European Journal of Mathematics (2022) 8:514–522
https://doi.org/10.1007/s40879-021-00488-3

RESEARCH ARTICLE

Holomorphic Lagrangian subvarieties in holomorphic symplectic manifolds with Lagrangian fibrations and special Kähler geometry

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Received: 22 March 2021 / Accepted: 21 June 2021 / Published online: 8 July 2021

Abstract

Let M be a holomorphic symplectic Kähler manifold equipped with a Lagrangian fibration π with compact fibers. The base of this manifold is equipped with a *special Kähler structure*, that is, a Kähler structure (I, g, ω) and a symplectic flat connection ∇ such that the metric g is locally the Hessian of a function. We prove that any Lagrangian subvariety $Z \subset M$ which intersects smooth fibers of π and smoothly projects to $\pi(Z)$ is a torus fibration over its image $\pi(Z)$ in B, and this image is also special Kähler. This answers a question of Nigel Hitchin related to Kapustin–Witten BBB/BAA duality.

Keywords Holomorphic symplectic manifolds · Lagrangian fibrations · Lagrangian subvarieties · Special Kähler structure · BBB/BAA duality

Mathematics Subject Classification 14J42 · 53D12 · 14J33 · 53D37

Ljudmila Kamenova was partially supported by a grant from the Simons Foundation/SFARI (522730, LK). Misha Verbitsky was partially supported by the Russian Academic Excellence Project '5-100', FAPERJ E-26/202.912/2018 and CNPq—Process 313608/2017-2.

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1 Introduction

The present paper is motivated by the observations made by Nigel Hitchin [11] who worked on the Kapustin–Witten version of the geometric Langlands correspondence, interpreted as Montonen–Olive generalization of electric-magnetic duality. This theory originates in 1977, when Peter Goddard, Jean Nuyts and David Olive discovered that magnetic sources in gauge theory with gauge group G are classified by irreducible representations of the Langlands dual group ${}^{\mathrm{L}}G$ [9]. Then Claus Montonen and Olive conjectured that the Yang–Mills theories with the gauge groups G and ${}^{\mathrm{L}}G$ are isomorphic on the quantum level. The Montonen–Olive duality can be regarded as a quantum field generalization of the usual electric-magnetic duality.

Michael Atiyah suggested that the Montonen–Olive conjecture [19] might be related to the Langlands duality, but it took many years until 2006, when Anton Kapustin and Edward Witten explained this conjectural relation.

In their celebrated paper [16], Kapustin and Witten produced a rich dictionary of the correspondence between the geometric Langlands program and S-duality in the 4-dimensional $N = 4$ gauge theory. This approach is based on the comparison between two Hitchin systems (the spaces of Higgs bundles on a curve) with values in Langlands dual groups. Both of these Hitchin systems are equipped with a Lagrangian fibration. Reminiscent of the Strominger–Yau–Zaslow interpretation of the Mirror Symmetry, the Langlands duality is interpreted as a correspondence between certain categories on these two spaces, associated with the duality of their fibers. For a less technical survey of the Kapustin–Witten program, see [15].

The Kapustin–Witten interpretation of Montonen–Olive/geometric Langlands duality can be understood as SYZ Mirror symmetry on the Hitchin space, but it is firmly based on the hyperkähler geometry of the Hitchin space. In place of the Fukaya category on the symplectic side of Mirror Symmetry, one has a category associated with the holomorphic Lagrangian subvarieties (BAA, ABA and AAB branes). In place of the derived category of coherent sheaves on the complex side of Mirror Symmetry one has a category which has pairs (trianalytic subvariety, hyperholomorphic bundle on it) as objects; these are called BBB branes. Since the fiberwise duality should somehow exchange these two categories, Hitchin argued, the fibers of the BAA brane under the Hitchin fibration map should be tori, and its image should retain the special Kähler structure which exists on the base of the Hitchin fibration. We define all these notions and state this result rigorously in Sect. 2.

