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RESEARCH ARTICLE

Bimeromorphic automorphism groups of certain $\mathbb{P}^1$-bundles

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Abstract
We call a group G very Jordan if it contains a normal abelian subgroup G_0 such that the orders of finite subgroups of the quotient G/G_0 are bounded by a constant depending on G only. Let Y be a complex torus of algebraic dimension 0. We prove that if X is a non-trivial holomorphic $\mathbb{P}^1$-bundle over Y then the group $\mathrm{Bim}(X)$ of its bimeromorphic automorphisms is very Jordan (contrary to the case when Y has positive algebraic dimension). This assertion remains true if Y is any connected compact complex Kähler manifold of algebraic dimension 0 without rational curves or analytic subsets of codimension 1.

Keywords Automorphism groups of compact complex manifolds · Algebraic dimension 0 · Complex tori · Conic bundles · Jordan properties of groups

Mathematics Subject Classification 14E05 · 14E07 · 14J50 · 32L05 · 32M05 · 32J27 · 32Q15

1 Introduction

Let X be a compact complex connected manifold. We denote by $\mathrm{Aut}(X)$ and $\mathrm{Bim}(X)$ the groups of automorphisms and bimeromorphic selfmaps of X, respectively. If X is

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projective, $\mathrm{Bir}(X)$ denotes the group of birational automorphisms of X. As usual, $\mathbb{P}^n$ stands for the n-dimensional complex projective space; $a(X)$ stands for the algebraic dimension of X. All manifolds in this paper are assumed to be *complex compact and connected* unless otherwise stated.

Vladimir L. Popov in [24] defined the Jordan property of a group and raised the following question: *When the groups* $\mathrm{Aut}(X)$ *and* $\mathrm{Bir}(X)$ *are Jordan?*

Definition 1.1
- A group G is called *bounded* if the orders of its finite subgroups are bounded by a universal constant that depends only on G [24, Definition 2.9].
- A group G is called *Jordan* if there is a positive integer J such that every finite subgroup B of G contains an abelian subgroup A that is normal in B and such that the index $[B:A] \leqslant J$ [24, Definition 2.1].

In this paper we are interested in the following property of groups.

Definition 1.2 We call a group G *very Jordan* if there exist a commutative normal subgroup G_0 of G and a bounded group F that sit in a short exact sequence

$$1 \to G_0 \to G \to F \to 1.$$

Remark 1.3
1. Every finite group is bounded, Jordan, and very Jordan.
2. Every commutative group is Jordan and very Jordan.
3. Every finitely generated commutative group is bounded.
4. A subgroup of a very Jordan group is very Jordan.
5. "Bounded" implies "very Jordan", "very Jordan" implies "Jordan".

The first goal of the paper is to find complex manifolds with very Jordan group $\mathrm{Aut}(X)$ or $\mathrm{Bim}(X)$. To this end we prove the following generalization of [19, Lemma 2.5] and [16, Lemma 3.1].

Proposition 2.1 *Let* X *be a connected compact complex Kähler manifold and* $F = \mathrm{Aut}(X)/\mathrm{Aut}_0(X)$*, where* $\mathrm{Aut}_0(X)$ *is the connected identity component of* $\mathrm{Aut}(X)$. *Then* F *is bounded.*

It follows that if the group $\mathrm{Aut}_0(X)$ is commutative, then $\mathrm{Aut}(X)$ is very Jordan.

Example 1.4 If X is a compact complex Kähler manifold of non-negative Kodaira dimension, then $\mathrm{Aut}(X)$ is very Jordan ([13, Proposition 5.11] and Corollary 2.3 below).

We also study another wide and interesting class of complex manifolds with very Jordan group of automorphisms, namely, compact uniruled manifolds that are equidimensional rational fibrations (i.e., all components of all the fibers are one-dimensional and the general fiber is $\mathbb{P}^1$) over complex tori of algebraic dimension zero.

In order to demonstrate the role of such manifolds, we shall survey Jordan properties of $\mathrm{Aut}(X)$ and $\mathrm{Bim}(X)$ for various types of compact complex manifolds X.

The group $\mathrm{Aut}(X)$ of any connected complex compact manifold X carries a natural structure of a complex (not necessarily connected) Lie group such that the action map $\mathrm{Aut}(X) \times X \to X$ is holomorphic (a theorem of Bochner–Montgomery [6]). It is known, for example, to be Jordan if

- X is projective (Meng, Zheng [19]);
- X is a compact complex Kähler manifold (Kim [16]).

Moreover, the connected identity component $\mathrm{Aut}_0(X) \subset \mathrm{Aut}(X)$ of $\mathrm{Aut}(X)$ is Jordan for every compact complex space X [25, Theorems 5 and 7].

Groups $\mathrm{Bir}(X)$ and $\mathrm{Bim}(X)$ of birational and bimeromorphic transformations, respectively, are more complicated.

Example 1.5 In the case of projective varieties X, it was proven by Popov [24] that $\mathrm{Bir}(X)$ is Jordan if $\dim(X) \leqslant 2$ and X is not birational to a product of an elliptic curve and $\mathbb{P}^1$. (The case of $X = \mathbb{P}^2$ was done earlier by Serre [32].)

Consider the following **LIST** of manifolds:

- E an elliptic curve;
- A_n an abelian variety of dimension n;
- $T := T_{n,a}$ a complex torus with dimension $\dim(T) = n$ and algebraic dimension $a(T) = a$;
- S_b a bielliptic surface;
- S_{K1} a surface of Kodaira dimension 1;
- S_K a Kodaira surface (it is not a Kähler surface).

Example 1.6 1. If S is a projective surface with non-negative Kodaira dimension then $\mathrm{Bir}(S)$ is bounded unless it appears on the **LIST** [28, Theorem1.1].

2. If X is a non-uniruled projective variety with irregularity $q(X) = 0$, then $\mathrm{Bir}(X)$ is bounded [26, Theorem1.8].

Example 1.7 1. $\mathrm{Bir}(X)$ is Jordan for a projective variety X if either X is not uniruled or $X = \mathbb{P}^N$ (proven in [26] modulo the Borisov–Alekseev–Borisov conjecture that was later established by Birkar [4]).

2. If X is a uniruled smooth projective variety that is a non-trivial conic bundle over a non-uniruled smooth projective variety Y then $\mathrm{Bir}(X)$ is Jordan [1].

3. If X is a projective threefold then $\mathrm{Bir}(X)$ is Jordan unless X is birational to a direct product $E \times \mathbb{P}^2$ or $S \times \mathbb{P}^1$, where a surface S appears on the **LIST** [27].

4. If X is a non-algebraic compact uniruled complex Kähler threefold then $\mathrm{Bim}(X)$ is Jordan unless X is either the projectivization of a rank 2 vector bundle over $T_{2,1}$ (and $a(X) = 1$) or X is bimeromorphic to $\mathbb{P}^1 \times T_{2,1}$ (and $a(X) = 2$) [8,29].

Example 1.8 1. If X is a projective variety, birational to $\mathbb{P}^m \times A_n$, $n, m > 0$, then $\mathrm{Bir}(X)$ is not Jordan [36].

2. The group $\mathrm{Bim}(X)$ is not Jordan for a certain class of $\mathbb{P}^1$-bundles (including the trivial ones) over complex tori of positive algebraic dimension [37].

These examples show that the worst case scenario for Jordan properties of $\mathrm{Bim}(X)$ or $\mathrm{Bir}(X)$ occurs when X is a uniruled variety (Kähler manifold) that is fibered over a torus of positive algebraic dimension.

The second goal of this paper is to check what happens in a similar situation when a compact complex manifold is uniruled and fibered over a torus of algebraic dimension

0. It appears that the Jordan properties are drastically different from the situation when the torus has positive algebraic dimension.

Let X, Y be compact connected complex manifolds endowed with a holomorphic map $p\colon X \to Y$. Assume that

- Dimension of the fiber $P_y := p^{-1}(y)$ is 1 for every point $y \in Y$;
- There is an analytic subset $Z \subsetneq Y$ such that for every point $y \notin Z$ the fiber $p^*(y)$ is reduced and isomorphic to $\mathbb{P}^1$.

In this situation we call P_y, $y \notin Z$, a general fiber and X (or a triple (X, p, Y)) an equidimensional rational bundle over Y. (Such bundles appear naturally in the classification of non-projective smooth compact Kähler uniruled threefolds [8].) If X is a holomorphically locally trivial fiber bundle over Y with fiber $\mathbb{P}^1$ we call it a $\mathbb{P}^1$-bundle. If X is a projectivization $\mathbb{P}(E)$ of a rank 2 holomorphic vector bundle E over Y, we will say that X is a linear $\mathbb{P}^1$-bundle over Y. We consider manifolds Y with $a(Y) = 0$ meeting certain additional conditions.

Definition 1.9 We say that a compact connected complex manifold Y of positive dimension is *poor* if it enjoys the following properties:

- The algebraic dimension $a(Y)$ of Y is 0.
- Y does not contain analytic subspaces of codimension 1.
- Y does not contain rational curves, i.e., it is meromorphically hyperbolic in the sense of Fujiki [14].

A complex torus T with $\dim(T) \geqslant 2$ and $a(T) = 0$ is a poor Kähler manifold. Indeed, a complex torus T is a Kähler manifold that does not contain rational curves. If $a(T) = 0$, it contains no analytic subsets of codimension 1 [5, Chapter 2, Corollary 6.4]. An explicit example of such a torus is given in [5, Example 7.4]. Another example of a poor manifold is provided by a non-algebraic $K3$ surface S with $\mathrm{NS}(S) = 0$ (see [3, Chapter VIII, Proposition 3.6]).

Remark 1.10 1. Clearly, the complex dimension of a poor manifold is at least 2.
2. A *generic* complex torus of given dimension $\geqslant 2$ has algebraic dimension 0 and therefore is poor.

