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RESEARCH ARTICLE

Some remarks on associated varieties of vertex operator superalgebras

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Abstract

We study several families of vertex operator superalgebras from a jet (super)scheme point of view. We provide new examples of vertex algebras which are "chiral-quantizations" of their C_2-algebras R_V. Our examples come from affine $C_\ell^{(1)}$-series vertex algebras, $\ell \geqslant 1$, certain $N = 1$ superconformal vertex algebras, Feigin–Stoyanovsky principal subspaces, Feigin–Stoyanovsky type subspaces, graph vertex algebras W_Γ, and extended Virasoro vertex algebras. We also give a counterexample to the chiral-quantization property for the $N = 2$ superconformal vertex algebra with central charge 1.

Keywords Vertex superalgebras · Jet superalgebras · Hilbert series · Characters · Principal subspaces

Mathematics Subject Classification 17B69 · 17B65 · 17B68

1 Introduction

Beilinson, Feigin and Mazur in [18] introduced the notions of singular support and lisse representation in order to study Virasoro (vertex) algebras. Arakawa later extended these notions to any finitely strongly generated, non-negatively graded vertex algebra V. More precisely, via a canonical decreasing filtration $\{F_p(V)\}$ introduced in [38], one can associate to V a positively graded vertex Possion algebra $\mathrm{gr}^F(V)$. The spectrum of $\mathrm{gr}^F(V)$ is called the singular support of V, and is denoted by $\mathrm{SS}(V)$. With respect to this filtration, $V/F_1(V)$ is the Zhu C_2-algebra R_V. The reduced spectrum $X_V = \mathrm{Specm}(R_V)$ is a Poisson variety which is called the associated variety of V. A large

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body of work has been devoted to descriptions of associated varieties for various vertex operator algebras [5,10,12,13]. Certainly the most prominent examples from this point of view are well-known lisse, or C_2-cofinite, vertex algebras characterized by $\dim(X_V) = 0$. Arakawa and Kawasetsu relaxed this condition to quasi-lisse in [7] which requires that X_V has finitely many symplectic leaves. Associated varieties are important in the geometry of Higgs branches in 4d/2d dualities in physics [17].

According to [5, Proposition 2.5.1], the embedding

$$R_V \hookrightarrow \mathrm{gr}^F(V)$$

can be extended to a surjective homomorphism of vertex Poisson algebras

$$\psi\colon J_\infty(R_V) \twoheadrightarrow \mathrm{gr}^F(V),$$

where $J_\infty(R_V)$ is the (infinite) jet algebra of R_V. The map ψ induces an injection from the singular support into the (infinite) jet scheme of the associated scheme of V, $\widetilde{X}_V = \mathrm{Spec}(R_V)$, i.e.,

$$\phi\colon \mathrm{SS}(V) \hookrightarrow J_\infty(\widetilde{X}_V).$$

In [11], authors showed that ϕ is an isomorphism of varieties if V is quasi-lisse. It was shown in [52] that if the map ψ is an isomorphism, then one can compute Hochschild homology of the Zhu algebra via the chiral homology of elliptic curves. Proving that ψ is an isomorphsim or finding the kernel of ψ turns out to be subtle. In [9,14,52], authors provided examples for which ψ is not an isomorphism, including the $\mathbb{Z}_2$-orbifold of the rank one Heisenberg algebra, affine vertex algebra $L_{\widehat{\mathfrak{sl}_4}}(-1,0)$, most Virasoro algebras, etc. However, a full description of the kernel, if non-trivial, of the map ψ is an interesting and difficult problem. Very recently, Andrews, van Ekeren and Heluani [4] found a remarkable q-series identity that allowed them to describe the kernel of ψ for the $c = 1/2$ Ising Virasoro vertex algebra.

For a vertex algebra V, where ψ is an isomorphism, one obtains a very interesting (and important) consequence

$$\mathrm{ch}[V](q) = \mathrm{HS}_q(J_\infty(R_V)),$$

where the left-hand side is the character of V and the right-hand side is the Hilbert series of the jet algebra of R_V. The left-hand side often has combinatorial interpretations which in turn can provide a non-trivial information about the jet scheme.

In Sects. 2–3, we recall some basics of vertex (super)algebras and generalize the notion of jet algebras to the super case. Then in Sect. 4 we investigate the map ψ in the cases of affine vertex algebras, rank one lattice vertex superalgebras including the simple $N = 2$ superconformal vertex algebra at level one. For the later case the map ψ is not an isomorphism, and we make a conjecture about its kernel. In Sect. 5 we analyze in great depth principal subspaces of lattice vertex algebras and affine vertex algebras, and show that the map ψ is an isomorphism for many examples. In particular, the principal subspaces are closely related to the jet algebras coming from graphs. Interestingly, in some examples their Hilbert series are (mixed) mock modular forms. In Sect. 6 we show that the map ψ is an isomorphism for the simple $N = 1$ vertex superalgebra associated with the $N = 1$ superconformal (p, p')-minimal model if and only if $(p, p') = (2, 4k)$, $k \in \mathbb{Z}_+$. We also study extended Virasoro vertex algebras in Sect. 7.

2 Definitions and preliminary results

Definition 2.1 Let V be a superspace, i.e., a $\mathbb{Z}_2$-graded vector space, $V = V_{\bar{0}} \oplus V_{\bar{1}}$, where $\{\bar{0}, \bar{1}\} = \mathbb{Z}_2$. If $a \in V_{p(a)}$, we say that the element a has *parity* $p(a) \in \mathbb{Z}_2$.

A *field* is a formal series of the form $a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$ where $a_{(n)} \in \mathrm{End}(V)$ and for each $v \in V$ one has

$$a_{(n)}v = 0 \quad \text{for} \quad n \gg 0.$$

We say that a field $a(z)$ has *parity* $p(a) \in \mathbb{Z}_2$ if

$$a_{(n)}V_\alpha \in V_{\alpha+p(a)} \quad \text{for all} \quad \alpha \in \mathbb{Z}_2, \quad n \in \mathbb{Z}.$$

A *vertex superalgebra* contains the following data: a vector space of states V, the vacuum vector $\mathbf{1} \in V_{\bar{0}}$, derivation T, and state-field correspondence map

$$a \mapsto Y(a, z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1},$$

satisfying the following axioms:

- (*translation coinvariance*): $[T, Y(a, z)] = \partial Y(a, z)$,
- (*vacuum*): $Y(\mathbf{1}, z) = \mathrm{Id}_V$, $Y(a, z)\mathbf{1}|_{z=0} = a$,
- (*locality*): $(z - w)^N Y(a, z)Y(b, w) = (-1)^{p(a)p(b)}(z - w)^N Y(b, w)Y(a, z)$ for $N \gg 0$.

In particular, a vertex superalgebra V is called *supercommutative* if $a_{(n)} = 0$ for $n \in \mathbb{N}$. It is well known that the category of commutative vertex superalgebras is equivalent with the category of unital commutative associative superalgebras equipped with an even derivation.

We say that a vertex superalgebra V *is generated by a subset* $\mathfrak{U} \subset V$ if any element of V can be written as a finite linear combination of terms of the form

$$b^1_{(i_1)} b^2_{(i_2)} \cdots b^n_{(i_n)} \mathbf{1}$$

for $b^k \in \mathfrak{U}$, $i_k \in \mathbb{Z}$, and $n \in \mathbb{N}$. If every element of V can be written with $i_k \in \mathbb{Z}_-$, we write $V = \langle \mathfrak{U} \rangle_S$, and say that V is *strongly generated* by $\mathfrak{U}$.

Example 2.2 ([53]) Let $\mathfrak{g}$ be a finite-dimensional Lie superalgebra with a non-degenerate even supersymmetric invariant bilinear form $(\cdot, \cdot)$. We can associate the affine Lie superalgebra $\widehat{\mathfrak{g}}$ to the pair $(\mathfrak{g}, (\cdot, \cdot))$.

Its universal vacuum representation of level k, $V_{\hat{\mathfrak{g}}}(k,0)$, is a vertex superalgebra. In particular, when $\mathfrak{g}$ is a simple Lie superalgebra, $V_{\hat{\mathfrak{g}}}(k,0)$ has a unique maximal ideal $I_{\hat{\mathfrak{g}}}(k,0)$, and $L_{\hat{\mathfrak{g}}}(k,0) = V_{\hat{\mathfrak{g}}}(k,0)/I_{\hat{\mathfrak{g}}}(k,0)$ is also a vertex superalgebra.

Example 2.3 ([36]) To any n-dimensional superspace A with a non-degenerate anti-supersymmetric bilinear form $\langle\cdot,\cdot\rangle$, we can associate a Lie superalgebra C_A. If we fix a basis of A,

$$\{\phi^1, \dots, \phi^n\},$$

the free fermionic vertex algebra $\mathcal{F}$ associated to A is a vertex superalgebra strongly generated by $\phi^i_{(-1/2)}\mathbf{1}$, $i = 1, \dots, n$, where $Y(\phi^i_{(-1/2)}\mathbf{1}, z) = \sum_{n\in 1/2+\mathbb{Z}} \phi^i_{(n)} z^{-n-1/2}$.

Definition 2.4 A vertex superalgebra V is called a *vertex operator superalgebra* if it is $\frac{1}{2}\mathbb{Z}$-graded,

$$V = \coprod_{n\in\frac{1}{2}\mathbb{Z}} V_{(m)},$$

with a conformal vector ω such that the set of operators $\{L_{(n)}, \mathrm{id}_V\}_{n\in\mathbb{Z}}$ with $L_{(n)} = \omega_{(n+1)}$ defines a representation of the Virasoro algebra on V; that is

$$[L_{(n)}, L_{(m)}] = (m-n)L_{(m+n)} + \frac{m^3-m}{12}\delta_{m+n,0}\, c_V, \quad m, n \in \mathbb{Z}.$$

We call c_V the *central charge* of V. We require that $L_{(0)}$ is diagonalizible and it defines the $\frac{1}{2}\mathbb{Z}$ grading — its eigenvalues are called (conformal) *weights*. In several examples we will encounter $\frac{1}{2}\mathbb{Z}$-graded vertex superalgebras without a conformal vector. For this reason, we define the *character* or graded dimension as

$$\mathrm{ch}[V](q) = \sum_{m\in\frac{1}{2}\mathbb{Z}} \dim(V_{(m)})\, q^m.$$

As we do not care about modularity here, we suppress the $q^{-c_V/24}$ factor and also view q as a formal variable.

Example 2.5 ([37]) Let Vir denote the Virasoro Lie algebra. Then the universal Vir-module $V_{\mathrm{Vir}}(c,0)$ has a natural vertex operator algebra with central charge c.

Example 2.6 ([36]) The universal vertex superalgebra associated with the $N = 1$ Neveu–Schwarz Lie superalgebra will be denoted by $V_c^{N=1}$, where c is the central charge. It is a vertex operator superalgebra strongly generated by an odd vector $G_{(-3/2)}\mathbf{1}$ and the conformal vector $L_{(-2)}\mathbf{1}$.

Example 2.7 ([36]) The universal vertex superalgebra associated with the $N = 2$ superconformal Lie algebra will be denoted by $V_c^{N=2}$. It is a vertex operator superalgebra strongly generated by two odd vectors $G^+_{(-3/2)}\mathbf{1}$, $G^-_{(-3/2)}\mathbf{1}$, and two even vectors $L_{(-2)}\mathbf{1}$, $J_{(-1)}\mathbf{1}$.

Definition 2.8 A commutative vertex superalgebra V is called a *vertex Poisson superalgebra* if it is equipped with a linear operation,

$$V \to \mathrm{Hom}(V, z^{-1}V[z^{-1}]), \quad a \mapsto Y_-(a,z) = \sum_{n\in\mathbb{N}} a_{(n)} z^{-n+1},$$

such that

- $(Ta)_n = -na_{(n-1)}$,
- $a_{(n)}b = \sum_{j\in\mathbb{N}} (-1)^{n+j+1} \frac{(-1)^{p(a)p(b)}}{j!} T^j(b_{(n+j)}a)$,
- $[a_{(m)}, b_{(n)}] = \sum_{j\in\mathbb{N}} \binom{m}{j} (a_{(j)}b)_{(m+n-j)}$,
- $a_{(n)}(b\cdot c) = (a_{(n)}b)\cdot c + (-1)^{p(a)p(b)} b\cdot(a_{(n)}c)$,

for $a, b, c \in V$ and $n, m \in \mathbb{N}$.

A vertex Lie superalgebra structure on V is given by (V, Y_-, T). So one can also say that a vertex Poisson superalgebra is a commutative vertex superalgebra equipped with a vertex Lie superalgebra structure. In fact, one can obtain a vertex Poisson superalgebra from any vertex superalgebra through standard increasing filtration or Li's filtration. Following [38], one can define a decreasing sequence of subspaces $\{F_n(V)\}$ of the superalgebra V, where for $n \in \mathbb{Z}$, $F_n(V)$ is linearly spanned by the vectors

$$u^{(1)}_{(-1-k_1)} \cdots u^{(r)}_{(-1-k_r)} \mathbf{1}$$

for $r \in \mathbb{Z}_+$, $u^{(1)}, \ldots, u^{(r)} \in V$, $k_1, \ldots, k_r \in \mathbb{N}$ with $k_1 + \cdots + k_r \geqslant n$. Then Li's filtration of V is given by

$$V = F_0(V) \supset F_1(V) \supset \cdots$$

satisfying

$$\begin{aligned} &u_{(n)}v \in F_{r+s-n-1}(V) && \text{for } u \in F_r(V),\ v \in F_s(V),\ r, s \in \mathbb{N},\ n \in \mathbb{Z}, \\ &u_{(n)}v \in F_{r+s-n}(V) && \text{for } u \in F_r(V),\ v \in F_r(V),\ r, s, n \in \mathbb{N}. \end{aligned}$$

The corresponding associated graded algebra $\mathrm{gr}^F(V) = \coprod_{n\in\mathbb{N}} F_n(V)/F_{n+1}(V)$ is a vertex Poisson superalgebra. Its vertex Lie superalgebra structure is given by

$$\begin{aligned} T(u + F_{r+1}(V)) &= Tu + F_{r+2}(V), \\ Y_-(u + F_{r+1}(V), z)(v + F_{s+1}(V)) &= \sum_{n\in\mathbb{N}} (u_{(n)}v + F_{r+s-n+1}(V))\, z^{-n-1}, \end{aligned}$$

for $u \in F_r(z)$, $v \in F_s(z)$ with $r, s \in \mathbb{N}$. For the standard increasing filtration $\{G_n(V)\}$, we also have the associated graded vertex superalgebra $\mathrm{gr}^G(V)$. In [5, Proposition 2.6.1], it was shown that

$$\mathrm{gr}^F(V) \cong \mathrm{gr}^G(V)$$

as vertex Poisson superalgebras. Thus we sometimes drop the upper index F or G for brevity.

According to [38], we know that

$$F_n(V)=\big\{u_{(-1-i)}v \mid u\in V, i\geqslant 1, v\in F_{n-i}(V)\big\}.$$

In particular, $F_0(V)/F_1(V)=V/C_2(V)=R_V$ is a Poisson superalgebra by [54]. Its Poisson structure is given by

$$\overline{u}\cdot\overline{v}=\overline{u_{(-1)}v},\quad \{\overline{u},\overline{v}\}=\overline{u_{(0)}v},$$

for $u,v\in V$ where $\overline{u}=u+C_2(V)$. It was shown in [38, Corallary 4.3] that $\mathrm{gr}^F(V)$ is generated by R_V as a differential algebra.

Next, let us compute the C_2-algebras for some simple examples.

Example 2.9 Following notations in Example 2.3, let $\mathcal{F}$ be a free fermionic vertex superalgebra associated with an n-dimensional superspace A. Clearly, the C_2-algebra of $\mathcal{F}$ is

$$R_{\mathcal{F}}=\mathbb{C}\big[\overline{\phi^1_{(-1/2)}\mathbf{1}},\ldots,\overline{\phi^n_{(-1/2)}\mathbf{1}}\big],$$

where $\overline{\phi^i_{(-1/2)}\mathbf{1}}$ is even (resp. odd) if ϕ^i is even (resp. odd) in A.

