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RESEARCH ARTICLE

Revisiting linear Weingarten hypersurfaces immersed into a locally symmetric Riemannian manifold

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Abstract
We deal with complete linear Weingarten hypersurfaces immersed into locally symmetric Riemannian manifolds whose sectional curvature obeys certain standard constraints. Under an assumption that such a hypersurface satisfies a suitable Okumura type inequality, we apply a version of the Omori–Yau maximum principle to prove that it must be either totally umbilical or isometric to an isoparametric hypersurface having two distinct principal curvatures. When the ambient space is Einstein, we also use a technique recently developed by Alías and Meléndez (Mediterr J Math 17(2): 61, 2020 [6]) to establish a sharp integral inequality for compact linear Weingarten hypersurfaces.

Keywords Locally symmetric Riemannian manifolds · Riemannian space forms · Linear Weingarten hypersurfaces · Okumura type inequality

Mathematics Subject Classification 53C24 · 53C40 · 53C42

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1 Introduction

The study of geometric properties of hypersurfaces immersed into Riemannian manifolds, under certain curvature constraints, constitutes an important topic of Differential Geometry. In this field, Brasil Jr., Colares and Palmas [9] used the Omori–Yau maximum principle to characterize complete hypersurfaces with constant normalized scalar curvature in the unit Euclidean sphere $\mathbb{S}^{n+1}$. In [3], Alías and García-Martínez used a weak Omori–Yau maximum principle due to Pigola, Rigoli and Setti [20] to study the behavior of the normalized scalar curvature R of a complete hypersurface with constant mean curvature immersed into a space form $\mathbb{Q}_c^{n+1}$ of constant sectional curvature c, deriving a sharp estimate for the infimum of R. Afterwards, the same authors jointly with Rigoli [4] obtained another suitable weak maximum principle for complete hypersurfaces with constant scalar curvature in $\mathbb{Q}_c^{n+1}$, and gave some applications of it in order to estimate the norm of the traceless part of its second fundamental form and, in particular, they improved the result of [9].

In [16], Li, Suh and Wei obtained characterization results concerning compact (without boundary) linear Weingarten hypersurfaces immersed into $\mathbb{S}^{n+1}$ (that is, compact hypersurfaces of $\mathbb{S}^{n+1}$ whose mean and normalized scalar curvatures are linearly related). Later on, the second author jointly with Aquino and Velásquez [7,8] established another characterization results related to complete linear Weingarten hypersurfaces immersed into $\mathbb{Q}_c^{n+1}$, under appropriate constraints on the norm of the traceless part of the second fundamental form. Next, the second author jointly with Alías, Meléndez and dos Santos [1] extended these results to the context of complete linear Weingarten hypersurfaces immersed into locally symmetric Riemannian manifolds obeying certain standard curvature conditions (in particular, in Riemannian spaces with constant sectional curvature). We recall that a Riemannian manifold is said to be *locally symmetric* if all covariant derivative components of its curvature tensor vanish identically. Under appropriate constrains on the scalar curvature function, they proved that such a hypersurface must be either totally umbilical or isometric to an isoparametric hypersurface with two distinct principal curvatures, one of them being simple.

More recently, the first and second authors [14] obtained a sharp estimate on the norm of the traceless second fundamental form of complete hypersurfaces with constant scalar curvature immersed into a locally symmetric Riemannian manifold obeying the same curvature constraints as assumed in [1]. When the equality holds, they proved that these hypersurfaces must be isoparametric with two distinct principal curvatures. Their approach involved a suitable Okumura type inequality which was introduced by Meléndez in [17], corresponding to a weaker hypothesis when compared with the assumption that these hypersurfaces have a priori at most two distinct principal curvatures. We point out that the same authors [12] had already used this Okumura type inequality to prove a sharp estimate on the scalar curvature of stochastically complete hypersurfaces with constant mean curvature immersed into a locally symmetric Riemannian manifold (see also [15] for other characterization results concerning complete hypersurfaces immersed into such an ambient space and having two distinct principal curvatures). Meanwhile, Alías and Meléndez [6] studied the rigidity of closed hypersurfaces with constant scalar curvature isometrically immersed into the unit Euclidean sphere $\mathbb{S}^{n+1}$. In particular, they established a sharp integral inequality for the behavior of the norm of the traceless second fundamental form, with the equality characterizing the totally umbilical hypersurfaces and certain Clifford tori.

Motivated by these works, here we deal with complete linear Weingarten hypersurfaces immersed into a locally symmetric Riemannian manifold, which is supposed to obey standard curvature constraints already considered in [1,12,14,15]. Assuming that such a hypersurface satisfies the Okumura type inequality introduced in [17], we apply a version of the Omori–Yau maximum principle to prove that it must be either totally umbilical or isometric to an isoparametric hypersurface having two distinct principal curvatures (see Theorem 4.1). When the ambient space is Einstein, we use the ideas and techniques recently developed by Alías and Meléndez in [6] to establish a sharp integral inequality for compact linear Weingarten hypersurfaces (see Theorem 5.1).

