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RESEARCH ARTICLE

Bivariate Beta distribution and multiplicative functions

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Abstract
We prove that two-dimensional Dirichlet distributions for any collection of positive parameters can be modeled by means of a sequence of distributions defined via non-negative valued multiplicative functions which satisfy some regularity conditions on prime powers.

Keywords Natural divisor · Multiplicative function · Dirichlet distribution

Mathematics Subject Classification 11N60 · 11K65

1 Introduction

We consider the question of how to get any two-dimensional Dirichlet distribution as a limit of the sequence of discrete distributions constructed by multiplicative functions. Actually, the Dirichlet distribution is a multivariate generalization of the Beta distribution and the two-dimensional Dirichlet distribution is usually called the bivariate Beta distribution [1].

Let a, b, c be positive constants and

$$E(u, v) := \{(s, t) \mid 0 \leqslant s \leqslant u,\ 0 \leqslant t \leqslant v,\ s + t \leqslant 1\}.$$

The bivariate Beta distribution $\mathcal{D}(a, b, c)$ concentrated on the triangle $E(1, 1)$ is defined by the distribution function

$$D(u, v; a, b, c) = \frac{\Gamma(a+b+c)}{\Gamma(a)\Gamma(b)\Gamma(c)} \iint\limits_{E(u,v)} \frac{\mathrm{d}t\,\mathrm{d}s}{s^{1-a}t^{1-b}(1-t-s)^{1-c}},$$

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where Γ denotes the Gamma function. The one-dimensional Dirichlet distribution is the well-known Beta law $\mathcal{B}(a, b)$ with distribution function

$$B(u; a, b) := \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^u \frac{ds}{s^{1-a}(1-s)^{1-b}}, \quad u \in [0; 1].$$

We note, that Beta distributions appear as finite dimensional distributions of the Poisson–Dirichlet process (see [19]). In this paper we generate two-dimensional Dirichlet vectors using the arguments of probabilistic number theory. We construct a sequence of two-dimensional vectors, defined via multiplicative functions, whose average distribution functions converge to two-dimensional Dirichlet distribution and estimate the convergence rate.

In the sequel we will use the following notations: p is prime, $d, q, k, m, n \in \mathbb{N}$. In the asymptotic relations it is assumed that $x \to \infty$. The letters c and C with or without subscripts denote constants.

Let $f_i : \mathbb{N} \to \mathbb{R}, i = 1, \ldots, k-1$, be non-negative multiplicative functions. Define the multiplicative function T_k by

$$T_k(n) := \sum_{j_1 j_2 \cdots j_{k-1} | n} f_1(j_1) \cdots f_{k-1}(j_{k-1}),$$

where the sum is taken over all ordered collections $(j_1, j_2, \ldots, j_{k-1})$. In case of all multiplicative functions $f_i \equiv 1$, this function coincides with the classical function τ_k which counts the number of ordered factorisations of $n \in \mathbb{N}$ into k factors

$$\tau_k(n) := \sum_{l_1 l_2 \cdots l_{k-1} | n} 1.$$

The first attempt to simulate the Arcsine law, that is $\mathcal{B}(1/2, 1/2)$, by means of the divisor function was made in [12]. Later other authors considered this problem on various subsets of natural numbers, for example, on the set of numbers free of large prime factors (see [7,18]), in short intervals (see [2,8,9,13,15]), on the set of square-free natural numbers in short intervals [14].

In [16] it was pointed out that using sequences of discrete distributions, constructed by multiplicative functions from some classes, one can simulate the Beta distribution. This idea was realized in papers [4,6,10,11].

For the first time the bivariate Beta distribution as a limit of the sequence of discrete distributions defined via multiplicative functions was considered by Nyandwi and Smati. They proved in [17] that using the divisor function $\tau_3(n)$ one can model some distribution which, as was noted in [5], turned out to be the two-dimensional Dirichlet distribution $\mathcal{D}(1/3, 1/3, 1/3)$. In the paper [11] it was shown that using the divisor function $\tau_2(n)$ one can model the Dirichlet distribution $\mathcal{D}(1/2, 1/4, 1/4)$. In [5] we showed that by means of multiplicative functions one can model one-parameter Dirichlet distributions $\mathcal{D}(a, a, 1-2a)$, $0 < a < 1/2$.

In this paper we show that taking a different construction of distribution function and using some ideas of [5,10] we can model the bivariate Beta distribution for any collection of positive parameters a, b, c. We note that new interesting questions arise if this problem is extended to some special subsets of natural numbers.

In the following we will need the multiplicative function T_3. Note that

$$T_3(p^m) = \sum_{i=0}^{m} f_1(p^i) \sum_{k=i}^{m} f_2(p^{m-k}). \tag{1.1}$$

Let us introduce the random vectors $(X_n; Y_n)$, which take values

$$\left(\frac{\ln d_1}{\ln n}; \frac{\ln d_2}{\ln n}\right),$$

when d_1, d_2 run through all divisors of n with uniform probability $1/T_3(n)$. The distribution function of vector $(X_n; Y_n)$ is

$$F_n(u, v) := \frac{1}{T_3(n)} \sum_{\substack{qm \mid n \\ q \leqslant n^u,\, m \leqslant n^v}} f_1(q) f_2(m).$$

It is easy to check that the sequence of distributions F_n does not converge pointwise on $[0, 1] \times [0, 1]$ (see [17]). Therefore, following [4] we consider the corresponding Cesàro mean

$$S_x(u, v) := \frac{1}{G(x)} \sum_{n \leqslant x} g(n) F_n(u, v), \tag{1.2}$$

here g is some multiplicative function and

$$G(x) := \sum_{n \leqslant x} g(n).$$

In this paper we show, that if the multiplicative functions g, f_1, f_2 satisfy some conditions of regularity, then the corresponding Cesàro mean (1.2) approaches a Dirichlet distribution function $D(u, v; a, b, c)$. We note, that any Dirichlet distribution $\mathcal{D}(a, b, c)$ can be modeled by a suitable choice of multiplicative functions.

2 Results

Definition 2.1 We say that a multiplicative function $g \colon \mathbb{N} \to [0; \infty)$ belongs to the class $\mathcal{K}(\varkappa, \delta)$, for some constants $\varkappa, \delta \geqslant 0$, if the function

$$L(s) := \sum_{p} \frac{g(p) - \varkappa}{p^s}, \quad s = \sigma + i\tau, \quad \sigma > 1,$$

for some $0 < c \leqslant 1/2$, has an analytic continuation $P(s)$ into the region

$$\sigma \geqslant \sigma(\tau) := 1 - \frac{c}{\ln(|\tau| + 3)},$$

where $P(s)$ is holomorphic and $|P(s)| \leqslant \delta \log(|\tau| + 1) + c_0$ for some $c_0 \geqslant 0$.

Definition 2.2 We say that a pair of non-negative multiplicative functions (φ, g) belongs to the class $\mathcal{G}(\varkappa, \delta)$ if $\varphi \cdot g \in \mathcal{K}(\varkappa, \delta)$ and $\varphi(p^j)g(p^k) \leqslant C$ for some $C > 0$ and all integers $0 \leqslant j \leqslant k$.

Note We say that a multiplicative function $g \in \mathcal{G}(\varkappa, \delta)$ if $(1, g) \in \mathcal{G}(\varkappa, \delta)$.

The aim of this paper is to prove the following result.