Example 2.10 According to [52, 16.16], for simple affine vertex algebras $L_{\hat{\mathfrak{g}}}(k,0)$, $k\in\mathbb{N}$, where $\mathfrak{g}$ is a simple Lie algebra, we have

$$R_{L_{\hat{\mathfrak{g}}}(k,0)}=\mathbb{C}\big[u^1_{(-1)}\mathbf{1},u^2_{(-1)}\mathbf{1},\cdots,u^n_{(-1)}\mathbf{1}\big]/\big\langle U(\mathfrak{g})\circ((e_\theta)_{(-1)})^{k+1}\mathbf{1}\big\rangle,$$

where $\{u^1,u^2,\ldots,u^n\}$ is a basis of $\mathfrak{g}$, θ is the highest root of $\mathfrak{g}$, and $\circ$ represents the adjoint action. In particular, when $\mathfrak{g}=\mathrm{sl}_2$, we have

$$R_{L_{\widehat{\mathrm{sl}_2}}(k,0)}\cong\mathbb{C}[e,f,h]/\langle f^i\circ e^{k+1}\mid i=0,\ldots,2k+2\rangle,$$

where e,f,h correspond to $e_{(-1)}\mathbf{1}$, $f_{(-1)}\mathbf{1}$, $h_{(-1)}\mathbf{1}$.

Example 2.11 For any simple Virasoro algebras $L_{\mathrm{Vir}}(c_{(p,p')},0)$, where $c_{(p,p')}=1-6(p-p')^2/(pp')$ with $p>p'\geqslant 2$ and p,p' coprime, according to [18,52] its C_2-algebra is isomorphic to $\mathbb{C}[x]/\langle x^{(p-1)(p'-1)/2}\rangle$, where x corresponds to $\omega=L_{(-2)}\mathbf{1}$.

Example 2.12 The C_2-algebra of $V_c^{N=1}$ is $R_{V_c^{N=1}}=\mathbb{C}[x,\theta]$, where x and θ correspond to the even vector $L_{(-2)}\mathbf{1}$ and odd vector $G_{(-3/2)}\mathbf{1}$, respectively.

Example 2.13 The C_2-algebra of $V_c^{N=2}$ is $\mathbb{C}[x,y,\theta_1,\theta_2]$ where x,y,θ_1,θ_2 correspond to $L_{(-2)}\mathbf{1}$, $J_{(-1)}\mathbf{1}$, $G^+_{(-3/2)}\mathbf{1}$ and $G^-_{(-3/2)}\mathbf{1}$, respectively. Here θ_1,θ_2 are odd variables.

3 Affine jet superalgebras

Inspired by the definition of a jet algebra, we may give an analogous definition of a jet superalgebra in the affine case. Here, we closely follow [5].

Let $\mathbb{C}[x^1, \ldots, x^n, \theta^1, \ldots, \theta^m]$ be a polynomial superalgebra, where

$$x^1, \ldots, x^n$$

are ordinary variables and

$$\theta^1, \ldots, \theta^m$$

are odd variables, i.e., $(\theta^i)^2 = 0$ for $1 \leqslant i \leqslant m$. Let $f_1, f_2, \ldots, f_n$ be $\mathbb{Z}_2$-homogeneous elements in the polynomial superalgebra. We will define the jet superalgebra of the quotient superalgebra as

$$R = \frac{\mathbb{C}[x^1, \ldots, x^n, \theta^1, \ldots, \theta^m]}{\langle f_1, \ldots, f_r \rangle}.$$

Firstly, let us introduce new even variables $x^j_{(-\Delta_j - i)}$ and odd variables $\theta^{j'}_{(-\Delta_{j'} - i)}$ for $0 \leqslant i \leqslant m$, where Δ_j and $\Delta_{j'}$ are degrees of x^j and $\theta^{j'}$. In most cases, we will assume that the degree of each variable is 1, although in some cases the odd degree can be shifted by $1/2$. We define an even derivation T on

$$\mathbb{C}\left[x^j_{(-\Delta_j - i)}, \theta^{j'}_{(-\Delta_{j'} - i)} \mid 0 \leqslant i \leqslant m, 1 \leqslant j \leqslant n, 1 \leqslant j' \leqslant m\right]$$

as

$$T\left(x^j_{(-\Delta_j - i)}\right) = \begin{cases} (-\Delta_j - i)\, x^j_{(-\Delta_j - i - 1)} & \text{for } \ 0 \leqslant i \leqslant m-1, \\ 0 & \text{for } \ i = m, \end{cases}$$

and

$$T\left(\theta^{j'}_{(-\Delta_{j'} - i)}\right) = \begin{cases} (-\Delta_{j'} - i)\, \theta^{j'}_{(-\Delta_{j'} - i - 1)} & \text{for } \ 0 \leqslant i \leqslant m-1, \\ 0 & \text{for } \ i = m. \end{cases}$$

Here we identify x^j and $\theta^{j'}$ with $x^j_{(-\Delta_j)}$ and $\theta^{j'}_{(-\Delta_{j'})}$, respectively. Set

$$J_m(R) = \frac{\mathbb{C}\left[x^j_{(-\Delta_j - i)}, \theta^j_{(-\Delta_{j'} - i)} \mid 0 \leqslant i \leqslant m, 1 \leqslant j \leqslant n, 1 \leqslant j' \leqslant m\right]}{\langle T^j f_i \mid 1 \leqslant i \leqslant n, j \in \mathbb{N} \rangle}.$$

Then the m-jet superscheme of $V = \mathrm{Spec}(R)$ is defined as $\mathrm{Spec}(J_m(R))$. The infinite jet superalgebra of R_V is defined as

$$\begin{aligned} J_\infty(R) &= \lim_{\overleftarrow{m}} J_m(R) \\ &= \frac{\mathbb{C}\left[x^j_{(-\Delta_j-i)}, \theta^{j'}_{(-\Delta_{j'}-i)} \mid i \in \mathbb{N}, 1 \leqslant j \leqslant n, 1 \leqslant j' \leqslant m\right]}{\langle T^j f_i \mid 1 \leqslant i \leqslant n, j \in \mathbb{N}\rangle}. \end{aligned}$$

We often omit “infinite” and call it jet superalgebra for brevity. The jet superalgebra is a differential commutative superalgebra. We denote the ideal

$$\langle T^j f_i \mid 1 \leqslant i \leqslant n, j \in \mathbb{N}\rangle$$

by $\langle f_1, \dots, f_n\rangle_\partial$. Later, we sometimes write $x_{(j)}$ as $x(j)$. The infinite jet superscheme, or arc space, of V is defined as

$$J_\infty(V) = \mathrm{Spec}(J_\infty(R)).$$

We define the degree of each variable $u_{(-\Delta-j)}$ to be $\Delta + j$, where $u = x$ or θ. Then $J_\infty(R) = \coprod_{m\in\frac{1}{2}\mathbb{Z}}(J_\infty(R))_{(m)}$, where $(J_\infty(R))_{(m)}$ is the set of all elements in the jet superalgebra with degree m. We define the Hilbert series of $J_\infty(R)$ as

$$\mathrm{HS}_q(J_\infty(R)) = \sum_{m\in\frac{1}{2}\mathbb{Z}} \dim((J_\infty(R))_{(m)})q^m.$$

Following [5], $J_\infty(R)$ has a unique vertex Poisson superalgebra structure such that

$$u_{(n)}v = \begin{cases} \{u, v\} & \text{if } n = 0, \\ 0 & \text{if } n \in \mathbb{Z}_+ \end{cases}$$

for $u, v \in R \in J_\infty(R)$.

Furthermore, one can extend the embedding $R_V \hookrightarrow \mathrm{gr}^F(V)$ to a surjective differential superalgebra homomorphism $J_\infty(R_V) \twoheadrightarrow \mathrm{gr}^F(V)$. It is obvious that the map is a differential superalgebra homomorphism. It is surjective, since $\mathrm{gr}^F(V)$ is generated by R_V as a differential algebra. Moreover, it was shown in [5] that this map is actually a vertex Poisson superalgebra epimorphism. From now on, we call this map ψ. The map ψ is not necessarily injective, and it is an open problem to characterize rational vertex algebras for which ψ is injective.

3.1 Complete lexicographic ordering

Following [30], we define the complete lexicographic ordering on a basis or spanning set of the jet superalgebra. Given a jet superalgebra

$$J_\infty(\mathbb{C}[y^1, y^2, \dots, y^n]/I) = \mathbb{C}\big[y^1_{(-\Delta_1-i)}, \dots, y^n_{(-\Delta_n-i)} \,|\, i \in \mathbb{N}\big]/\langle I\rangle_\partial,$$

where Δ_i is the degree of y^i, we can first define an ordering of all variables in the following way:

$$y^1_{(-\Delta_1)} < y^2_{(-\Delta_1)} < \cdots < y^n_{(-\Delta_n)} < y^1_{(-\Delta_1-1)} < y^2_{(-\Delta_2-1)} < \cdots .$$

Definition 3.1 A monomial u of $J_\infty(\mathbb{C}[y^1, y^2, \dots, y^n]/I)$ is called an *ordered monomial* if it is of the form

$$\big(y^n_{(-\Delta_n-m)}\big)^{a^n_{m+1}} \cdots \big(y^1_{(-\Delta_1-m)}\big)^{a^1_{m+1}} \cdots \big(y^n_{(-\Delta_n)}\big)^{a^n_1} \cdots \big(y^2_{(-\Delta_1)}\big)^{a^2_1} \big(y^1_{(-\Delta_1)}\big)^{a^1_1},$$

where $m \in \mathbb{Z}_+$ and $a^i_j \in \mathbb{N}$.

It should be clear that all ordered monomials form a spanning set of the jet superalgebra. Then let us define the multiplicity of an ordered monomial as

$$\mu(u) = \sum_{i=1}^{m+1} (a^1_i + a^2_i + \cdots + a^n_i).$$

Given two arbitrary ordered monomials

$$u = \big(y^n_{(-\Delta_n-m)}\big)^{a^n_{m+1}} \cdots \big(y^1_{(-\Delta_1-m)}\big)^{a^1_{m+1}} \cdots \big(y^n_{(-\Delta_n)}\big)^{a^n_1} \cdots \big(y^2_{(-\Delta_1)}\big)^{a^2_1} (y^1_{(-\Delta_1)})^{a^1_1},$$
$$v = \big(y^n_{(-\Delta_n-m)}\big)^{b^n_{m+1}} \cdots \big(y^1_{(-\Delta_1-m)}\big)^{b^1_{m+1}} \cdots \big(y^n_{(-\Delta_n)}\big)^{b^n_1} \cdots \big(y^2_{(-\Delta_1)}\big)^{b^2_1} \big(y^1_{(-\Delta_1)}\big)^{b^1_1},$$

we define a complete lexicographic ordering as follows. If $\mu(u) < \mu(v)$, we say that $u < v$. If $\mu(u) = \mu(v)$, we compare exponents of

$$y^1_{(-\Delta_1)},\ y^2_{(-\Delta_1)},\ \dots,\ y^n_{(-\Delta_n)},\ \dots,\ y^1_{(-\Delta_1-m)},\ y^n_{(-\Delta_n-m)}$$

in this order. Namely, we say $v < u$ if $a^1_1 < b^1_1$; if they are equal, we then compare a^2_1 and b^2_1, and so on. Given a polynomial f, we call the greatest monomial among all its terms with respect to the complete lexicographic ordering the *leading term* of f.

4 Affine and lattice vertex algebras

In this section we analyze the C_2-algebra R_V and the injectivity of the map ψ for some familiar examples of affine and lattice vertex algebras.

Example 4.1 It was shown in ([5, Proposition 2.7.1]) that for any simple Lie algebra $\mathfrak{g}$, we have $J_\infty(R_{V_{\hat{\mathfrak{g}}(k,0)}}) \cong \mathrm{gr}^F(V_{\hat{\mathfrak{g}}(k,0)})$.

Proposition 4.2 ([6, Example 4.10]) *For the free fermionic vertex superalgebra, we have $J_\infty(R_{\mathcal{F}}) \cong \mathrm{gr}^F(\mathcal{F})$ as vertex Poisson superalgebras.*

Proof We use Arakawa's argument in [5, Proposition 2.7.1]. We include the proof for completeness. Here we still follow the notations from Example 2.3. According to [36, Section 3.6], we can choose a conformal vector such that $\mathcal{F}$ is $\frac{1}{2}\mathbb{N}$-graded. We consider the standard filtration G on F. Firstly, we have $\mathcal{F} \cong U(A[t^{-1}]t^{-1})$ as vector superspaces. Moreover,

$$G^m(\mathcal{F}) = \left\{u^1_{(-k_1)} \cdots u^r_{(-k_r)}\mathbf{1} \mid k_i \in \tfrac{1}{2} + \mathbb{N}, r \in \mathbb{N}, r \leqslant 2m\right\},$$

where $m \in \frac{1}{2}\mathbb{N}$ and $u^i \in \{\phi^1, \ldots, \phi^n\}$. So $\mathrm{gr}^G(\mathcal{F}) \cong S(A[t^{-1}]t^{-1}) \cong J_\infty(R_{\mathcal{F}})$ as vertex Poissson superalgebras. Therefore, $\mathrm{gr}^G(\mathcal{F}) \cong \mathrm{gr}^F(\mathcal{F}) \cong J_\infty(R_{\mathcal{F}})$. $\square$

Similarly, we can show that ψ is an isomorphism for the vertex superalgebra $V_{\hat{\mathfrak{g}}}(k, 0)$, where $\mathfrak{g}$ is a Lie superalgebra satisfying conditions in Example 2.2, and for superconformal vertex algebras $V_c^{N=1}$ and $V_c^{N=2}$.

Let

$$V_{\sqrt{p}\mathbb{Z}} = M(1) \otimes \mathbb{C}\big[\sqrt{p}\mathbb{Z}\big]$$

be a rank one lattice vertex algebra (resp. superalgebra) constructed from an integral lattice $L = \mathbb{Z}\alpha \cong \sqrt{p}\mathbb{Z}$, where $\langle \alpha, \alpha \rangle = p$ is even (resp. odd). It has a conformal vector $\omega = \frac{1}{2p}\alpha^2_{(-1)}\mathbf{1}$. As usual, we denote the extremal lattice vectors by $e^{n\alpha}$, $n \in \mathbb{Z}$.

Proposition 4.3 *For the lattice vertex algebra $V_{\sqrt{p}\mathbb{Z}}$ we have*

$$R_{V_{\sqrt{p}\mathbb{Z}}} \cong \mathbb{C}[x, y, z]/\langle x^2, y^2, xy = z^p, xz, yz\rangle.$$

When p is odd, x and y are odd vectors.

Proof According to the following calculations:

$$\begin{aligned}
&(e^\alpha)_{(-2)}(e^{-\alpha}) - \frac{(\alpha_{(-1)})^{p+1}\mathbf{1}}{(p+1)!} \in C_2\big(V_{\sqrt{p}\mathbb{Z}}\big),\\
&(e^\alpha)_{(-2)}(\mathbf{1}) - \alpha_{(-1)}e^\alpha \in C_2\big(V_{\sqrt{p}\mathbb{Z}}\big),\\
&(e^\alpha)_{(-p-1)}(e^\alpha) - e^{2\alpha} \in C_2\big(V_{\sqrt{p}\mathbb{Z}}\big),\\
&(e^{-\alpha})_{(-2)}(\mathbf{1}) - \alpha_{(-1)}e^{-\alpha} \in C_2\big(V_{\sqrt{p}\mathbb{Z}}\big),\\
&(e^{-\alpha})_{(-p-1)}(e^{-\alpha}) - 2e^{-2\alpha} \in C_2\big(V_{\sqrt{p}\mathbb{Z}}\big),
\end{aligned}$$

we know that all vectors except for $\alpha_{(-1)}\mathbf{1}, \ldots, \alpha^p_{(-1)}\mathbf{1}$, e^α, $e^{-\alpha}$ and $\mathbf{1}$ are zero in $R_{V_{\sqrt{p}\mathbb{Z}}}$.