Hitchin stated his theorem in bigger generality than required by the Kapustin–Witten theory: he expected it to be true for any hyperkähler manifold equipped (such as the Hitchin system) with a $\mathbb{C}^*$-action rotating the complex structures within the twistor family. We prove the same result without a $\mathbb{C}^*$-action. Our main theorem is the following.

Theorem 1.1 (Theorem 3.2) *Let* (M, Ω) *be a holomorphic symplectic Kähler manifold, and let* $\pi : M \longrightarrow B$ *be a proper Lagrangian fibration. Consider an irreducible Lagrangian subvariety* $Z \subset M$ *such that* $\pi(Z)$ *does not lie in the discriminant locus* D *of* π*. Then for any smooth point* $x \in \pi(Z)\backslash D$ *which is a regular value of* $\pi : Z \longrightarrow \pi(Z)$*, the fiber* $\pi^{-1}(x) \cap Z$ *is a union of translation equivalent subtori* *in the complex torus $\pi^{-1}(x)$, and the regular part of $\pi(Z)$ is a special Kähler submanifold in $B\backslash D$.*

2 Special Kähler manifolds

2.1 Special Kähler manifolds and Hessian manifolds

Special Kähler manifolds first appeared in physics [6,7] as allowed targets for the scalars of the vector multiplets of field theories with $N = 2$ supersymmetry on a 4-dimensional Minkowski space-time. Originally they came in two flavours, the affine special Kähler manifolds associated with rigid supersymmetry, and projective special Kähler manifolds associated with the local supersymmetry. In the present paper we are interested only in the affine version.

The first comprehensive mathematical exposition of this theory is due to Dan Freed [8]. After this geometric structure was presented to the general mathematical readership, special Kähler manifolds became prominent in differential geometry. In [2], Baues and Cortés showed that special Kähler manifolds can be interpreted as "affine hyperspheres". This classical concept, going back to the work of Blaschke in affine geometry, is described by solutions of real Monge–Ampère equation. This interpretation leads to a classification of special Kähler manifolds. For more details on the differential geometry of special Kähler manifolds, the reader is directed to the survey [5].

Definition 2.1 A *special complex manifold* is a complex manifold (M, I) equipped with a flat, torsion-free connection ∇ such that the tensor $\nabla(I) \in \Lambda^1(M) \otimes \Lambda^1(M) \otimes TM$ is symmetric in the first two variables. A special complex manifold is *special Kähler* if it is equipped with a Kähler form ω which satisfies $\nabla(\omega) = 0$.

Let $(M, I, \nabla, g, \omega)$ be a special Kähler manifold. Since $\nabla(\omega) = 0$, and $\nabla(I)$ is symmetric in the first two variables, the tensor

$$\nabla(g) = \nabla(I \circ \omega) \in \Lambda^1(M) \otimes \Lambda^1(M) \otimes \Lambda^1(M) \tag{2.1}$$

is symmetric in the first two variables. This tensor is symmetric in the last two variables, because g is symmetric. Therefore, ∇g is a symmetric 3-tensor.

Definition 2.2 Let (M, ∇) be a manifold equipped with a flat torsion-free connection, and g a Riemannian metric. It is called *Hessian* if $\nabla(g)$ is symmetric in all three variables.

Remark 2.3 It is not hard to see that the Riemannian metric g on (M, ∇) is Hessian if and only if g is locally the Hessian of a function, which is called the *potential* of the Hessian metric. A priori the potential exists only locally, but when M is simply connected, it can be defined globally on M.

This construction is due to Hitchin [10] who exhibited the special Kähler structure on the moduli space of holomorphic Lagrangian subvarieties in a hyperkähler manifold, and exhibited many interesting differential-geometric properties of special Kähler manifolds.

Remark 2.4 Let $(M, I, \nabla, g, \omega)$ be a special Kähler manifold, Vol M the Riemannian volume form, and f the potential of its Hessian metric. Since $\mathrm{Vol}\, M = \omega^n$, and $\nabla(\omega) = 0$, the function f is a solution of the real Monge–Ampère equation $\det \frac{d^2 f}{dx_i dx_j} = \mathrm{const}$. In the paper [3], Cheng and Yau studied Hessian manifolds with a prescribed Riemannian volume form, and proved an analogue of Calabi–Yau's theorem for such manifolds.