Theorem 2.3 *Let multiplicative functions* $g, f_1, f_2 \colon \mathbb{N} \to [0, \infty)$ *be such that* $g \in \mathcal{G}(\alpha; \delta_1)$, $g \cdot f_1/T_3 \in \mathcal{K}(\beta; \delta_2)$, $g \cdot f_2/T_3 \in \mathcal{K}(\gamma; \delta_3)$ *for some* $\beta, \gamma > 0$ *and* $\beta + \gamma < \alpha$, $0 \leqslant \delta_1 + \delta_2 + \delta_3 < 1$. *Then for all* $u, v \in [0, 1]$,

$$S_x(u, v) = D(u, v; \beta, \gamma, \alpha - \beta - \gamma) + \mathrm{O}(\xi_x(u + v; \alpha, \beta, \gamma)).$$

Here

$$\xi_x(z; \alpha, \beta, \gamma) := \frac{1}{\ln^{\beta} x} + \frac{1}{\ln^{\gamma} x} + \frac{l_x(\beta, \gamma)}{\ln x} + \frac{\mathbb{1}(z)}{\ln^{\alpha-\beta-\gamma} x} + \frac{l_x(\alpha - \beta - \gamma)}{\ln x}\mathbb{1}(z),$$

$$l_x(t_1, t_2, \ldots, t_k) := \begin{cases} \ln\ln x & \text{if } t_i = 1 \text{ for some } 1 \leqslant i \leqslant k, \\ 1 & \text{otherwise,} \end{cases}$$

and

$$\mathbb{1}(z) := \begin{cases} 1 & \text{if } z > 1, \\ 0 & \text{if } z \leqslant 1. \end{cases}$$

Unless otherwise indicated, here and in what follows we assume that the implicit constants in $\ll$ or $\mathrm{O}(\cdot)$ depend at most on the parameters and constants involved in the definitions of the corresponding classes $\mathcal{G}$ and $\mathcal{K}$.

Example 2.4 Consider a Dirichlet distribution $\mathcal{D}(a, b, c)$ with any positive parameters a, b, c. Let us find multiplicative functions g, f_1, f_2 such that $S_x(u, v) \to D(u, v; a, b, c)$ as $x \to \infty$. Assume that these functions are strongly multiplicative with non-negative constant values on prime numbers, say $g(p) = z_0$, $f_1(p) = z_1$, $f_2(p) = z_2$. Then $g \in \mathcal{G}(z_0, 0)$,

$$\frac{g \cdot f_1}{T_3} \in \mathcal{K}\left(\frac{z_0 z_1}{1 + z_1 + z_2}, 0\right), \quad \frac{g \cdot f_2}{T_3} \in \mathcal{K}\left(\frac{z_0 z_2}{1 + z_1 + z_2}, 0\right).$$

By Theorem 2.3 the limit distribution of S_x becomes $\mathcal{D}(a, b, c)$ provided $z_0 = a+b+c$, $z_1 = a/c$, $z_2 = b/c$.

Example 2.5 Suppose that $g(n) = \mu^2(n)$ and $f_1(n) = f_2(n) \equiv 1$. In this case we have that $T_3(n) \equiv \tau_3(n)$, and $g \in \mathcal{G}(1, 0)$, $g \cdot f_1/\tau_3 \in \mathcal{K}(1/3, 0)$, $g \cdot f_2/\tau_3 \in \mathcal{K}(1/3, 0)$. By the classical formula for the number of square-free integers (see e.g. [20, Theorem 3.10])

$$\sum_{n\leqslant x} \mu^2(n) = \frac{6}{\pi^2} x + \mathrm{O}(\sqrt{x}).$$

Then Theorem 2.3 yields

$$\frac{\pi^2}{6x} \sum_{\substack{n\leqslant x \\ n \text{ is square-free}}} \frac{1}{\tau_3(n)} \sum_{\substack{qm\,|\,n \\ q\leqslant n^u,\, m\leqslant n^v}} 1 = D\left(u, v; \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) + \mathrm{O}\left(\frac{1}{\sqrt[3]{\ln x}}\right).$$

3 Preliminaries

For $\varkappa > 0$ and any multiplicative function θ set

$$A(\varkappa, \theta) := \frac{1}{\Gamma(\varkappa)} \prod_p \left(1 - \frac{1}{p}\right)^{\varkappa} \sum_{k=0}^{\infty} \frac{\theta(p^k)}{p^k}.$$

Note, that $A(\varkappa, \theta) > 0$, when $\theta \in \mathcal{G}(\varkappa, \delta)$, $0 \leqslant \delta < 1$.

Lemma 3.1 in [3] and Lemma 1 in [5] imply

Lemma 3.1 *Assume that* $(\varphi, g) \in \mathcal{G}(\varkappa, \delta)$, $\varkappa > 0$ *and* $0 \leqslant \delta < 1$. *Then, uniformly for all* $x \geqslant 1$ *and* $d \in \mathbb{N}$,

$$\sum_{n\leqslant x} \varphi(n) g(nd) = \frac{x}{\ln^{1-\varkappa}(\mathrm{e}x)} \left(A(\varkappa, \varphi \cdot g) \cdot \widetilde{h}(d \,|\, \varphi, g) + \mathrm{O}\left(\frac{\widehat{h}(d \,|\, \varphi, g)}{\ln(\mathrm{e}x)}\right)\right),$$

where the multiplicative functions $\widetilde{h}$ *and* $\widehat{h}$ *are defined by*

$$\begin{aligned} \widetilde{h}(p^k \,|\, \varphi, g) &:= \left(\sum_{j=0}^{\infty} \frac{\varphi(p^j) g(p^j)}{p^j}\right)^{-1} \sum_{j=0}^{\infty} \frac{\varphi(p^j) g(p^{k+j})}{p^j}, \\ \widehat{h}(p^k \,|\, \varphi, g) &:= \left(1 + \frac{c_1}{p^{\sigma_0}}\right) \sum_{j=0}^{\infty} \frac{\varphi(p^j) g(p^{k+j})}{p^{j\sigma_0}}. \end{aligned} \tag{3.1}$$

Here $\sigma_0 = \sigma(0)$ *and* $c_1 \geqslant 0$ *is a constant, depending on the parameters* c, $\varkappa$ *and* C *of the classes* $\mathcal{G}$ *and* $\mathcal{K}$. *Moreover*

$$\sum_{n\leqslant x} \varphi(n) g(nd) \ll x \cdot \widehat{h}(d \,|\, \varphi, g) \ln^{\varkappa-1}(\mathrm{e}x). \tag{3.2}$$

Remark 3.2 If $(\varphi, g) \in \mathcal{G}(\varkappa, \delta)$, then

$$\begin{aligned} \widetilde{h}(p^k \mid \varphi, g) &= g(p^k) + \mathrm{O}\biggl(\frac{g(p^k)}{p} + \sum_{j=1}^{\infty} \frac{\varphi(p^j)g(p^{k+j})}{p^j}\biggr), \\ \widehat{h}(p^k \mid \varphi, g) &= g(p^k) + \mathrm{O}\biggl(\frac{g(p^k)}{p^{\sigma_0}} + \sum_{j=1}^{\infty} \frac{\varphi(p^j)g(p^{k+j})}{p^{j\sigma_0}}\biggr), \end{aligned} \tag{3.3}$$

for any $k \in \mathbb{N}$. Hence $(\varphi, \widetilde{h}) \in \mathcal{G}(\varkappa, \delta)$ and $(\varphi, \widehat{h}) \in \mathcal{G}(\varkappa, \delta)$. In the sequel we will often use this property.

For $0 \leqslant u \leqslant w \leqslant 1$, $x \geqslant 1$, $b \in \mathbb{R}$ we set

$$\mathfrak{S}_x(u, w, b) := \sum_{x^u < m \leqslant x^w} \frac{a_m}{m \ln^b(\mathrm{e}x/m)}, \quad a_m \geqslant 0.$$

This sum may be evaluated in terms of the integral

$$I(u, w; a, b, \eta) := \int_u^w \frac{\mathrm{d}v}{(\eta + v)^a (\eta + 1 - v)^b}, \tag{3.4}$$

provided some information about the behaviour of the sum

$$M(v) := \sum_{m \leqslant v} a_m$$

is given.