We will show that all these vectors are indeed non-zero in $R_{V_{\sqrt{p}\mathbb{Z}}}$. Suppose there exist $u, v \in V_{\sqrt{p}\mathbb{Z}}$ such that $u_{(-2)}v = e^{\alpha}$. Then $\mathrm{wt}(a_{(-2)}b) = \mathrm{wt}(a) + \mathrm{wt}(b) + 1 = p/2$, which implies that $u, v \in \pi_0$, where π_0 is the Heisenberg subalgebra $\mathbb{C}[\alpha_{(n)}]_{n\in\mathbb{Z}_-} \cdot \mathbf{1}$. This is a contradiction. So the equivalent class $\overline{e^{\alpha}}$ is non-zero in $R_{V_{\sqrt{p}\mathbb{Z}}}$. Using a similar weight argument, we can show that equivalence classes

$$\overline{e^{-\alpha}},\ \overline{\mathbf{1}},\ \overline{\alpha_{(-1)}\mathbf{1}},\ \ldots,\ \overline{\alpha^{p}_{(-1)}\mathbf{1}}$$

are all non-zero in $R_{V_{\sqrt{p}\mathbb{Z}}}$. Moreover, we have

$$(e^{\alpha})_{(-1)}(e^{-\alpha}) - \frac{\alpha^{p}_{(-1)}\mathbf{1}}{p!} \in C_2\big(V_{\sqrt{p}\mathbb{Z}}\big).$$

Then the map $\phi\colon R_{V_{\sqrt{p}\mathbb{Z}}} \to \mathbb{C}[x, y, z]/\langle x^2, y^2, xy = z^p, xz, yz\rangle$, sending $\overline{e^{\alpha}}$ to x, $\overline{e^{-\alpha}}$ to y, $\overline{\mathbf{1}}$ to 1, and $\sqrt[p]{\frac{1}{p!}}\,\overline{\alpha_{(-1)}\mathbf{1}}$ to z, is an isomorphism. □

Remark 4.4 By the Frenkel–Kac construction, we know that $V_{\sqrt{2}\mathbb{Z}} \cong L_{\widehat{\mathrm{sl}_2}}(1, 0)$. Following Proposition 4.3, we have $R_{L_{\widehat{\mathrm{sl}_2}}(1,0)} \cong \mathbb{C}[e, f, h]/\langle e^2, f^2, ef = h^2, eh, fh\rangle$. According to [52, 16.16], one can also compute $R_{L_{\widehat{\mathrm{sl}_2}}(1,0)}$ directly.

Given a vertex superalgebra $V = \coprod_{n\in\frac{1}{2}\mathbb{Z}} V_{(n)}$ where $V_{\bar{0}} = \coprod_{n\in\mathbb{Z}} V_{(n)}$ and $V_{\bar{1}} = \coprod_{n\in\frac{1}{2}+\mathbb{Z}} V_{(n)}$, there are two binary operations defined as follows: for homegeneous $a, b \in V$,

$$a * b = \begin{cases} \sum_{i\in\mathbb{N}} \binom{\mathrm{wt}(a)}{i} a_{(i-1)}b & \text{if } a, b \in V_{\bar{0}}, \\ 0 & \text{if } a \text{ or } b \in V_{\bar{1}}, \end{cases}$$

and

$$a \circ b = \begin{cases} \sum_{i\in\mathbb{N}} \binom{\mathrm{wt}(a)}{i} a_{(i-2)}b & \text{if } a \in V_{\bar{0}}, \\ \sum_{i\in\mathbb{N}} \binom{\mathrm{wt}(a)-1/2}{i} a_{(i-1)}b & \text{if } a \in V_{\bar{1}}. \end{cases}$$

Let $O(V)$ be the linear span of elements of the form $a \circ b$ in V. Then Zhu's algebra $A(V)$ is defined as the quotient space $V/O(V)$ with the mutiplication from $*$. According to [54, Theorem 2.1.1], there is a filtration $\{\overline{F}_k(A(V))\}$ on $A(V)$, where $\overline{F}_k(A(V)) := \big(\bigoplus_{i\in\frac{1}{2}\mathbb{Z}, i\leqslant k} V_{(i)} + O(V)\big)/O(V)$. Moreover, [54, Lemma 2.1.3] implies its associated graded algebra,

$$\mathrm{gr}^{\overline{F}}(A(V)) = \bigoplus_{i=0}^{\infty} \overline{F}_k(A(V))/\overline{F}_{k-1}(A(V)),$$

is a commutative algebra with respect to the multiplication $u * v$ and the commutation $u * v - v * u$. Note, by definition of Zhu's algebra we have $A(V) \cong V_{\bar{0}}/(V_{\bar{0}} \cap O(V))$.

Following a similar argument to [25, Proposition 2.17 (c)] or [8, Propostion 3.3], we can define a surjective homomorphism of graded commutative Poisson algebras:

$$f\colon (R_V)_{\overline{0}} \twoheadrightarrow \mathrm{gr}^{\overline{F}}(A_V) \tag{1}$$

given by $f(\overline{a}\cdot\overline{b}) = [a*b] + \overline{F}^{k+l-1}(A(V))$ for $a \in V_{(k)}$ and $b \in V_{(l)}$. Now we can prove:

Corollary 4.5 *Let p be a positive odd integer, then the even part of $R_{V_{\sqrt{p}\mathbb{Z}}}$, i.e., $(R_{V_{\sqrt{p}\mathbb{Z}}})_{\overline{0}}$, is isomorphic to the associated graded algebra $\mathrm{gr}^{\overline{F}}(A_{V_{\sqrt{p}\mathbb{Z}}})$.*

Proof According to [45, Theorem 3.3], we know that $A_{V_{\sqrt{p}\mathbb{Z}}} \cong \mathbb{C}[x]/(F_p(x))$, where $F_p(x) = x(x+1)(x-1)\cdots(x+(p-1)/2)(x-(p-1)/2)$ in which x corresponds to $[\alpha_{(-1)}\mathbf{1}]$ in $A_{V_{\sqrt{p}\mathbb{Z}}}$. Then according to Proposition 4.3, we have

$$\left(R_{V_{\sqrt{p}\mathbb{Z}}}\right)_{\overline{0}} \cong \mathrm{gr}^{\overline{F}}\left(A_{V_{\sqrt{p}\mathbb{Z}}}\right) \cong \mathbb{C}[x]/\langle x^p\rangle$$

via f. □

Remark 4.6 If $L = \sqrt{2k}\mathbb{Z}$, $k \in \mathbb{Z}_+$, is an even lattice, the above result is true only for $k = 1$. Indeed, according to [26], z^{p-1} is a non-trivial element in the kernel of f in (1).

In [52], authors proved that the map ψ is an isomorphism for $L_{\widehat{\mathfrak{sl}_2}}(k,0)$ using a PBW-type basis of $L_{\widehat{\mathfrak{sl}_2}}(k,0)$ from [42] and Gröbner bases. In [32], the author essentially proved the same result using a technique called the “degeneration procedure". In the following, we briefly explain how his results imply the isomorphism.

Proposition 4.7 *The map $\psi\colon J_\infty(R_{L_{\widehat{\mathfrak{sl}_2}}(k,0)}) \cong \mathrm{gr}^F(L_{\widehat{\mathfrak{sl}_2}}(k,0))$ is an isomorphism of vertex Poisson algebras.*

Proof According to Example 2.10, the C_2-algebra $R_{L_{\widehat{\mathfrak{sl}_2}}(k,0)}$ is isomorphic to

$$\begin{aligned}&\mathbb{C}[e,f,h]/\langle f^i\circ e^{k+1} \mid 0 \leqslant i \leqslant 2k+2\rangle\\ &\quad = \mathbb{C}[e,f,h]/\left\langle e^{k+1}, e^k h, e^{k-1}h^2 - 2e^k f, \ldots, f^{k+1}\right\rangle.\end{aligned}$$

It is clear that $\psi(u_{(-i)}) = \overline{u_{(-i)}\mathbf{1}}$ for $u \in \{e,f,h\}$ and $i \in \mathbb{Z}_+$. Let $u(z) = \sum_{n\in\mathbb{Z}_-} u_{(n)}z^{-n-1}$ where $u \in \{e,f,h\}$. Now we consider $e(z)^{k+1}$. The coefficient of z^n equals $T^n(e_{(-1)}^{k+1})$ up to a scalar multiple for $n \in \mathbb{N}$. We have similar results for $e^k h, e^{k-1}h^2 - 2e^k f, \ldots, f^{k+1}$. Thus

$$J_\infty\left(R_{L_{\widehat{\mathfrak{sl}_2}}(k,0)}\right) \cong \frac{\mathbb{C}[e_{(-1-i)}, f_{(-1-i)}, h_{(-1-i)} \mid i \in \mathbb{N}]}{\langle e(z)^{k+1}, e(z)^k h(z), e(z)^{k-1}h(z)^2 - 2e(z)^k f(z), \ldots, f(z)^{k+1}\rangle},$$

where

$$\left\langle e(z)^{k+1}, e(z)^k h(z), e(z)^{k-1}h(z)^2 - 2e(z)^k f(z), \ldots, f(z)^{k+1}\right\rangle$$

stands for the ideal generated by the Fourier coefficients of

$$e(z)^{k+1},\ e(z)^k h(z),\ e(z)^{k-1}h(z)^2 - 2e(z)^k f(z),\ \ldots,\ f(z)^{k+1}.$$

It is sufficient to show that

$$\mathrm{HS}_q\big(J_\infty\big(R_{L_{\widehat{\mathfrak{sl}_2}}(k,0)}\big)\big) = \mathrm{ch}\big[L_{\widehat{\mathfrak{sl}_2}}(k,0)\big](q). \tag{2}$$

To this end, we use results from [32]. Feigin constructed the following three quotient polynomial algebras, B_k, C_k and D_k:

- The quotient of the algebra B_k in variables $e_{-1-i}, h_{-1-i}, f_{-1-i}, i \in \mathbb{N}$, is generated by Fourier coefficients of the series:

  $$e(z)^i h(z)^{k+1-i}, \quad i = 1, \ldots, k+1,$$

  and

  $$h(z)^i f(z)^{k+1-i}, \quad i = 0, \ldots, k+1.$$

- Let $u^{[l]}(z) = \sum_{i \in \mathbb{Z}_-} z^{-1-l} u_i^{[l]}$, where $u = e, h, f$ and $u_i^{[l]} = 0$ for $i > -l$. Then the quotient of the polynomial algebra C_k in variables $u_i^{[l]}$, $l = 1, \ldots, k$, is generated by Fourier coefficients of the series:

  $$\begin{aligned} &u^{[l]}(z)^{(\alpha)} u^{[m]}(z)^{(\beta)} && \text{for } u = e, f, h, \text{ and } \alpha + \beta < \min(l, m), \\ &e^{[l]}(z)^{(\alpha)} h^{[m]}(z)^{(\beta)} && \text{for } \alpha + \beta < \max(0, l + m - k), \\ &h^{[l]}(z)^{(\alpha)} f^{[m]}(z)^{(\beta)} && \text{for } \alpha + \beta < \max(0, l + m - k). \end{aligned}$$

- Define a lattice Q generated by vectors $p_i, q_i, r_i \in \mathbb{R}^N$, $i = 1, \ldots, k$, with scalar products:

  $$\begin{aligned} &\langle p_i, p_j\rangle = \langle q_i, q_j\rangle = \langle r_i, r_j\rangle = 2\delta_{i,j}, \quad \langle p_i, q_j\rangle = \langle q_i, r_j\rangle = \delta_{i,k+1-j}, \\ &\langle p_i, r_j\rangle = 0. \end{aligned}$$

  The algebra D_k is generated from the highest weight vector with the Fourier coefficients of

  $$\sum_{i=1}^k Y(e^{p_i}, z), \quad \sum_{i=1}^k Y(e^{q_i}, z), \quad \sum_{i=1}^k Y(e^{r_i}, z).$$

By using certain filtrations [32, Lemmas 3.2, 3.4], one gets

$$\mathrm{HS}_q\big(J_\infty\big(R_{L_{\widehat{\mathfrak{sl}_2}}(k,0)}\big)\big) \leqslant \mathrm{HS}_q(B_k) \leqslant \mathrm{HS}_q(C_k). \tag{3}$$

The "degenerate procedures" [32, Lemma 3.5, Proposition 3.1] give us

$$\mathrm{HS}_q(D_k) \geqslant \mathrm{HS}_q(C_k) \quad \text{and} \quad \mathrm{ch}\big[L_{\widehat{\mathfrak{sl}_2}}(k,0)\big](q) \geqslant \mathrm{HS}_q(D_k). \tag{4}$$

Combining (3), (4) and the fact that ψ is surjective, we get (2). □

Before we prove the next result, let us fix some notations. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra of type $C_n, n \geqslant 2$. Here we assume that $\mathfrak{g}$ has a basis $\{x^i \mid 1 \leqslant i \leqslant (2n+1)n\}$. Let θ be the maximal root of $\mathfrak{g}$, and x_θ the corresponding maximal root vector. Let $\widehat{\mathfrak{g}}$ be the affine Lie algebra associated with $\mathfrak{g}$ and its universal vacuum representation be $V_{\hat{\mathfrak{g}}}(1,0)$ for $k \in \mathbb{Z}_+$. Set

$$R = U(\mathfrak{g}) \circ (x_\theta)^2_{(-1)}\mathbf{1}, \quad \overline{R} = \mathrm{Span}_{\mathbb{C}}\{r_{(n)} \mid r \in R, n \in \mathbb{Z}\},$$

where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$, and $\circ$ is the adjoint action. Then the $\widehat{\mathfrak{g}}$-module $V_{\hat{\mathfrak{g}}}(1,0)$ has a maximal submodule $I_{\hat{\mathfrak{g}}}(1,0)$ generated by $\overline{R}\cdot\mathbf{1}$. Let $L_{\hat{\mathfrak{g}}}(1,0)$ denote the simple quotient $V_{\hat{\mathfrak{g}}}(1,0)/I_{\hat{\mathfrak{g}}}(1,0)$. Now we are ready to prove:

Theorem 4.8 *The map ψ is an isomorphism for the affine vertex algebra $L_{\hat{\mathfrak{g}}}(1,0)$.*

Proof It is clear that the C_2-algebra of $L_{\hat{\mathfrak{g}}}(1,0)$ is $R_{L_{\hat{\mathfrak{g}}}(1,0)} = S(\mathfrak{g})/\langle U(\mathfrak{g}) \circ e_\theta^2\rangle$, where $S(\mathfrak{g})$ is the symmetric algebra of $\mathfrak{g}$. We denote the algebra

$$\mathbb{C}[x^i_{(-j)} \mid j \in \mathbb{Z}_+]/\langle U(\mathfrak{g}) \circ e_\theta^2(z)\rangle$$

by Q, where $e_\theta(z) = \sum_{n\in\mathbb{Z}_-}(e_\theta)_{(n)}z^{-n-1}$. Following a similar argument to Proposition 4.7, we see that $J_\infty(R_{L_{\hat{\mathfrak{g}}}(1,0)}) \cong Q$. In order to show that ψ is an isomorphism, it is enough to prove that $\mathrm{gr}^F(L_{\hat{\mathfrak{g}}}(1,0))$ and Q have the same basis. Notice that

$$I = \overline{R} \cap \mathbb{C}\big[x^i_{(-j)} \mid j \in \mathbb{Z}_+\big] = \langle U(\mathfrak{g}) \circ e_\theta^2(z)\rangle.$$

We can define an order on all monomials of $\mathbb{C}[x^i_{(-j)} \mid j \in \mathbb{Z}_+]$ in the sense of [49, Section 8]. From the same paper, we know that every non-zero homogeneous polynomial $\mathbb{C}[x^i_{(-j)} \mid j \in \mathbb{Z}_+]$ has a unique largest monomial. For an arbitrary non-zero polynomial u, we define the leading term $\mathrm{lt}(u)$ as the largest monomial of the non-zero homogeneous component of the smallest degree, which is unique. We denote all monomials in $\mathbb{C}[x^i_{(-j)} \mid j \in \mathbb{Z}_+]$ by $\mathcal{P}$. We clearly have $\mathcal{P}$ as a spanning set of Q. Since $u = 0$ in Q if $u \in I$, the leading term $\mathrm{lt}(u)$ equals the linear combination of other terms. Therefore, $\mathcal{P} \setminus \langle \mathrm{lt}(U)\rangle$ is a smaller spanning set of Q. We denote it by $\mathcal{RR}$. Meanwhile according to [49, Theorem 11.3], we know that $\psi(\mathcal{RR})$ is a basis of $\mathrm{gr}(L_{\hat{\mathfrak{g}}}(1,0))$. Together with the surjectivity of ψ, we have that $\mathcal{RR}$ is a basis of Q. So ψ is an isomorphism. □

4.1 $N = 2$ vertex superalgebra at $c = 1$

In this section we study the simple $N = 2$ superconformal vertex algebra with central charge $c = 1$, denoted by $L_1^{N=2}$. The odd lattice vertex algebra $V_{\sqrt{3}\mathbb{Z}}$ is known to be isomorphic to $L_1^{N=2}$. Here we identify $\frac{1}{3}\alpha_{(-1)}\mathbf{1}$ with $J_{(-1)}\mathbf{1}$, $\frac{1}{\sqrt{3}}e^{\pm\alpha}$ with $G_{(\pm 3/2)}\mathbf{1}$, and $\frac{1}{6}(\alpha_{(-1)}\alpha_{(-1)}\mathbf{1})_{(-1)}\mathbf{1}$ with $L_{(-2)}\mathbf{1}$.