Let (X, p, Y) be an equidimensional rational bundle over a poor manifold Y. Since Y contains no rational curves, there are no non-constant holomorphic maps $\mathbb{P}^1 \to Y$. It follows that every map $f \in \mathrm{Bim}(X)$ is p-fiberwise, i.e., there exists a group homomorphism $\tau\colon \mathrm{Bim}(X) \to \mathrm{Aut}(Y)$ (see Lemma 3.3) such that for all $f \in \mathrm{Bim}(X)$

$$p \circ f = \tau(f) \circ p.$$

We denote by $\mathrm{Bim}(X)_p$ ($\mathrm{Aut}(X)_p$) the kernel of τ, i.e., the subgroup of all those $f \in \mathrm{Bim}(X)$ (respectively, $f \in \mathrm{Aut}(X)$) that leave every fiber $P_y := p^{-1}(y)$, $y \in Y$, fixed. We prove the following

Theorem 1.11 *Let (X, p, Y) be an equidimensional rational bundle over a poor manifold Y. Then:*

- *(X, p, Y) is a $\mathbb{P}^1$-bundle (see Proposition 3.6).*
- $\mathrm{Bim}(X) = \mathrm{Aut}(X)$ *(see Corollary 4.1).*
- *The restriction homomorphism $\mathrm{Aut}(X)_p \to \mathrm{Aut}(P_y)$, $f \to f|_{P_y}$ is a group embedding. Here $P_y = p^{-1}(y)$ for any point $y \in Y$ (Lemmas 4.3, 4.4, and Case* ***C***(h) *of Sect. 4).*

Assume additionally that Y is Kähler and X is not bimeromorphic to the direct product $Y \times \mathbb{P}^1$. Then:

- *The connected identity component $\mathrm{Aut}_0(X)$ of the complex Lie group $\mathrm{Aut}(X)$ is* commutative *(Theorem 5.4).*
- *The group $\mathrm{Aut}(X)$ is very Jordan. Namely, there is a short exact sequence*

$$1 \to \mathrm{Aut}_0(X) \to \mathrm{Aut}(X) \to F \to 1,$$

  where F is a bounded group (Theorem 5.4).
- *The commutative group $\mathrm{Aut}_0(X)$ sits in a short exact sequence of complex Lie groups*

$$1 \to \Gamma \to \mathrm{Aut}_0(X) \to H \to 1,$$

  where H is a complex torus and one of the following conditions holds (Proposition 4.8 and (15)):

  - $\Gamma = \{\mathrm{id}\}$, *the trivial group;*
  - $\Gamma \cong \mathbb{C}^+$, *the additive group of complex numbers;*
  - $\Gamma \cong \mathbb{C}^*$, *the multiplicative group of complex numbers.*

The paper is organized as follows. Section 2 contains preliminary results about automorphisms of equidimensional rational bundles and meromorphic groups in a sense of Fujiki. In Sect. 3, we deal with equidimensional rational bundles over poor manifolds and prove that every such equidimensional rational bundle is a $\mathbb{P}^1$-bundle. In Sect. 4, we study $\mathbb{P}^1$-bundles X over a poor manifold T and classify their non-trivial fiberwise automorphisms in terms of the corresponding fixed points sets. In particular, we prove that $\mathrm{Bim}(X) = \mathrm{Aut}(X)$. In Sect. 5, assuming that our poor manifold T is Kähler we prove that the connected identity component $\mathrm{Aut}_0(X)$ of $\mathrm{Aut}(X)$ is commutative. In Sect. 6, we provide a class of examples of $\mathbb{P}^1$-bundles X over complex tori T of algebraic dimension 0 that do not admit a section but admit a bisection that coincides with the set of fixed points of a certain equivariant automorphism.

2 Preliminaries and notation

We assume that all complex manifolds under consideration are connected and compact. We use the following notation and assumptions.

Notation and Assumptions 1

- $\mathrm{Aut}_0(X)$ stands for the connected identity component of $\mathrm{Aut}(X)$ (as a complex Lie group).

- If $p\colon X\to Y$ is a morphism of complex manifolds, then $\mathrm{Bim}(X)_p$ (respectively, $\mathrm{Aut}(X)_p$) is the subgroup of all $f\in\mathrm{Bim}(X)$ (respectively, $f\in\mathrm{Aut}(X)$) such that $p\circ f=p$.
- $\cong$ stands for "isomorphic groups" (or isomorphic complex Lie groups if the groups involved are the ones), and $\sim$ for biholomorphically isomorphic complex manifolds.
- id stands for the identity automorphism.
- $\mathbb{P}^n_{(x_0:\cdots:x_n)}$ stands for a complex projective space $\mathbb{P}^n$ with homogeneous coordinates $(x_0:\ldots:x_n)$.
- $\mathbb{C}_z$ (respectively, $\overline{\mathbb{C}}_z\sim\mathbb{P}^1$) is the complex line (extended complex line, respectively) with coordinate z.
- For an element $m\in\mathrm{PGL}(2,\mathbb{C})$ we define $\mathrm{DT}(m):=\mathrm{tr}^2(M)/\det(M)$ where $M\in\mathrm{GL}(2,\mathbb{C})$ is any matrix representing m, $\mathrm{tr}(M)$ and $\det(M)$ are the trace and the determinant of M, respectively. $\mathrm{DT}(m)=4$ if and only if m is proportional either to the identity matrix or to a unipotent matrix.
- $\mathbb{C}^+$ and $\mathbb{C}^*$ stand for the complex Lie groups $\mathbb{C}$ and $\mathbb{C}^*$ with additive and multiplicative group structure, respectively.
- $\dim(X)$, $a(X)$ are the complex and algebraic dimensions of a compact complex manifold X, respectively.
- Let X, Y be two compact connected irreducible reduced analytic complex spaces. A meromorphic map $f\colon X\to Y$ relates to every point $x\in X$ a subset $f(x)\subset Y$ (the image of x) such that the following conditions are met:
  - The graph $G_f:=\{(x,y)\,|\,y\in f(x)\subset X\times Y\}$ is a connected irreducible complex analytic subspace of $X\times Y$ with $\dim(G_f)=\dim(X)$.
  - There exists an open dense subset $X_0\subset X$ such that $f(x)$ consists of one point for every $x\in X_0$.
- We say that a compact complex manifold Y contains no rational curves if there are no *non-constant* holomorphic maps $\mathbb{P}^1\to Y$.
- Following Fujiki, we call a compact complex manifold *meromorphically hyperbolic* if it contains no rational curves [14].
- According to Fujiki [13, Definition 2.1], a *meromorphic structure* on a complex Lie group G is a compactification G^* of G such that *the group multiplication* $\mu\colon G\times G\to G$ *extends to a meromorphic map* $\mu^*\colon G^*\times G^*\to G^*$ and μ^* *is holomorphic on* $G^*\times G\cup G\times G^*$.
- Following Fujiki, we say that a complex Lie group G *acts meromorphically* on a complex manifold Z if
  - G acts biholomorphically on Z;
  - there is a meromorphic structure G^* on G such that the G-action $\sigma\colon G\times Z\to Z$ extends to a meromorphic map $\sigma^*\colon G^*\times Z\to Z$ (see [13, Definition 2.1] for details).

It was proven in [14] that if a manifold Y is meromorphically hyperbolic then

- Every meromorphic map $f\colon X\to Y$ is holomorphic for any complex manifold X.

- If, in addition, Y is Kähler then
  - Every connected component of the set $H(Y, Y)$ of all holomorphic maps $Y \to Y$ (regarded as a certain subspace of the Douady complex analytic space $D_{Y\times Y}$) is compact.
  - In particular, $\mathrm{Aut}_0(Y)$ is a compact complex Lie group, that is isomorphic to a certain complex torus $\mathrm{Tor}(Y)$ (see also [13, Corollary 3.7]).
  - Actually, $\mathrm{Aut}_0(Y)$ is isomorphic to a complex torus for any compact complex Kähler manifold X of non-negative Kodaira dimension [13, Proposition 5.11].

In general, let Z be a compact complex connected Kähler manifold. The group $\mathrm{Aut}_0(Z)$ acts meromorphically on Z, and the analogue of the Chevalley decomposition for algebraic groups is valid for the complex Lie group $\mathrm{Aut}_0(Z)$:

$$1 \to L(Z) \to \mathrm{Aut}_0(Z) \to \mathrm{Tor}(Z) \to 1 \tag{1}$$

where $L(Z)$ is bimeromorphically isomorphic to a linear group, and $\mathrm{Tor}(Z)$ is a complex torus ([13, Theorem 5.5], [18, Theorem 3.12], [7, Theorem 3.28]).

If $L(Z)$ in (1) is not trivial, Z contains a rational curve. Moreover, according to [13, Corollary 5.10], Z is bimeromorphic to a fiber space whose general fiber is $\mathbb{P}^1$.

The next proposition is similar to [16, Lemma 3.1].

Proposition 2.1 *Let X be a connected complex compact Kähler manifold and* $F = \mathrm{Aut}(X)/\mathrm{Aut}_0(X)$. *Then the group F is bounded.*

Proof By functoriality, there is a natural group homomorphism

$$\phi\colon F \to \mathrm{Aut}(H^2(X, \mathbb{Q})), \quad \mathrm{Aut}(X) \ni f \mapsto f^* \in \mathrm{Aut}_{\mathbb{Q}}(H^2(X, \mathbb{Q})).$$

The connected Lie group $\mathrm{Aut}_0(X)$ is arcwise connected. Hence, f^* is the identity map for all $f \in \mathrm{Aut}_0(X)$. The image $\phi(\mathrm{Aut}(X))$ is bounded, since it is a subgroup of a bounded (thanks to Minkowski's theorem [33, Section 9.1]) group $\mathrm{Aut}_{\mathbb{Q}}(H^2(X, \mathbb{Q})) \cong \mathrm{GL}(b_2(X), \mathbb{Q})$. (Here $b_2(X) = \dim_{\mathbb{Q}} H^2(X, \mathbb{Q})$ is the second Betti number of X.) On the other hand, if $f \in \ker(\phi)$ then its action on $H^2(X, \mathbb{R}) = H^2(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R}$ is trivial as well. Thus, if ω is a Kähler form on X, and $\overline{\omega}$ is its cohomology class in $H^2(X, \mathbb{R})$, and if $f \in \ker(\phi)$, then

$$f^*(\overline{\omega}) = \overline{\omega}. \tag{2}$$

Let $\mathrm{Aut}(X)_{\overline{\omega}} \subset \mathrm{Aut}(X)$ be the subgroup of all automorphisms meeting condition (2). We have $\mathrm{Aut}_0(X) \subset \ker(\phi) \subset \mathrm{Aut}(X)_{\overline{\omega}}$. Since the quotient group $\mathrm{Aut}(X)_{\overline{\omega}}/\mathrm{Aut}_0(X)$ is finite ([13, Theorem 4.8], [18, Proposition 2.2]), the quotient $\ker(\phi)/\mathrm{Aut}_0(X) \subset \mathrm{Aut}(X)_{\overline{\omega}}/\mathrm{Aut}_0(X)$ is a finite group. Thus, we have a short exact sequence of groups

$$1 \to \ker(\phi)/\mathrm{Aut}_0(X) \to (\mathrm{Aut}(X)/\mathrm{Aut}_0(X) = F) \to \phi(\mathrm{Aut}(X)) \to 1.$$

The group $\ker(\phi)/\mathrm{Aut}_0(X)$ is finite, the group $\phi(\mathrm{Aut}(X))$ is bounded, thus the quotient group $\mathrm{Aut}(X)/\mathrm{Aut}_0(X)$ is also bounded. □

Remark 2.2 Our proof was inspired by the proofs of [16, Lemma 3.1] and [19, Lemma 2.5]. Namely, [16, Lemma 3.1] states the following. Let X be a normal compact Kähler variety. Then there exists a positive integer l, depending only on X, such that for any finite subgroup G of $\mathrm{Aut}(X)$ acting biholomorphically and meromorphically on X we have $[G : G \cap \mathrm{Aut}_0(X)] \leqslant l$. We cannot use straightforwardly this lemma since a finite subgroup of $\mathrm{Aut}(X)/\mathrm{Aut}_0(X)$ may not be isomorphic to a quotient $G/(G \cap \mathrm{Aut}_0(X))$ where G is a finite subgroup of $\mathrm{Aut}(X)$.