Claim 2.5 *Let $(M, I, \nabla, g, \omega)$ be a Kähler manifold equipped with a flat connection ∇ which satisfies $\nabla(\omega) = 0$. Then $(M, I, \nabla, g, \omega)$ is special Kähler if and only if the metric g is Hessian.*

Proof Follows immediately from (2.1). □

2.2 Special Kähler structure on the base of a complex Lagrangian fibration

Special Kähler manifolds naturally occur in many situations associated with the geometry of Calabi–Yau and hyperkähler subvarieties. For the present paper, the following construction is most significant.

Definition 2.6 Let (M, Ω) be a holomorphic symplectic manifold. A *Lagrangian subvariety* of M is a subvariety such that its smooth part is a Lagrangian submanifold in M. A (holomorphic) *Lagrangian fibration* on M is a proper holomorphic map $\pi : M \longrightarrow B$ with general fibers being Lagrangian submanifolds in (M, Ω).

The following claim is well known in classical mechanics.

Claim 2.7 *A smooth fiber of a holomorphic Lagrangian fibration is always a torus.*

Proof For any fibration $\pi : M \longrightarrow B$, any smooth fiber F has trivial normal bundle NF. However, NF is dual to the tangent bundle TF whenever π is a Lagrangian fibration. Therefore, the bundle TF is also trivial. For any function on B, its Hamiltonian gives a section of TF. Choose a collection of holomorphic functions such that their Hamiltonians give a basis in TF. Since these Hamiltonians commute, the corresponding vector fields in TF also commute. This gives a locally free action of an abelian Lie group on F, and therefore F is a quotient of an abelian group by a lattice. □

Definition 2.8 Let $\pi : M \longrightarrow B$ be a proper fibration. Consider the first derived direct image $R^1\pi_*(\mathbb{R}_M)$, where $\mathbb{R}_M$ is the trivial sheaf on M. This is a constructible sheaf; at any point $x \in B$, the fiber of $R^1\pi_*(\mathbb{R}_M)$ is equal to the first cohomology of the fiber $\pi^{-1}(x)$. Outside of singularities of π, this sheaf is locally constant. The flat connection on the corresponding vector bundle is called the *Gauss–Manin connection*. This connection is defined in the complement to the set $\mathrm{Disc}(\pi)$ of all critical values of π; this set is called the *discriminant locus* of π.

Definition 2.9 Let $\pi : M \longrightarrow B$ be a Lagrangian fibration, and let F be the fiber over $x \in B$. Then $\pi^* TB = NF = T^* F$. Identifying $H^0(NF) = T_x B$ with $H^0(T^* F) = H^1(F, \mathbb{R})$, we obtain an identification of TB and the bundle $R^1\pi_*(\mathbb{R}_M)$ of the first cohomology constructed above. Therefore, TB is equipped with a natural flat connection, also called the *Gauss–Manin connection*.

Remark 2.10 Let $\pi : M \longrightarrow B$ be a holomorphic Lagrangian fibration. A Kähler form ω on M restricted to a smooth fiber F of π defines a cohomology class $[\omega] \in H^2(F)$. Since F is a torus, we can consider $[\omega]$ as a 2-form on $R^1\pi_*(\mathbb{R}_M) = TB$. This form is clearly parallel under the Gauss–Manin connection. Abusing the notation, we denote this 2-form by the same letter ω.

Theorem 2.11 ([8, Theorem 3.4], [10, Theorem 3]) *Let $\pi : M \longrightarrow B$ be a holomorphic Lagrangian fibration on a Kähler holomorphic symplectic manifold and let $B_0 \subset B$ be the complement to the discriminant locus of B. Consider the 2-form ω on B constructed in Remark 2.10, and the Gauss–Manin connection ∇ on TB defined in Definition 2.9. Then (B, ∇, ω) is a special Kähler manifold.*

3 Special Kähler geometry and holomorphic Lagrangian subvarieties

3.1 Holomorphic Lagrangian subvarieties: main theorem

Recall that *projective special Kähler manifold* [17] is a special Kähler manifold (M, g, I, ω) equipped with a vector field v acting on (M, g) by homotheties which preserve the complex structure, such that the vector field $I(v)$ acts by isometries.