According to [2,3], the maximal submodule of $V_1^{N=2}$ is generated by

$$G^+_{(-5/2)}G^+_{(-3/2)}\mathbf{1} \quad \text{and} \quad G^-_{(-5/2)}G^-_{(-3/2)}\mathbf{1}.$$

Identifying G^+ with $G^+_{(-3/2)}\mathbf{1}$, G^- with $G^-_{(-3/2)}\mathbf{1}$, and h with $J_{(-1)}\mathbf{1}$, we have

$$R_{L_1^{N=2}} \cong \mathbb{C}[G^+, G^-, h]/\big\langle (G^+)^2, (G^-)^2, G^+G^- = h^3, G^+h, G^-h\big\rangle.$$

For $J_\infty(R_{L_1^{N=2}})$ we identify G^+, G^-, h with $G^+(-3/2)$, $G^-(-3/2)$, $h(-1)$. We have

$$J_\infty\big(R_{L_1^{N=2}}\big) \cong \frac{\mathbb{C}[G^+(-3/2-i), G^-(-3/2-i), h(-1-i) \mid i \in \mathbb{N}]}{\langle (G^+(z))^2, (G^-(z))^2, G^+(z)G^-(z) = h(z)^3, G^+(z)h(z), G^-(z)h(z)\rangle},$$

where $G^\pm(z) = \sum_{n\in -\frac{1}{2}+\mathbb{Z}_-} G^\pm(n)z^{-n-3/2}$, $h(z) = \sum_{n\in\mathbb{Z}_-} h(n)z^{-n-1}$. The map ψ is not an isomorphism in this case because the images of non-zero elements

$$G^+\left(-\frac{5}{2}\right)G^+\left(-\frac{3}{2}\right) \quad \text{and} \quad G^-\left(-\frac{5}{2}\right)G^-\left(-\frac{3}{2}\right)$$

in the jet superalgebra under ψ, i.e., $G^+_{(-5/2)}G^+_{(-3/2)}\mathbf{1}$ and $G^-_{(-5/2)}G^-_{(-3/2)}\mathbf{1}$, are null vectors. Thus

$$\langle a, b\rangle_\partial = \left\langle T^i\left(G^+\left(-\frac{5}{2}\right)G^+\left(-\frac{3}{2}\right)\right), T^i\left(G^-\left(-\frac{5}{2}\right)G^-\left(-\frac{3}{2}\right)\right) \;\middle|\; i \in \mathbb{N}\right\rangle \subset \ker(\psi),$$

where $a = G^+(-5/2)G^+(-3/2)$ and $b = G^-(-5/2)G^-(-3/2)$. Let us consider

$$J_\infty\big(R_{L_1^{N=2}}\big)/\langle a, b\rangle_\partial.$$

We will write down a spanning set of $J_\infty(R_{L_1^{N=2}})/\langle a, b\rangle_\partial$. We let the ordered monomial be a monomial of the form

$$G^-\left(-n-\frac{1}{2}\right)^{a_n} h(-n)^{b_n} G^+\left(-n-\frac{1}{2}\right)^{c_n} \cdots G^-\left(-\frac{5}{2}\right)^{a_2} h(-2)^{b_2} G^+\left(-\frac{5}{2}\right)^{c_2} \cdot G^-\left(-\frac{3}{2}\right)^{a_1} h(-1)^{b_1} G^+\left(-\frac{3}{2}\right)^{c_1}.$$

Then we have a complete lexicographic ordering on the set of ordered monomials in the sense of Sect. 3.1. Now let us find the leading terms of the Fourier coefficients of

$$G^+(z)G^-(z) = h^3(z), \quad G^+(z)h(z), \quad G^-(z)h(z), \quad T^i(a), \quad T^i(b).$$

(a) The leading term of $G^+(z)h(z)$:

- n is even, the leading term of the coefficient of z^n is

$$h\left(\frac{-2-n}{2}\right)G^+\left(-\frac{3+n}{2}\right).$$

- n is odd, the leading term of the coefficient of z^n is

$$G^+\left(\frac{-4-n}{2}\right)h\left(\frac{-1-n}{2}\right).$$

(b) The leading term of $G^-(z)h(z)$:

- n is even, the leading term of the coefficient of z^n is

$$G^-\left(-\frac{3+n}{2}\right)h\left(\frac{-2-n}{2}\right).$$

- n is odd, the leading term of the coefficient of z^n is

$$h\left(\frac{-3-n}{2}\right)G^-\left(\frac{-2-n}{2}\right).$$

(c) The leading term of $G^+(z)G^-(z) = h^3(z)$:

- The leading term of the constant term is

$$h(-1)h(-1)h(-1).$$

- n is even and not equal to 0, the leading term of the coefficient of z^n is

$$G^-\left(-\frac{3+n}{2}\right)G^+\left(-\frac{3+n}{2}\right).$$

- n is odd, the leading term of the coefficient of z^n is

$$G^+\left(\frac{-n-4}{2}\right)G^-\left(\frac{-n-2}{2}\right).$$

(d) The leading term of $T^n(a)$ or $T^n(b)$:

- n is even, the leading term is

$$G^{\pm}\left(\frac{-n-5}{2}\right)G^{\pm}\left(\frac{-n-3}{2}\right).$$

- n is odd, the leading term is

$$G^{\pm}\left(\frac{-n-6}{2}\right)G^{\pm}\left(\frac{-n-2}{2}\right).$$

Clearly all ordered monomials constitute a spanning set of $J_{\infty}(R_{L_1^{N=2}})/\langle a, b\rangle_{\partial}$. Since all polynomials we considered above equal zero in $J_{\infty}(R_{L_1^{N=2}})/\langle a, b\rangle_{\partial}$, the leading term of each can be written as a linear combination of all other terms. Thus if we want to get a “smaller" spanning set, all above leading terms cannot appear as segments of an ordered monomial. Therefore, we can impose some difference conditions on ordered monomials using these leading terms to get a new spanning set.

Definition 4.9 We call an ordered monomial a *Gh-monomial*, if it satisfies the following conditions:

(i) Either b_i or c_i is 0 and either b_i or c_{i+1} is 0,
(ii) either a_i or b_i is 0 and either a_i or b_{i+1} is 0,
(iii) $b_1 \leqslant 2$, $a_1 + c_2 \leqslant 1$, and $a_i + c_i + c_{i+1} \leqslant 1$ for $i \geqslant 2$,
(iv) $c_i + c_{i+1} + c_{i+2} \leqslant 1$ and $a_i + a_{i+1} + a_{i+2} \leqslant 1$.

Here constraints (i)–(iv) come from the leading terms in (a)–(d), respectively. Then we have the following:

Proposition 4.10 *Gh-monomials form a spanning set of*

$$A = J_{\infty}\big(R_{L_1^{N=2}}\big)/\langle a, b\rangle_{\partial}.$$

Let us write down the first few terms of the Hilbert series of A.

Example 4.11 For $i \leqslant 5$, Gh-monomials give us a basis of A_i:

$$\begin{aligned}
A_1 &: h(-1),\\
A_{3/2} &: G^+\Big(-\frac{3}{2}\Big), G^-\Big(-\frac{3}{2}\Big),\\
A_2 &: h(-1)^2, h(-2),\\
A_{5/2} &: G^+\Big(-\frac{5}{2}\Big), G^-\Big(-\frac{5}{2}\Big),\\
A_3 &: G^-\Big(-\frac{3}{2}\Big)G^+\Big(-\frac{3}{2}\Big), h(-1)h(-2), h(-3),\\
A_{7/2} &: G^-\Big(-\frac{5}{2}\Big)h(-1), G^+\Big(-\frac{7}{2}\Big), G^-\Big(-\frac{7}{2}\Big), G^+\Big(-\frac{3}{2}\Big)h(-2),\\
A_4 &: G^+\Big(-\frac{3}{2}\Big)G^-\Big(-\frac{5}{2}\Big), h(-2)^2, h(-1)^2h(-2), h(-3)h(-1), h(-4),\\
A_{9/2} &: G^-\Big(-\frac{5}{2}\Big)h(-1)^2, G^+\Big(-\frac{9}{2}\Big), G^-\Big(-\frac{9}{2}\Big),\\
&\quad G^-\Big(-\frac{7}{2}\Big)h(-1), G^+\Big(-\frac{7}{2}\Big)h(-1), h(-3)G^-\Big(-\frac{3}{2}\Big), h(-3)G^+\Big(-\frac{3}{2}\Big),\\
A_5 &: G^-\Big(-\frac{3}{2}\Big)G^+\Big(-\frac{7}{2}\Big), G^+\Big(-\frac{3}{2}\Big)G^-\Big(-\frac{7}{2}\Big), h(-1)h(-4),\\
&\quad h(-1)h(-2)^2, h(-1)^2h(-3), h(-2)h(-3), h(-5).
\end{aligned}$$

We have $\mathrm{HS}_q(A) = 1 + q + 2q^{3/2} + 2q^2 + 2q^{5/2} + 3q^3 + 4q^{7/2} + 5q^4 + 7q^{9/2} + 7q^5 + \mathrm{O}(q^{11/2})$. Meanwhile

$$\begin{aligned}
\mathrm{ch}\big[L_1^{N=2}\big](q) = \mathrm{ch}\big[V_{\sqrt{3}\mathbb{Z}}\big](q) &= \frac{\sum_{n\in\mathbb{Z}} q^{3n^2/2}}{\prod_{n\in\mathbb{Z}_+}(1-q^n)}\\
&= 1 + q + 2q^{3/2} + 2q^2 + 2q^{5/2} + 3q^3\\
&\qquad + 4q^{7/2} + 5q^4 + 6q^{9/2} + 7q^5 + \mathrm{O}(q^{11/2}).
\end{aligned}$$

Since in degree $9/2$ the dimension of A is bigger than the dimension of $V_{\sqrt{3}\mathbb{Z}}$ by 1, the induced map

$$\overline{\psi}\colon J_\infty\big(R_{L_1^{N=2}}\big)/\langle a, b\rangle_\partial \to \mathrm{gr}\big(L_1^{N=2}\big)$$

is not injective. It is not hard to see that the 1-dimensional kernel of $\overline{\psi}$ in degree $9/2$ is spanned by

$$c = G^-\Big(-\frac{9}{2}\Big) - \frac{1}{3}h(-3)G^-\Big(-\frac{3}{2}\Big) - G^-\Big(-\frac{7}{2}\Big)h(-1) + \frac{1}{3}G^-\Big(-\frac{5}{2}\Big)h(-1)^2.$$

We make the following conjecture:

Conjecture 4.12 *The induced map* $\widehat{\psi}\colon J_\infty(R_{L_1^{N=2}})/\langle a,b,c\rangle_\partial \to \mathrm{gr}(L_1^{N=2})$ *is an isomorphism.*

5 Principal subspaces

Principal subspaces of affine vertex algebras (at least in a special case) were introduced by Feigin and Stoyanovsky [31] and further studied by several people; see [21–24,27] and references therein. In [47,48], Primc studied Feigin–Stoyanovsky *type* subspaces which are analogs of principal subspaces but easier to analyze. They were further investigated for many integral levels and types [16,35,50,51]. Here we follow notations from [44], where principal subspaces are defined for general integral lattices (not necessarily positive definite). As in [44], we let $V_L = M(1)\otimes\mathbb{C}[L]$ denote a lattice vertex algebra. We fix a $\mathbb{Z}$-basis $\mathcal{B}=\{\alpha_1,\ldots,\alpha_n\}$ of L. Let e^{α_i} be an element in the group algebra $\mathbb{C}[L]$. Then the principal subspace associated to $\mathcal{B}$ and L is defined as

$$W_L(\mathcal{B}) := \langle e^{\alpha_1},\ldots,e^{\alpha_n}\rangle,$$

that is the smallest vertex algebra that contains extremal vectors e^{α_i}. Once $\mathcal{B}$ is fixed, we shall drop $\mathcal{B}$ in the parentheses and write W_L for convenience.

Let $\mathfrak{g}$ be a simple finite-dimensional complex Lie algebra of type A, D or E, and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. We choose simple roots $\{\alpha_1,\ldots,\alpha_n\}$ of $(\mathfrak{g},\mathfrak{h})$, and let Δ^+ denote the set of positive roots. Let $(\cdot,\cdot)$ be a rescaled Killing form on $\mathfrak{g}$ such that $(\alpha_i,\alpha_i)=2$ for $i=1,\ldots,n$ (as usual we identify $\mathfrak{h}$ and $\mathfrak{h}^*$ via the Killing form). Fundamental weights of $\mathfrak{g}$, $\{\omega_1,\ldots,\omega_n\}\subset\mathfrak{h}^*$, are defined by $(\omega_i,\alpha_j)=\delta_{i,j}$, $1\leqslant i,j\leqslant n$.

Let $\mathfrak{n}_+$ be $\coprod_{\alpha\in\Delta^+}\mathbb{C}x_\alpha$, where x_α is a corresponding root vector, and $\widehat{\mathfrak{n}_+}=\mathfrak{n}_+\otimes\mathbb{C}[t,t^{-1}]$ is its affinization. For an affine vertex algebra $L_{\hat{\mathfrak{g}}}(k,0)$, $k\neq -h^\vee$, which is isomorphic to $L(k\Lambda_0)$ as $\hat{\mathfrak{g}}$-modules, we define the (FS)-principal subspace of the simple $\widehat{\mathfrak{g}}$-module $L_{\hat{\mathfrak{g}}}(k,0)$ as

$$W_{\Lambda_{k,0}} := U(\widehat{\mathfrak{n}}_+)\cdot\mathbb{1},$$

where $\mathbb{1}$ is the vacuum vector. It is easy to see that this is a vertex algebra (without conformal vector). For $k=1$, we have $W_L\cong W_{\Lambda_{1,0}}$, where L is the root lattice spanned by simple roots.