For any $y > 0$ and $a \in \mathbb{R}$ let us define

$$\lambda(y, a) := \begin{cases} y^{1-a}, & a \neq 1; \\ |\ln y|, & a = 1. \end{cases}$$

In addition we assume that $\lambda(0, a) = \infty$, when $a \geqslant 1$, and $\lambda(0, a) = 0$ otherwise.

For $x \geqslant \mathrm{e}$, $0 \leqslant w \leqslant 1$, $0 < t \leqslant \min\{x^{1-w}, x/\mathrm{e}\}$, $a_1, a_2 \in \mathbb{R}$ we set $\eta_x := \ln^{-1} x$,

$$\begin{aligned} r_x(a_1, a_2) &:= \frac{l_x(a_1 + 1, a_2)}{\ln^{a_1 + a_2} x}, \\ \rho_x(t, w; a_1, a_2) &:= \frac{(1 + w \ln x)^{-a_1}}{(1 + \ln \frac{x}{t} - w \ln x)^{a_2 - 1} \ln \frac{x}{t}} + \left(\ln \frac{x}{t}\right)^{-a_2} + r_{x/t}(a_1, a_2). \end{aligned}$$

Note that $\lambda(\eta_x, a) = l_x(a) \ln^{a-1} x$.

The following consequence from [4, Lemmas 3 and 4] will be applied to evaluate the sum $\mathfrak{S}_x(0, w, b)$.

Lemma 3.3 *Assume that* $x \geqslant \mathrm{e}$ *and*

$$\left| M(v) - \frac{Av}{\ln^a(\mathrm{e}v)} \right| \leqslant \frac{Bv}{\ln^{a+1}(\mathrm{e}v)}$$

for some $A, a \in \mathbb{R}$, $B \geqslant 0$, *and all* $1 \leqslant v \leqslant x$. *Then*

$$\mathfrak{S}_x(0, w; b) = \frac{A}{(\ln x)^{a+b-1}} I(0, w; a, b, \eta_x) + \mathrm{O}\big((|A| + B)\rho_x(1, w; a, b)\big). \quad (3.5)$$

The implicit constant in $\mathrm{O}(\cdot)$ *depends at most on* a *and* b.

We will need some estimates of the integrals

$$J_1(\varepsilon, \eta, u, v, a, b, c) := \int_\varepsilon^u \int_\varepsilon^v \frac{\mathrm{d}z\mathrm{d}s}{(\eta + s)^a(\eta + z)^b(\eta + 1 - s - z)^c},$$

$$J_2(\varepsilon, \eta, u, a, b, c) := \int_\varepsilon^u \int_\varepsilon^{1-s-\varepsilon} \frac{\mathrm{d}z\mathrm{d}s}{(\eta + s)^a(\eta + z)^b(\eta + 1 - s - z)^c}.$$

It is easy to see that

$$\begin{aligned} J_2(0, 0, u, a, b, c) &= \int_0^u \frac{\mathrm{d}s}{s^a(1-s)^{b+c-1}} \int_0^1 \frac{\mathrm{d}t}{t^b(1-t)^c} \\ &= B(u; 1-a, 2-b-c)\,\frac{\Gamma(1-a)\Gamma(1-b)\Gamma(1-c)}{\Gamma(3-a-b-c)}. \end{aligned} \quad (3.6)$$

The following four lemmas can be proved by repeating the corresponding arguments in the proofs of [5, Lemmas 4–7].

Lemma 3.4 *If* $a, b, c \in (-\infty; 1)$ *and* $0 \leqslant \eta \leqslant 1$, *then*

$$\begin{aligned} J_1(0, \eta, u, v, a, b, c) &\ll \min\big((u+\eta)^{1-a}, (v+\eta)^{1-b}\big), \\ J_2(0, \eta, u, a, b, c) &\ll (u+\eta)^{1-a} \end{aligned} \quad (3.7)$$

uniformly for $u, v \in [0, 1]$, $u + v \leqslant 1$. *Constants in* $\ll$ *depend on* a, b, c *only.*

Lemma 3.5 *Let* $0 \leqslant \varepsilon \leqslant 1/4$, $0 \leqslant \eta \leqslant 1$, $\lambda(a) = \lambda(\varepsilon + \eta, a)$. *If* $c < 2$, *then*

$$J_1(\varepsilon, \eta, u, v, a, b, c) \ll 1 + \lambda(a)\lambda(b) + (\lambda(a) + \lambda(b))(\lambda(c) + 1)$$

uniformly for $u, v \in [\varepsilon, 1]$, $u + v \leqslant 1$. *Constants in* $\ll$ *depend at most on* a, b, c.

Lemma 3.6 *Let* $0 \leqslant \varepsilon \leqslant 1/4$, $0 \leqslant \eta \leqslant 1$, $\lambda(a) = \lambda(\varepsilon + \eta, a)$. *Then*

$$J_2(\varepsilon, \eta, u, a, b, c) \ll \lambda(a)\lambda(b) + (1 + \lambda(a) + \lambda(b))(1 + \lambda(c))$$

uniformly for $u \in [\varepsilon, 1 - 2\varepsilon]$. *The constant in* $\ll$ *depends at most on* a, b, c.

Lemma 3.7 *Let* $0 \leqslant \eta \leqslant 1$ *and* $a, b, c \in (-\infty, 1)$. *Then*

$$\begin{aligned} J_1(0, 0, u, v, a, b, c) - J_1(0, \eta, u, v, a, b, c) &\ll \eta^{1-a} + \eta^{1-b} + \eta, \\ J_2(0, 0, u, a, b, c) - J_2(0, \eta, u, a, b, c) &\ll \eta^{1-a} + \eta^{1-b} + \eta^{1-c} + \eta \end{aligned} \tag{3.8}$$

uniformly for $u, v \in [0, 1]$, $u + v \leqslant 1$. *Constants in* $\ll$ *depend at most on* a, b, c.

Note Evaluating $\lambda(a) \cdot \lambda(b)$ in Lemmas 3.5 and 3.6 we assume that $0 \cdot \infty = \infty$.

Combining Lemmas 3.3 and 3.1 we obtain the following result.

Lemma 3.8 *Assume that* $b \in \mathbb{R}$ *and* $(\varphi, \theta) \in \mathcal{G}(1 - a, \delta)$ *for some* $a < 1$, $0 \leqslant \delta < 1$. *Then for* $q \in \mathbb{N}$, $x \geqslant \mathrm{e}$, $\eta_x \leqslant w \leqslant 1$, $0 < t \leqslant x^{1-w}$, *we have*

$$\begin{aligned} &Z_x(q, t, w, b; \varphi, \theta) \\ &\quad := \sum_{m \leqslant x^w} \frac{\varphi(m)\theta(qm)}{m(\ln(\mathrm{e}x/(mt)))^b} \\ &\quad = A(1 - a, \varphi \cdot \theta)\, \widetilde{h}(q \mid \varphi, \theta) \left(\frac{1}{\ln x}\right)^{a+b-1} \\ &\quad\quad \times \int_0^w \frac{\mathrm{d}s}{(\eta_x + s)^a \left(\eta_x + 1 - \frac{\ln t}{\ln x} - s\right)^b} + \mathrm{O}\big(\rho_x(t, w; a, b)\, \widehat{h}(q \mid \varphi, \theta)\big). \end{aligned}$$

Moreover,

$$Z_x(q, t, w, b; \varphi, \theta) \ll \rho_x(t, w; a - 1, b)\, \widehat{h}(q \mid \varphi, \theta). \tag{3.9}$$

The multiplicative functions $\widetilde{h}$ *and* $\widehat{h}$ *are defined in* (3.1). *The implicit constants in* $\ll$ *and* $\mathrm{O}(\cdot)$ *depend at most on* b *and the parameters of class* $\mathcal{G}$.