2 Preliminaries

Let us denote by Σ^n an orientable and connected hypersurface isometrically immersed into an arbitrary $(n+1)$-dimensional Riemannian manifold $\overline{M}^{n+1}$. Let $\{e_1, \ldots, e_{n+1}\}$ be a local orthonormal frame on $\overline{M}^{n+1}$ with dual coframe $\{\omega_1, \ldots, \omega_{n+1}\}$ such that at each point of Σ^n, $e_1, \ldots, e_n$ are tangent to Σ^n and e_{n+1} is normal to Σ^n. We will use the following convention for the indices:

$$1 \leqslant A, B, C, \ldots \leqslant n+1 \quad \text{and} \quad 1 \leqslant i, j, k, \ldots \leqslant n.$$

Restricting all the tensors to Σ^n, we see that $\omega_{n+1} = 0$ on Σ^n. Hence, $0 = d\omega_{n+1} = -\sum_i \omega_{n+1i} \wedge \omega_i$ and as it is well known we get

$$\omega_{n+1i} = \sum_j h_{ij}\omega_j, \quad h_{ij} = h_{ji}.$$

This gives the second fundamental form of Σ^n, $A = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j e_{n+1}$, and its squared norm, $|A|^2 = \sum_{i,j} h_{ij}^2$. Furthermore, the mean curvature H of Σ^n is defined by $H = \frac{1}{n}\,\mathrm{tr}(A) = \frac{1}{n}\sum_i h_{ii}$.

Denoting by $\overline{R}_{ABCD}$, $\overline{R}_{AC}$ and $\overline{R}$, respectively, the Riemannian curvature tensor, the Ricci tensor and the scalar curvature function of the Riemannian manifold $\overline{M}^{n+1}$, we have

$$\overline{R}_{AC} = \sum_B \overline{R}_{ABCB} \quad \text{and} \quad \overline{R} = \sum_A \overline{R}_{AA}.$$

It follows from the Gauss equation that the normalized scalar curvature R of Σ^n is given by

$$n(n-1)R = \sum_{i,j} \overline{R}_{ijij} + n^2H^2 - |A|^2. \tag{2.1}$$

The traceless second fundamental form is given (in local coordinates) by $\Phi_{ij} = h_{ij} - H\delta_{ij}$, which corresponds to the symmetric tensor

$$\Phi = \sum_{i,j} \Phi_{ij}\omega_i \otimes \omega_j.$$

Thus, we have $|\Phi|^2 = \sum_{i,j} \Phi_{ij}^2$ is the squared norm of Φ. Moreover, from (2.1) we obtain

$$n(n-1)R = \sum_{i,j} \overline{R}_{ijij} + n(n-1)H^2 - |\Phi|^2. \tag{2.2}$$

The Cheng–Yau operator [10], here denoted by L, is defined as being the second order linear differential operator $L\colon C^\infty(\Sigma) \to C^\infty(\Sigma)$ given by

$$Lu = \operatorname{tr}(P \circ \operatorname{hess} u), \tag{2.3}$$

where $P\colon \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ is the first Newton transformation of Σ^n, which is defined as the operator $P = nHI - A$, where I is the identity in the algebra of smooth vector fields on Σ^n, and $\operatorname{hess} u\colon \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ denotes the self-adjoint linear operator metrically equivalent to the Hessian of u, $\operatorname{Hess} u$, which are given by

$$\operatorname{hess} u(X) = \nabla_X \nabla u \quad \text{and} \quad \operatorname{Hess}(X, Y) = \langle \operatorname{hess} u(X), Y\rangle,$$

respectively, for all $X, Y \in \mathfrak{X}(\Sigma)$. It is not difficult to see that P is a self-adjoint operator which commutes with the second fundamental form A and satisfies $\operatorname{tr}(P) = n(n-1)H$.

3 Setup and key lemmas

In what follows, we assume $\overline{M}^{n+1}$ to be a locally symmetric Riemannian manifold, which means that all covariant derivative components $\overline{R}_{ABCD;E}$ of its curvature tensor vanish identically.

Let Σ^n be a hypersurface immersed into $\overline{M}^{n+1}$ and let $\overline{K}$ be the sectional curvature of $\overline{M}^{n+1}$. Following the ideas of Nishikawa [18], Choi et al. [11,21], Gomes et al. [15], and Alías et al. [1], among others, we will assume in our main result that there exist constants c_1 and c_2 such that the following relations hold:

$$\overline{K}(e_{n+1}, e_i) = \frac{c_1}{n}, \tag{3.1}$$

and

$$\overline{K}(e_i, e_j) \geqslant c_2 \tag{3.2}$$

for all $i, j = 1, \ldots, n$.

Moreover, we say that a hypersurface Σ^n is linear Weingarten when its normalized scalar curvature and mean curvature are linearly related. That said, we are going to consider Σ^n, a linear Weingarten hypersurface immersed into $\overline{M}^{n+1}$.

It is worth to point out that the Riemannian space forms $\mathbb{Q}_c^{n+1}$ of constant sectional curvature $c \in \{0, 1, -1\}$ satisfy conditions (3.1) and (3.2) for $\frac{c_1}{n} = c_2 = c$. Just to mention other spaces having these properties, a standard computation proves that the Riemannian products $\mathbb{R}^{n-k} \times \mathbb{S}^{k+1}$ and $\mathbb{R}^{n-k} \times \mathbb{H}^{k+1}$ are locally symmetric Riemannian manifolds which also satisfy the above conditions for a wide class of hypersurfaces (for more details, see [1, Remark 3.1]).