We fix a fundamental weight $\omega=\omega_m$ and set

$$\Gamma=\{\alpha\in\Delta \mid (\omega,\alpha)=1\},$$

where Δ is the root system of $\mathfrak{g}$, and $\mathfrak{g}_1 := \coprod_{\alpha\in\Gamma}\mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ is the α-root space. This Lie algebra is commutative. We let $\mathfrak{g}_1\otimes\mathbb{C}[t,t^{-1}]$ be $\widehat{\mathfrak{g}_1}$. Then we can define the so-called Feigin–Stoyanovsky type subspace of $L_{\hat{\mathfrak{g}}}(k,0)$ as

$$W'_{\Lambda_{k,0}} := U(\widehat{\mathfrak{g}}_1)\cdot v_{k\Lambda_0}.$$

Unlike the FS subspace, this vertex subalgebra is commutative. We denote

$$\widetilde{\Gamma}^- = \{x_\gamma(-r) \mid \gamma \in \Gamma, r \in \mathbb{Z}_+\}, \quad \widetilde{\Gamma} = \{x_\gamma(-r) \mid \gamma \in \Gamma, r \in \mathbb{Z}\}.$$

Notice that $U(\widehat{\mathfrak{g}}_1) \cong \mathbb{C}[\widetilde{\Gamma}]$. Therefore, we can identify the elements in $W'_{\Lambda_{k,0}}$ with the elements in $\mathbb{C}[\widetilde{\Gamma}^-]$. For any element in $W'_{\Lambda_{k,0}}$,

$$v = x_{\beta_1}(m_1) \cdots x_{\beta_l}(m_l), \quad \beta_i \in \Gamma,$$

we define the colored weight as

$$\mathrm{cwt}(v) = \sum_{i=1}^{l} \beta_i$$

for the later use.

5.1 Root lattices of type A

Following the notations in [23], we can prove the following result.

Proposition 5.1 *For $\mathfrak{g} = \mathrm{sl}_2$, we have $W_{\Lambda_{k,0}} \cong \mathrm{gr}(W_{\Lambda_{k,0}}) \cong J_\infty(\mathbb{C}[x]/\langle x^{k+1}\rangle)$ for $k \in \mathbb{Z}_+$.*

Proof It is clear that $R_{W_{\Lambda_{k,0}}} = \mathbb{C}[x]/\langle x^{k+1}\rangle$. The result follows from [23, Theorem 3.1]. □

Remark 5.2 When $k = 1$, $W_{\Lambda_{1,0}}$ of type A is isomorphic to $J_\infty(\mathbb{C}[x]/\langle x^2\rangle)$. Using different methods to calculate the Hilbert–Poincaré series, see [20] and [15], one can derive the famous Rogers–Ramanujan identities.

For the rest of this subsection, we let L be the A_{n-1} root lattice with the rescaled Killing form $(\cdot, \cdot)$ such that $(\alpha, \alpha) = 2$ for any root and the standard $\mathbb{Z}$-basis $\alpha_1, \ldots, \alpha_{n-1}$ of simple roots. We are going to prove that ψ is an isomorphism for the principal subspace W_L corresponding to this basis. In the following, we will identify W_L and $W_{\Lambda_{1,0}}$.

Proposition 5.3 *Given elements α, β, γ and τ in the lattice L, we have*

$$(e^\alpha)_{(-1)} e^\beta = 0 \quad \text{if} \quad (\alpha, \beta) \in \mathbb{Z}_+, \tag{5}$$

$$(e^\alpha)_{(-1)} e^\beta = \frac{\epsilon(\alpha, \beta)}{\epsilon(\gamma, \tau)} (e^\gamma)_{(-1)} e^\tau \quad \text{if} \quad (\alpha, \beta) = (\gamma, \tau) \ \text{and} \ \alpha + \beta = \gamma + \tau. \tag{6}$$

Proof From the definition of vertex operators in [36], we have

$$Y(e^\alpha, z) e^\beta = \epsilon(\alpha, \beta) z^{(\alpha,\beta)} \mathrm{Exp}\left(\sum_{n \in \mathbb{Z}_-} \frac{-\alpha(n)}{n} z^{-n} \right) e^{\alpha+\beta},$$

where $\epsilon(\alpha, \beta)$ is a 2-cocycle constant. We have

$$(e^{\alpha})_{(-1)}e^{\beta} = \mathrm{Coeff}_{z^0}(Y(e^{\alpha}, z)e^{\beta}) = 0,$$

since the minimal power of z above is greater than 0. The coefficients of z^0 of $Y(e^{\alpha}, z)e^{\beta}$ and $Y(e^{\gamma}, z)e^{\tau}$ are $\epsilon(\alpha, \beta)e^{\alpha+\beta}$ and $\epsilon(\gamma, \tau)e^{\gamma+\tau}$. The identity (6) follows from this fact and the given condition. $\square$

It is clear that all quotient relations in $R_{L_{\widehat{\mathrm{sl}_n}}(1,0)}$ come from (5) and (6). Thus $R_{L_{\widehat{\mathrm{sl}_n}}}(1, 0) \cap \mathbb{C}[E_{i,j} \mid 1 \leqslant i < j \leqslant n] = R_{W_L}$.

We let $E_{i,j}$ be the (i, j)-th elementary matrix. Therefore, $\{E_{i,j}\}_{1\leqslant i<j\leqslant n}$ is the set of all positive root vectors. It is not hard to see that the C_2-algebra R_{W_L} equals $\mathbb{C}[E_{i,j} \mid 1 \leqslant i < j \leqslant n]/I$, where we denote the equivalence class of $(E_{i,j})_{(-1)}\mathbf{1}$ by $E_{i,j}$. In [28, Corollary 2.7] (see also [31] for $\mathfrak{g} = \mathrm{sl}_3$), authors wrote down the graded decomposition of $R_{L_{\widehat{\mathrm{sl}_n}}(1,0)}$. By restricting it to its principal subspace, we have:

Proposition 5.4 *The C_2-algebra of W_L equals*

$$\mathbb{C}[E_{i,j} \mid 1 \leqslant i < j \leqslant n]\Big/\Big\langle \sum_{\sigma\in S_2} E_{i_1,j_{\sigma_1}} E_{i_2,j_{\sigma_2}} \mid j_1 > i_2 \Big\rangle$$

where $1 \leqslant i_1 \leqslant i_2 \leqslant n$ and $1 \leqslant j_1 \leqslant j_2 \leqslant n$.

Moreover, we have the following combinatorial q-identity which will be proven in a joint work with Milas [39], where we also establish more general identities.

Theorem 5.5 (Li–Milas) *Let A be the Cartan matrix $((\alpha_i, \alpha_j))_{1\leqslant i,j\leqslant n-1}$ of type A_{n-1}, $n \geqslant 2$, and*

$$\mathbf{n} = (n_{1,2}, \ldots, n_{n-1,n}) = (n_{i,j})_{1\leqslant i<j\leqslant n}.$$

Then we have

$$\sum_{\mathbf{n}\in\mathbb{N}^{n(n-1)/2}} \frac{q^{B(\mathbf{n})}}{\prod_{1\leqslant i<j\leqslant n}(q)_{n_{i,j}}} = \sum_{\boldsymbol{k}=(k_1,\ldots,k_{n-1})\in\mathbb{N}^{n-1}} \frac{q^{\boldsymbol{k}A\mathbf{k}^{\top}}}{(q)_{k_1}(q)_{k_2}\cdots(q)_{k_{n-1}}}, \tag{7}$$

where

$$B(\mathbf{n}) = \sum_{\substack{1\leqslant i_1<j_1\leqslant n\\ 1\leqslant i_2<j_2\leqslant n\\ 1\leqslant i_1\leqslant i_2\leqslant n\\ 1\leqslant j_1\leqslant j_2\leqslant n\\ j_1>i_2}} n_{i_1,j_1}n_{i_2,j_2}.$$

Example 5.6 For sl_4, we have the following q-series identity:

$$\sum_{\mathbf{n}\in\mathbb{N}^6} \frac{q^{n_1^2+n_2^2+n_3^2+n_4^2+n_5^2+n_6^2+n_1n_4+n_1n_6+n_2n_4+n_2n_5+n_3n_5+n_3n_6+n_4n_6+n_5n_6+a_4a_5}}{(q)_{n_1}(q)_{n_2}(q)_{n_3}(q)_{n_4}(q)_{n_5}(q)_{n_6}}$$
$$= \sum_{\mathbf{k}\in\mathbb{N}^3} \frac{q^{k_1^2-k_1k_2+k_2^2-k_2k_3+k_3^2}}{(q)_{k_1}(q)_{k_2}(q)_{k_3}},$$

where we use multiindices $\mathbf{n} = (n_1, n_2, \ldots, n_6)$ and $\mathbf{k} = (k_1, k_2, k_3)$. Upon the following replacement:

$$\begin{aligned} &n_{1,2} \leftrightarrow n_1, \quad n_{2,3} \leftrightarrow n_2, \quad n_{3,4} \leftrightarrow n_3,\\ &n_{1,3} \leftrightarrow n_4, \quad n_{2,4} \leftrightarrow n_5, \quad n_{1,4} \leftrightarrow n_6, \end{aligned}$$

we recover the formula in Theorem 5.5.

Now we are ready to prove:

Theorem 5.7 *The map ψ is an isomorphism between $J_\infty(R_{W_L})$ and $\mathrm{gr}(W_L)$.*

Proof From Proposition 5.4, we know that $J_\infty(R_{W_L})$ is isomorphic to

$$\mathbb{C}\big[E_{i,j}(n) \mid n \leqslant -1, 1 \leqslant i < j \leqslant n\big] \Big/ \Big\langle \sum_{\sigma\in S_2} E_{i_1,j_{\sigma_1}}(z)E_{i_2,j_{\sigma_2}}(z) \,\Big|\, j_1 > i_2 \Big\rangle,$$

where $E_{i,j}(z) = \sum_{n\leqslant -1} E_{i,j}(n)z^{-n-1}$ and $1 \leqslant i_1 \leqslant i_2 \leqslant n$, $1 \leqslant j_1 \leqslant j_2 \leqslant n$. In order to simplify notation, we first order $\{E_{i,j}\}_{1\leqslant i<j\leqslant n}$ as

$$E_{1,2}, \ E_{1,3}, \ \ldots, \ E_{1,n}, \ E_{2,3}, \ \ldots, \ E_{2,n}, \ \ldots, \ E_{n-1,n},$$

and denote this sequence by $\{E_m\}_{1\leqslant m\leqslant n(n-1)/2}$ (i.e., $E_1 = E_{1,2}$, $E_2 = E_{1,3}$ etc.). We then have a spanning set of jet algebras with each element of the form

$$E_1\big(-n_1^1\big)\cdots E_1\big(-n_1^{k_1}\big)E_2\big(-n_2^1\big)\cdots E_2\big(-n_2^{k_2}\big)\cdots,$$

where $1 \leqslant n_m^{k_m} \leqslant \cdots \leqslant n_m^1$ for $1 \leqslant m \leqslant n(n-1)/2$. Here $k_s = 0$ when we do not have terms involving E_s. Now we can reduce this spanning set by using quotient relations as follows:

- (*difference two condition at distance* 1) If we have $E_m(z)^2 = 0$ in the quotient of the jet algebra, then we can impose the condition $n_m^p \geqslant n_m^{p+1}+2$, $1 \leqslant p \leqslant k_m-1$, on the above spanning set.
- (*boundary condition*) If we have $E_s(z)E_t(z) + \cdots = 0$, $s < t$, we can impose the condition $n_s^{k_s} \geqslant k_t + 1$.

Therefore, we have a reduced spanning set which implies

$$\mathrm{HS}_q(J_\infty(R_{W_L})) \leqslant \sum_{\mathbf{n}\in\mathbb{N}^{n(n-1)/2}} \frac{q^{B(\mathbf{n})}}{\prod_{1\leqslant i<j\leqslant n}(q)_{n_{i,j}}}.$$

It is well known that

$$\mathrm{ch}[\mathrm{gr}(W_L)](q) = \sum_{\mathbf{k}=(k_1,\ldots,k_n)\in\mathbb{N}^n} \frac{q^{\mathbf{k}A\mathbf{k}^\top}}{(q)_{k_1}(q)_{k_2}\cdots(q)_{k_n}}.$$

Surjectivity of ψ and identity (7) together imply that ψ is an isomorphism and the image of above spanning set under ψ is a basis of W_L. □

Remark 5.8 By a result in [44], we can write down a basis of W_L using $(e^{\alpha_i})_{(j)}$, where α_i is a simple root of sl_n and j can be greater than or equal to 0. If we want the subscript j to be always less than 0, we have to include $(e^{\beta})_{(j)}$, where β is a positive root. It is clear that $E_m = E_{i_m,j_m}$ is a root vector of a positive root

$$\beta_m := \alpha_{i_m} + \alpha_{i_m+1} + \cdots + \alpha_{j_m-1}.$$

The above proposition gives us a new basis of W_L:

$$\left(e^{\beta_1}\right)_{(-n_1^1)}\cdots\left(e^{\beta_1}\right)_{(-n_1^{k_1})}\left(e^{\beta_2}\right)_{(-n_2^1)}\cdots\left(e^{\beta_2}\right)_{(-n_2^{k_2})}\cdots\left(e^{\beta_M}\right)_{(-n_M^{k_M})}\mathbf{1},$$

where $M = n(n-1)/2, n_M^{k_M} \in \mathbb{Z}_+, n_m^p \geqslant n_m^{p+1}+2, 1 \leqslant p \leqslant k_m-1$, and $n_s^{k_s} \geqslant k_t+1$ if $1 \leqslant s < t \leqslant M, i_t < j_s \leqslant j_t$.

5.2 Feigin–Stoyanovsky type subspaces

In this section, we consider Feigin–Stoyanovsky type subspaces of affine vertex algebras of type A_n at level 1. We first consider the special case when $\omega = \omega_1$. For any element of the A_n root lattice, $\alpha = m_1\alpha_1 + m_2\alpha_2 + \cdots + m_n\alpha_n$, we define a subspace of $W'_{\Lambda_{1,0}}$ as $(W'_{\Lambda_{1,0}})^\alpha := \{v \in W'_{\Lambda_{1,0}} \mid \mathrm{cwt}(v) = \alpha\}$. It is not hard to see that $(W'_{\Lambda_{1,0}})^\alpha$ is non-trivial if and only if $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_n \geqslant 0$. According to [51, (3.8)], we have

$$\mathrm{ch}\left[(W'_{\Lambda_{1,0}})^\alpha\right](q) = \frac{q^{\sum_{i=1}^n m_i^2 - \sum_{i=1}^{n-1} m_i m_{i+1}}}{(q)_{m_n}(q)_{m_{n-1}-m_n}\cdots(q)_{m_1-m_2}}.$$

Then

$$\begin{aligned}\mathrm{ch}\big[W'_{\Lambda_{1,0}}\big](q) &= \sum_{0\leqslant m_n\leqslant\cdots\leqslant m_1} \frac{q^{\sum_{i=1}^n m_i^2-\sum_{i=1}^{n-1} m_i m_{i+1}}}{(q)_{m_n}(q)_{m_{n-1}-m_n}\cdots(q)_{m_1-m_2}}\\ &= \sum_{(l_1,\ldots,l_n)\in\mathbb{N}^n} \frac{q^{\sum_i^n l_i^2+\sum_{1\leqslant i<j\leqslant n} l_i l_j}}{(q)_{l_1}(q)_{l_2}\cdots(q)_{l_n}}.\end{aligned}$$

Moreover, in this case,

$$\Gamma = \big\{\beta_1 := \alpha_1, \beta_2 := \alpha_1+\alpha_2, \ldots, \beta_n := \alpha_1+\cdots+\alpha_n\big\}.$$

Note that

$$L = \mathbb{Z}\beta_1 \oplus \cdots \oplus \mathbb{Z}\beta_n$$

is a lattice with basis $\{\beta_1, \ldots, \beta_n\}$. Then we have

$$W_L \cong W'_{\Lambda_{1,0}}.$$

It is not hard to see that

$$\begin{aligned}\langle\beta_i, \beta_i\rangle &= 2 \quad \text{if} \ \ i = 1, \ldots, n,\\ \langle\beta_i, \beta_j\rangle &= 1 \quad \text{if} \ \ 1 \leqslant i \neq j \leqslant n.\end{aligned}$$

According to Proposition 5.3, we have that the C_2-algebra of W_L is

$$\mathbb{C}[x_1, \ldots, x_n]/\langle x_i x_j \mid 1 \leqslant i \leqslant j \leqslant n\rangle.$$

By a similar argument as in the previous section, we get

$$\mathrm{HS}_q\Big(J_\infty\Big(\mathbb{C}[x_1, \ldots, x_n]/\langle x_i x_j \mid 1 \leqslant i \leqslant j \leqslant n\rangle\Big)\Big) = \mathrm{ch}[W_L](q),$$

which implies isomorphism between $J_\infty(R_{W'_{\Lambda_{1,0}}})$ and $\mathrm{gr}(W'_{\Lambda_{1,0}})$. Similarly we can prove isomorphism in cases where $\omega = \omega_i$, $2 \leqslant i \leqslant n$, using [51, (3.21)].