Proof Using notations of Lemma 3.3 and taking $a_m = \varphi(m)\theta(qm)$, we can write

$$Z_x(q, t, w, b; \theta) = \mathfrak{S}_{x/t}(0, z, b) + \theta(q)\left(\ln \frac{\mathrm{e}x}{t}\right)^{-b}, \tag{3.10}$$

where $z = w \frac{\ln x}{\ln(x/t)}$. By Lemma 3.1 with $d = q$, having in mind that $(\varphi, \theta) \in \mathcal{G}(1 - a, \delta)$, we get

$$\sum_{n \leqslant x} \varphi(n)\theta(nq) = \frac{x}{\ln^a(\mathrm{e}x)} \left(A(1 - a, \varphi \cdot \theta) \cdot \widetilde{h}(q \mid \varphi, \theta) + \mathrm{O}\left(\frac{\widehat{h}(q \mid \varphi, \theta)}{\ln(\mathrm{e}x)}\right)\right).$$

Therefore we may evaluate $\mathfrak{S}_{x/t}(0,z,b)$ by means of Lemma 3.3. Then (3.10) becomes

$$Z_x(q,t,w,b;\theta)=\frac{A(1-a,\varphi\cdot\theta)\widetilde{h}(q\,|\,\varphi,\theta)}{(\ln(x/t))^{a+b-1}}\,I(0,z;a,b,\eta_{x/t})\\ +\mathrm{O}\big(\rho_x(t,w;a,b)\widehat{h}(q\,|\,\varphi,\theta)\big),$$

where the integral I is defined in (3.4). Changing the integration variable gives

$$I(0,z;a,b,\eta_{x/t})=\left(\frac{\ln(x/t)}{\ln x}\right)^{a+b-1}\int\limits_0^w\frac{\mathrm{d}s}{(\eta_x+s)^a\big(\eta_x+1-\frac{\ln t}{\ln x}-s\big)^b}.$$

This proves the first relation of the lemma.

It remains to prove the estimate (3.9). By (3.2) we have

$$\sum_{n\leqslant x}\varphi(n)\theta(nd)\ll x\cdot\widehat{h}(d\,|\,\varphi,\theta)\ln^{-a}(\mathrm{e}x).$$

Therefore (3.9) follows from (3.10) and (3.5) by taking $A=0$, $B=\mathrm{O}(\widehat{h}(d\,|\,\varphi,\theta))$ and choosing $a-1$ instead of a. □

In the next two lemmas we consider the triplet of non-negative multiplicative functions $(\varphi_1,\varphi_2,\theta)$ that satisfy the conditions

$$(\varphi_1,\theta)\in\mathcal{G}(1-a,\delta')\quad\text{and}\quad(\varphi_2,\theta)\in\mathcal{G}(1-d,\delta'') \tag{3.11}$$

for some $a<1$, $d<1$, $0\leqslant\delta',\delta''<1$;

$$\varphi_1(p^i)\varphi_2(p^j)\theta(p^k)\leqslant C_1 \tag{3.12}$$

for some $C_1>0$ and all non-negative integers i,j,k such that $i+j\leqslant k$.

Remark 3.9 Note, that (3.1), (3.3), (3.11) and (3.12) imply

$$(\varphi_1,\widetilde{h}(\cdot\,|\,\varphi_2,\theta))\in\mathcal{G}(1-a,\delta')\quad\text{and}\quad(\varphi_2,\widetilde{h}(\cdot\,|\,\varphi_1,\theta))\in\mathcal{G}(1-d,\delta'').$$

The same relations hold for $\widehat{h}$ instead of $\widetilde{h}$.

Lemma 3.10 *Assume that* $x\geqslant\mathrm{e}$, $u,v\geqslant 0$, $u+v\leqslant 1$ *and the multiplicative functions* $(\varphi_1,\varphi_2,\theta)$ *satisfy* (3.11) *and* (3.12).

If $b<2$, *then*

$$E_x(u,v,b;\varphi_1,\varphi_2,\theta):=\sum_{q\leqslant x^u}\frac{\varphi_2(q)}{q}Z_x(q,q,v,b;\varphi_1,\theta)\\ \ll\frac{1}{\ln^{a+b+d-2}x}+\frac{l_x(d+1,b)l_x(d+b)}{\ln^a x}+\frac{l_x(a+1,b)}{\ln^d x}. \tag{3.13}$$

Moreover, if $b<1$*, then*

$$E_x(u,v,b;\varphi_1,\varphi_2,\theta)=A^*\frac{J_1(0,\eta_x,u,v,d,a,b)}{(\ln x)^{a+b+d-2}}+R_E, \tag{3.14}$$

here $A^*=A(1-a,\varphi_1\theta)A(1-d,\varphi_2\widetilde{h}(\cdot\,|\,\varphi_1,\theta))$*, and*

$$R_E\ll\frac{1}{\ln^{a+b-1}x}+\frac{1}{\ln^{b+d-1}x}+\frac{l_x(a+1)l_x(a+b)+l_x(d+1)}{\ln^{a+b+d-1}x}. \tag{3.15}$$

The implicit constants in $\ll$ *depend at most on* b*,* C_1 *and the parameters of classes* $\mathcal{G}$*.*

Proof Let $b\in\mathbb{R}$. Firstly assume that $u\leqslant\eta_x$. Then using (3.9), (3.3) and (3.12) we have

$$\begin{aligned}E_x(u,v,b;\varphi_1,\varphi_2,\theta)&\ll Z_x(1,1,1,b;\varphi_1,\theta)+\varphi_2(2)Z_x(2,2,1,b;\varphi_1,\theta)\\&\ll\rho_x(1,1;a-1,b).\end{aligned}$$

If $v\leqslant\eta_x$, then

$$E_x(u,v,b;\varphi_1,\varphi_2,\theta)\ll\rho_x(1,1;d-1,b),$$

since $E_x(u,v,b;\varphi_1,\varphi_2,\theta)=E_x(v,u,b;\varphi_2,\varphi_1,\theta)$. Thus

$$E_x(u,v,b;\varphi_1,\varphi_2,\theta)\ll\rho_x(1,1;a-1,b)+\rho_x(1,1;d-1,b) \tag{3.16}$$

uniformly in $\{(u,v)\,|\,0\leqslant u,v\leqslant 1,\min(u,v)\leqslant\eta_x\}$.

Assume that $\min(u,v)>\eta_x$. Then $u,v\in(\eta_x;1-\eta_x)$, since $u+v\leqslant 1$.