We also observe that, in the case where $\overline{M}^{n+1}$ satisfies condition (3.1), its scalar curvature $\overline{R}$ is such that

$$\overline{R} = \sum \overline{R}_{AA} = \sum \overline{R}_{ijij} + 2\sum \overline{R}_{(n+1)i(n+1)i} = \sum \overline{R}_{ijij} + 2c_1. \tag{3.3}$$

Since the scalar curvature of a locally symmetric Riemannian manifold is constant, from (3.3) we see that $\sum_{i,j} \overline{R}_{ijij}$ is a constant naturally attached to $\overline{M}^{n+1}$. In this context, for the sake of simplicity, we will consider the constant $\overline{\mathcal{R}} := \frac{1}{n(n-1)} \sum_{i,j} \overline{R}_{ijij}$ and, assume that $\overline{M}^{n+1}$ also satisfies condition (3.2), we denote $c_0 := 2c_2 - \dfrac{c_1}{n}$ for convenience.

Our first key lemma is an Okumura type result due to Meléndez [17] which is closely related to the total umbilicity tensor (for more details, see [17, Lemma 2.2]; see also [19]).

Lemma 3.1 *Let $\kappa_1, \ldots, \kappa_n$, $n \geqslant 3$, be real numbers such that $\sum_i \kappa_i = 0$ and $\sum_i \kappa_i^2 = \beta^2$, where $\beta \geqslant 0$. Then, the equation*

$$\sum_i \kappa_i^3 = \frac{(n-2p)}{\sqrt{np(n-p)}}\beta^3 \quad \left(\sum_i \kappa_i^3 = -\frac{(n-2p)}{\sqrt{np(n-p)}}\beta^3\right), \qquad 1 \leqslant p \leqslant n-1,$$

holds if and only if p of the numbers κ_i are nonnegative (resp. nonpositive) and equal, and the rest $n-p$ of the numbers κ_i are nonpositive (resp. nonnegative) and equal.

Next, we recall our second auxiliary result, which corresponds to [1, Lemma 3.2].

Lemma 3.2 *Let Σ^n be a linear Weingarten hypersurface immersed into a locally symmetric Riemannian manifold $\overline{M}^{n+1}$ satisfying curvature conditions (3.1) and (3.2), with $R = aH + b$. Suppose that*

$$(n-1)^2a^2 + 4n(n-1)(b - \overline{\mathcal{R}}) \geqslant 0, \tag{3.4}$$

then

$$|\nabla A|^2 - n^2|\nabla H|^2 \geqslant 0. \tag{3.5}$$

Moreover, if inequality (3.4) is strict and equality in (3.5) holds on Σ^n, then H is constant on Σ^n.

For our purposes, it will be crucial to consider the following Cheng–Yau modified operator

$$\mathcal{L} = L - \frac{n-1}{2} a\Delta.$$

So, for any $u \in C^2(\Sigma^n)$, from (2.3) we have that

$$\mathcal{L}(u) = \mathrm{tr}(\mathcal{P} \circ \mathrm{hess}\, u) \tag{3.6}$$

with

$$\mathcal{P} = \left(nH - \frac{n-1}{2} a\right) I - A. \tag{3.7}$$

Besides that, since $R = aH + b$, equation (2.2) becomes

$$|\Phi|^2 = |A|^2 - nH^2 = n(n-1)H^2 - n(n-1)aH - n(n-1)(b - \overline{\mathcal{R}}). \tag{3.8}$$

In our last key lemma, we collect some analytic properties of Cheng–Yau's modified operator, namely: sufficient conditions for the ellipticity property of $\mathcal{L}$ and the validity of a generalized version of the Omori–Yau maximum principle on Σ^n, i.e., for any function $u \in C^2(\Sigma^n)$ with $u^* = \sup u < +\infty$, there exists a sequence of points $\{p_j\} \subset \Sigma^n$ satisfying

$$u(p_j) > u^* - \frac{1}{j}, \quad |\nabla u(p_j)| < \frac{1}{j} \quad \text{and} \quad \mathcal{L}u(p_j) < \frac{1}{j},$$

for every $j \in \mathbb{N}$. For the proof of these properties see Lemma 3.4, the beginning of the proof of Lemma 3.6, and Proposition 4.2 in [1].

Lemma 3.3 *Let Σ^n be a complete linear Weingarten hypersurface immersed into a locally symmetric Riemannian manifold $\overline{M}^{n+1}$ satisfying curvature conditions (3.1) and (3.2), with $R = aH + b$ satisfying $b > \overline{\mathcal{R}}$ (resp. $b \geqslant \overline{\mathcal{R}}$). In the case where $b = \overline{\mathcal{R}}$, assume in addition that the mean curvature function H does not change sign on Σ^n. Then:*

(i) *If we choose a local orthonormal frame $\{e_1, \ldots, e_n\}$ on Σ^n such that $h_{ij} = \lambda_i \delta_{ij}$, the formula*

$$\begin{aligned} \mathcal{L}(nH) = {} & |\nabla A|^2 - n^2|\nabla H|^2 + nH\mathrm{tr}(A^3) - |A|^4 \\ & + \sum_i \overline{R}_{(n+1)i(n+1)i}(nH\lambda_i - |A|^2) + \sum_{i,j} (\lambda_i - \lambda_j)^2 \overline{R}_{ijij} \end{aligned} \tag{3.9}$$

holds on Σ^n, where $\lambda_1, \ldots, \lambda_n$ denote the principal curvatures of Σ^n.