5.3 Principal subspaces and jet algebras from graphs

In this part we study principal subspaces and jet algebras coming from graphs. We begin from any graph G with k vertices and possibly with loops (and for simplicity we assume no double edges). We denote the vertices of G by $\{v_1, v_2, \ldots, v_k\}$. We denote by $\Gamma := \Gamma(G)$ the (symmetric) incidence matrix of G and by $(L(\Gamma), \langle\cdot,\cdot\rangle)$ the rank k

lattice with basis $\alpha_1, \ldots, \alpha_k$ such that $\langle \alpha_i, \alpha_j \rangle = (\Gamma)_{i,j}$. The incidence matrix of the graph induces a quadratic form

$$\Gamma \to \frac{1}{2} Q(x_1, \ldots, x_k), \quad Q(x_1, \ldots, x_k) = \sum_{\substack{i,j=1 \\ v_i v_j \in E(G)}}^{k} x_i x_j,$$

where we sum over all edges $E(G)$. Out of monomials appearing in the sum we form the jet algebra $J_\infty(R_\Gamma)$, where

$$R_\Gamma = \mathbb{C}[x_1, \ldots, x_k] \Big/ \Big\langle \bigcup_{v_i v_j \in E(G)} x_i x_j \Big\rangle.$$

We let $W_{L(\Gamma)} \subset V_{L(\Gamma)}$ be the principal subspace corresponding to $\{e^{\alpha_i}\}_{i=1}^{k}$ inside the lattice vertex algebra $V_{L(\Gamma)}$. For simplicity we write W_Γ for $W_{L(\Gamma)}$.

Example 5.9 Consider the graph $\circ - \circ - \circ$. Then $\Gamma = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, and $W_\Gamma = \langle e^{\alpha_1}, e^{\alpha_2}, e^{\alpha_3} \rangle$ where $L = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \oplus \mathbb{Z}\alpha_3$ with $\langle \alpha_1, \alpha_2 \rangle = \langle \alpha_2, \alpha_3 \rangle = 1$ (zero otherwise), $R_\Gamma = \mathbb{C}[x_1, x_2, x_3]/(x_1x_2, x_2x_3)$, and $Q(x_1, x_2, x_3) = x_1x_2 + x_2x_3$.

Theorem 5.10 *If the bilinear form associated with Γ is non-degenerate, that is Γ is invertible, then there exists a unique conformal vector in the lattice vertex algebra such that eigenvalue of $L_{(0)}$ defines a grading such that*

$$\mathrm{wt}(e^{\alpha_i}) = \frac{3}{2} \quad \textit{if} \ \langle \alpha_i, \alpha_i \rangle = 1,$$
$$\mathrm{wt}(e^{\alpha_i}) = 1 \quad \textit{if} \ \langle \alpha_i, \alpha_i \rangle = 0.$$

Moreover, the character is given by

$$\mathrm{ch}[W_\Gamma](q) = \sum_{n_1, \ldots, n_k \in \mathbb{N}} \frac{q^{n_1 + n_2 + \cdots + n_k + \frac{1}{2} Q(n_1, \ldots, n_k)}}{(q)_{n_1} \cdots (q)_{n_k}}.$$

Proof Clearly, we have the standard conformal vector in the lattice vertex algebra given by $\omega_{\mathrm{st}} = \frac{1}{2} \sum_{i=1}^{n} u^{(i)}_{(-1)} u^{(i)}_{(-1)} \mathbf{1}$, where $\{u^{(1)}, \ldots, u^{(n)}\}$ is an orthonormal basis with respect to the bilinear form associated with Γ. We know that

$$L_{\mathrm{st}\,(0)}(e^{\alpha_i}) = \frac{\langle \alpha_i, \alpha_i \rangle}{2}.$$

It is clear that by adding a linear combination of $\{(\alpha_i)_{(-2)} \mathbf{1}\}_{i=1}^{n}$, we will still get a conformal vector. Now assume that $\omega_{\mathrm{st}} + \sum_{i=1}^{n} a_i (\alpha_i)_{(-2)} \mathbf{1}$, where $a_i \in \mathbb{C}$, would give us expected weights. Then we have a system of linear equations. The non-degeneracy of the bilinear form implies that there is a unique solutions set. Thus we always have a conformal vector with the grading:

$$\begin{aligned}\mathrm{wt}(e^{\alpha_i}) &= \frac{3}{2} \quad \text{if} \ \ \langle \alpha_i, \alpha_i \rangle = 1,\\ \mathrm{wt}(e^{\alpha_i}) &= 1 \quad \text{if} \ \ \langle \alpha_i, \alpha_i \rangle = 0.\end{aligned}$$

Applying [44, Corollary 4.14], we can write a combinatorial basis of W_Γ. Now let us use this basis to write down the character. Firstly, the generating function of a colored partition into $(n_1, n_2, \ldots, n_k)$ parts is $\frac{1}{(q)_{n_1}\cdots(q)_{n_k}}$. It is clear that

$$\mathrm{ch}[W_\Gamma](q) \ = \sum_{k_1,\ldots,k_k\in\mathbb{N}} \frac{q^{\mathrm{wt}(f_{(n_1,\ldots,n_k)})}}{(q)_{n_1}\cdots(q)_{n_k}},$$

where $f_{(n_1,\ldots,n_k)}$ is the vector in W_Γ of charge $(n_1, \ldots, n_k)$ with the minimal weight. For the n_i-th part, there is a unique element u_{n_i} of the minimal weight which is

$$e^{\alpha_i}_{(-1-\sum_{j=1}^{i-1}\langle\alpha_i,\alpha_j\rangle n_j-(n_i-1)\langle\alpha_i,\alpha_i\rangle)} \cdots e^{\alpha_i}_{(-1-\sum_{j=1}^{i-1}\langle\alpha_i,\alpha_j\rangle n_j)}\mathbf{1}.$$

The weight of u_{n_i} is

$$\begin{aligned}&\frac{n_i}{2}\left(2\Big(\sum_{j=1}^{i-1}\langle\alpha_i,\alpha_j\rangle n_j + \mathrm{wt}\big((e^{\alpha_i})_{(-1)}\mathbf{1}\big)\Big) + (n_i-1)\langle\alpha_i,\alpha_i\rangle\right)\\ &\quad = \sum_{j=1}^{i-1}\langle\alpha_i,\alpha_j\rangle n_i n_j + \frac{n_i^2}{2}\langle\alpha_i,\alpha_i\rangle + \left(-\frac{\langle\alpha_i,\alpha_i\rangle}{2} + \mathrm{wt}\big((e^{\alpha_i})_{(-1)}\mathbf{1}\big)\right)n_i.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{wt}(f_{(n_1,\ldots,n_k)}) &= \sum_{i=1}^{k}\mathrm{wt}(u_{n_i})\\ &= \sum_{i=1}^{k}\sum_{j=1}^{i-1}\langle\alpha_i,\alpha_j\rangle n_i n_j + \frac{n_i^2}{2}\langle\alpha_i,\alpha_i\rangle + \left(-\frac{\langle\alpha_i,\alpha_i\rangle}{2} + \mathrm{wt}\big((e^{\alpha_i})_{(-1)}\mathbf{1}\big)\right)n_i\\ &= n_1 + n_2 + \cdots + n_k + \frac{1}{2}Q(n_1,\ldots,n_k).\end{aligned}$$

Thus we proved the claimed identity. □

Remark 5.11 If the lattice L is degenerate, then V_L has no conformal vector which can give us expected weights. But we can still view W_L as a graded vertex algebra, if we define the degree of e^{α_i} as above. Then the character formula is still valid for singular Γ.

Before we prove the next result, let us generalize [46, Theorem 4.3.1].

Proposition 5.12 *We have an isomorphism*

$$\mathrm{gr}(W_\Gamma) \cong \frac{\mathbb{C}[x_i(p) \,|\, i=1,\ldots,k,\ p\in\mathbb{Z}_-]}{\left\langle \sum_{m=0}^{-l-1} \frac{(m+\langle\alpha_i,\alpha_j\rangle-1)!}{m!}\langle\alpha_i,\alpha_j\rangle x_i(-\langle\alpha_i,\alpha_j\rangle-m)x_j(l+m) \,|\, 1\leqslant i,j\leqslant k, l\leqslant -1\right\rangle}.$$

Proof First, we define a map π from

$$\mathbb{C}[x_i(p)\,|\,i=1,\ldots,k,\ p\in\mathbb{Z}_-]$$

to $\mathrm{gr}(W_\Gamma)$ by sending $x_i(p)$ to $e^{\alpha_i}_{(p)}\mathbf{1}$. We denote the ideal

$$\left\langle \sum_{m=0}^{-l-1} \frac{(m+\langle\alpha_i,\alpha_j\rangle-1)!}{m!}\langle\alpha_i,\alpha_j\rangle x_i(-\langle\alpha_i,\alpha_j\rangle-m)x_j(l+m) \,|\, 1\leqslant i,j\leqslant k, l\in\mathbb{Z}_-\right\rangle$$

by I_Γ. We can use the same argument as in [46] to show that $I_\Gamma\subset\ker(\pi)$.

We prove that $\ker(\pi)\subset I_\Gamma$ by contradiction. Suppose there exists an element $a\in\mathbb{C}[x_i(p)\,|\,i=1,\ldots,k,\ p\in\mathbb{Z}_-]$ such that $a\in\ker(\pi)$ and $a\notin I_\Gamma$. Suppose a is homogeneous with respect to weight and charge. Choose r such that a contains some element $x_r(p)$ as a factor. We assume that a has the minimum weight among all elements that satisfy the above conditions. Again by the same argument as in [46], this a can be written as $bx_r(-1)$, where $b\in\mathbb{C}[x_i(p)\,|\,i=1,\ldots,k,\ p\in\mathbb{Z}_-]$. We shall prove the case when $\langle\alpha_r,\alpha_r\rangle=0$. For other cases, it is proved in [46]. Firstly we define a map $\mathbf{e}^{\alpha_\mathbf{r}}\colon W_\Gamma\to W_\Gamma$ as

$$\mathbf{e}^{\alpha_\mathbf{r}}\left((e^{\alpha_j})_{(m)}\mathbf{1}\right)=(e^{\alpha_j})_{(m)}(e^{\alpha_r})_{(-1)}\mathbf{1}.$$

Then we lift this map to

$$\mathbf{x_r}\colon\mathbb{C}[x_i(p)\,|\,i=1,\ldots,k,\ p\in\mathbb{Z}_-]\to\mathbb{C}[x_i(p)\,|\,i=1,\ldots,k,\ p\in\mathbb{Z}_-],$$

which is defined as

$$\mathbf{x_r}(x_i(j))=x_i(j)x_r(-1).$$

Since $a\in\ker(\pi)$, $\pi(a)=\pi(bx_r(-1))=0$. Then

$$\mathbf{e}^{-\alpha_\mathbf{r}}(\pi(bx_r(-1)))=\pi(b)=0,$$

which implies that $b\in\ker(\pi)$. If $b\in I_\Gamma$, then $a=\mathbf{x_r}(b)\in\mathbf{x_r}I_\Gamma\in I_\Gamma$ which contradicts our assumption. If $b\notin I_\Gamma$, then b is an element such that $b\in\ker(\pi)$ and $b\notin I_\Gamma$ but with the weight strictly less than the weight of a. This also contradicts our assumption. Thus we proved the claim. □

Theorem 5.13 *We have*

$$\mathrm{gr}(W_\Gamma) \cong J_\infty\big(\mathbb{C}[y_1, y_2, \ldots, y_k]/\langle\langle\alpha_i, \alpha_j\rangle y_i y_j \mid 1 \leqslant i, j \leqslant k\rangle\big).$$

Proof From the definition of a jet superalgebra, we know that

$$T^{(-l-1)}(\langle\alpha_i, \alpha_j\rangle y_i y_j) = \sum_{m=0}^{-l-1} c_m^l \langle\alpha_i, \alpha_j\rangle y_i(-\langle\alpha_i, \alpha_j\rangle - m) y_j(l+m),$$

where c_m^l is a constant coefficient. Therefore,

$$J_\infty\big(\mathbb{C}[y_1, y_2, \ldots, y_k]/\langle\langle\alpha_i, \alpha_j\rangle y_i y_j \mid 1 \leqslant i, j \leqslant k\rangle\big)$$

has quotient relation

$$\left\langle \sum_{m=0}^{-l-1} c_m^l \langle\alpha_i, \alpha_j\rangle y_i(-\langle\alpha_i, \alpha_j\rangle - m) y_j(l+m) \,\Big|\, 1 \leqslant i, j \leqslant k, l \in \mathbb{Z}_- \right\rangle.$$

Together with Proposition 5.12, we get an isomorphism of differential algebras induced from the map $\psi : x_i(-1) \to y_i(-1)$. □

When $\langle\alpha_i, \alpha_i\rangle = 1$, we increase the degree of $y_i(-1)$ by 1/2. Then clearly we have

$$\mathrm{HS}_q(J_\infty(R_\Gamma)) = \mathrm{ch}[W_\Gamma](q).$$

5.4 Positive lattices

Given a lattice L of rank n with a $\mathbb{Z}$-basis $\{\alpha_i\}_{i=1}^n$, we say that the basis is positive if we have $\langle\alpha_i, \alpha_j\rangle \in \mathbb{N}$ for $1 \leqslant i \leqslant j \leqslant n$. In this part, we study principal subspaces associated with positive bases. The examples we studied in the previous two sections are such principal subspaces. Now let us prove a more general result about the map ψ and these principal subspaces.

Theorem 5.14 *For a lattice L of rank n with a positive basis, the map ψ is an isomorphism for W_L if and only if its positive basis satisfies $\langle\alpha_i, \alpha_i\rangle = 0$ or 1 or 2, and $\langle\alpha_i, \alpha_j\rangle = 0$ or 1.*

Proof First let us assume that the positive basis of the lattice L satisfies given conditions. According to Theorem 5.13, we know that when $\langle\alpha_i, \alpha_i\rangle = 0$ or 1, and $\langle\alpha_i, \alpha_j\rangle = 0$ or 1, the map ψ is an isomorphism for the principal subspace. Now the only case we need to consider is the positive basis for which $\langle\alpha_i, \alpha_j\rangle = 2\delta_{i,j}$. It is not hard to see that $J_\infty(\mathbb{C}[x]/\langle x^2\rangle)$ has a basis

$$\big\{x_{(m_1)}x_{(m_2)} \cdots x_{(m_k)} \mid m_{j-1} \leqslant m_j - 2, k \in \mathbb{N}\big\}.$$

Thus $J_\infty(\mathbb{C}[x_1, x_2, \ldots, x_n]/\langle x_1^2, x_2^2, \ldots, x_n^2\rangle)$ has a basis

$$\left\{ \begin{array}{c|c} (x_{i_1})_{(m_1^1)}(x_{i_1})_{(m_2^1)}\cdots(x_{i_1})_{(m_{k_1}^1)} & m_{j-1}^i \leqslant m_j^i - 2 \\ \cdots(x_{i_n})_{(m_1^n)}(x_{i_n})_{(m_2^n)}\cdots(x_{i_n})_{(m_{k_n}^n)} & 1 \leqslant j \leqslant k_i - 1 \end{array} \right\}.$$

Note that the C_2-algebra of W_L is

$$\mathbb{C}[x_1, \ldots, x_n]/\langle x_1^2, \ldots, x_n^2\rangle.$$

Now, the map ψ sends $(x_i)_{(-1)}$ to $(e^{\alpha_i})_{(-1)}\mathbf{1}$. According to [44, Corollary 4.14], the image of the basis of $J_\infty(R_{W_L})$ is a basis of $\mathrm{gr}(W_L)$. Thus the map ψ is an isomorphism.

