(K) = \{\text{diag}(I_{k-4}, \pm I_4)\}$ is contained in $\tilde{H} = \text{SO}(k-4) \times \text{Sp}(1)^+$. We now transfer this statement to $G = \text{Spin}(k)$: Denote the universal covering map by $\pi: \text{Spin}(k) \rightarrow \text{SO}(k)$. First, we observe that the connected subgroup of $\text{Spin}(r)$ that maps onto a block-diagonally embedded $\text{SO}(r-1)$ is $\text{Spin}(r-1)$, for $r \geq 4$: this is because $S^{r-1} = \text{SO}(r)/\text{SO}(r-1)$ is 2-connected for $r \geq 4$, hence equal to $\text{Spin}(r)/\text{Spin}(r-1)$ by the long exact sequence in homotopy. Thus, for $k \geq 7$, the subgroups $\text{SO}(k-4)$ and $\text{SO}(4)$ lift to $\text{Spin}(k-4)$ and $\text{Spin}(4)$, respectively. As π is a 2:1-covering, the group covering $\text{SO}(k-4) \times \text{SO}(4)$ is $\text{Spin}(k-4) \times_{\mathbb{Z}_2} \text{Spin}(4)$, where the \mathbb{Z}_2 quotient means that the nontrivial elements in the kernels of the respective projections are identified. Now, $\text{Spin}(k-4) \times_{\mathbb{Z}_2} \text{Spin}(4) \cong \text{Spin}(k-4) \times_{\mathbb{Z}_2} (\text{Sp}(1)^+ \times \text{Sp}(1)^-)$. This implies that the center of $\text{Sp}(1)^-$ is contained in $H = \text{Spin}(k-4) \times \text{Sp}(1)^+$.

$G = G_2, F_4, E_6, E_7, E_8$: ISHITOYA and TODA showed that the subgroup U of the corresponding symmetric base space G/U has to be of the form $U = (H \times \text{Sp}(1))/\mathbb{Z}_2$ and that the center $Z(\text{Sp}(1))$ is contained in H [12, Theorem 2.1]. \square

11 Deriving the classification of homogeneous positive qK manifolds

We end this article by showing that the classification of homogeneous positive qK manifolds in Sect. 1.3, which had originally been the stepping stone for the classification of homogeneous 3-Sasakian manifolds, can in turn be derived from our results.

Let B be a positive qK manifold. We recall that qK manifolds may be characterized by a subbundle $Q \subset \text{End } TB$ of the endomorphism bundle which admits a local frame satisfying the multiplication rules of the quaternions. In her 1975 article [15], KONISHI showed that the $\text{SO}(3)$ -principal fibre bundle $P \rightarrow B$ of oriented orthonormal frames of Q admits a 3-Sasakian structure. This construction is known as the *Konishi bundle* over B and constitutes the inverse of the fibration introduced in Proposition 2.7. Another natural and interesting bundle over B is the unit sphere bundle $Z := S(Q)$ in Q , known as the *twistor fibration*. Its total space Z is both a complex contact manifold and a Fano variety [9, Chapters 12, 13].

A *qK automorphism of B* is an isometry $\phi: B \rightarrow B$ such that conjugation with its differential $d\phi$ leaves the bundle Q invariant. We call B a *homogeneous qK manifold* if there is a Lie group G acting transitively on B by qK automorphisms. We first show the following

Proposition 11.1 *The Konishi bundle over a simply connected homogeneous positive qK manifold is a homogeneous 3-Sasakian manifold.*

Let B be a simply connected homogeneous positive qK manifold, so that we may write $B = G/U$, where G is simply connected and U is connected. Because G acts on B by qK automorphisms, the G -action lifts to the Konishi bundle P . In particular, the isotropy group U acts on the fiber F of P over the identity coset $eU \in B$.

Choose and fix a frame $p \in F$. This allows us to identify F with $\text{SO}(3)$ via the orbit bijection $\theta_p: \text{SO}(3) \rightarrow F, g \mapsto p \cdot g$. The $\text{SO}(3)$ -left action on itself by left multiplication now induces a left action on F (which depends on the choice of p), viz. $g \cdot_p q := \theta_p(g\theta_p^{-1}(q))$. Since the U -action commutes with the $\text{SO}(3)$ -right action on F , there exists a homomorphism $\rho: U \rightarrow V$ onto a subgroup $V \subset \text{SO}(3)$ (again, all depending on p) such that $u \cdot q = \rho(u) \cdot_p q$ for all $q \in F$, namely $\rho(u) := \theta_p^{-1}(u \cdot p)$. Clearly, $d := \dim V \in \{0, 1, 3\}$.

Lemma 11.2 $d = 3$.

Proof Let us first assume that $d = 0$. Then, the connected group U would act trivially on F . Hence, we would obtain a well-defined global section $B \rightarrow P, gU \mapsto g \cdot p$, meaning that the principal fiber bundle P was trivial. But the first Pontryagin class $p_1(P)$ of P is (up to a factor) given by the class of the fundamental 4-form $\Omega \in \Omega^4(B)$ of B and is therefore non-trivial [4, Proposition 14.92].

If we suppose that $d = 1$, then V is a connected one-dimensional subgroup of $\text{SO}(3)$ and is thus comprised of rotations around a fixed axis $L \subset \mathbb{R}^3$. Choose a point $x \in L \cap S^2$. We view the frame $p \in F$ as a linear isometry $\mathbb{R}^3 \rightarrow Q$ and consider the mapping $B \rightarrow Z, gU \mapsto (g \cdot p)(x)$. This map is well-defined because

$$(u \cdot p)(x) = (p \cdot \theta_p^{-1}(u \cdot p))(x) = (p \cdot \rho(u))(x) = p(\rho(u)(x)) = p(x) \text{ for all } u \in U.$$

We would thus obtain a global section of the twistor fibration, which is impossible on compact positive qK manifolds [3, Theorem 3.8]. □

Proof of Proposition 11.1 From the previous lemma, we know that U acts transitively on F and consequently, G acts transitively on P . This action preserves the 3-Sasakian structure, since the Reeb vector fields are (by construction of the Konishi bundle) the infinitesimal generators of the $SO(3)$ -action, which commutes with G . □

Proof of Corollary 1.3 By Proposition 11.1, the Konishi bundle P over a simply connected homogeneous positive qK manifold B is a homogeneous 3-Sasakian manifold, i.e. one of the manifolds listed in Corollary 1.2. Dividing P by the action of the group $K \subset G$ from the previous sections, we obtain the list in Corollary 1.3. The statement about the Riemannian metric and quaternionic structure follows from the fact that the Konishi bundle is a Riemannian fibration.

Let us now assume that $\overline{B} = G/U$ was a non-simply connected homogeneous positive qK manifold, where \overline{U} is disconnected. Then, \overline{B} is finitely covered by $B \cong G/U$, where U denotes the identity component of \overline{U} . The qK structure lifts from \overline{B} to B and the Konishi bundles P, \overline{P} over B, \overline{B} form a diagram

$$\begin{array}{ccc} P & \dashrightarrow & \overline{P} \\ \downarrow & & \downarrow \\ B & \longrightarrow & \overline{B} \end{array}$$

We obtain the existence of the dashed equivariant map $P \rightarrow \overline{P}$, so that \overline{P} is a non-simply connected homogeneous 3-Sasakian manifold. By Theorem 10.1, \overline{P} can only be $\mathbb{R}P^{4n+3}$, which leads to the same quotient $\overline{B} = Sp(n + 1)/(Sp(n) \times Sp(1))$ as $P = S^{4n+3}$. □

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