(ii) *The operator $\mathcal{L}$ is an elliptic (resp. semi-elliptic) operator or, equivalently, $\mathcal{P}$ is positive definite (resp. semi-definite), for an appropriate choice of the orientation of Σ^n.*

(iii) *If* $\sup_\Sigma |\Phi| < +\infty$, *then the Omori–Yau maximum principle holds on* Σ^n *for the operator* $\mathcal{L}$.

4 Rigidity of complete linear Weingarten hypersurfaces

In the setup of the previous section, we have the following rigidity result:

Theorem 4.1 *Let* Σ^n *be a complete linear Weingarten hypersurface immersed into a locally symmetric Riemannian manifold* $\overline{M}^{n+1}$ *satisfying curvature conditions* (3.1) *and* (3.2), *with* $R = aH + b$ *such that* $a \geqslant 0$ *and* $b > \max\{\overline{\mathcal{R}} - c_0, \overline{\mathcal{R}}\}$. *If its total umbilicity tensor* Φ *satisfies*

$$\operatorname{tr}(\Phi^3) \geqslant -\frac{(n-2p)}{\sqrt{np(n-p)}}|\Phi|^3, \tag{4.1}$$

for some $1 \leqslant p \leqslant \frac{n-\sqrt{n}}{2}$, *then*

(i) *either* $\sup|\Phi| = 0$ *and* Σ^n *is a totally umbilical hypersurface, or*

(ii)

$$\sup_\Sigma |\Phi| \geqslant \alpha(a, b, n, p, \overline{\mathcal{R}}, c_0) > 0,$$

where $\alpha(a, b, n, p, \overline{\mathcal{R}}, c_0)$ *is a positive constant depending only on* a, b, n, p, $\overline{\mathcal{R}}$ *and* c_0. *Moreover, if the equality* $\sup_\Sigma |\Phi| = \alpha(a, b, p, n, \overline{\mathcal{R}}, c_0)$ *holds and this supremum is attained at some point of* Σ^n, *then* Σ^n *is an isoparametric hypersurface with two distinct principal curvatures of multiplicities* p *and* $n-p$.

Proof Initially we must obtain a suitable lower bound for the operator $\mathcal{L}$ acting on the squared norm of the total umbilicity tensor Φ of Σ^n. Since $\overline{\mathcal{R}}$ is constant, we get from (3.8) that

$$\begin{aligned}\frac{1}{2(n-1)}\mathcal{L}(|\Phi|^2) &= \frac{1}{2}\mathcal{L}(nH^2) - \frac{a}{2}\mathcal{L}(nH)\\ &= H\mathcal{L}(nH) + n\langle \mathcal{P}\nabla H, \nabla H\rangle - \frac{a}{2}\mathcal{L}(nH).\end{aligned} \tag{4.2}$$

By Lemma 3.3 (ii), we have that the operator $\mathcal{P}$ is positive definite. In particular, from (4.2) we obtain

$$\frac{1}{2(n-1)}\mathcal{L}(|\Phi|^2) \geqslant \left(H - \frac{a}{2}\right)\mathcal{L}(nH). \tag{4.3}$$

Without loss of generality we can choose the orientation of Σ^n so that $H > 0$, occurring the strict inequality because of $b > \overline{\mathcal{R}}$. From this, we claim that $H - a/2 > 0$. Indeed, it is enough to see that we can rewrite equation (3.8) as

$$nH(nH - (n-1)a) = |A|^2 + n(n-1)(b - \overline{\mathcal{R}}) > 0.$$

So, formula (3.9) and inequality (4.3) jointly with Lemma 3.2 give

$$\begin{aligned}
&\frac{1}{2(n-1)}\mathcal{L}(|\Phi|^2)\\
&\geqslant \left(H-\frac{a}{2}\right)\left(|\nabla A|^2 - n^2|\nabla H|^2 + nH\mathrm{tr}(A^3) - |A|^4\right)\\
&\quad + \left(H-\frac{a}{2}\right)\left(\sum_i \overline{R}_{(n+1)i(n+1)i}(nH\lambda_i - |A|^2) + \sum_{i,j}(\lambda_i-\lambda_j)^2\overline{R}_{ijij}\right) \qquad (4.4)\\
&\geqslant \left(H-\frac{a}{2}\right)\left(nH\mathrm{tr}(A^3) - |A|^4\right)\\
&\quad + \left(H-\frac{a}{2}\right)\left(\sum_i \overline{R}_{(n+1)i(n+1)i}(nH\lambda_i - |A|^2) + \sum_{i,j}(\lambda_i-\lambda_j)^2\overline{R}_{ijij}\right).
\end{aligned}$$