It is easy to see that

$$\Big(1+\ln\frac{x}{q}-v\ln x\Big)^{b-1}\ln\frac{x}{q}\geqslant\Big(\ln\frac{ex^{1-v}}{q}\Big)^{b}.$$

Having this in mind and using Lemma 3.8 we get

$$\begin{aligned}&E_x(u,v,b;\varphi_1,\varphi_2,\theta)\\&\quad=\frac{A(1-a,\varphi_1\theta)}{\ln^{a-1}x}\int\limits_0^v\frac{Z_{x^{1-s}}(1,1,u/(1-s),b;\varphi_2,\widetilde{h}(\cdot\,|\,\varphi_1,\theta))}{(\eta_x+s)^a}\,\mathrm{d}s\\&\qquad+\mathrm{O}\big(Z_x\big(1,1,1,b;\varphi_2,\widehat{h}(\cdot\,|\,\varphi_1,\theta)\big)\\&\qquad+(1+v\ln x)^{-a}Z_{x^{1-v}}\big(1,1,1,b;\varphi_2,\widehat{h}(\cdot\,|\,\varphi_1,\theta)\big)\\&\qquad+Z_x\big(1,1,1,a+b;\varphi_2,\widehat{h}(\cdot\,|\,\varphi_1,\theta)\big)\cdot l_x(a+1,b)\big)=:S_1+\mathrm{O}(R_1).\end{aligned} \tag{3.17}$$

By Remark 3.9, $(\varphi_2, \widetilde{h}(\cdot\,|\,\varphi_1,\theta)) \in \mathcal{G}(1-d, \delta'')$. Then using Lemma 3.8 once again we get

$$Z_{x^{1-s}}\left(1, 1, \frac{u}{1-s}, b; \varphi_2, \widetilde{h}(\cdot\,|\,\varphi_1,\theta)\right)$$
$$= \frac{A(1-d, \varphi_2\widetilde{h}(\cdot\,|\,\varphi_1,\theta))}{(\ln x^{1-s})^{d+b-1}} \int\limits_0^{\frac{u}{1-s}} \frac{(\eta_{x^{1-s}}+t)^{-d}}{(\eta_{x^{1-s}}+1-t)^b}\,\mathrm{d}t$$
$$+ \mathrm{O}\big(((1-s)\ln x)^{-b} + ((1-s)\ln x)^{-d-b} + ((1-s)\ln x)^{-d-1} + r_{x^{1-s}}(d,b)\big).$$

Therefore the term S_1 in (3.17) becomes

$$S_1 \cdot \ln^{a-1} x$$
$$= A(1-a, \varphi_1\theta) A(1-d, \varphi_2\widetilde{h}(\cdot\,|\,\varphi_1,\theta)) \int\limits_0^v \frac{\mathrm{d}s}{((1-s)\ln x)^{d+b-1}(\eta_x+s)^a}$$
$$\times \int\limits_0^{\frac{u}{1-s}} \frac{\mathrm{d}t}{(\eta_{x^{1-s}}+t)^d(\eta_{x^{1-s}}+1-t)^b} + \mathrm{O}(L_1), \tag{3.18}$$

where $\gamma_0 = \min\{b, b+d, d+1\}$,

$$L_1 = \int\limits_0^v \left(\frac{1}{((1-s)\ln x)^{\gamma_0}} + r_{x^{1-s}}(d,b)\right)\frac{\mathrm{d}s}{(\eta_x+s)^a}. \tag{3.19}$$

We have

$$\int\limits_0^v \frac{\mathrm{d}s}{(\eta_x+s)^a(1-s)^{\gamma_0}} \ll 1 + \lambda(\eta_x, \gamma_0). \tag{3.20}$$

Similarly,

$$\int\limits_0^v \frac{r_{x^{1-s}}(d,b)\,\mathrm{d}s}{(\eta_x+s)^a} \ll \frac{l_x(d+1,b)}{\ln^{d+b} x}\,(1+\lambda(\eta_x, d+b)).$$

This estimate together with (3.20), (3.19) and (3.18) yield

$$S_1 = \frac{A(1-a,\varphi_1\theta) A(1-d, \varphi_2\widetilde{h}(\cdot\,|\,\varphi_1,\theta))}{\ln^{a+b+d-2} x} J_1(0, \eta_x, u, v, d, a, b)$$
$$+ \mathrm{O}\left(\frac{1}{\ln^{\gamma_0+a-1} x} + \frac{l_x(\gamma_0) + l_x(d+1,b)l_x(d+b)}{\ln^a x} + \frac{l_x(d+1,b)}{\ln^{a+b+d-1} x}\right). \tag{3.21}$$

Since $(\varphi_2, \widehat{h}(\cdot\,|\,\varphi_1, \theta)) \in \mathcal{G}(1-d, \delta'')$ (see Remark 3.9), we can employ (3.9) to estimate the remainder term R_1 in (3.17). We have

$$Z_x\big(1, 1, 1, b; \varphi_2, \widehat{h}(\cdot\,|\,\varphi_1, \theta)\big) \ll \frac{1}{\ln^d x} + \frac{l_x(b)}{\ln^{b+d-1} x}, \tag{3.22}$$

$$\begin{aligned}(1 + v \ln x)^{-a} & Z_{x^{1-v}}\big(1, 1, 1, b; \varphi_2, \widehat{h}(\cdot\,|\,\varphi_1, \theta)\big) \\ &\ll \frac{1}{\ln^a x} + \frac{1}{\ln^d x} + \frac{1}{\ln^{a+d} x} + \frac{l_x(b)}{\ln^{b+d-1} x} + \frac{l_x(b)}{\ln^{a+b+d-1} x},\end{aligned} \tag{3.23}$$

and

$$\begin{aligned}Z_x\big(1, 1, 1, a+b; \varphi_2, \widehat{h}(\cdot\,|\,\varphi_1, \theta)\big)\cdot l_x(a+1, b) \\ \ll \frac{l_x(a+1, b)}{\ln^d x} + \frac{l_x(a+1, b)\, l_x(a+b)}{\ln^{a+b+d-1} x}.\end{aligned} \tag{3.24}$$

Taking into account the last three estimates we obtain that the remainder term in (3.17) can be estimated by

$$R_1 \ll \frac{1}{\ln^a x} + \frac{1}{\ln^{a+d} x} + \frac{l_x(a+1, b)}{\ln^d x} + \frac{l_x(b)}{\ln^{b+d-1} x} + \frac{l_x(a+1, b)\, l_x(a+b)}{\ln^{a+b+d-1} x}. \tag{3.25}$$

Note that $l_x(\gamma_0) \leqslant l_x(d+1, b) l_x(d+b)$. When $\min(u, v) > \eta_x$, from (3.17), (3.21) and (3.25) we deduce that the remainder term in (3.14) is

$$\begin{aligned}R_E \ll \frac{1}{\ln^{a+b-1} x} + \frac{1}{\ln^{a+d} x} + \frac{l_x(d+1, b)\, l_x(d+b)}{\ln^a x} + \frac{l_x(a+1, b)}{\ln^d x} \\ + \frac{l_x(b)}{\ln^{b+d-1} x} + \frac{l_x(a+1, b)\, l_x(a+b) + l_x(d+1)}{\ln^{a+b+d-1} x}.\end{aligned} \tag{3.26}$$

Therefore the estimate (3.13) follows from (3.26) by means of (3.16) and Lemma 3.5 provided $b < 2$.

When $b < 1$ the estimate (3.26) implies (3.15). Note that in this case (3.15) easily follows from (3.16) and (3.7) if $\min(u, v) \leqslant \eta_x$. □

Lemma 3.11 *Assume that $b \in \mathbb{R}$ and the multiplicative functions $(\varphi_1, \varphi_2, \theta)$ satisfy (3.11) and (3.12). Then uniformly for $0 \leqslant u \leqslant 1 - \eta_x$,*

$$\begin{aligned}E_x^*(u, b; \varphi_1, \varphi_2, \theta) \\ := \sum_{q \leqslant x^u} \frac{\varphi_2(q)}{q} Z_{x/q}(q, 1, 1, b; \varphi_1, \theta) \\ = \frac{A(1-a, \varphi_1\theta) A(1-d, \varphi_2 \widetilde{h}(\cdot\,|\,\varphi_1, \theta))}{(\ln x)^{a+d+b-2}} J_2(0, \eta_x, u, d, a, b) \\ + \mathrm{O}\bigg(\frac{1}{\ln^{a+d-1} x} + \frac{l_x(b)}{\ln^{b+d-1} x} + \frac{l_x(d+1)}{\ln^{a+b-1} x} + \frac{l_x(a+1, b)\, l_x(a+b)}{\ln^{a+b+d-1} x}\bigg).\end{aligned}$$

The implicit constant in $\mathrm{O}(\cdot)$ depends at most on b, C_1 and the parameters of classes $\mathcal{G}$.