In his talk [11] at the SCGP in October 2018, Hitchin stated the following theorem.

Theorem 3.1 (Hitchin, [11]) *Let $\pi : M \longrightarrow B$ be an algebraically integrable system with a $\mathbb{C}^*$-action defining a projective special Kähler structure. Then any $\mathbb{C}^*$-invariant holomorphic Lagrangian submanifold has an open set with the structure of a fibration over a projective special Kähler submanifold of B, and each fiber is a disjoint union of translates of an abelian subvariety.*

Hitchin asked whether there is an analogue of his result in the affine special Käher setting. Here we prove it, and give examples of holomorphic Lagrangian submanifolds projecting to special Kähler submanifolds.

Theorem 3.2 *Let (M, Ω) be a holomorphic symplectic Kähler manifold, and let $\pi : M \longrightarrow B$ be a proper Lagrangian fibration. Consider an irreducible Lagrangian subvariety $Z \subset M$ such that $\pi(Z)$ does not lie in the discriminant locus D of π. Then for any smooth point $x \in \pi(Z)\backslash D$ which is a regular value of $\pi : Z \longrightarrow \pi(Z)$, the fiber $\pi^{-1}(x) \cap Z$ is a union of translation equivalent subtori in the complex torus $\pi^{-1}(x)$, and the smooth part of $\pi(Z)\backslash D$ is a special Kähler submanifold in $B_0 := B\backslash D$.*

Before we prove Theorem 3.2, we state the following elementary linear-algebraic lemma.

Lemma 3.3 *Suppose $V \subset W \oplus W^*$ is a Lagrangian vector subspace in $W \oplus W^*$ with standard symplectic structure, and $\pi : W \oplus W^* \longrightarrow W$ the projection. Then $\pi(V)^\perp = V \cap W^*$, where $R^\perp \subset W^*$ denotes the annihilator of a subspace $R \subset W$, and W^* is considered as a subspace in $W \oplus W^*$.*

Proof of Theorem 3.2 **Step 1:** Let $Z_x := \pi^{-1}(x) \cap Z$, where $x \in \pi(Z)\backslash D$ is a regular value of $\pi : Z \longrightarrow \pi(Z)$. Denote by $T^\pi M$ the fiberwise tangent bundle, and let $T_z^\pi M$ be its fiber over $z \in Z_x$. The holomorphic symplectic form induces non-degenerate pairing between $T_z^\pi M$ and $T_x B$. Lemma 3.3 applied to $V = T_z Z$ and $W \oplus W^* = T_z M$ implies $T_z Z_x = \pi(T_z Z)^\perp$. Indeed, in this case $T_z Z_x = V \cap W^*$ and $\pi(T_z Z) = \pi(V)$. However, $\dim \pi(Z) = \dim Z - \dim Z_x = \dim \pi(T_z Z)$, hence in all points $x \in B\backslash D$ and all $z \in \pi^{-1}(x)$, one has $T_x\pi(Z) = \pi(T_z Z)$. This gives

$$T_z Z_x = T_x(\pi(Z))^\perp. \tag{3.1}$$

Step 2: From (3.1), we obtain that $\pi(T_z Z)^\perp$ is constant: for different $z, z' \in Z_x$, the spaces $T_z Z_x$ and $T_{z'} Z_x$ are obtained by a translation within the torus $\pi^{-1}(x)$. In other words, the space $T_z Z_x$ is constant in the standard coordinates on the torus, and $Z_x \subset \pi^{-1}(x)$ is a union of subtori which are translates of each other.