Corollary 2.3 *Let X be a compact complex Kähler manifold of Kodaira dimension $\varkappa(X) \geqslant 0$. Then $\mathrm{Aut}(X)$ is very Jordan.*

Proof In view of Proposition 2.1 it is sufficient to prove that $\mathrm{Aut}_0(X)$ is commutative. But this follows from [13, Proposition 5.11] that asserts that $\mathrm{Aut}_0(X)$ in this case is a complex torus. □

3 Equidimensional rational bundles over poor manifolds

We will use the following property of poor manifolds.

Lemma 3.1 *Let X, Y be connected compact manifolds and let $f : X \to Y$ be a unramified finite holomorphic covering. Then*

(i) *If Y is Kähler, so is X.*
(ii) *If Y contains no rational curves, so does X.*
(iii) *If Y contains no analytic subsets of codimension 1, so does X.*
(iv) *If Y is poor, so is X.*

Proof Indeed, (i) If ω is a Kähler form on Y, then its pullback $f^*\omega$ is a Kähler form on X, thus X is a Kähler manifold. (ii) If X contained a rational curve C then $f(C)$ would be a rational curve in Y. (iii) If X contained an analytic subset Z of codimension 1, then $f(Z)$ would be a codimension-1 analytic subset in Y. Thus if Y is poor, according (i) and (ii), X contains neither rational curves nor analytic subsets of codimension 1. In particular, $a(X) = 0$. Thus, X is poor. □

An equidimensional rational bundle (X, p, Y) defines the holomorphically locally trivial fiber bundle with fiber $\mathbb{P}^1$ over a certain open dense subset $U \subset Y$. Indeed, by definition, there is an open dense subset $U \subset Y$ of points $y \in Y$ such that for all $y \in U$ the fiber $P_y = p^{-1}(y) \sim \mathbb{P}^1$. By a theorem of Fischer and Grauert [12], the triple $(p^{-1}(U), p, U)$ is a holomorphically locally trivial fiber bundle. Actually, we may (and will) assume that U is a complement of an analytic subset of Y, since the image of the set of points where p is singular is an analytic subset (see, for example, [23, Theorem 1.22]).

Definition 3.2 (cf. [2]) Let X, Y, Z be three complex manifolds, $f : X \to Y$, $g : Z \to Y$ be holomorphic maps, and $h : X \to Z$ be a meromorphic map. We say that h is *f, g-fiberwise* if there exists a holomorphic map $\tau(h) : Y \to Y$ that may be included into the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{h} & Z \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{\tau(h)} & Y. \end{array}$$

If $X = Z$ and $f = g$ we say that h is *f-fiberwise* (or *equivariant*). If $\tau(h) = \mathrm{id}$ we say that h is *fiberwise*.

Lemma 3.3 (cf. [1, Lemma 3.4] for the algebraic case) *Assume that Y is a connected compact complex meromorphically hyperbolic manifold, and let (X, p_X, Y) and (Z, p_Z, Y) be two equidimensional rational bundles over Y. Then any surjective meromorphic map $f: X \dashrightarrow Z$ is p_X, p_Z-fiberwise.*

Proof Consider the meromorphic map $g_f := p_Z \circ f : X \to Y$. It is holomorphic [14, Proposition 1] since Y has no rational curves. Let $U \subset Y$ be a dense Zariski open subset of Y such that $(p_X^{-1}(U), p_X, U)$ is a holomorphically locally trivial fiber bundle. Take a fiber $P_u = p_X^{-1}(u)$, $u \in U$. Since g_f is holomorphic, the image $g_f(P_u)$ may be either a point or a rational curve. Since Y contains no rational curves, the restriction map

$$g_f|_{P_u} : P_u \to Y \text{ is a constant map.}$$

Since U is dense and the set of points $y \in Y$ such that $g_f|_{P_y}$ is a constant, is closed, we get that $g_f|_{P_y}$ is constant for any $y \in Y$. Put $\tau(f)(y) := g_f|_{P_y}$.

For a fiber P_u with $u \in U$, there exists an open neighborhood W of u in U such that $V = p_X^{-1}(W)$ is p_X, p_1-fiberwise isomorphic to $W \times \mathbb{P}^1_{(x:y)}$, where p_1 stands for the natural projection to the second factor. Then for $w \in W$ we have $\tau(f)(w) = p_Z \circ f(w, (0{:}1))$, hence is a holomorphic function on w. Thus, $\tau(f)$ is holomorphic on U, defined and continuous on Y. Let $y \in Y \setminus U$ and $z = \tau(f)(y)$. Let us choose open neighbourhoods $W_y, W_z \subset Y$ of y, z, respectively, such that

- both W_y and W_z are biholomorphic to an open ball in $\mathbb{C}^n$ with induced coordinates $y_1, \ldots, y_n$ and $z_1, \ldots, z_n$, respectively;
- $\tau(f)(W_y) \subset W_z$.

Then the induced functions $\tau(f)^*(z_i)$ are holomorphic in $W_y \cap U$, defined and locally bounded in W_y, thus, by the first Riemann continuation theorem ([15, Chapter 1, C, 3], [11, Section 2.23]) are holomorphic in W_y. Hence, $\tau(f)$ is a holomorphic map. □

Corollary 3.4 *For an equidimensional rational bundle (X, p, Y) over a meromorphically hyperbolic (complex connected compact) manifold Y there is a group homomorphism $\tau : \mathrm{Bim}(X) \to \mathrm{Aut}(Y)$ such that*

$$p \circ f = \tau(f) \circ p$$

for every $f \in \mathrm{Bim}(X)$. Thus every $f \in \mathrm{Bim}(X)$ is p-fiberwise.

Remark 3.5 If, in addition, Y is Kähler, then $\tau(\mathrm{Aut}_0(X))$ has a natural meromorphic structure and the group homomorphism

$$\tau|_{\mathrm{Aut}_0(X)} : \mathrm{Aut}_0(X) \to \tau(\mathrm{Aut}_0(X))$$

is a meromorphic map, in particular, τ is a holomorphic homomorphism of complex Lie groups and $\tau(\mathrm{Aut}_0(X))$ is a complex Lie subgroup of $\mathrm{Aut}(Y)$ [13, Lemma 2.4, (3)].

Proposition 3.6 *Let (X, p, Y) be an equidimensional rational bundle. Assume that Y contains no analytic subsets of codimension 1. Then (X, p, Y) is a $\mathbb{P}^1$-bundle.*

Proof Let $\dim(Y) = n$, and

$$S = \{x \in X \mid \mathrm{rk}(dp)(x) < n\}$$

be the set of all points in X where the differential dp of p does not have the maximal rank. Then S and $\widetilde{S} = p(S)$ are analytic subsets of X and Y, respectively (see, for instance, [20, Chapter VII, Theorem 2], [23, Theorem 1.22], [31]). Moreover, codim $\widetilde{S} = 1$ [30]. Since Y contains no analytic subsets of codimension 1, we obtain: $\widetilde{S} = \varnothing$. Thus the holomorphic map p has no singular fibers. □

Remark 3.7 We used the following theorem of Ramanujam [30]. Let X and Y be connected complex manifolds, $f : X \to Y$ a proper flat holomorphic map such that the general fiber is Riemann sphere, D the set of points in X such that df is not of maximal rank, and $E = f(D)$. Then E is pure of codimension 1 in Y. In the algebraic case this result was proven by Dolgachev [10].

Consider now a $\mathbb{P}^1$-bundle over a compact complex connected manifold Y, i.e., a triple (X, p, Y) such that X is a holomorphically locally trivial fiber bundle over Y with fiber $\mathbb{P}^1$ and with the corresponding projection $p : X \to Y$. Let us fix some notation.

Notation and Assumptions 2
- P_y stands for the fiber $p^{-1}(y)$.
- We call the covering $Y = \bigcup U_i$ by open subsets of Y *fine* if for every i there exist an isomorphism ϕ_i of $V_i = \pi^{-1}(U_i)$ to direct product of U_i and $\mathbb{P}^1_{(x_i:y_i)}$ that is compatible with the natural projection pr on the second factor (i.e., p, pr-fiberwise). In other words $V_i \subset X$ stands for $p^{-1}(U_i)$: we have an induced isomorphism $\phi_i : V_i \to U_i \times \mathbb{P}^1_{(x_i:y_i)}$ and $(y, (x_i : y_i))$ are coordinates in V_i.
  In $(U_i \cap U_j) \times \mathbb{P}^1_{(x_i:y_i)}$ defined is a holomorphic map $\Phi_{i,j} = (\mathrm{id}, A_{i,j}(y))$:

$$(y, (x_i : y_i)) \mapsto (y, (x_j : y_j))$$

  such that

  - ⋆ $A_{i,j} \in \mathrm{PGL}(2, \mathbb{C})$ with representative

$$\widetilde{A}_{i,j}(t) = \begin{bmatrix} a_{i,j}(t) & b_{i,j}(t) \\ c_{i,j}(t) & d_{i,j}(t) \end{bmatrix} \in \mathrm{GL}(2, \mathbb{C}).$$

⋆ $(x_j : y_j) = A_{i,j}(y)((x_i : y_i)) = ((a_{i,j}(y)\, x_i + b_{i,j}(y)\, y_i) : (c_{i,j}(y)\, x_i + d_{i,j}(y)\, y_i))$.
⋆ $A_{i,j}(y) = A_{j,i}(y)^{-1}$.
⋆ The following diagram commutes:

$$\begin{array}{ccc} V_i \cap V_j & \xrightarrow{\phi_i} & (U_i \cap U_j) \times \mathbb{P}^1_{x_i:y_i} \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \Phi_{i,j}} \\ V_i \cap V_j & \xrightarrow{\phi_j} & (U_i \cap U_j) \times \mathbb{P}^1_{x_j:y_j}. \end{array}$$

⋆ $A_{i,j}(y)$ depend holomorphically on y in $U_i \cap U_j$.
⋆ $A_{i,j}(y) A_{j,k}(y) = A_{i,k}(y)$.