The curvature constraints (3.1) and (3.2) yield

$$\sum_i \overline{R}_{(n+1)i(n+1)i}(nH\lambda_i - |A|^2) = -c_1|\Phi|^2 \tag{4.5}$$

and

$$\sum_{i,j}(\lambda_i-\lambda_j)^2\overline{R}_{ijij} \geqslant 2nc_2|\Phi|^2. \tag{4.6}$$

Thus, plugging (4.5) and (4.6) into (4.4), we obtain

$$\frac{1}{2(n-1)}\mathcal{L}(|\Phi|^2) \geqslant \left(H-\frac{a}{2}\right)\left(nH\mathrm{tr}(A^3) - |A|^4 + nc_0|\Phi|^2\right). \tag{4.7}$$

On the other hand, it is not difficult to see that

$$\mathrm{tr}(A^3) = \mathrm{tr}(\Phi^3) + 3H|\Phi|^2 + nH^3. \tag{4.8}$$

Putting (4.8) into (4.7) we find

$$\frac{1}{2(n-1)}\mathcal{L}(|\Phi|^2) \geqslant \left(H-\frac{a}{2}\right)\left(-|\Phi|^4 + nH\mathrm{tr}(\Phi^3) + n(H^2+c_0)|\Phi|^2\right). \tag{4.9}$$

Now, taking into account the Okumura type inequality (4.1), from (4.9) we get

$$\begin{aligned}
\frac{1}{2(n-1)}\mathcal{L}(|\Phi|^2) \geqslant &\left(H-\frac{a}{2}\right)|\Phi|^2\\
&\left(-|\Phi|^2 - \frac{n(n-2p)}{\sqrt{np(n-p)}}H|\Phi| + n(H^2+c_0)\right). \qquad (4.10)
\end{aligned}$$

Since $H - a/2 > 0$, we observe that equation (3.8) implies that the mean curvature can be written as

$$H - \frac{a}{2} = \frac{1}{\sqrt{n(n-1)}} \sqrt{|\Phi|^2 + n(n-1)\left(\frac{a^2}{4} + b - \overline{\mathcal{R}}\right)}. \tag{4.11}$$

Thus, substituting (4.11) into (4.10) we get

$$\begin{aligned} \frac{1}{2}\mathcal{L}(|\Phi|^2) \geqslant {} & \frac{(n-1)}{\sqrt{n(n-1)}}|\Phi|^2 \\ & \cdot \Bigg\{ -|\Phi|^2 - \frac{n(n-2p)}{\sqrt{np(n-p)}} \left(\frac{1}{\sqrt{n(n-1)}} \sqrt{|\Phi|^2 + n(n-1)\left(\frac{a^2}{4} + b - \overline{\mathcal{R}}\right)} + \frac{a}{2} \right) |\Phi| \\ & + n \left[\left(\frac{1}{\sqrt{n(n-1)}} \sqrt{|\Phi|^2 + n(n-1)\left(\frac{a^2}{4} + b - \overline{\mathcal{R}}\right)} + \frac{a}{2} \right)^2 + c_0 \right] \Bigg\} \\ & \cdot \sqrt{|\Phi|^2 + n(n-1)\left(\frac{a^2}{4} + b - \overline{\mathcal{R}}\right)}. \end{aligned} \tag{4.12}$$

After some straightforward computations, inequality (4.12) gives us the next one

$$\begin{aligned} \frac{1}{2}\mathcal{L}(|\Phi|^2) \geqslant {} & \frac{1}{\sqrt{n(n-1)}}|\Phi|^2 \\ & \cdot \Bigg\{ -(n-1)|\Phi|^2 - \frac{n(n-1)(n-2p)a}{2\sqrt{np(n-p)}}|\Phi| - \frac{(n-1)(n-2p)}{\sqrt{(n-1)p(n-p)}}|\Phi| \\ & \cdot \sqrt{|\Phi|^2 + n(n-1)\left(\frac{a^2}{4} + b - \overline{\mathcal{R}}\right)} + n(n-1)\left(\frac{a^2}{4} + b - \overline{\mathcal{R}}\right) + |\Phi|^2 \\ & + a\sqrt{n(n-1)}\sqrt{|\Phi|^2 + n(n-1)\left(\frac{a^2}{4} + b - \overline{\mathcal{R}}\right)} + n(n-1)\frac{a^2}{4} + n(n-1)c_0 \Bigg\}. \end{aligned} \tag{4.13}$$

So, inequality (4.13) lead us to the following estimate:

$$\begin{aligned} \frac{1}{2}\mathcal{L}(|\Phi|^2) \geqslant \frac{1}{\sqrt{n(n-1)}} & |\Phi|^2 Q_{a,b,n,p,\overline{\mathcal{R}},c_0}(|\Phi|) \\ & \sqrt{|\Phi|^2 + n(n-1)\left(\frac{a^2}{4} + b - \overline{\mathcal{R}}\right)}, \end{aligned} \tag{4.14}$$

where the function $Q_{a,b,n,p,\overline{\mathcal{R}},c_0}$ is given by

$$\begin{aligned} & Q_{a,b,n,p,\overline{\mathcal{R}},c_0}(x) \\ & \quad = -(n-2)x^2 - \frac{n(n-1)(n-2p)a}{2\sqrt{np(n-p)}}x \\ & \qquad - \left((n-2p)\frac{\sqrt{n-1}}{\sqrt{p(n-p)}}x - a\sqrt{n(n-1)} \right) \end{aligned}$$

$$\sqrt{x^2+n(n-1)\left(\frac{a^2}{4}+b-\overline{\mathcal{R}}\right)}$$
$$+n(n-1)\left(\frac{a^2}{2}+b-\overline{\mathcal{R}}+c_0\right). \tag{4.15}$$