Step 3: Since $\pi(Z) \subset B$ is a complex subvariety, in order to prove that it is special Kähler it suffices to show that it is totally geodesic (that is, constant) with respect to the Gauss–Manin connection ∇ on TB_0. However, the connection ∇ is identified with the Gauss–Manin connection under the identification $TB_0 = R^1\pi_*(\mathbb{R}_M)$, and it preserves any sublattice in $T_x B_0 = H^1(F, \mathbb{R})$, where $F = \pi^{-1}(x)$.

Since $Z_x \subset \pi^{-1}(x)$ is a subtorus, it corresponds to a sublattice $H_1(Z_x) \subset H_1(\pi^{-1}(x))$ in homology, and in a neighbourhood $U \ni x$, all fibers $Z_{x'} \subset \pi^{-1}(x')$ correspond to the same sublattice. Therefore, its orthogonal complement in $R^1\pi_*(\mathbb{R}_M)$ is constant. However, by (3.1), this orthogonal complement generates $T_x(\pi(Z))$. This implies that $T\pi(Z)$ is constant with respect to the Gauss–Manin connection ∇ on B_0. □

4 Examples

Many (or most) holomorphic Lagrangian tori in hyperkähler manifolds occur as fibers of Lagrangian fibrations. Indeed, in [12] it was shown that any Lagrangian subtorus in a hyperkähler manifold is a fiber of a holomorphic Lagrangian fibration. However, for any two distinct Lagrangian fibrations over a maximal holonomy hyperkähler manifold, the intersection index of their fibers is positive ([14], second paragraph of the proof of Theorem 2.11). Therefore, any fiber of the first fibration is projected to the base of the second one surjectively and finitely in the general point. In this case, Theorem 3.2 is tautologically true, because the fibers of $\pi\big|_Z$ are 0-dimensional, and its base coincides with B.

It is much harder to find examples where the special Kähler geometry of $\pi(Z)$ is non-trivial. This is easy to explain. Indeed, $\pi(Z)$ gives a flat submanifold in the special Kähler manifold $B_0 = B\backslash D$. Therefore, the tangent space $T_x\pi(Z)$ to any smooth point is fixed by the monodromy of the Gauss–Manin connection on B_0. However, the monodromy representation is quite often irreducible, or has very few subrepresentaions.

The Hitchin system (moduli of Higgs bundles over a curve) is equipped with a Lagrangian fibration ("Hitchin fibration"), which has abelian varieties (Jacobians of the "spectral curve") as its fibers. For some examples of the Hitchin system, the corresponding monodromy representation was computed in [1]. From these computations it follows that the monodromy representation is reducible [1, Corollary 4.23]. This suggests that some interesting Lagrangian subvarieties, not transversal to fibers of the Hitchin system, might exist in this case. Two of the first papers to study the monodromy for the Hitchin fibration are [4,21].

Holomorphic Lagrangian fibrations on a deformation of the second Hilbert scheme of a K3 were studied by Markushevich and by Kamenova [13,18]. They split into two distinct cases. In the first case, studied by Markushevich, the Abelian fibers are Jacobians of smooth genus two curves. If the fibers of $\pi : M \longrightarrow B$ have no elliptic curve then all Lagrangian subvarieties of M would either project to B surjectively or would lie in the fibers of π. In the second case, studied by Kamenova, the fibers of π are products of two elliptic curves, i.e., the fibers are Jacobians of singular genus two curves. This situation occurs, for example, when one takes the punctual Hilbert scheme of two points on an elliptic K3 surface $S \longrightarrow \mathbb{P}^1$. Then the Hilbert scheme $S^{[2]}$ is fibered over the base $(\mathbb{P}^1)^{[2]} = \mathbb{P}^2$ with general fibers that are products of the fibers of the ellitic fibration $S \longrightarrow \mathbb{P}^1$. As shown in [13], under some "genericity" hypotheses, all deformations of the second Hilbert scheme of a K3, fibered with the fiber that is a product of two elliptic curves, are obtained in this way.