Lemma 3.8 *Let $Z \subset Y$ be an analytic subset of Y with* $\operatorname{codim} Z \geqslant 2$, *and $\widetilde{Z} := p^{-1}(Z) \subset X$. Let $f \in \operatorname{Bim}(X)$ be p-fiberwise. If f is defined at every point $x \in X \setminus \widetilde{Z}$ then $f \in \operatorname{Aut}(X)$.*

Proof Let $\{U_i\}$ be a fine covering of Y. Since $f^{-1} \in \operatorname{Bim}(X)$ is p-fiberwise as well, $g := \tau(f)$ is a biholomorphic map. Let $z \in \widetilde{Z}$ and $W \subset U_i$ be an open neighborhood of $p(z)$ such that $g(W) \subset U_j$ for some j. Let $B := W \cap Z$, $A := W \setminus B$. For every $t \in A$ the restriction $f|_{P_t}$ is an isomorphism of P_t with $P_{g(t)}$ defined in corresponding coordinates by an element of $\operatorname{PSL}(2, \mathbb{C})$. Thus, we have a holomorphic map

$$\psi_{W,f} \colon A \to \operatorname{PSL}(2, \mathbb{C})$$

such that

$$f(t, (x_i : y_i)) = (g(t), \psi_{W,f}(t)((x_i : y_i))).$$

Since $\operatorname{PSL}(2, \mathbb{C})$ is an affine variety, and $\operatorname{codim} B \geqslant 2$, by Levi's continuation theorem ([17], see also [20, Chapter VII, Theorem 4] or [11, Section 4.8]) there exists a holomorphic extension $\widetilde{\psi}_{W,f} \colon W \to \operatorname{PSL}(2, \mathbb{C})$. We define

$$\tilde{f}(t, (x_i : y_i)) = (g(t), \tilde{\psi}_{W,f}(t)((x_i : y_i))) \quad \text{in} \quad p^{-1}(W).$$

Thus we can extend f holomorphically at any point $z \in \widetilde{Z}$. Since outside $\widetilde{Z}$ all the extensions coincide, this global extension is uniquely defined. □

Definition 3.9 An *n-section* S of p is a codimension 1 analytic subset $D \subset X$ such that the intersection $X \cap P_y$ is finite for every $y \in Y$ and consists of n distinct points for the general $y \in Y$. A *bisection* is a 2-section. A *section* S of p is a 1-section.

Remark 3.10 For a section S of p there is a holomorphic map $\sigma : Y \to X$ such that the section $S = \sigma(Y)$ and $p \circ \sigma = \mathrm{id}$ on Y. In every U_i the map σ is defined by a function $\sigma_i : U_i \to V_i$ such that

$$A_{i,j}(t) \circ \sigma_i(y) = \sigma_j(t), \quad t \in U_i \cap U_j.$$

Lemma 3.11 *If Y contains no analytic subsets of codimension 1, then*

(i) *any two distinct sections of p in X are disjoint;*
(ii) *an n-section has no ramification points (i.e., the intersection $X \cap P_y$ consists of n distinct points for every $y \in Y$).*

Proof (i) If a section $S = \sigma(Y)$ meets a section $R = \rho(Y)$ then the intersection $S \cap R$ is either empty or has codimension 2 in X. Since none of sections contains a fiber, $p(S \cap R)$ is either empty or has codimension 1 in Y. Since Y carries no analytic subsets of codimension 1, $p(S \cap R) = \varnothing$.

(ii) Let R be an n-section of p, let A be the set of all points $x \in R$ where the restriction $p|_R : R \to Y$ of p onto R is not locally biholomorphic. Then the image $p(A)$ is either empty or has pure codimension 1 in Y([9, Sections 1, 9], [22, Theorem 1.6], [31]). Since Y carries no analytic subsets of codimension 1, $p(A) = \varnothing$. Hence, $A = \varnothing$. □

4 $\mathbb{P}^1$-bundles over poor manifolds

We now fix a poor complex manifold T and consider a $\mathbb{P}^1$-bundle over T, i.e., a triple (X, p, T) such that

- X and T are connected compact complex manifolds;
- T contains neither a rational curve nor an analytic subspace of codimension 1, and algebraic dimension $a(T) = 0$;
- X is a holomorphically locally trivial fiber-bundle over T with fiber $\mathbb{P}^1$ and with the corresponding projection map $p : X \to T$.

Corollary 4.1 $\mathrm{Bim}(X) = \mathrm{Aut}(X)$.

Proof Since T contains no rational curves, by Lemma 3.3, every $f \in \mathrm{Bim}(X)$ is p-fiberwise. For $f \in \mathrm{Bim}(X)$ let $\widetilde{S}_f$ be the indeterminancy locus of f that is an analytic subspace of X of codimension at least 2 [31, p. 369]. Let $S_f = p(\widetilde{S}_f)$, which is an analytic subset of Y ([31], [20, Chapter VII, Theorem 2]). Since T contains no analytic subsets of codimension 1, codim $S_f \geqslant 2$. Moreover, f is defined at all points of $X \setminus p^{-1}(S_f)$. By Lemma 3.8, both $f \in \mathrm{Bim}(X)$ and, similarly, $f^{-1} \in \mathrm{Bim}(X)$ may be holomorphically extended to X, hence we get $\mathrm{Bim}(X) = \mathrm{Aut}(X)$. □

Recall that by $\mathrm{Aut}(X)_p$ we denote the kernel of the group homomorphism $\tau : \mathrm{Bim}(X) = \mathrm{Aut}(X) \to \mathrm{Aut}(T)$, i.e., the subgroup of automorphisms of X that leave every fiber of p invariant.

Let $f \in \mathrm{Aut}(X)_p$, $f \neq \mathrm{id}$. By Lemma 3.1 of [34] we know that the set of fixed points of f is a divisor in X. The following consideration shows that this divisor is either a smooth section of p, or a union of two disjoint sections of p, or a smooth 2-section.

Proposition 4.2 *Assume that $X \sim T \times \mathbb{P}^1$. Let $f \in \mathrm{Aut}(X)_p$, $f \neq \mathrm{id}$, and let S be the set of all fixed points of f. Then one of three following cases holds:*

A. *$S = S_1 \cup S_2$ is a union of two disjoint sections S_1 and S_2 of p.*
B. *S is a section of p.*
C. *S is a 2-section of p (meeting every fiber at two distinct points).*

Proof Let $\{U_i\}$ be a fine covering of T. Let $\mathrm{id} \neq f \in \mathrm{Aut}(X)_p$ be defined (see Notation and Assumptions 2) in V_i with coordinates $(t, (x_i : y_i))$ by $f_i(t, (x_i : y_i)) = (t, F_i(t)(x_i : y_i))$, where

1. $F_i(t)(x_i : y_i) = (f_{i,11}(t)x_i + f_{i,12}(t)y_i) : f_{i,21}(t)x_i + f_{i,22}(t)y_i)$;
2. $\widetilde{F}_i(t) = \begin{bmatrix} f_{i,11}(t) & f_{i,12}(t) \\ f_{i,21}(t) & f_{i,22}(t) \end{bmatrix}$ represents $F_i(t) := \psi_{U_i,f}(t) \in \mathrm{PGL}(2,\mathbb{C})$ (see the proof of Lemma 3.8);
3. the set of fixed points of $F_i(t)$ is the analytic subset of X defined by the equation

$$(f_{i,11}(t)x_i + f_{i,12}(t)y_i) : (f_{i,21}(t)x_i + f_{i,22}(t)y_i) = (x_i : y_i),$$

that is

$$f_{i,12}(t)y_i^2 + (f_{i,11}(t) - f_{i,22}(t))x_i y_i - f_{i,21}(t)x_i^2 = 0. \tag{3}$$

It is obviously an analytic subset of X. In every U_i the function

$$\mathrm{TD}_i(t) = \mathrm{TD}(F_i(t)) = \frac{\mathrm{tr}(\widetilde{F}_i(t))^2}{\det(\widetilde{F}_i)}$$

is defined and holomorphic. Since $F_i(t)$ represent the globally defined map $f \in \mathrm{Aut}(X)_p$, we get (see Notation and Assumptions 2)

$$F_j(t) \circ A_{i,j} = A_{i,j} \circ F_i(t),$$

which means that

$$\widetilde{F}_j(t)\widetilde{A}_{i,j} = \lambda_{i,j}(t)\widetilde{A}_{i,j}\widetilde{F}_i(t), \tag{4}$$

where $\lambda_{i,j}(t) \neq 0$ are some complex functions in $U_i \cap U_j$. From (4) we have

(i) $\mathrm{TD}(t) := \mathrm{TD}_i(t)$ for $t \in U_i$, is holomorphic and globally defined on T, hence constant, we denote this number by TD_f.
(ii) If $\delta_f = \mathrm{TD}_f - 4 \neq 0$, then fix a square root $A_f := \sqrt{\mathrm{TD}_f - 4} \in \mathbb{C}^*$ and define $\lambda_f = \frac{T_f + A_f}{T_f - A_f}$ as the ratio of the eigenvalues of $\widetilde{F}_i(t)$ (it does not depend on i). Then for every i one can define coordinates (t, u_i), $u_i \in \overline{\mathbb{C}}$, in $V_i = p^{-1}(U_i)$ in such a way that $f(t, u_i) = (t, \lambda_f u_i)$. The set $S \cap V_i$ of fixed points of f in V_i is $\{u_i = 0\} \cup \{u_i = \infty\}$. Thus S is an unramified double cover of T: it may be either a union of two disjoint sections or one bisection (see Cases **A**, **C** below for details).
(iii) If $\delta_f = \mathrm{TD}_f - 4 = 0$ then $\widetilde{F}_i(t)$ is proportional to a unipotent matrix and for every i one can define in $V_i = p^{-1}(U_i)$ coordinates (t, w_i), $w_i \in \overline{\mathbb{C}}$, in such a way that $f(t, w_i) = (t, w_i + a_i(t))$ where $a_i(t)$ are holomorphic functions in

U_i. The set S of fixed points in V_i is thus the union of the section $\{w_i = \infty\}$ and $p^{-1}(R_f)$, where

$$R_f = \bigcup \{a_i(t) = 0\} = \{t \in T : f|_{P_t} = \text{id}\}.$$

Since it has codimension 1, it has to be empty (see Case **B** below for details).

In other words, for every $t \in T$ the selfmap $F_i(t)$ of P_t is either the identity map, or has two fixed points, or has one fixed point. If $\text{TD}_f - 4 \neq 0$ then (3) defines a smooth analytic subset S of X and $p^{-1}(t) \cap S$ contains precisely two distinct points for any $t \in T$. Therefore, S is either an unramified smooth double cover of T or a union of two smooth disjoint sections of p. If $\text{TD}_f - 4 = 0$ then (3) defines a smooth section of p over the complement to an analytic subset R_f of T (that has to be empty) or holds identically on X. □

Thus we have the following three cases.