Now, we are going to finish the proof by applying the Omori–Yau maximum principle to the operator $\mathcal{L}$ acting on the function $|\Phi|^2$. We note that if $\sup_\Sigma|\Phi|=+\infty$, then claim (ii) of Theorem 4.1 trivially holds and there is nothing to prove. Otherwise, if $\sup_\Sigma|\Phi|<+\infty$, item (iii) of Lemma 3.3 says that there exists a sequence of points $\{p_j\}$ in Σ^n such that

$$\lim|\Phi|(p_j)=\sup|\Phi| \quad\text{and}\quad \mathcal{L}(|\Phi|^2)(p_j)<\frac{1}{j}.$$

Hence, estimate (4.14) implies that

$$\frac{1}{j}>\mathcal{L}(|\Phi|^2)(p_j)\geqslant\frac{2}{\sqrt{n(n-1)}}|\Phi|^2(p_j)Q_{a,b,n,p,\overline{\mathcal{R}},c_0}(|\Phi|(p_j))$$
$$\cdot\sqrt{|\Phi|^2(p_j)+n(n-1)\left(\frac{a^2}{4}+b-\overline{\mathcal{R}}\right)},$$

and, taking the limit as $j\to+\infty$, we infer

$$\left(\sup_\Sigma|\Phi|\right)^2Q_{a,b,n,p,\overline{\mathcal{R}},c_0}\left(\sup_\Sigma|\Phi|\right)\sqrt{\left(\sup_\Sigma|\Phi|\right)^2+n(n-1)\left(\frac{a^2}{4}+b-\overline{\mathcal{R}}\right)}\leqslant 0.$$

It follows that either $\sup_\Sigma|\Phi|=0$, which means that $|\Phi|\equiv 0$ and the hypersurface is totally umbilical, or $\sup_\Sigma|\Phi|>0$ and then $Q_{a,b,n,p,\overline{\mathcal{R}},c_0}(\sup_\Sigma|\Phi|)\leqslant 0$. In the latter case, since $b>\max\{\overline{\mathcal{R}}-c_0,\overline{\mathcal{R}}\}$, we have that

$$Q_{a,b,n,p,\overline{\mathcal{R}},c_0}(0)=n(n-1)a\sqrt{\frac{a^2}{4}+b-\overline{\mathcal{R}}}+n(n-1)\left(\frac{a^2}{2}+b-\overline{\mathcal{R}}+c_0\right)>0.$$

Moreover, since $1\leqslant p\leqslant\frac{n-\sqrt{n}}{2}$, we can reason as in [13, Remark 3.3] to conclude that the function $Q_{a,b,n,p,\overline{\mathcal{R}},c_0}(x)$ is strictly decreasing for $x\geqslant 0$.

Hence, we guarantee existence of a unique positive real number $pg\alpha(a,b,n,p,\overline{\mathcal{R}},c_0)>0$, depending only on $a,b,n,p,\overline{\mathcal{R}}$ and c_0, such that $Q_{a,b,n,p,\overline{\mathcal{R}},c_0}(\alpha(a,b,n,p,\overline{\mathcal{R}},c_0))=0$. Therefore, we must have

$$\sup_\Sigma|\Phi|\geqslant\alpha(a,b,n,p,\overline{\mathcal{R}},c_0)>0,$$

concluding the proof of the first part of Theorem 4.1.

Finally, let us assume that equality $\sup_\Sigma |\Phi| = \alpha(a, b, n, p, \overline{\mathcal{R}}, c_0)$ holds. In particular, we get

$$Q_{a,b,n,p,\overline{\mathcal{R}},c_0}(|\Phi|) \geqslant 0$$

on Σ^n and then (4.14) assures that $|\Phi|^2$ is a $\mathcal{L}$-subharmonic function on Σ^n, that is,

$$\mathcal{L}(|\Phi|^2) \geqslant 0 \quad \text{on} \quad \Sigma^n. \tag{4.16}$$

Furthermore, since $b > \overline{\mathcal{R}}$, item (ii) of Lemma 3.3 asserts that the operator $\mathcal{L}$ is elliptic. Thus, since Σ^n is complete and taking into account that we are assuming the existence of a point $p \in \Sigma^n$ such that $|\Phi(p)| = \alpha(a, b, n, p, \overline{\mathcal{R}}, c_0) = \sup_\Sigma |\Phi|$, from (4.16) we can apply Hopf's strong maximum principle for the elliptic operator $\mathcal{L}$ acting on the function $|\Phi|^2$ to conclude that it must be constant, that is, $|\Phi| = \alpha(a, b, n, p, \overline{\mathcal{R}}, c_0)$. Hence, the equality in (4.14) holds, namely,

$$\begin{aligned}\frac{1}{2}\mathcal{L}(|\Phi|^2) = 0 = \frac{1}{\sqrt{n(n-1)}}\,|\Phi|^2 Q_{a,b,n,p,\overline{\mathcal{R}},c_0}(|\Phi|)\\ \cdot\sqrt{|\Phi|^2 + n(n-1)\left(\frac{a^2}{4} + b - \overline{\mathcal{R}}\right)}.\end{aligned}$$