The main (and, so far, the only) non-trivial example of the geometric construction obtained in this paper is given by the Hilbert scheme of an elliptic K3 surface as follows. Let $\pi : M \longrightarrow S = \mathbb{C}P^1$ be an elliptic fibration on a K3 surface. Consider the corresponding fibration $\pi^{[n]} : M^{[n]} \longrightarrow S^{[n]} = \mathbb{C}P^n$ on its Hilbert scheme. A *multisection* of π is a curve which is transversal to the fibers of π; a multisection exists if and only if M is projective. Fix points $s_1, \ldots, s_k \in S$, and let $C_{k+1}, \ldots, C_n \subset M$ be multisections. Denote by $\widehat{L}_k(s_1, \ldots, s_k, C_{k+1}, \ldots, C_n) \subset \operatorname{Sym}^n M$ the set of n-tuples of points $(e_1, \ldots, e_k, c_{k+1}, \ldots, c_n) \in \operatorname{Sym}^n M$, such that $e_i \in \pi^{-1}(s_i)$ and $c_j \in C_j$. Since the holomorphic symplectic form on $\operatorname{Sym}^n M$ is locally a product of the holomorphic symplectic form on M, and the curves $\pi^{-1}(s_i)$ and $C_j \subset M$ are Lagrangian, the subvariety

$$\widehat{L}_k(s_1, \ldots, s_k, C_{k+1}, \ldots, C_n) \subset \operatorname{Sym}^n M$$

is Lagrangian. Then its proper preimage $L_k(s_1, \ldots, s_k, C) \subset M^{[n]}$ is also Lagrangian. Under the natural map $\pi^{[n]} : M^{[n]} \longrightarrow S^{[n]} = \mathbb{C}P^n$, this subvariety is projected to a subset of $\operatorname{Sym}^n S = \mathbb{C}P^n$ consisting of all n-tuples which contain $(s_1, \ldots, s_k)$.

Another example is due to Richard Thomas (private communication). In the early versions of this paper, we did not specify the behaviour of the restriction $\pi\big|_Z$ outside of its regular values, and this example shows that it can be pretty wild.

Let S be a compact complex torus, $\dim_{\mathbb{C}} S = n$, and $M = T^*S$ the total space of its cotangent bundle. Since T^*S admits a natural trivialization, the manifold M is equipped with a Lagrangian fibration $M \longrightarrow \mathbb{C}^n$, with the fibers obtained as translates of S.

Let $X \subset S$ be a complex submanifold, and $Z := NS^\perp \subset M$ the total space of its conormal bundle. It is always Lagrangian, and in many situations the projection of Z to the base $\mathbb{C}^n$ is n-dimensional. To illustrate it, let us identify the base $B = \mathbb{C}^n$ of π with $T^*_s S$, for some $s \in S$. A vector $v \in B$ belongs to $\pi(Z)$ if and only if $v \in T_x X^\perp$ for some $x \in X$, where $T_x X^\perp = \{\zeta \in T^*_x S \mid \langle \zeta, T_x X\rangle = 0\}$. Then $\pi(Z)$ is a union of subspaces $T_x X^\perp$ parametrized by the family of $x \in X$. If, for example, X is a curve, and $T_x X$ is not constant, this is a union of a non-constant family of hyperplanes, hence it is Zariski dense in B.

Unless the tangent space $T_x X$ stays constant as we vary $x \in X$, the image $\pi(Z)$ is n-dimensional, and the corresponding fiber is 0-dimensional. However, the central fiber $\pi^{-1}(0)$ is X, not a torus and of different dimension from the general fiber. The proof of Theorem 3.2 fails for the central fiber, because $\pi(Z)$ is not smooth at 0, and the identification $T_z Z_x = \pi(T_z Z)^\perp \cap T^\pi_z M$ does not hold.

Acknowledgements Great many thanks to Richard Thomas for his comments, questions and examples. We are indebted to Nigel Hitchin for his interest and inspiration, and to Laura Schaposnik for her comments and expertise. We express our gratitude to Stony Brook University and to the SCGP, where this paper was prepared, and for their hospitality. We thank the referees for their suggestions. Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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