Case A. The set of all fixed points of a non-identity map $f \in \text{Aut}(X)_p$ is the union of two disjoint sections S_1 and S_2 of p. We will say that f has type **A** with Data (S_1, S_2) (an ordered pair). Changing Data (S_1, S_2) to Data (S_2, S_1) would lead to changing λ_f to $1/\lambda_f$.

Lemma 4.3 *Assume that* $f \in \text{Aut}(X_p)$, $f \neq \text{id}$, *has type* **A** *with Data* (S_1, S_2) *and* $X \not\sim T \times \mathbb{P}^1$. *Then*

- $X \setminus S_2$ *is the total body of a holomorphic line bundle* $\mathcal{L}_f$ *with zero section* S_1*;*
- $\mathcal{L}_f$ *has no other sections;*
- $\text{Aut}(X)_p$ *contains a subgroup* $\Gamma_A \cong \mathbb{C}^*$ *of all* $g \in \text{Aut}(X)_p$ *with the same Data* (S_1, S_2)*;*
- *any automorphism* $g \in \text{Aut}(X)_p$ *of type* **A** *belongs to* Γ_A*;*
- *an automorphism* $g \in \Gamma_A$ *is uniquely determined by its restriction to any fiber* P_t *with* $t \in T$ (cf. [34, Lemma 4.3]).

Proof Similarly to the proof of Proposition 4.2, let $\{U_i\}$ be a fine covering of T. Let $f \in \text{Aut}(X)_p$ be defined in $V_i = p^{-1}(U_i)$ with coordinates $(t, (x_i : y_i))$ by $f_i(t, (x_i : y_i)) = (t, F_i(t)(x_i : y_i))$. Let $z_i = y_i/x_i \in \overline{\mathbb{C}}$ and

$$S_1 \cap U_i = \{(t, z_i = a_i(t)\}, \quad S_2 \cap U_i = \{(t, z_i = b_i(t)\}.$$

Since $F_i(t) = \psi_{U_i, f}(t)$ depend on t holomorphically, $a_i(t)$ and $b_i(t)$ are meromorphic functions in U_i. Since $S_1 \cap S_2 = \varnothing$, $a_i(t) \neq b_i(t)$ for all $t \in U_i$ and all i. The holomorphic coordinate change in V_i introduced in item (ii) of the proof of Proposition 4.2 is

$$(t, z_i) \to \left(t, \frac{z_i - a_i(t)}{z_i - b_i(t)}\right) = (t, u_i).$$

In these coordinates $S_1 \cap V_i = \{u_i = 0\}$, and $S_2 \cap V_i = \{u_i = \infty\}$. Since both sections are globally defined and f-invariant, there are holomorphic functions $\mu_{i,j} \in U_i \cap U_j$,

$\mu_{i,j} \neq 0$, such that

$$(t, u_j) = \Phi_{i,j}(t, u_i) = (t, \mu_{i,j} u_i).$$

Since $u_j = \mu_{i,j} u_i = \mu_{k,j} u_k$ in $U_i \cap U_j \cap U_k$, we have $\mu_{i,k} = \mu_{j,k}\mu_{i,j}$, that is we have a cocycle. It defines a holomorphic line bundle $\mathcal{L}_f$ on T with transition functions $\mu_{i,j}$ such that $X \setminus S_2$ is the total body of $\mathcal{L}_f$ and S_1 is the zero section of $\mathcal{L}_f$. Moreover (see item (ii) of the proof of Proposition 4.2),

$$f(t, u_i) = (t, \lambda_f u_i), \quad \lambda_f \neq 0.$$

If $\mathcal{L}_f$ had another section, then the $\mathbb{P}^1$-bundle X would have three disjoint sections, thus would be isomorphic to $T \times \mathbb{P}^1$ (the excluded case). Since every $g \in \mathrm{Aut}(X)_p$ of type **A** has sections as the set of fixed points, it has to have the same Data (S_1, S_2).

The maps having the same Data differ only by the coefficient $\lambda_f \in \mathbb{C}^*$. It follows that an automorphism of type **A** is uniquely defined by its restriction to any fiber P_t, $t \in T$ (cf. [34, Lemma 4.3]). On the other hand, for every $\lambda \in \mathbb{C}^*$ one can define an automorphism $f_\lambda \in \mathrm{Aut}(X)_p$ of type **A** on X by the formula

$$f_\lambda(t, u_i) = (t, \lambda u_i).$$

Thus all automorphisms of type **A** on X form a subgroup $\Gamma_A \cong \mathbb{C}^*$ of $\mathrm{Aut}(X)_p$. □

Case B. If the set of all fixed points of a non-identity $f \in \mathrm{Aut}(X)_p$ is a section S of (X, p, T) we will say that f has type **B** with Data S.

Lemma 4.4 *Assume that $f \in \mathrm{Aut}(X_p)$, $f \neq \mathrm{id}$, has type **B** with Data S and $X \nsim T \times \mathbb{P}^1$. Then*

(i) *$X \setminus S$ is an $\mathbb{A}^1$-bundle $\mathcal{A}_f$ over T;*
(ii) *$\mathcal{A}_f$ has no sections;*
(iii) *$\mathrm{Aut}(X)_p$ contains a subgroup $\Gamma_B \cong \mathbb{C}$ of all $g \in \mathrm{Aut}(X)_p$ with the same Data S;*
(iv) *any automorphism $g \in \mathrm{Aut}(X)_p$ of type **B** belongs to Γ_B;*
(v) *an automorphism $g \in \Gamma_B$ is uniquely determined by its restriction to any fiber P_t with $t \in T$;*
(vi) *$\mathrm{Aut}(X)_p$ contains no automorphisms of type **A**.*

Proof In notation of Proposition 4.2 in this case $\delta_f = TD_f - 4 = 0$. Thus (3) has the set of solutions $S = \{2y_i f_{i,12} + x_i(f_{i,11} - f_{i,22}) = 0\} \subset X$ of fixed points of f. Consider the set $R_f \subset T$ defined locally by the conditions

$$f_{i,12}(t) = f_{i,21}(t) = 0, \quad f_{i,11}(t) = f_{i,22}(t).$$

Since $f \neq \mathrm{id}$, R_f is an analytic subset of T, and codim $R_f \geqslant 2$. Note that $p^{-1}(R_f) \subset S$. Consider the function

$$g_i(t) = \frac{f_{i,22}(t) - f_{i,11}(t)}{2f_{i,21}(t)} = \frac{2f_{i,12}(t)}{f_{i,11}(t) - f_{i,22}(t)}$$

(the equality follows from $\delta_f = 0$). The function g_i is meromorphic in $U_i \setminus R_f$. Since codim $R_f \geqslant 2$, by Levi's theorem ([17], [20, Chapter VII, Theorem 4], [11, Section 4.8]) g_i may be extended to a meromorphic function to U_i. Define $w_i = \frac{y_i}{x_i + y_i g_i(t)} \in \overline{\mathbb{C}}$. The direct computation shows that $f(t, w_i) = w_i + a_i(t)$, where $a_i = \frac{2 f_{i,21}(t)}{\mathrm{tr}(\tilde{F}_i(t))}$. Since $\delta_f = 0$, the denominator never vanishes, thus $a_i(t)$ is a holomorphic function in U_i. The set $\{a_i(t) = 0\} = R_f \cap U_i$ has codimension 1 in U_i, which is impossible if $R_f \neq \varnothing$. It follows that $R_f = \varnothing$. Thus, $f|_{P_t} \neq \mathrm{id}$ for any $t \in T$ and $a_i(t)$ does not vanish in U_i.

Since $S = \{w_i = \infty\}$ is globally defined, we have $w_j = A_{i,j}(w_i) = \nu_{i,j} w_i + \tau_{i,j}$, where $\nu_{i,j}$ and $\tau_{i,j}$ are holomorphic functions in $U_i \cap U_j$. Since f is globally defined

$$\nu_{i,j}(w_i + a_i(t)) + \tau_{i,j} = (\nu_{i,j} w_i + \tau_{i,j}) + a_j(t),$$

we have

- $\{\nu_{i,j}\}$ do not vanish in $U_i \cap U_j$ and form a cocycle, thus define a holomorphic line bundle $\mathcal{M}_f$ on T;
- $\{a_i(t)\}$ is a section of $\mathcal{M}_f$.

Since a non-trivial holomorphic line bundle on T has no nonzero sections, either $a_i(t) \equiv 0$ and $f = \mathrm{id}$ (the excluded case), or $\mathcal{M}_f$ is trivial and we have a global holomorphic, hence constant function

$$a_i(t) \equiv a_f.$$

Thus,

- $X \setminus S$ is an $\mathbb{A}^1$-bundle $\mathcal{A}_f$ with transition holomorphic functions τ_{ij} in $U_i \cap U_j$;
- for every $b \in \mathbb{C}$ there is $f_b \in \mathrm{Aut}(X)_p$ defined in each V_i by

$$f_b(t, w_i) = (t, w_i + b);$$

- the subgroup Γ_B of all f_b, $b \in \mathbb{C}$, is isomorphic to $\mathbb{C}^+$.

Let us show that $\mathcal{A}_f$ has no sections. If it had a section S_1, then S, S_1, $f(S_1)$ would be three disjoint sections of X. Since $X \not\sim T \times \mathbb{P}^1$, this is impossible. It follows that $\mathrm{Aut}(X)_p$ contains neither an automorphism of type **A** nor an automorphism of type **B** with Data different from S. The maps having the same Data S differ only by the summand $a_f \in \mathbb{C}$. It follows that an automorphism of type **B** is uniquely defined by its restriction to a fiber P_t for every $t \in T$. □

Case C. Assume that $X \not\sim T \times \mathbb{P}^1$ and the set $S \subset X$ of all fixed points of a non-identity map $f \in \mathrm{Aut}(X)_p$ is a smooth unramified double cover of T. We will call such an f an automorphism of type **C** defined by Data S. Consider

$$\widetilde{X} := \widetilde{X}_f := S \times_T X = \{(s, x) \in S \times X \subset X \times X \mid p(s) = p(x)\}.$$

We denote the restriction of p to S by the same letter p, while p_X and $\tilde{p}$ stand for the restrictions to $\widetilde{X}$ of natural projections $S \times X \to X$ and $S \times X \to S$, respectively. We write inv: $S \to S$ for an involution (the only non-trivial deck transformation for $p|_S$). We have

(a) The following diagram commutes:

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{p_X} & X \\ \downarrow{\tilde{p}} & & \downarrow{p} \\ S \subset X & \xrightarrow{p} & T. \end{array} \tag{5}$$

(b) $p_X : \widetilde{X} \to X$ is an unramified double cover of X.
(c) Every fiber $\widetilde{p}^{-1}(s)$, $s \in S$, is isomorphic to

$$P_{p(s)} = p^{-1}(p(s)) \sim \mathbb{P}^1.$$

(d) The $\mathbb{P}^1$-bundle $\widetilde{X}$ over S has two sections

$$S_+ := S_+(f) := \{(s, s) \in \widetilde{X} \mid s \in S \subset X\}$$

and

$$S_- := S_-(f) := \{(s, \text{inv}(s)) \in \widetilde{X} \mid s \in S \subset X\}.$$