Consequently, all the inequalities along the proof of (4.14) must be, in fact, equalities. In particular, we obtain that equation (4.3) must be an equality, which jointly with the positiveness of the operator $\mathcal{P}$ imply that the mean curvature H is constant. Moreover, also equality in (4.4) holds, that is,

$$|\nabla A|^2 = n^2|\nabla H|^2 = 0.$$

Therefore, the principal curvatures of Σ^n must be constant and Σ^n is an isoparametric hypersurface. Besides, Eq. (4.10) is equality too, which implies by Lemma 3.1 that Σ^n has exactly two distinct constant principal curvatures, with multiplicities p and $n - p$. □

Remark 4.2 When $\overline{M}^{n+1} = \mathbb{Q}_c^{n+1}$ is a Riemannian space form of constant sectional curvature c, the constants $\overline{\mathcal{R}}$ and c_0 in Theorem 4.1 just agree with c. For this reason, Theorem 4.1 can be regarded as a natural generalization of [13, Theorem 1].

5 A sharp integral inequality

In this last section, we will establish a sharp integral inequality concerning compact (without boundary) linear Weingarten hypersurface immersed into a locally symmetric Einstein manifold. This will be done by applying the ideas and techniques introduced by Alías and Meléndez in reference [6] for the case of hypersurfaces with constant scalar curvature in the Euclidean sphere.

We observe that the operator $\mathcal{L}$ defined in (3.6) is a divergence-type operator when the ambient space is an Einstein manifold. Indeed, choosing a local orthonormal frame $\{e_1, \ldots, e_n\}$ on Σ^n and using the standard notation $\langle \cdot, \cdot \rangle$ for the (induced) metric of Σ^n, from (2.3) we get

$$L(u) = \sum_{i=1}^{n} \langle P(\nabla_{e_i} \nabla u), e_i \rangle. \tag{5.1}$$

Thus, from (5.1) with a straightforward computation we have

$$\operatorname{div}(P(\nabla u)) = \langle \operatorname{div} P, \nabla u \rangle + L(u), \tag{5.2}$$

where

$$\operatorname{div} P = \operatorname{tr}(\nabla P) = \sum_{i=1}^{n} (\nabla_{e_i} P) e_i.$$

Hence, from [5, Lemma 25] (see also [2, Lemma 3.1]) we have

$$\langle \operatorname{div} P, \nabla u \rangle = -\overline{\operatorname{Ric}}(N, \nabla u), \tag{5.3}$$

where $\overline{\operatorname{Ric}}$ stands for the Ricci tensor of $\overline{M}^{n+1}$ and N denotes the orientation of Σ^n. Assuming that the ambient space $\overline{M}^{n+1}$ is an Einstein manifold, from (5.3) we get

$$\langle \operatorname{div} P, \nabla u \rangle = 0.$$

Consequently, from (5.2) we conclude that

$$L(u) = \operatorname{div}(P(\nabla u)).$$

So, returning to the operator $\mathcal{L}$, we get

$$\mathcal{L}(u) = \operatorname{div}(\mathcal{P}(\nabla u)), \tag{5.4}$$

where $\mathcal{P}$ is defined in (3.7).

Finally, we are able to establish the following sharp integral inequality.

Theorem 5.1 *Let Σ^n be a compact linear Weingarten hypersurface immersed into a locally symmetric Riemannian manifold $\overline{M}^{n+1}$ satisfying curvature conditions (3.1) and (3.2), with $R = aH + b$ such that $b \geqslant \overline{\mathcal{R}}$. In the case where $b = \overline{\mathcal{R}}$, suppose that $a > 0$. If its totally umbilical tensor Φ satisfies (4.1), for some $1 \leqslant p \leqslant \frac{n-\sqrt{n}}{2}$, then*

$$\int_{\Sigma} |\Phi|^{q+2} Q_{a,b,n,p,\overline{\mathcal{R}},c_0}(|\Phi|)\, d\Sigma \leqslant 0, \tag{5.5}$$

for every $q \geqslant 2$, where the real function $Q_{a,b,n,p,\overline{\mathcal{R}},c_0}$ is defined in (4.15). Moreover, assuming $b > \overline{\mathcal{R}}$, the equality in (5.5) holds if and only if

(i) *either Σ^n is a totally umbilical hypersurface, or*

(ii)

$$|\Phi|^2 = \alpha(a, b, n, p, \overline{\mathcal{R}}, c_0) > 0,$$

where $\alpha(a, b, n, p, \overline{\mathcal{R}}, c_0)$ is a positive constant depending only on $a, b, n, p, \overline{\mathcal{R}}$ and c_0, and Σ^n is an isoparametric hypersurface with two distinct principal curvatures of multiplicities p and $n - p$.