Proof By Lemma 3.8 we get

$$\begin{aligned} E_x^*(u,b;\varphi_1,\varphi_2,\theta) &= S_2 + \mathrm{O}(R_2)\\ &=: A(1-a,\varphi_1\cdot\theta)\sum_{q\leqslant x^u}\frac{\varphi_2(q)}{q}\widetilde{h}(q\,|\,\varphi_1,\theta)\Big(\ln\frac{x}{q}\Big)^{1-a-b}\int_0^1\frac{\mathrm{d}s}{(\eta_{x/q}+s)^a(\eta_{x/q}+1-s)^b}\\ &\quad+\mathrm{O}\Big(\sum_{q\leqslant x^u}\frac{\varphi_2(q)}{q}\widehat{h}(q\,|\,\varphi_1,\theta)\Big(\Big(\ln\frac{x}{q}\Big)^{-b}+\Big(\ln\frac{x}{q}\Big)^{-a-1}+r_{x/q}(a,b)\Big)\Big). \end{aligned} \tag{3.27}$$

We have that $(\varphi_2,\widehat{h}(\cdot\,|\,\varphi_1,\theta))\in\mathcal{G}(1-d,\delta'')$ (see Remark 3.9). Applying Lemma 3.8 and relations (3.22), (3.24) we obtain

$$\begin{aligned} R_2 &\ll Z_x\big(1,1,1-\eta_x,b;\varphi_2,\widehat{h}(\cdot\,|\,\varphi_1,\theta)\big)+Z_x\big(1,1,1-\eta_x,1+a;\varphi_2,\widehat{h}(\cdot\,|\,\varphi_1,\theta)\big)\\ &\qquad+Z_x\big(1,1,1-\eta_x,a+b;\varphi_2,\widehat{h}(\cdot\,|\,\varphi_1,\theta)\big)\,l_x(a+1,b)\\ &\ll\frac{l_x(b)}{\ln^{b+d-1}x}+\frac{l_x(a+1)}{\ln^{a+d}x}+\frac{l_x(a+1,b)}{\ln^{d}x}+\frac{l_x(a+1,b)l_x(a+b)}{\ln^{a+b+d-1}x}. \end{aligned}$$

Set $\varphi_3(q):=\varphi_2(q)\widetilde{h}(q\,|\,\varphi_1,\theta)$ and

$$I_3(t):=\int_0^{\omega(t)}\frac{\mathrm{d}z}{(\eta_x+z)^a(\eta_x+\omega(t)-z)^b},$$

where $\omega(t)=1-\ln t/\ln x$. The main term in (3.27) can be written as

$$S_2=\frac{A(1-a,\varphi_1\cdot\theta)}{(\ln x)^{a+b-1}}\sum_{q\leqslant x^u}\varphi_3(q)\,\frac{I_3(q)}{q}. \tag{3.28}$$

Partial summation yields

$$\begin{aligned} \sum_{q\leqslant x^u}\varphi_3(q)\,\frac{I_3(q)}{q}&=\frac{I_3(x^u)}{x^u}\sum_{q\leqslant x^u}\varphi_3(q)+\int_{1-}^{x^u}\sum_{q\leqslant s}\varphi_3(q)\,\frac{I_3(s)}{s^2}\,\mathrm{d}s\\ &\quad-\int_{1-}^{x^u}\sum_{q\leqslant s}\varphi_3(q)\,\frac{I_3'(s)}{s}\,\mathrm{d}s=:S_{21}+S_{22}+S_{23}. \end{aligned} \tag{3.29}$$

For $0\leqslant u\leqslant 1-\eta_x$ and $a<1$ we have

$$(1-u)^{a+b-1}I_3(x^u)=I\Big(0,1,a,b,\frac{\eta_x}{1-u}\Big)\ll 1+\lambda\Big(\frac{\eta_x}{1-u},b\Big).$$

It follows from Remark 3.9 that $\varphi_3 \in \mathcal{G}(1-d, \delta'')$. Therefore the last estimate and Lemma 3.1 yield

$$S_{21} \ll \eta_x^d (\eta_x + u)^{-d} (1-u)^{1-a-b} \left(1 + \lambda\left(\frac{\eta_x}{1-u}, b\right)\right).$$

If $b \neq 1$, then this estimate becomes

$$S_{21} \ll \eta_x^d \big(1 + \eta_x^{-d} + \eta_x^{1-a-b}\big) + \eta_x^{1+d-b}\big(1 + \eta_x^{-d} + \eta_x^{-a}\big).$$

If $b = 1$, then similarly

$$S_{21} \ll 1 + \eta_x^d + \eta_x^{d-a} + \eta_x^d (\eta_x + u)^{-d} (1-u)^{-a} \ln \frac{1-u}{\eta_x}.$$

Separately estimating the last summand for $u \in [0, 1/2]$ and $u \in [1/2, 1-\eta_x]$ we obtain

$$S_{21} \ll \eta_x^{d-a} + (1 + \eta_x^d) \ln \frac{1}{\eta_x}.$$

Thus for any $b \in \mathbb{R}$,

$$S_{21} \ll (\ln x)^{a+b-d-1} + (1 + \ln^{-d} x)(1 + l_x(b) \ln^{b-1} x). \tag{3.30}$$

Evaluating $I_3'(s)$ and using Lemma 3.1 we deduce

$$S_{23} \ll I(0, u, d, a, \eta_x) \ln^{b-d} x + b J_2(0, \eta_x, u, d, a, b+1) \ln^{-d} x.$$

In view of Lemma 3.6 we have

$$b J_2(0, \eta_x, u, d, a, b+1) \ll 1 + \ln^b x.$$

Therefore

$$S_{23} \ll (\ln x)^{b-d} + (\ln x)^{-d}. \tag{3.31}$$

To evaluate the second term in (3.29) we use Lemma 3.1 once again. Since $\varphi_3 \in \mathcal{G}(1-d, \delta'')$, we get

$$\begin{aligned} S_{22} &= A(1-d, \varphi_3) \int_1^{x^u} \frac{I_3(s)}{s \ln^d(\mathrm{e}s)}\, \mathrm{d}s + \mathrm{O}\left(\int_1^{x^u} \frac{I_3(s)}{s \ln^{d+1}(\mathrm{e}s)}\, \mathrm{d}s\right) \\ &= \frac{A(1-d, \varphi_3)}{\ln^{d-1} x} J_2(0, \eta_x, u, d, a, b) + \mathrm{O}\left(\frac{J_2(0, \eta_x, u, d+1, a, b)}{\ln^d x}\right). \end{aligned}$$

Using Lemma 3.6 we obtain

$$S_{22} = \frac{A(1-d,\varphi_3)}{\ln^{d-1} x} J_2(0,\eta_x,u,d,a,b) \\ + \mathrm{O}\Big(l_x(d+1) + \frac{1}{\ln^d x} + \frac{l_x(b)}{\ln^{d-b+1} x} + \frac{l_x(d+1)l_x(b)}{\ln^{1-b} x}\Big). \tag{3.32}$$

Combining the estimates (3.30), (3.31), (3.32) together with (3.29), (3.28), (3.27) and having in mind the estimate of R_2 we obtain assertion of the lemma. □

4 Proof of the main theorem

Let us start the proof of Theorem 2.3 with the following

Remark 4.1 The conditions of Theorem 2.3 and (1.1) imply $(f_1, g/T_3) \in \mathcal{G}(\beta; \delta_2)$ and $(f_2, g/T_3) \in \mathcal{G}(\gamma; \delta_3)$, moreover there exists a constant $C_2 \geqslant 0$ such that

$$f_1(p^i) f_2(p^j) \frac{g}{T_3}(p^k) \leqslant C_2, \tag{4.1}$$

for all $i, j, k \geqslant 0$, $i + j \leqslant k$.