Next, let us prove that if a basis does not satisfy the given conditions, the map ψ is not an isomorphism. We will consider two cases:

- Suppose that for one simple root α_i, we have $\langle\alpha_i, \alpha_i\rangle \geqslant 3$. Without loss generality, we prove that ψ is not an isomorphism when lattice $L = \mathbb{Z}\alpha_i$. In this case, from [44, Corollary 4.14], the basis of $\mathrm{gr}^F(W_L)$ is

$$\left\{(e^{\alpha_i})_{(m_1)}(e^{\alpha_i})_{(m_2)}\cdots(e^{\alpha_i})_{(m_k)}\mathbf{1} \mid m_{j-1} \leqslant m_j - \langle\alpha_i, \alpha_i\rangle,\ m_k \in \mathbb{Z}_-,\ k \in \mathbb{N}\right\}. \tag{8}$$

  It is clear that neither $J_\infty(\mathbb{C}[x]/(x^2))$ nor $J_\infty(\wedge[x])$ has the same corresponding basis (here $\wedge$ denotes the exterior algebra). Indeed, for the jet algebra $J_\infty(\mathbb{C}[x]/(x^2))$ we have two monomials of degree 4 with two variables, i.e., $x_{(-1)}x_{(-3)}$ and $x_{(-2)}x_{(-2)}$. But we only have one quotient relation of degree 4 involving these two monomials, i.e., $x_{(-2)}x_{(-2)} + x_{(-1)}x_{(-3)} = 0$. Therefore, either $x_{(-1)}x_{(-4)}$ or $x_{(-2)}x_{(-2)}$ should be a basis element. But we do not have such corresponding element in (8). A similar argument works for $J_\infty(\wedge[x])$.
- Suppose that there exist two distinct roots α_i, α_j, $i < j$, such that $\langle\alpha_i, \alpha_j\rangle \geqslant 2$. Without loss of generality, we assume $L = \mathbb{Z}\alpha_i \oplus \mathbb{Z}\alpha_j$, then the basis of $J_\infty(W_L)$ is

$$\left\{ \begin{array}{c|c} (x_i)_{(-1-m_1)}(x_i)_{(-1-m_2)}\cdots(x_i)_{(-1-m_k)}(x_j)_{(-1-n_1)} & m_1 - m_2 \geqslant \langle\alpha_i, \alpha_i\rangle, \\ (x_j)_{(-1-n_2)}\cdots(x_j)_{(-1-n_l)} & n_1 - n_2 \geqslant \langle\alpha_j, \alpha_j\rangle \\ & m_k \geqslant l, n_l \in \mathbb{N} \end{array} \right\}.$$

  Meanwhile, according to [44, Corollary 4.14], the image of this basis under ψ strictly contains the basis of W_L. We do not have an isomorphism.

Thus we proved the statement. □

5.5 New character formulas for ch$[W_\Gamma]$

If the graph Γ is a Dynkin diagram of type A_k or C_k (cycle of length k) we expect that the generating series $\mathrm{HS}_q(J_\infty(R_\Gamma))$ has much better combinatorial behavior and perhaps even mock modular properties. We now present “sum of tails" formulas for $\mathrm{HS}_q(J_\infty(R_{A_k}))$ for several low “rank" cases. To simplify notation we let

$$A_k(q) := \mathrm{HS}_q(J_\infty(R_{A_k})).$$

From Theorem 5.10 we have a fermionic formula

$$A_k(q) = \sum_{n_1,n_2,\ldots,n_k\in\mathbb{N}} \frac{q^{n_1+n_2+\cdots+n_k+n_1n_2+n_2n_3+\cdots+n_{k-1}n_k}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_k}}.$$

The next formulas have been established recently by Jennings-Shaffer and Milas [34].

Theorem 5.15 *We have*

- $A_2(q) = \frac{1}{(1-q)(q)_\infty}$,
- $A_3(q) = q^{-1}\left(\frac{1}{(q)_\infty^2} - \frac{1}{(q)_\infty}\right)$,
- $A_4(q) = \frac{q^{-1}}{(q)_\infty^2}\sum_{n\geqslant 1}\frac{q^n}{1-q^n}$,
- $A_5(q) = \frac{1}{(q)_\infty^2}\sum_{n\in\mathbb{N}}\frac{q^n}{(q)_n(1-q^{n+1})^2}$,
- $A_6(q) = \frac{1}{(q)_\infty^2}\sum_{n,m\in\mathbb{N}}\frac{q^{n+m+nm}}{(q)_{n+1}(q)_{m+1}}$.

Moreover, for cyclic C_k-graphs we have fermionic formulas for $C_k(q) := \mathrm{HS}_q(J_\infty(R_{C_k}))$ valid for $k \geqslant 3$,

$$C_k(q) = \sum_{n_1,n_2,\ldots,n_k\in\mathbb{N}} \frac{q^{n_1+n_2+\cdots+n_k+n_1n_2+n_2n_3+\cdots+n_{k-1}n_k+n_kn_1}}{(q)_{n_1}(q)_{n_2}\cdots(q)_{n_k}}.$$

Again we have partial results for “bosonic" representations for 3- and 5-cycle graphs [34].

Proposition 5.16 ([34, Proposition 6.1 and Section 7]) *We have*

$$C_3(q) = \frac{1}{(q)_\infty}\sum_{n\in\mathbb{N}}\frac{q^n}{(q^{n+1})_{n+1}}, \quad C_5(q) = \frac{q^{-1}}{(q)_\infty^2}\sum_{n\in\mathbb{Z}_+}\frac{nq^n}{1-q^n}.$$

5.6 Combinatorial interpretation

Next we present combinatorial interpretations of formulas in Theorem 5.15 and Proposition 5.16. For simplicity, in several formulas we factored out a (power of) Euler factor which can be easily interpreted as the number of (colored) partitions.

Theorem 5.17 *We have:*

- *$A_2(q)$ counts the number of partitions of $2n$ with all parts either even or equal to* 1.

- *$qA_3(q)$ counts the number of partitions of $n+1$ into two kinds of parts with the first kind of parts used in each partition.*
- *$q(q)_\infty A_4(q)$ counts the total number of parts in all partitions of n, which is also the sum of largest parts of all partitions of n.*
- *$(q)_\infty^2 A_5(q)$ is the sum of the numbers of times that the largest part appears in each partition of n.*
- *$q(q)_\infty^2 A_6(q)$ counts twice the total number of parts in all partitions of n minus the number of partitions of n.*
- *$(q)_\infty C_3(q)$ counts the number of partitions of n such that twice the least part is bigger than the greatest part.*
- *$q(q)_\infty C_5(q)$ counts the sum of all parts of all partitions of n, also known as $np(n)$.*

Proof For $A_2(q)$, observe that $\mathrm{Coeff}_{q^n} A_2(q) = p(1)+p(2)+\cdots+p(n)$, where $p(i)$ is the number of partitions of i. The number of 1's must be even, say $2k$, so we have to compute the number of partitions of $2n-2k$ where all parts are even. This is given by $p(n-k)$. Then summing over k gives the claim.

The interpretation for the $A_3(q)$ series is clear because we can also write

$$qA_3(q) = \frac{1}{(q)_\infty}\left(\frac{1}{(q)_\infty} - 1\right).$$

Extracting the coefficient on the right-hand side gives $p_2(n) - p(n)$, where $p_2(i)$ denotes the number of two colored partitions.

For $A_4(q)$, this can be seen from the identity

$$\frac{\sum_{n\in\mathbb{Z}_+} \frac{q^n}{1-q^n}}{(q)_\infty} = \sum_{n\in\mathbb{Z}_+} \frac{nq^n}{(q)_n},$$

which follows by taking the $\left(x\frac{d}{d}\right)$ derivative of $\frac{1}{(xq;q)_\infty} = \sum_{n\in\mathbb{N}} \frac{x^n q^n}{(q)_n}$. This clearly counts the total number of parts in all partitions of n.

The $(q)_\infty^2 A_5(q)$ case has already been discussed in [34].

For $(q)_\infty^2 A_6(q)$, this follows from another identity given in [34]:

$$\frac{1}{(q)_\infty^2}\sum_{n,m\in\mathbb{N}} \frac{q^{n+m+nm}}{(q)_{n+1}(q)_{m+1}} = \frac{q^{-1}}{(q)_\infty^2}\left(2\sum_{n\in\mathbb{Z}_+} \frac{q^n}{(1-q^n)(q)_\infty} + 1 - \frac{1}{(q)_\infty}\right),$$

together with a previous observation that $\frac{\sum_{n\in\mathbb{Z}_+} \frac{q^n}{1-q^n}}{(q)_\infty}$ counts the total number of parts in all partitions of n.

For $(q)_\infty C_3(q)$ we use a well-known interpretation for the fifth order mock theta function, and finally for $(q)_\infty C_5(q)$ we observe the formula

$$\left(q\frac{d}{dq}\right)\frac{1}{(q)_\infty}=\frac{1}{(q)_\infty}\sum_{n\in\mathbb{Z}_+}\frac{nq^n}{1-q^n}=\sum_{n\in\mathbb{Z}_+}np(n)q^n$$

as claimed. □

Remark 5.18 It is interesting to observe that the numerators of $C_3(q)$ and $C_5(q)$ are mock modular forms, and thus $C_3(q)$ and $C_5(q)$ are *mixed* mock. Completion of the Ramanujan fifth order mock theta function $\sum_{n\in\mathbb{N}}\frac{q^n}{(q^{n+1})_{n+1}}$ is well-documented [19]. For $\sum_{n\in\mathbb{Z}_+}\frac{nq^n}{1-q^n}$ we only have to observe that adding $-1/24$ to the numerator gives $E_2(\tau)$, the weight 2 quasimodular Eisenstein series, which is known to be mock.

6 $N=1$ superconformal vertex algebras

In this section we consider rational $N=1$ vertex superalgebras $L^{N=1}_{c_{2,4k}}$, $k\in\mathbb{Z}_+$, associated to $N=1$ superconformal $(2,4k)$-minimal models [1]. Here the central charge is $c_{2,4k}=\frac{3}{2}\big(1-\frac{2(4k-1)^2}{8k}\big)$.

According to [41,43], the normalized character of $L^{N=1}_{c_{2,4k}}$ (without the $q^{-c/24}$ factor) is

$$\begin{aligned}\mathrm{ch}\big[L^{N=1}_{c_{2,4k}}\big](q) &= \prod_{\substack{n=1\\ n\not\equiv 2 (\mathrm{mod}\ 2)\\ n\not\equiv 0,\ \pm1 (\mathrm{mod}\ 4k)}}^{\infty}\frac{1}{(1-q^{n/2})}\\ &= \sum_{m_1,\ldots,m_{k-1}\in\mathbb{N}}\frac{(-q^{1/2})_{N_1}q^{N_1^2/2+N_2^2+\cdots+N_{k+1}^2+N_{(s+1)/2}+N_{(s+3)/2}+\cdots+N_{k-1}}}{(q)_{m_1}(q)_{m_2}\cdots(q)_{m_{k-1}}}.\end{aligned}$$

The fermionic character formula is the generating function (cf. [41])

$$\mathrm{ch}\big[L^{N=1}_{c_{2,4k}}\big](q)=\sum_{n=0}^{\infty}D_{k,1}(n)q^{n/2}$$

of the number of partitions of $D_{k,1}(n)$ of $n/2$ in the form $n/2=b_1+\cdots+b_m$, $b_j\in\frac{1}{2}\mathbb{Z}_+$, where $b_1,\ldots,b_m$ satisfy the following conditions:

- no half-odd integer is repeated,
- $b_j\geqslant b_{j+1}$, $b_m\in\frac{3}{2}\mathbb{Z}_+$,
- $b_j-b_{j+k-1}\geqslant 1$ if $b_j\in\frac{1}{2}+\mathbb{Z}$,
- $b_j-b_{j+k-1}>1$ if $b_j\in\mathbb{Z}$.

Since the $N=1$ vertex superalgebra $L^{N=1}_{c_{2,4}}$ is isomorphic to $\mathbb{C}$, we only need to consider $L^{N=1}_{c_{2,4k}}$, where $k>1$. First let us find the C_2-algebra of $L^{N=1}_{c_{2,4k}}$. According to [43, Section 4], the null vector in the universal algebra which survives inside the C_2-algebra is $L^{k-1}_{(-2)}G_{(-3/2)}\mathbf{1}$. Moreover, if we let $G_{(-1/2)}$ act on the null vector, we get another null vector which survives in the C_2-algebra, i.e., $L^{k}_{(-2)}\mathbf{1}$. These two null vectors in the vacuum algebra generate the whole quotient ideal of $R_{L^{N=1}_{c_{2,4k}}}$. Thus $R_{L^{N=1}_{c_{2,4k}}}$ is isomorphic to the superalgebra $\mathbb{C}[l,g]/\langle l^k, l^{k-1}g\rangle$, where g is an odd element.

We are going to prove that ψ is an isomorphism. We identify l, g with $l(-2)$, $g(-3/2)$, respectively, inside the jet superalgebra.

It is clear that $J_\infty(\mathbb{C}[l,g]/\langle l^k, l^{k-1}g\rangle)$ is isomorphic to

$$\mathbb{C}\left[l(-2-i), g\left(-\frac{3}{2}-j\right) \,\Big|\, i,j\in\mathbb{N}\right]\Big/\langle l(z)^k, l(z)^{k-1}g(z)\rangle,$$

$l(z) = \sum_{n\in\mathbb{N}} l(-2-n)z^n$, $g(z) = \sum_{n\in\mathbb{N}} g(-3/2-n)z^n$ and $\langle l(z)^k, l(z)^{k-1}g(z)\rangle$ is the ideal generated by the Fourier coefficients of $l(z)^k, l(z)^{k-1}g(z)$. We define an ordered monomial in $J_\infty(\mathbb{C}[l,g]/\langle l^k, l^{k-1}g\rangle)$ to be a monomial of the form

$$l(-2-n)^{a_1} g\left(-\frac{3}{2}-n\right)^{b_1} l(-1-n)^{a_2} g\left(-\frac{3}{2}-n+1\right)^{b_2} \cdots l(-2)^{a_{n+1}} g\left(-\frac{3}{2}\right)^{b_{n+1}},$$

where $n\in\mathbb{N}$. Then we have a complete lexicographic ordering on all ordered monomials according to Sect. 3.1.