Proof Taking $u = |\Phi|^2$, we can rewrite inequality (4.14) as follows:

$$\mathcal{L}(u) \geqslant \frac{1}{\sqrt{n(n-1)}}\, u Q_{a,b,n,p,\overline{\mathcal{R}},c_0}(\sqrt{u})\sqrt{4u + n(n-1)(4(b-\overline{\mathcal{R}}) + a^2)}.$$

Since $u \geqslant 0$ and $a > 0$ when $b = \overline{\mathcal{R}}$, we obtain

$$u^{(q+2)/2} Q_{a,b,n,p,\overline{\mathcal{R}},c_0}(\sqrt{u}) \leqslant \sqrt{n(n-1)}\, \frac{u^{q/2}}{\sqrt{4u + n(n-1)(4(b-\overline{\mathcal{R}}) + a^2)}}\, \mathcal{L}(u),$$

for every real number q. Besides that, the compactness of Σ^n guarantees that we can integrate both sides of the previous equation getting

$$\begin{aligned} &\int_\Sigma u^{(q+2)/2} Q_{a,b,n,p,\overline{\mathcal{R}},c_0}(\sqrt{u})d\Sigma \\ &\qquad \leqslant \sqrt{n(n-1)} \int_\Sigma \frac{u^{q/2}}{\sqrt{4u + n(n-1)(4(b-\overline{\mathcal{R}}) + a^2)}}\, \mathcal{L}(u)\, d\Sigma. \end{aligned} \tag{5.6}$$

But, from (5.4) we deduce that

$$f(u)\mathcal{L}(u) = \operatorname{div}(f(u)\mathcal{P}(\nabla u)) - f'(u)\langle \mathcal{P}(\nabla u), \nabla u\rangle,$$

for every smooth function $f \in C^1(\mathbb{R})$. So, integrating both sides and using Stoke's Theorem, we arrive at

$$\int_\Sigma f(u)\mathcal{L}(u)\, d\Sigma = -\int_\Sigma f'(u)\langle \mathcal{P}(\nabla u), \nabla u\rangle\, d\Sigma,$$

for every smooth function f. In our case, we choose

$$f(t) = \frac{t^{q/2}}{\sqrt{4t + n(n-1)(4(b-\overline{\mathcal{R}}) + a^2)}}, \quad \text{for } t \geqslant 0. \tag{5.7}$$

With this choice, we get

$$f'(t) = \frac{(q-1)4t^{q/2} + n(n-1)(4(b-\overline{\mathcal{R}}) + a^2)qt^{(q-2)/2}}{2(4t + n(n-1)(4(b-\overline{\mathcal{R}}) + a^2))^{3/2}} \geqslant 0, \tag{5.8}$$

for all real numbers $q \geqslant 2$ and $t \geqslant 0$. Putting (5.7) and (5.8) into (5.6), we obtain

$$\int_\Sigma u^{(q+2)/2} Q_{a,b,n,p,\overline{\mathcal{R}},c_0}(\sqrt{u})\, d\Sigma \leqslant -\sqrt{n(n-1)} \int_\Sigma f'(u)\langle \mathcal{P}(\nabla u), \nabla u\rangle\, d\Sigma \leqslant 0, \tag{5.9}$$

since $\mathcal{P}$ is positive semi-defined by item (ii) of Lemma 3.3. Therefore,

$$\int_\Sigma u^{(q+2)/2} Q_{a,b,n,p,\overline{\mathcal{R}},c_0}(\sqrt{u})\, d\Sigma \leqslant 0, \tag{5.10}$$

proving inequality (5.5).

For the second part of Theorem 5.1, assuming that the equality in (5.10) holds, from (5.9) we obtain

$$\int_\Sigma f'(u)\langle \mathcal{P}(\nabla u), \nabla u\rangle\, d\Sigma = 0. \tag{5.11}$$

Consequently, we get from (5.8) that

$$f'(u) = \frac{(q-1)4u^{q/2} + n(n-1)(4(b-\overline{\mathcal{R}}) + a^2)qu^{(q-2)/2}}{2(4u + n(n-1)(4(b-\overline{\mathcal{R}}) + a^2))^{3/2}} \geqslant 0,$$

with equality if and only if $u = 0$ and $q \geqslant 2$. Moreover, since $b > \overline{\mathcal{R}}$, we know from item (ii) of Lemma 3.3 that

$$\langle \mathcal{P}(\nabla u), \nabla u\rangle \geqslant 0,$$

with equality if and only if $\nabla u = 0$. Thus, from (5.11) we have

$$f'(u)\langle \mathcal{P}(\nabla u), \nabla u\rangle = 0.$$

Hence, the function $u = |\Phi|^2$ must be constant, either $u = 0$ or $\nabla u = 0$. In the case that $|\Phi|^2 = 0$, Σ^n must be totally umbilical. Otherwise, $|\Phi|^2$ is a positive constant and the equality in (5.5) implies $Q_{a,b,n,p,\overline{\mathcal{R}},c_0}(|\Phi|) = 0$. Therefore, we can reason as in the last part of the proof of Theorem 4.1 to conclude that Σ^n is an isoparametric hypersurface with two distinct principal curvatures of multiplicities p and $n-p$. □

Remark 5.2 With the same argumentation made in Remark 4.2, we conclude that Theorem 5.1 corresponds to an extension of [6, Theorem 4.1].

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