Setting

$$K_i(q,m,d) := \frac{g(qmd) f_i(q) f_{3-i}(m)}{T_3(qmd)}, \quad i = 1, 2,$$

we have

$$\begin{aligned} S_x(u,v) &= \frac{1}{G(x)} \sum_{d\leqslant x} \sum_{\substack{q\leqslant n^u,\, m\leqslant n^v \\ n:=qmd\leqslant x}} K_1(q,m,d) \\ &= \frac{1}{G(x)} \sum_{d\leqslant x} \Bigg(\sum_{\substack{q\leqslant x^u \\ m\leqslant x^v \\ qm\leqslant x/d}} - \sum_{\substack{n^u<q\leqslant x^u \\ m\leqslant n^v \\ n:=qmd\leqslant x}} - \sum_{\substack{q\leqslant n^u \\ n^v<m\leqslant x^v \\ n:=qmd\leqslant x}} - \sum_{\substack{n^u<q\leqslant x^u \\ n^v<m\leqslant x^v \\ n:=qmd\leqslant x}} \Bigg) K_1(q,m,d) \\ &=: H(u,v) - H_1(u,v) - H_2(u,v) - H_3(u,v). \end{aligned} \tag{4.2}$$

For $i = 1, 2$ set

$$R_i(u,v) := \frac{1}{G(x)} \sum_{d\leqslant x} \sum_{q\leqslant x^u} \sum_{m\leqslant \min(x^v,\, x^{1-u}/d)} K_i(q,m,d).$$

Then

$$H_1(u,v) \leqslant \frac{1}{G(x)} \sum_{d\leqslant x} \sum_{\substack{n^u<q\leqslant x^u,\, m\leqslant x^v \\ md\leqslant n^{1-u},\, n:=qmd}} K_1(q,m,d) \leqslant R_1(u,v). \tag{4.3}$$

Similarly,

$$H_2(u,v) \leqslant R_2(v,u), \quad H_3(u,v) \leqslant \min\{R_1(u,v), R_2(v,u)\}. \tag{4.4}$$

By Remark 4.1 we have $(f_2, g/T_3) \in \mathcal{G}(\gamma, \delta_3)$. Then using Lemma 3.1 we get

$$\begin{aligned} R_1(u,v) &\ll \frac{1}{G(x)} \sum_{d \leqslant x^{1-u}} \sum_{q \leqslant x^u} f_1(q) \sum_{m \leqslant x^{1-u}/d} \frac{g(qmd) f_2(m)}{T_3(qmd)} \\ &\ll \frac{x^{1-u}}{G(x)} \sum_{d \leqslant x^{1-u}} \frac{1}{d\left(\ln \frac{ex^{1-u}}{d}\right)^{1-\gamma}} \sum_{q \leqslant x^u} f_1(q) h_1(qd), \end{aligned}$$

where

$$h_1(p^k) := \widehat{h}\left(p^k \mid f_2, \frac{g}{T_3}\right).$$

Remarks 3.9 and 4.1 give us $(f_1, h_1) \in \mathcal{G}(\beta, \delta_2)$. According to Lemma 3.1 we have

$$R_1(u,v) \ll \frac{x}{G(x)(1+u\ln x)^{1-\beta}} \sum_{d \leqslant x^{1-u}} \frac{h_2(d)}{d\left(\ln \frac{ex^{1-u}}{d}\right)^{1-\gamma}}, \tag{4.5}$$

where the multiplicative function h_2 is defined by $h_2(p^k) := \widehat{h}(p^k \mid f_1, h_1)$. We note that (3.3) and (4.1) imply

$$h_2(p^k) = \frac{g}{T_3}(p^k) + \mathrm{O}\left(\frac{1}{p^{\sigma_0}}\right).$$

Then having in mind the assumptions of the theorem one can show that g/T_3 and $h_2 \in \mathcal{G}(\alpha - \beta - \gamma, \delta_1 + \delta_2 + \delta_3)$. Moreover by Lemma 3.1,

$$G(x) = \frac{x}{\ln^{1-\alpha}(ex)}\left(A(\alpha, g) + \mathrm{O}\left(\frac{1}{\ln(ex)}\right)\right), \tag{4.6}$$

since $g \in \mathcal{G}(\alpha, \delta_1)$. Thus (4.5) becomes

$$R_1(u,v) \ll (\ln x)^{1-\alpha}(1+u\ln x)^{\beta-1} Z_{x^{1-u}}(1,1,1,1-\gamma;1,h_2). \tag{4.7}$$

Therefore in (3.23) taking $a = 1-\beta$, $b = 1-\gamma$, $d = 1-\alpha+\beta+\gamma$ we obtain

$$R_1(u,v) \ll \ln^{\beta-\alpha} x + \ln^{-\beta} x + \ln^{-1} x \tag{4.8}$$

uniformly for $0 \leqslant u \leqslant 1 - \eta_x$.

If $1-\eta_x < u \leqslant 1$, then we estimate $Z_{x^{1-u}}(\cdot)$ in (4.7) using (3.9) with $a = 1-\alpha+\beta+\gamma$, $b = 1-\gamma$ and get

$$R_1(u,v) \ll \ln^{\beta-\alpha} x.$$

Thus (4.8) is valid uniformly for $u, v \in [0,1]$. Similarly,

$$R_2(u,v) \ll \ln^{\gamma-\alpha} x + \ln^{-\gamma} x + \ln^{-1} x$$

uniformly for $u, v \in [0,1]$.

Taking into account that $\alpha-\beta-\gamma > 0$, the estimates for R_1 and R_2 together with (4.2), (4.3) and (4.4) yield

$$S_x(u,v) = H(u,v) + \mathrm{O}(\ln^{-\kappa} x) \tag{4.9}$$

uniformly for $u, v \in [0,1]$. Here $\kappa := \min\{\gamma, \beta, 1\}$.

Consider the first summand of (4.2). Changing the order of summation we have

$$H(u,v) = \frac{1}{G(x)} \sum_{q \leqslant x^u} f_1(q) \sum_{m \leqslant x^v} f_2(m) \sum_{d \leqslant \frac{x}{qm}} \frac{g(qmd)}{T_3(qmd)}. \tag{4.10}$$

Since $g/T_3 \in \mathcal{G}(\alpha-\beta-\gamma, \delta_1+\delta_2+\delta_3)$, applying Lemma 3.1 we obtain

$$\begin{aligned}\sum_{d \leqslant \frac{x}{qm}} \frac{g(qmd)}{T_3(qmd)} &= \frac{x}{qm\left(\ln\left(\frac{\mathrm{e}x}{qm}\right)\right)^{1-\alpha+\beta+\gamma}} \\ &\quad \times \left(g_1(qm) \cdot A\left(\alpha-\beta-\gamma, \frac{g}{T_3}\right) + \mathrm{O}\left(\frac{g_2(qm)}{\ln\left(\frac{\mathrm{e}x}{qm}\right)}\right)\right),\end{aligned} \tag{4.11}$$

where

$$g_1(\cdot) := \widetilde{h}\left(\cdot \,|\, 1, \frac{g}{T_3}\right), \quad g_2(\cdot) := \widehat{h}\left(\cdot \,|\, 1, \frac{g}{T_3}\right).$$

By Remark 3.2 we see that $g_1, g_2 \in \mathcal{G}(\alpha-\beta-\gamma, \delta_1+\delta_2+\delta_3)$.