We know that all ordered monomials constitute a spanning set of the jet superalgebra. Following an argument similar to the one in Sect. 4.1, we can make use of the quotient relation to impose some conditions on the spanning set to get a smaller spanning set. Firstly, since all variables $g(k)$'s are odd, no two $g(k)$ can appear in the ordered monomial. The leading term of any coefficient of z^{nk} in $l(z)^k$ is $l(-2-n)^k$. Thus $l(-2-n)^k$ should not appear as a segment of any element in the spanning set. Similarly we can list further leading terms in the quotient:

- The leading term of the coefficient of z^{nk} in $l(z)^{k-1}g(z)$ is
$$l(-2-n)^{k-1} g\left(-\frac{3}{2}-n\right).$$
- The leading term of the coefficient of $z^{n(k-1-i)+(n-1)i+n}$ in $l(z)^{k-1}g(z)$ is
$$l(-2-n)^{k-1-i} g\left(-\frac{3}{2}-n\right) l(-2-n+1)^i, \quad i=1,\dots,k-1.$$

Now we obtain a smaller spanning set, where the above three type leading terms cannot appear inside any ordered monomial. More precisely, any element in this spanning set is of the form

$$w(b_1)w(b_2)\cdots w(b_m),$$

where $b_i \geqslant b_{i+1}$, $w(a) = l(a)$ if $a\in\mathbb{Z}$ and $w(a) = g(a)$ if $a\in\frac{1}{2}\mathbb{Z}$. The fact that $g(a)$ is odd implies that no half-odd-integer is repeated in $\{b_1, b_2, \dots, b_m\}$. Moreover, we have a condition

$$b_j - b_{j-k+1} > 1 \quad \text{if} \quad b_j \in \mathbb{Z},$$

because

$$l(-2-n)^k, \quad l(-2-n)^{k-1}g\left(-\frac{3}{2}-n\right),$$
$$l(-2-n)^{k-1-i}g\left(-\frac{3}{2}-n\right)l(-2-n+1)^i, \quad i=1,\ldots,k-1,$$

are leading terms of some elements in the quotient ideal. We also have a condition

$$b_j - b_{j-k+2} \geqslant 1 \quad \text{if} \quad b_j \in \tfrac{1}{2}\mathbb{Z},$$

because

$$g\left(-\frac{3}{2}-n\right)l(-2-n+1)^{k-1}$$

is the leading term of some element in the quotient ideal. So we have

$$\mathrm{HS}_q\big(J_\infty(\mathbb{C}[l,g]/\langle l^k, l^{k-1}g\rangle)\big) \leqslant \sum_{n=0}^{\infty} D_{k,1}(n)\, q^{n/2} = \mathrm{ch}\big[\mathrm{gr}\big(L^{N=1}_{c_{2,4k}}\big)\big](q).$$

Meanwhile, the surjectivity of ψ implies that

$$\mathrm{HS}_q\big(J_\infty(\mathbb{C}[l,g]/\langle l^k, l^{k-1}g\rangle)\big) \geqslant \mathrm{ch}\big[\mathrm{gr}\big(L^{N=1}_{c_{2,4k}}\big)\big](q).$$

Thus $\mathrm{HS}_q(J_\infty(\mathbb{C}[l,g]/\langle l^k, l^{k-1}g\rangle)) = \mathrm{ch}[\mathrm{gr}(L^{N=1}_{c_{2,4k}})](q)$, and ψ is an isomorphism. It implies that the above spanning set is a basis of the jet superalgebra. The image of the basis of jet superalgebra under the map ψ is a basis of $\mathrm{gr}(L^{N=1}_{c_{2,4k}})$ [41].

We have following result which is a super-analog of [52, Theorem 16.13]:

Theorem 6.1 *Let $p' > p \geqslant 2$ satisfy that $(p'-p)/2$ and p are coprime positive integers. Let $L^{N=1}_{c_{p,p'}}$ be the simple $N=1$ vertex superalgebra associated with the $N=1$ superconformal (p,p')-minimal model of central charge $c_{p,p'} = \frac{3}{2}\big(1-\frac{2(p'-p)^2}{pp'}\big)$. Then the map ψ is an isomorphism if and only if $(p,p') = (2,4k)$, $k \in \mathbb{Z}_+$.*

Proof We first consider the C_2-algebra of $L^{N=1}_{c_{p,p'}}$. We let

$$|c_{p,p'}| = \frac{(p-1)(p'-1)}{4} + \frac{1+(-1)^{pp'}}{8} \in \mathbb{N}.$$

When p and p' are both even, according to [43, Section 4], there are two null vectors which survive in $R_{V^{N=1}_{c_{p,p'}}}$, i.e., $L_{(-2)}^{|c_{p,p'}|}\mathbf{1}$ and $L_{(-2)}^{|c_{p,p'}|-1}G_{-3/2}\mathbf{1}$. They generate the quotient ideal of $R_{V^{N=1}_{c_{p,p'}}}$ in the vacuum algebra. In this case, the C_2-algebra $R_{L^{N=1}_{c_{p,p'}}}$ is isomorphic to

$$\mathbb{C}[l,g]/\langle l^{|c_{p,p'}|}, l^{|c_{p,p'}|-1}g\rangle.$$

When p and p' are both odd, again from [43, Section 4], the null vector $L_{(-2)}^{|c_{p,p'}|}\mathbf{1}$ generates the quotient ideal of $R_{L^{N=1}_{c_{p,p'}}}$. The C_2-algebra is isomorphic to

$$\mathbb{C}[l,g]/\langle l^{|c_{p,p'}|}\rangle.$$

Suppose p and p' are both odd. Then

$$\mathrm{HS}_q\big(J_\infty(\mathbb{C}[l,g]/\langle l^{|c_{p,p'}|}\rangle)\big) = \mathrm{HS}_q(J_\infty(\mathbb{C}[g]))\,\mathrm{HS}_q(J_\infty(\mathbb{C}[l]/\langle l^{|c_{p,p'}|}\rangle)).$$

It is clear that $\mathrm{HS}_q(J_\infty(\mathbb{C}[g])) = \prod_{i\in\mathbb{Z}_+}(1+q^{i+1/2})$. According to [29], we get

$$\mathrm{HS}_q\big(J_\infty(\mathbb{C}[l]/\langle l^{|c_{p,p'}|}\rangle)\big) \cong \mathrm{ch}\big[L_{\mathrm{Vir}}\big(c_{2,(p-1)(p'-1)/2+1},0\big)\big](q),$$

where $L_{\mathrm{Vir}}(c_{2,(p-1)(p'-1)/2+1},0)$ is the simple Virasoro vertex algebra coming from the $(2,(p-1)(p'-1)/2+1)$-minimal model. Using the character formula of $L_{\mathrm{Vir}}(c_{q,q'},0)$ from [29], the Hilbert series of $J_\infty(\mathbb{C}[l,g]/\langle l^{|c_{p,p'}|}\rangle)$ is

$$\begin{aligned}&\frac{\prod_{i\in\mathbb{Z}_+}(1+q^{i+1/2})}{\prod_{i\in\mathbb{Z}_+}(1-q^i)}\\ &\quad\cdot\sum_{j\in\mathbb{Z}}\Big(q^{j(j(p-1)(p'-1)+2j+\frac{(p-1)(p'-1)}{2}-1)} - q^{(2j+1)((\frac{(p-1)(p'-1)}{2}+1)j+1)}\Big).\end{aligned}\tag{9}$$

Meanwhile, by [41] the character of $L^{N=1}_{c_{p,p'}}$ is

$$\mathrm{ch}\big[L^{N=1}_{c_{p,p'}}\big](q) = \frac{\prod_{i\in\mathbb{Z}_+}(1+q^{i-1/2})}{\prod_{i\in\mathbb{Z}_+}(1-q^i)}\sum_{j\in\mathbb{Z}}\Big(q^{\frac{j(jpp'+p'-p)}{2}} - q^{\frac{(jp+1)(jp'+1)}{2}}\Big).\tag{10}$$

Comparing (9) and (10) we get that ψ is not an isomorphism in this case.

Let p and p' be both even. Suppose $(p,p')\notin\{(2,4k)\,|\,k\in\mathbb{Z}_+\}$ and ψ is an isomorphism for $L^{N=1}_{c_{p,p'}}$. Then

$$\mathrm{HS}_q\big(J_\infty\big(\mathbb{C}[l,g]/\langle l^{|c_{p,p'}|}, l^{|c_{p,p'}|-1}g\rangle\big)\big) = \mathrm{ch}\big[L^{N=1}_{c_{p,p'}}\big](q).$$

On the other hand, we have shown that

$$\mathrm{HS}_q(J_\infty\big(\mathbb{C}[l,g]/\langle l^k, l^{k-1}g\rangle)\big) = \mathrm{ch}\big[L^{N=1}_{c_{2,4k}}\big](q),\quad k\in\mathbb{Z}_+.$$

Therefore, the character of $L^{N=1}_{c_{p,p'}}$ must coincide with the character of $L^{N=1}_{c_{2,4k}}$ for some k. Note, (10) is also true when p and p' are both even, and it is easy to verify from the numerator that no two $N=1$ minimal vertex algebras have the same character. This is a contradiction. Thus the statement is proved. □

7 Extended Virasoro vertex algebras

For a simple Virasoro vertex algebra $L_{\mathrm{Vir}}(c_{2,2k+1},0)$ coming from the $(2,2k+1)$-minimal model, according to [29], we know that $R_{L_{\mathrm{Vir}}(c_{2,2k+1},0)} \cong \mathbb{C}[x]/(x^k)$, and ψ is an isomorphism. Let p and p' be two positive coprime integers satisfying $p > p' \geqslant 2$. It is easy to see that ψ is an isomorphism if and only if $(p,p') = (2,2k+1)$ (see [52, Theorem 16.13]). Recently, the authors displayed the kernel of ψ [4, Theorem 1] for the $c = 1/2$ Ising model vertex algebra $L_{\mathrm{Vir}}(c_{3,4},0)$, based on a new fermionic character formula for $L_{\mathrm{Vir}}(c_{3,4},0)$.

If we consider extended Virasoro vertex algebras associated with minimal model which is not necessarily a $(2,2k+1)$-minimal model, we might still have that ψ is an isomorphism. Our discussion is heavily motivated by [33], where the combinatorics of (super)extensions of $(3,p)$-minimal vertex algebras was discussed.

Example 7.1 For the free fermion model $\mathcal{F} = L_{\mathrm{Vir}}(c_{(3,4)},0) \oplus L_{\mathrm{Vir}}(c_{(3,4)},1/2)$, ψ is clearly an isomorphism as discussed in Proposition 4.2.

Example 7.2 The $L^{N=1}_{c_{2,8}}$ minimal vertex superalgebra has the following realization:

$$L^{N=1}_{c_{(2,8)}} \cong L_{\mathrm{Vir}}(c_{(3,8)},0) \oplus L_{\mathrm{Vir}}\big(c_{(3,8)},\tfrac{3}{2}\big).$$

This realization is called the extended algebra, and it was studied in [33]. The map ψ is not an isomorphism in the case of $L_{\mathrm{Vir}}(c_{(3,8)},0)$. But we have shown that for the extended algebra of $L_{\mathrm{Vir}}(c_{(3,8)},0)$, the map ψ is an isomorphism. This model was analyzed from a different perspective in [40].

Example 7.3 Next, let us consider $V = L_{\mathrm{Vir}}(c_{(3,10)},0) \oplus L_{\mathrm{Vir}}(c_{(3,10)},2)$. It is well known that

$$L_{\mathrm{Vir}}(c_{(2,5)},0) \otimes L_{\mathrm{Vir}}(c_{(2,5)},0) \cong L_{\mathrm{Vir}}(c_{(3,10)},0) \oplus L_{\mathrm{Vir}}(c_{(3,10)},2).$$

We let ω_1 and ω_2 be conformal vectors of the first factor and the second factor of $L_{\mathrm{Vir}}(c_{(2,5)},0) \otimes L_{\mathrm{Vir}}(c_{(2,5)},0)$. Then the isomorphism map f sends $\omega_1 + \omega_2$ to the conformal vector ω of $L_{\mathrm{Vir}}(c_{(3,10)},0)$, and $\omega_1 - \omega_2$ to the lowest weight vector ϕ of $L_{\mathrm{Vir}}(c_{(3,10)},2)$. Since we know that

$$J_\infty(R_{L_{\mathrm{Vir}}(c_{(2,5)},0)}) \cong J_\infty(\mathbb{C}[x]/\langle x^2\rangle) \cong \mathrm{gr}(L_{\mathrm{Vir}}(c_{2,5},0)),$$

the map ψ is an isomorphism for V, i.e.,

$$J_\infty(R_V) = J_\infty\big(R_{L_{\mathrm{Vir}}(c_{(2,5)},0)} \otimes R_{L_{\mathrm{Vir}}(c_{(2,5)},0)}\big) \cong J_\infty(\mathbb{C}[x,y]/\langle x^2,y^2\rangle) \cong \mathrm{gr}(V).$$

For $L_{\mathrm{Vir}}(c_{(3,10)}, 0) \oplus L_{\mathrm{Vir}}(c_{(3,10)}, 2)$, its C_2-algebra is isomorphic to

$$\mathbb{C}[u, v]/\langle uv, u^2 + v^2, u^3, v^3\rangle$$

after we identify $x + y$, $x - y$ in $\mathbb{C}[x, y]/\langle x^2, y^2\rangle$ with u and v, respectively.

Remark 7.4 We also know from [33] that the normalized parafermionic character of $V = L_{\mathrm{Vir}}(c_{(3,10)}, 0) \oplus L_{\mathrm{Vir}}(c_{(3,10)}, 2)$ is given by

$$\mathrm{ch}[V](q) = \sum_{n_1,n_2,m_1\in\mathbb{N}} \frac{q^{(n_1+n_2+m_1)(n_1+n_2)+n_2(n_2+m_1)+m_1^2+m_1+n_1+2n_2}}{(q)_{n_1}(q)_{n_2}(q)_{m_1}}.$$

Next, let us consider the jet algebra

$$J_\infty(\mathbb{C}[u, v]/\langle u^2, v^3, uv\rangle),$$

where degrees of u and v are both 2. Clearly, it has the following spanning set:

$$u_{(-n_1)} \cdots u_{(-n_N)} v_{(-m_1)} \cdots v_{(-m_M)}$$

subject to constraints:

(a) (*difference two condition at distance* 1) $n_i \geqslant n_{i+1} + 2$,
(b) (*difference two condition at distance* 2) $m_i \geqslant m_{i+2} + 2$,
(c) (*boundary condition*) $n_N \geqslant 2 + M$,

where conditions (a), (b), (c) come from $(u^2)_\partial$, $(v^3)_\partial$, $(uv)_\partial$ in the quotient ideal of the jet algebra. Meanwhile, according to Proposition 5.1 and Theorem 5.13, we know that

$$\begin{aligned} J_\infty(\mathbb{C}[u]/\langle u^2\rangle) &\cong \mathrm{gr}(W_{\Lambda_{1,0}}), \\ J_\infty(\mathbb{C}[v]/\langle v^3\rangle) &\cong \mathrm{gr}(W_{\Lambda_{2,0}}), \\ J_\infty(\mathbb{C}[u, v]/\langle uv\rangle) &\cong \mathrm{gr}(W_\Gamma), \end{aligned}$$

where Γ is the graph $\circ - \circ$. Using three realizations of jet algebras and the Gordon–Andrews character formulas from [23,31], it is not hard to see that the above spanning set, subject to constraints (a)–(c), would produce a basis of the jet algebra $J_\infty(\mathbb{C}[u, v]/\langle u^2, v^3, uv\rangle)$ whose Hilbert series is given by

$$\sum_{n_1,n_2,m_1\in\mathbb{N}} \frac{q^{(n_1+n_2+m_1)(n_1+n_2)+n_2(n_2+m_1)+m_1^2+m_1+n_1+2n_2}}{(q)_{n_1}(q)_{n_2}(q)_{m_1}}.$$

The normalized character formula for $V = L_{\mathrm{Vir}}(c_{(2,5)}, 0) \otimes L_{\mathrm{Vir}}(c_{(2,5)}, 0)$ is

$$\mathrm{ch}[V](q) = \sum_{n_1,n_2\in\mathbb{N}} \frac{q^{n_1^2+n_2^2+n_1+n_2}}{(q)_{n_1}(q)_{n_2}}.$$

Thus we have the following Hilbert series identities:

$$\begin{aligned}\mathrm{HS}_q(J_\infty(\mathbb{C}[x,y]/(x^2,y^2))) &= \mathrm{HS}_q\left(J_\infty(\mathbb{C}[u,v]/\langle uv, u^2+v^2, u^3, v^3\rangle)\right)\\ &= \mathrm{HS}_q\left(J_\infty(\mathbb{C}[u,v]/\langle u^2, v^3, uv\rangle)\right)\end{aligned}$$

and

$$\sum_{n_1,n_2,m_1\in\mathbb{N}} \frac{q^{(n_1+n_2+m_1)(n_1+n_2)+n_2(n_2+m_1)+m_1^2+m_1+n_1+2n_2}}{(q)_{n_1}(q)_{n_2}(q)_{m_1}} = \sum_{n_1,n_2\in\mathbb{N}} \frac{q^{n_1^2+n_2^2+n_1+n_2}}{(q)_{n_1}(q)_{n_2}}.$$

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