They are mapped onto S isomorphically by p_X.
(e) Every section $N = \{t, \sigma(t)\}$ of p in X induces the section $\widetilde{N} := \{(s, \sigma(p(s)))\}$ of $\tilde{p}$ in $\widetilde{X}$. We have $p_X(\widetilde{N}) = N$ is a section of p, thus $\widetilde{N}$ cannot coincide with S_+ or S_-.
(f) Every $h \in \text{Aut}(X)_p$ induces an automorphism $\tilde{h} \in \text{Aut}(\widetilde{X})_{\tilde{p}}$ defined by

$$\tilde{h}(s, x) = (s, h(x)).$$

(g) In particular, for the lift $\tilde{f}$ of f all the points of S_+ and S_- are fixed, hence $\tilde{f}$ is of type **A** with Data (S_+, S_-).
(h) The map $\tilde{f}$ is uniquely determined by its restriction to any fiber $\widetilde{P}_s = \tilde{p}^{-1}(s)$ (see Case **A**), hence f is uniquely determined by its restriction on the fiber $P_t = p^{-1}(t)$. Indeed, if $f|_{P_t} = \text{id}$, then

- $\tilde{f}|_{P_s} = \text{id}$, $t = p(s)$, hence
- $\tilde{f} = \text{id}$, hence
- $\tilde{f}|_{P_{s_1}} = \text{id}$ for every $s_1 \in S$, hence
- $f|_{P_{t_1}} = \text{id}$ for $t_1 = p(s_1) \in T$.

(i) It follows that $h \mapsto \tilde{h}$ is a group embedding of $\text{Aut}(X)_p$ to $\text{Aut}(\widetilde{X})_{\tilde{p}}$.
(j) The involution $s \to \text{inv}(s)$ may be extended from S to a holomorphic involution $\widetilde{X}$ by

$$\text{inv}(s, x) = (\text{inv}(s), x).$$

(k) S is a poor manifold by Lemma 3.1.

Clearly, the maps having the same Data differ only by the coefficient $\lambda_{\tilde{f}} \in \mathbb{C}^*$.

Corollary 4.5 *If X admits a non-identity automorphism of type **C** and $\widetilde{X}_f \nsim S \times \mathbb{P}^1$ then the $\mathbb{P}^1$-bundle $p\colon X \to T$ does not have a section. In particular, it does not admit automorphisms of type **A** or **B**.*

Proof Indeed, if X admitted an automorphism of type **A** or **B** then, by Proposition 4.2, there would be a section Σ of p in X. The preimage $p_X^{-1}(\Sigma) \subset \widetilde{X}$ would be a section $\widetilde{S}$ of $\tilde{p}$ in $\widetilde{X}_f$. Thus $\widetilde{X}_f$ would admit three disjoint sections: S_-, S_+, and $\widetilde{S}$. In this case $\widetilde{X}_f$ would be the direct product $S \times \mathbb{P}^1$. □

Lemma 4.6 *Assume that $f \in \mathrm{Aut}(X)_p$, $f \neq \mathrm{id}$, has type **A** with Data (S_1, S_2) and $X \nsim T \times \mathbb{P}^1$. Let $\mathcal{L}_f$ be defined by f (see Case **A** for the definition) holomorphic line bundle on T with transition functions μ_{ij} and such that S_1 is its zero section. Then one of the following holds:*

(i) $\mathrm{Aut}(X)_p = \Gamma_A \cong \mathbb{C}^*$ *and X admits only automorphisms of type **A** except* id*;*
(ii) $\mathrm{Aut}(X)_p$ *contains an automorphism h of type **C** with Data S. In this case $\mathcal{L}_f^{\otimes 2}$ is a trivial holomorphic line bundle and the corresponding to h double cover $\widetilde{X}_h \sim S \times \mathbb{P}^1$.*

Proof (i) By Lemma 4.4, we know that $\mathrm{Aut}(X)_p$ contains no automorphisms of type **B**. If there is no automorphism of type **C** then all $f \in \mathrm{Aut}(X)_p$ are of type **A** except id. By Lemma 4.3, in this case $\mathrm{Aut}(X_p) = \Gamma_A \cong \mathbb{C}^*$.

(ii) Let $h \in \mathrm{Aut}(X)_p$ be of type **C** with Data S. Let a point $t \in U_i \subset T$, where U_i is a fine covering of T, and let $(u, t_i), u \in U_i, t_i \in \overline{\mathbb{C}}$, be coordinates in $V_i = p^{-1}(U_i) \subset X$. Since $S_1 \cup S_2$ are the only sections of $p\colon X \to T$ and points of $S_1 \cup S_2$ are not fixed by h, we have

$$h(u, t_i) = \frac{\nu_i(u)}{t_i} \tag{6}$$

where ν_i are holomorphic in U_i functions. Since h is defined globally, we have

$$\mu_{ij}(u)\left(\frac{\nu_i(u)}{t_i}\right) = \frac{\nu_j(u)}{\mu_{ij}(u)t_i}.$$

Thus $\mu_{ij}(u)^2\nu_i(u) = \nu_j(u)$. Since a non-trivial line bundle over T has only zero section, it follows that $\mathcal{L}_f^{\otimes 2}$ is a trivial bundle and $\nu_i(u) = \nu$ is a constant function. The bisection S is defined locally by the equation $t_i^2 = \nu$. By Corollary 4.5, $\widetilde{X} \sim S \times \mathbb{P}^1$. □

Assume that $f \in \mathrm{Aut}(X)_p$, $f \neq \mathrm{id}$, and f is of type **C** defined by Data (bisection) S. Let $\widetilde{X} := \widetilde{X}_f$ be the corresponding double cover (see Case **C** in Sect. 4 and diagram (5)). Recall that S is poor and $\tilde{p}\colon \widetilde{X} \to S$ has two sections.

Lemma 4.7 *Assume that $f \in \mathrm{Aut}(X)_p$, $f \neq \mathrm{id}$, and f is of type **C** defined by Data (bisection) S.*

(i) *If the corresponding double cover* (*see Case **C***) $\widetilde{X} := \widetilde{X}_f$ *is not isomorphic to* $S \times \mathbb{P}^1$ *then* $\mathrm{Aut}(X)_p$ *has exponent* 2 *and consists of two or four elements.*

(ii) *If* $\widetilde{X} := \widetilde{X}_f$ *is isomorphic to* $S \times \mathbb{P}^1$ *then there are two sections* $S_1, S_2 \subset X$ *of* p*. Moreover,* $\mathrm{Aut}(X)_p$ *is a disjoint union of its abelian complex Lie subgroup* $\Gamma \cong \mathbb{C}^*$ *of index* 2 *and its coset* Γ'*. The subgroup* Γ *consists of those* $f \in \mathrm{Aut}(X)_p$ *that fix* S_1 *and* S_2*. The coset* Γ' *consists of those* $f \in \mathrm{Aut}(X)_p$ *that interchange* S_1 *and* S_2*.*

Proof Choose a point $a \in S$. Let $b = p(a) \in T$. It means that a is one of two points in $S \cap P_b$. The lift $\tilde{f}$ of f onto $\widetilde{X}$ has type **A**, and for the corresponding line bundle $\widetilde{\mathfrak{L}}_{\tilde{f}}$ we may assume that S_+ is a zero section. Let

- $\widetilde{U}_i$ be the fine covering of S;
- μ_{ij} be transition functions of $\widetilde{\mathfrak{L}}_{\tilde{f}}$ in $\widetilde{U}_i \cap \widetilde{U}_j$;
- $\widetilde{V}_i = \tilde{p}^{-1}(\widetilde{U}_i) \subset \widetilde{X}$;
- (u, z_i) be the local coordinates in $\widetilde{V}_i$ such that $z_j = \mu_{ij} z_i$ in $\widetilde{U}_i \cap \widetilde{U}_j$;
- $a \in \widetilde{U}_i$, $\mathrm{inv}(a) \in \widetilde{U}_k$ and $\widetilde{U}_k \cap \widetilde{U}_i = \varnothing$;
- $b = p(a) = p(\mathrm{inv}(a)) \in T$.

Since S_+ is the zero section, $z_i = 0$ on $S_+ \cap \widetilde{V}_i$, $z_i = \infty$ on $S_- \cap \widetilde{V}_i$, while $z_k = 0$ on $S_+ \cap \widetilde{V}_k$, and $z_k = \infty$ on $S_- \cap \widetilde{V}_k$. We have

$$\begin{aligned} &z_i(a, \mathrm{inv}(a)) = \infty, \quad z_i(a, a) = 0, \\ &z_k(\mathrm{inv}(a), a) = \infty, \quad z_k(\mathrm{inv}(a), \mathrm{inv}(a)) = 0. \end{aligned} \tag{7}$$

It may be demonstrated by the following diagram:

$$\begin{array}{ccccc} P_b & \xrightarrow{(a,\mathrm{id})} & a \times P_b & \xrightarrow{z_i} & \overline{\mathbb{C}}_{z_i} \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \downarrow{\scriptstyle \alpha} \\ P_b & \xrightarrow{(\mathrm{inv}(a),\mathrm{id})} & \mathrm{inv}(a) \times P_b & \xrightarrow{z_k} & \overline{\mathbb{C}}_{z_k}. \end{array}$$

Here the isomorphism $\alpha\colon \overline{\mathbb{C}}_{z_i} \to \overline{\mathbb{C}}_{z_k}$ is defined in such a way that the diagram is commutative.