Let us split the unit square $K = [0,1] \times [0,1]$ into two parts $K = K_1 \cup K_2$, here $K_1 := \{(u,v) \in K \,|\, u+v \leqslant 1\}$ and $K_2 := \{(u,v) \in K \,|\, u+v > 1\}$.

1. Firstly we consider the case where $(u, v) \in K_1$. Then (4.11) and (4.10) yield

$$\frac{G(x)}{x} H(u,v) = A\Big(1-b, \frac{g}{T_3}\Big) \sum_{q \leqslant x^u} \frac{f_1(q)}{q} \sum_{m \leqslant x^v} \frac{f_2(m) g_1(qm)}{m \ln^b\big(\frac{ex}{qm}\big)}$$
$$+ \mathrm{O}\bigg(\sum_{q \leqslant x^u} \frac{f_1(q)}{q} \sum_{m \leqslant x^v} \frac{f_2(m) g_2(qm)}{m \ln^{b+1}\big(\frac{ex}{qm}\big)}\bigg)$$
$$= A\Big(1-b, \frac{g}{T_3}\Big) E_x(u, v, b; f_2, f_1, g_1) + \mathrm{O}\big(E_x(u, v, b+1; f_2, f_1, g_2)\big),$$

here and below $b := 1-\alpha+\beta+\gamma$. Now applying (4.6), (3.8) and Lemma 3.10 with $a = 1-\gamma, d = 1-\beta, b = 1-\alpha+\beta+\gamma$ we deduce

$$H(u,v) = B \cdot J_1(0, 0, u, v, d, a, b) + \mathrm{O}\bigg(\frac{1}{\ln^{\min(\beta,\gamma)} x} + \frac{l_x(\beta, \gamma, \alpha-\beta)}{\ln x}\bigg) \tag{4.12}$$

uniformly for $(u, v) \in K_1$. Here

$$B := \frac{A(\alpha-\beta-\gamma, g/T_3)}{A(\alpha, g)} A(\gamma, f_2 g_1) A(\beta, f_1 \widetilde{h}(\cdot \,|\, f_2, g_1)).$$

Taking into account (1.1) we get

$$B \cdot \frac{\Gamma(\alpha-\beta-\gamma)\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\alpha)} = \frac{\prod_p \sum_{k=0}^{\infty} \frac{f_1(p^k)}{p^k} \sum_{j=0}^{\infty} \frac{f_2(p^j)}{p^j} \sum_{i=0}^{\infty} \frac{g(p^{i+j+k})}{p^i T_3(p^{i+j+k})}}{\prod_p \sum_{k=0}^{\infty} \frac{g(p^k)}{p^k}} = 1.$$

Thus

$$B = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta-\gamma)\Gamma(\beta)\Gamma(\gamma)}.$$

This together with (4.12) and (4.9) complete the proof of Theorem 2.3 in the region K_1.

2. Let $(u, v) \in K_2$. If $\min\{u, v\} \leqslant \eta_x$, taking into account Remarks 3.2, 3.9 and 4.1 from (4.10), (4.11), (3.9) and (4.6) we obtain

$$H(u,v) \ll \frac{x}{G(x)} \sum_{q \leqslant \mathrm{e}} \frac{1}{q} \big(f_1(q) Z_x(q, 1, 1, b; f_2, g_2) + f_2(q) Z_x(q, 1, 1, b; f_1, g_2)\big)$$
$$\ll \ln^{-\gamma} x + \ln^{-\beta} x. \tag{4.13}$$

Consider the case $u > \eta_x$ and $v > \eta_x$. For any $t \in [0, 1]$ we define

$$V_i(t) := \frac{1}{G(x)} \sum_{q \leqslant x^{1-t}} \sum_{m \leqslant \frac{x}{q}} \sum_{d \leqslant \frac{x}{qm}} K_i(q, m, d), \quad i = 1, 2.$$

Then

$$H(u, v) = H(1, 1) - V_1(v) + H(1 - v, v) - V_2(u) + H(u, 1 - u).$$

By definition, $S_x(1, 1) = 1$. Hence from (4.9) it follows that

$$H(u, v) = 1 - V_1(v) + H(1 - v, v) - V_2(u) + H(u, 1 - u) + \mathrm{O}(\ln^{-\kappa} x). \quad (4.14)$$

Since $g/T_3 \in \mathcal{G}(1 - b, \delta_1 + \delta_2 + \delta_3)$, by Lemma 3.1,

$$\frac{G(x)}{x} V_1(v) = A\left(1 - b, \frac{g}{T_3}\right) E_x^*(1 - v, b; f_2, f_1, g_1) \\ + \mathrm{O}\big(E_x^*(1 - v, 1 + b; f_2, f_1, g_2)\big),$$

where E_x^* is defined in Lemma 3.11. Therefore, having in mind (4.6) and using Lemmas 3.11, 3.6 and 3.7, we obtain

$$V_1(v) = B \cdot J_2(0, 0, 1-v, 1-\beta, 1-\gamma, b) + \mathrm{O}\left(\frac{1}{\ln^{\kappa_1} x} + \frac{l_x(\alpha - \beta - \gamma, \alpha - \beta, \beta, \gamma)}{\ln x}\right),$$

where $\kappa_1 = \min\{\beta, \gamma, \alpha - \beta - \gamma\}$. Analogously,

$$V_2(u) = B \cdot J_2(0, 0, 1-u, 1-\gamma, 1-\beta, b) + \mathrm{O}\left(\frac{1}{\ln^{\kappa_1} x} + \frac{l_x(\alpha - \beta - \gamma, \alpha - \gamma, \beta, \gamma)}{\ln x}\right).$$

From this, (4.12) and (4.14) we get

$$H(u, v) = 1 - B \cdot \big(J_2(0, 0, 1 - v, 1 - \beta, 1 - \gamma, b) \\ - J_1(0, 0, 1 - v, v, 1 - \beta, 1 - \gamma, b) + J_2(0, 0, 1 - u, 1 - \gamma, 1 - \beta, b) \\ - J_1(0, 0, u, 1 - u, 1 - \beta, 1 - \gamma, b)\big) + \mathrm{O}(\xi_x(u + v; \alpha, \beta, \gamma)). \quad (4.15)$$

From (3.6) we have

$$B J_2(0, 0, 1, 1 - \beta, 1 - \gamma, b) = 1.$$

Hence the main term in (4.15) equals to $D(u, v; \beta, \gamma, 1-b)$. Moreover by Lemma 3.4,

$$\begin{aligned} &D(u, v; \beta, \gamma, 1-b) \\ &\quad \leqslant B \cdot \min\big\{J_2(0,0,u,1-\beta,1-\gamma,b),\, J_2(0,0,v,1-\gamma,1-\beta,b)\big\} \\ &\quad \ll \min\{u^{\beta}, v^{\gamma}\}. \end{aligned}$$

Thus (4.13) and (4.15) yield

$$H(u, v) = D(u, v; \beta, \gamma, \alpha-\beta-\gamma) + \mathrm{O}(\xi_x(u+v; \alpha, \beta, \gamma))$$

uniformly for $(u, v) \in K_2$.

The proof of Theorem 2.3 follows now from this estimate, (4.12) and (4.9).

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