We get from (7) that $\alpha(0) = \infty$, $\alpha(\infty) = 0$. Hence

$$z_k = \alpha(z_i) = \frac{\nu}{z_i}$$

for some $\nu \neq 0$. By construction,

$$p_X(\mathrm{inv}(a), \alpha(z_i)) = p_X(a, z_i).$$

Consider an automorphism $h \in \mathrm{Aut}(X)_p$. Let $\tilde{h}$ be its pullback to $\mathrm{Aut}(\widetilde{X})_{\tilde{p}}$ defined by $\tilde{h}(s, x) = (s, h(x))$. Let $n_1(z_i) = \tilde{h}|_{\tilde{P}_a}$, which means that $h(a, z_i) = (a, n_1(z_i))$. Let

$n_2(z_k) = \tilde{h}|_{\tilde{P}_{\mathrm{inv}(a)}}$, which means that $h(\mathrm{inv}(a), z_k) = (a, n_2(z_k))$. Choose in P_b the coordinate z such that $z_i = p_X^*(z)$, i.e., $p_X(a, z_i) = (b, z_i)$ for a point $(a, z_i) \in \widetilde{P}_a$. By construction, $z(a) = 0$, $z(\mathrm{inv}(a)) = \infty$. We have the following commutative diagram:

$$\begin{array}{ccccccc}
P_b \ni (b, z_i) & \xrightarrow{(a,\mathrm{id})} & (a, z_i) & \xrightarrow{\tilde{h}} & (a, n_1(z_i)) & \xrightarrow{p_X} & (b, n_1(z_i)) \in P_b \\
\Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle (\mathrm{inv}(a),\alpha)} & & \Big\downarrow{\scriptstyle (\mathrm{inv}(a),\alpha)} & & \Big\| \\
P_b \ni (b, z_i) & \xrightarrow{(\mathrm{inv}(a),\mathrm{id})} & (\mathrm{inv}(a), \alpha(z_i)) & \xrightarrow{\tilde{h}} & (\mathrm{inv}(a), n_2(\alpha(z_i))) & \xrightarrow{p_X} & (b, \alpha(n_1(z_i))) \in P_b.
\end{array}$$

Hence

$$\frac{\nu}{n_1(z_i)} = \alpha(n_1(z_i)) = n_2(\alpha(z_i)) = n_2\left(\frac{\nu}{z_i}\right). \tag{8}$$

(i) Assume that $\widetilde{X} \nsim S \times \mathbb{P}^1$. It follows from (6) and the proof of Lemma 4.6 (applied to $\widetilde{X}$) that for every $\tilde{h} \in \mathrm{Aut}(\widetilde{X})_{\tilde{p}}$ in every U_j of our fine covering either $\tilde{h}(s, z_j) = \lambda z_j$, or $h(s, z_j) = \lambda/z_j$ for some $\lambda \in \mathbb{C}^*$, and λ does not depend on s or j. Thus, one of following two conditions holds:

- $n_1(z_i) = \lambda z_i, n_2(z_k) = \lambda z_k, z_k = \nu/z_i$ and from (8)

$$\frac{\nu}{\lambda z_i} = \lambda \frac{\nu}{z_i}.$$

- $n_1(z_i) = \lambda/z_i, n_2(z_k) = \lambda/z_k, z_k = \nu/z_i$ and from (8)

$$\frac{\nu z_i}{\lambda} = \frac{\lambda z_i}{\nu}.$$

  In the former case $\lambda = \pm 1$, in the latter $\lambda = \pm\nu$. Hence, at most four maps are possible. Clearly, the squares of all these maps are the identity map.

(ii) Assume that $\widetilde{X} \sim S \times \mathbb{P}^1$. Let $z\colon S \times \mathbb{P}^1 \to \mathbb{P}^1 \sim \overline{\mathbb{C}}_z$ be the natural projection. Since $S_+ = \{(s, s) \,|\, s \in S\}$ and $S_- = \{(s, \mathrm{inv}(s)) \,|\, s \in S\}$ have algebraic dimension 0, the rational function z is constant along these sections. We may assume that $z = 0$ on $S_+ = \{(s, s)\}$, $z = \infty$ on $S_- = \{(s, \mathrm{inv}(s))\}$, and all $z_j = z$. Moreover, in this case $n_1(z) := n(z) = n_2(z)$ and

$$\tilde{h}(s, z) = (s, z'), \quad \text{where } \; z' := n(z) := \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{C}. \tag{9}$$

On the other hand, it follows from (8) that the map $\tilde{h}(s, z)$ defined by (9) may be pushed down to X if and only if

$$\alpha(n(z)) = n(\alpha(z)).$$

In the expression $\alpha(z) = \nu/z$, we may assume that $\nu = 1$. (Indeed, choose a $\sqrt{\nu}$ and divide z by it). The map $\tilde{h}(s, z)$ defined by (9) may be pushed down to X if and only if

$$\frac{a\frac{1}{z}+b}{c\frac{1}{z}+d}=\frac{cz+d}{az+b}. \tag{10}$$

For every $(a{:}b) \in \mathbb{P}^1$, $a^2-b^2 \neq 0$, two types of $\tilde{h}$ with property (10) are possible:

$$z' = \frac{az+b}{bz+a} = \tilde{h}_{a,b}(z),$$
$$z' = -\frac{az+b}{bz+a} = -\tilde{h}_{a,b}(z) = \tilde{h}_{a,-b}(-z).$$

Note that the only non-trivial automorphism of $\widetilde{X}$ leaving $z=0$, $z=\infty$ invariant is $-\tilde{h}_{a,0}$, which is the lift $\tilde{f}$ of f. All the transformations $h_{a,b}$ form an abelian group $\widetilde{\Gamma}$ with

$$h_{a,b}h_{\alpha,\beta} = h_{c,d}, \quad c = a\alpha + b\beta, \quad d = a\beta + b\alpha.$$

The transformations $-h_{a,b}$ form a coset $\widetilde{\Gamma}' = -h(1{:}0)\widetilde{\Gamma}$. All the transformations from $\widetilde{\Gamma} \cup \widetilde{\Gamma}'$ may be pushed down to X. We have: $\mathrm{Aut}(X)_p$ is embedded into $\mathrm{Aut}(\widetilde{X})_{\tilde{p}}$ and its image is $\widetilde{\Gamma} \cup \widetilde{\Gamma}'$. Thus, $\mathrm{Aut}(X)_p$ is the disjoint union of a subgroup Γ and its coset Γ' corresponding to $\widetilde{\Gamma}$ and $\widetilde{\Gamma}'$, respectively. The index of Γ in $\mathrm{Aut}(X)_p$ is 2.

Note that the sets $\{z=1\}$ and $\{z=-1\}$ consist of fixed points of all the maps $\tilde{h}_{a,b}$ if $b \neq 0$. Moreover, they are invariant under the deck transformation $(s, z) \mapsto (\mathrm{inv}(s), 1/z)$. Their images provide two sections S_1, S_2 of the $\mathbb{P}^1$-bundle $p\colon X \to T$. Hence, in this case $X = \mathbb{P}(E)$ for some decomposable rank 2 vector bundle E over T. If we change coordinates $w = (z+1)/(z-1)$ then $w' = (z'+1)/(z'-1) = \tilde{h}_{a,b}(w) = w(a+b)/(a-b) := w\mu_{a,b}$ and $\tilde{h}_{a,b}(\tilde{h}_{\alpha,\beta}w)$ corresponds to $\mu_{a,b}\mu_{\alpha,\beta}$. The condition $a^2-b^2 \neq 0$ means that $\mu \neq 0, \infty$. Thus, $\Gamma \cong \mathbb{C}^*$ as a complex Lie group. In coordinates w we have $-\tilde{h}_{a,b}(w) = 1/(w\mu_{a,b})$, thus Γ' consists of maps interchanging the sections. □

Proposition 4.8 *Let (X, p, T) be a $\mathbb{P}^1$-bundle over a poor manifold T. Then one of the following holds:*

(i) $X \sim T \times \mathbb{P}^1$;
(ii) $\mathrm{Aut}(X)_p$ *has exponent at most 2 and consists of one, two or four elements;*
(iii) $\mathrm{Aut}(X)_p \cong \mathbb{C}^+$;
(iv) $\mathrm{Aut}(X)_p \cong \mathbb{C}^*$;
(v) $\mathrm{Aut}(X)_p = \Gamma \sqcup \Gamma'$ *where* $\Gamma \cong \mathbb{C}^*$ *is a complex Lie subgroup of* $\mathrm{Aut}(X)_p$ *and* Γ' *is its coset in* $\mathrm{Aut}(X)_p$.

Proof We use the following: assume that $X \nsim T \times \mathbb{P}^1$ and $f \in \mathrm{Aut}(X)_p$, $f \neq \mathrm{id}$. Then

- f being of type **A** implies the existence of exactly two sections of p (see Case **A**);
- f being of type **B** implies the existence of exactly one section of p (see Case **B**);
- f being of type **C** implies the existence of either no or exactly two sections of p (see Case **C** and Lemma 4.7).

Consider the cases.

(ii) If X contains no sections of p then either $\mathrm{Aut}(X)_p = \{\mathrm{id}\}$ or there is $f \in \mathrm{Aut}(X)_p$ of type **C**. Let S be a bisection of p that is the fixed points set of f. The corresponding to f double cover $\widetilde{X}_f$ of X cannot be isomorphic to $S \times \mathbb{P}^1$ by Lemma 4.7(ii), since there are no sections of p. Thus, by Lemma 4.7(i), $\mathrm{Aut}(X)_p$ has exponent at most 2 and consists of two or four elements.

(iii) Assume that X contains exactly one section S of p. Then $\mathrm{Aut}(X)_p = \{\mathrm{id}\}$ or there is non-identity $f \in \mathrm{Aut}(X)_p$ of type **B** only. Then $\mathrm{Aut}(X)_p \cong \mathbb{C}^+$ by combination of Lemma 4.4, Corollary 4.5, and Lemma 4.7.

(iv) Assume that X contains exactly two sections S_1 and S_2 of p. Then there are two options:

- $\mathrm{Aut}(X)_p$ consists of automorphisms of type **A** only (except id) and $\mathrm{Aut}(X)_p \cong \mathbb{C}^*$ according to Lemma 4.3;
- $\mathrm{Aut}(X)_p$ contains automorphisms of type **A** and **C**. By Lemma 4.7, $\mathrm{Aut}(X)_p = \Gamma \sqcup \Gamma'$ where $\Gamma \cong \mathbb{C}^*$ is a complex Lie subgroup of $\mathrm{Aut}(X)_p$ consisting of those maps that fix S_1 and S_2, and Γ' is its coset in $\mathrm{Aut}(X)_p$ that consists of maps that interchange the sections. □

Remark 4.9 Let us formulate a byproduct of the proof of Proposition 4.2. Assume that (V, p, U) is a $\mathbb{P}^1$-bundle over a connected complex (not necessarily compact) manifold U, and let $f \in \mathrm{Aut}(V)_p$, $f \neq \mathrm{id}$. Then

- The function $\mathrm{TD}(u)$ is globally defined.
- If $\mathrm{TD}(u) = \mathrm{const} \neq 4$ on U then the set of fixed points of f is an unramified (may be reducible) double cover of U.
- If $\mathrm{TD}(u) \equiv 4$ on U and U contains no analytic subset of codimension 1, then the set of fixed points of f is a section of p.

5 $\mathbb{P}^1$-bundles over poor Kähler manifolds

In this section we continue to consider a triple (X, p, T) that is a $\mathbb{P}^1$-bundle over a poor manifold T. Further on we assume that T is a *Kähler manifold*. Recall that this means that

- X and T are connected complex compact manifolds;
- T contains no rational curves and no analytic subspaces of codimension 1 (in particular, $a(T) = 0$);
- T is Kähler;
- $p : X \to T$ is a surjective holomorphic map;
- X is a holomorphically locally trivial fiber bundle over T with fiber $\mathbb{P}^1$ and projection p.

Lemma 5.1 *If T is a poor Kähler manifold and* $\mathrm{Aut}(X)_p \neq \{\mathrm{id}\}$ *then X is a Kähler manifold.*

Proof Let $f \in \mathrm{Aut}(X)_p$, $f \neq \mathrm{id}$. Then either X or its étale double cover $\widetilde{X}$ is $\mathbb{P}(E)$ where E is a holomorphic rank 2 vector bundle over a Kähler manifold T or its double cover, respectively (that is also Kähler, see Lemma 3.1). In both cases X is Kähler according to [35, Proposition 3.18]. □

Corollary 5.2 $\mathrm{Bim}(X) = \mathrm{Aut}(X)$ *is Jordan.*

Proof The statement follows from the result of [16]. □

Lemma 5.3 *Consider a short exact sequence of connected complex Lie groups:*

$$0 \to A \xrightarrow{i} B \xrightarrow{j} D \to 0.$$

Here i is a closed holomorphic embedding and j is surjective holomorphic. Assume that D is a complex torus and A is isomorphic as a Lie group either to $\mathbb{C}^+$ or to $\mathbb{C}^*$. Then B is commutative.

Proof *Step 1.* First, let us prove that A is a central subgroup in B. Take any element $b \in B$. Define a holomorphic map $\phi_b \colon A \to A$, $\phi_b(a) = bab^{-1} \in A$ for an element $a \in A$. Since it depends holomorphically on b, we have a holomorphic map $\xi \colon B \to \mathrm{Aut}(A)$, $b \to \phi_b$.

Since A is commutative, for every $a \in A$ we have $\phi_{ab} = \phi_b$. Thus there is a well-defined map ψ fitting into the following commutative diagram:

$$\begin{array}{ccc} & B & \\ {}^{j}\swarrow & & \searrow^{\xi} \\ D & \xrightarrow{\quad\psi\quad} & \mathrm{Aut}(A). \end{array}$$

The map $\psi = \xi \circ j^{-1}$ is defined at every point of D. It is holomorphic (see, for example, [21, Section 3]). Since D is a complex torus, we have $\psi(D)$ is $\{\mathrm{id}\}$. It follows that A is a central subgroup of B.

Step 2. Let us prove that B is commutative. Consider a holomorphic map $\mathrm{com} \colon B \times B \to A$ defined by $\mathrm{com}(x, y) = xyx^{-1}y^{-1}$. Since A is a central subgroup of B, similarly to Step 1 we get a holomorphic map $D \times D \to A$. It has to be constant, since D is a complex torus and A is either $\mathbb{C}^+$ or $\mathbb{C}^*$. □

Theorem 5.4 *Let X be a $\mathbb{P}^1$-bundle over a Kähler poor manifold T and $X \not\sim T \times \mathbb{P}^1$. Then the connected identity component $\mathrm{Aut}_0(X)$ of $\mathrm{Aut}(X)$ is commutative and the quotient $\mathrm{Aut}(X)/\mathrm{Aut}_0(X)$ is a bounded group.*

Proof From equation (1), applied to X and T, combined with Lemma 3.4 and Remark 3.5, we get the following commutative diagram of complex Lie groups and their holomorphic homomorphisms:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L(X) & \longrightarrow & \mathrm{Aut}_0(X) & \longrightarrow & \mathrm{Tor}(X) & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle \tau} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & \mathrm{Aut}_0(T) & \xrightarrow{\cong} & \mathrm{Tor}(T) & \longrightarrow & 0. \end{array}$$

Let us identify a complex torus with the group of its translations and put $H := \tau(\mathrm{Aut}_0(X))$. Then H is the image of a complex torus $\mathrm{Tor}(X) \cong \mathrm{Aut}_0(X)/L(X)$ under a holomorphic homomorphism, thus is a complex subtorus of $\mathrm{Tor}(T)$. Let G be the preimage of H in $\mathrm{Aut}(X)$, with respect to $\tau : \mathrm{Aut}(X) \to \mathrm{Aut}(T)$. By definition, G is a a complex Lie group that contains $\ker(\tau) = \mathrm{Aut}(X)_p$ as a closed complex Lie subgroup. Since $\mathrm{Aut}_0(X) \subset G \subset \mathrm{Aut}(X)$, the identity connected component of G coincides with $\mathrm{Aut}_0(X)$. One has the following short exact sequences of complex Lie groups:

$$1 \to \mathrm{Aut}(X)_p \to G \xrightarrow{\tau} H \to 1, \tag{11}$$

$$1 \to (\mathrm{Aut}(X)_p \cap \mathrm{Aut}_0(X)) \to \mathrm{Aut}_0(X) \xrightarrow{\tau} H \to 1. \tag{12}$$

According to Proposition 4.8 only the following cases may occur.
Case 1. $\mathrm{Aut}(X)_p$ is finite. Then $\mathrm{Aut}(X)_p \cap \mathrm{Aut}_0(X)$ is finite as well, hence $\mathrm{Aut}_0(X) \to H$ is a surjective holomorphic homomorphism of connected complex Lie groups with finite kernel, thus an unramified finite covering [21, Section 4.3]. It follows that $\mathrm{Aut}_0(X)$ is a complex torus, hence commutative.
Case 2. $\mathrm{Aut}(X)_p \cong \mathbb{C}^+$ or $\mathrm{Aut}(X)_p \cong \mathbb{C}^*$. In this case in the short exact sequence (11) both H and $\mathrm{Aut}(X)_p$ are connected. This implies that G is connected, hence $G = \mathrm{Aut}_0(X)$. According to Lemma 5.3, $\mathrm{Aut}_0(X) = G$ is commutative.
Case 3. $\mathrm{Aut}(X)_p$ has a closed subgroup $\Gamma \cong \mathbb{C}^*$ of index 2. According to Lemma 4.7 and Proposition 4.8, it happens when X admits precisely two sections S_1, S_2 of p, and these sections are disjoint. In addition, all automorphisms $f \in \Gamma$ leave invariant these sections as subsets of X. As for automorphisms from coset $\Gamma' = \mathrm{Aut}(X)_p \setminus \Gamma$ of Γ, they interchange S_1 and S_2.

Let us show that in this case

$$\mathrm{Aut}(X)_p \cap \mathrm{Aut}_0(X) = \Gamma. \tag{13}$$

(a) Every automorphism $f \in \mathrm{Aut}(X)$ moves a section S of p to a section of p. Indeed, since f is p-fiberwise, for every $t \in T$ we have

$$f(S \cap P_t) = f(S) \cap P_{\tau(t)}.$$

Thus, since S meets every fiber at one point, the same is valid for $f(S)$. Since there are only two sections of p,

$$f(S_1 \cup S_2) = S_1 \cup S_2. \tag{14}$$

(b) The action $\mathrm{Aut}(X) \times X \to X$, $(f, x) \mapsto f(x)$, on X is holomorphic, hence continuous. Thus the image S of a connected set $\mathrm{Aut}_0(X) \times S_1$ in X is connected. Since sections S_1, S_2 are disjoint, from (14) it follows that $S = S_1$ or $S = S_2$. On the other hand, $\mathrm{id} \in \mathrm{Aut}_0(X)$. It follows that $f(S_1) = S_1$, $f(S_2) = S_2$ for every $f \in \mathrm{Aut}_0(X)$, and $\Gamma' \cap \mathrm{Aut}_0(X) = \emptyset$. This proves (13).

Now, (12) maybe rewritten as a short exact sequence of holomorphic maps of complex Lie groups

$$1 \to \Gamma \to \mathrm{Aut}_0(X) \to H \to 1, \quad \text{where } \Gamma \cong \mathbb{C}^*. \tag{15}$$

Lemma 5.3 implies that $\mathrm{Aut}_0(X)$ is commutative.

Cases 1–3 give us that $\mathrm{Aut}_0(X)$ is commutative. The group $F := \mathrm{Aut}(X)/\mathrm{Aut}_0(X)$ is bounded according to Proposition 2.1. □

Now Theorem 1.11 follows from combination of Proposition 3.6, Corollary 4.1, Proposition 4.2, Theorem 5.4, equations (11), (15), and Proposition 4.8.

6 Examples of $\mathbb{P}^1$-bundles without sections

If S is a complex manifold then we write $\mathbf{1}_S$ for the trivial line bundle $S \times \mathbb{C}$ over S. In this section we construct a $\mathbb{P}^1$-bundle (X, p, T) such that

- T is a complex torus with $\dim(T) = n \geqslant 2$, $a(T) = 0$;
- the projection $p\colon X \to T$ has no section, i.e., there is no divisor $\Delta \subset X$ that meets every fiber P_t at a single point;
- $\mathrm{Aut}(X)_p$ contains no automorphisms of type **A** or **B**;
- $\mathrm{Aut}(X)_p$ contains an automorphism of type **C**;
- there exists a bisection of p that intersects every fiber P_t at two distinct points.

We use the fact that distinct sections of a $\mathbb{P}^1$-bundle over a torus T with $a(T) = 0$ do not intersect, thus our example is impossible with $\dim(T) = 1$.

Let S be a torus with

$$\dim(S) = n \geqslant 2, \quad a(S) = 0.$$

Let $\mathcal{L}$ be a *non-trivial* holomorphic line bundle over S such that

- $\mathcal{L} \in \mathrm{Pic}_0(S)$;
- $\mathcal{L}^{\otimes 2} = \mathbf{1}_S$.

Let Y be the total body of $\mathcal{L}$ and $q\colon Y \to S$ the corresponding surjective holomorphic map. Consider the rank 2 vector bundle $E := \mathcal{L} \oplus \mathbf{1}_S$ on S, let $\overline{Y} = \mathbb{P}(E)$ be the projectivization of E, and let $\overline{q}\colon \overline{Y} \to S$ be the holomorphic extension of q to $\overline{Y}$. The holomorphic map $\overline{q}$ has precisely two sections, namely, D_0 that is the zero section of $\mathcal{L}$ and $D_\infty = \overline{Y} \setminus Y$. Since $\mathcal{L}$ is a non-trivial line bundle and $a(S) = 0$, there are no other sections of $\overline{q}$. We may describe Y in the following way (see [5, Chapter 1, Section 2]). Let $S = V/\Gamma$, where $V = \mathbb{C}^n$ is the n-dimensional complex vector space, $n = \dim(S)$ and Γ is a discrete lattice of rank $2n$. Then there exists a *non-trivial* group homomorphism

$$\xi\colon \Gamma \to \{\pm 1\} \subset \mathbb{C}^*$$

such that Y is the quotient $(V \times \mathbb{C})/\Gamma$ with respect to the action of group Γ on $V \times \mathbb{C}$ by automorphisms

$$g_\gamma(v, z) = (v + \gamma, \xi(\gamma)z) \quad \text{for all} \;\; \gamma \in \Gamma, \quad (v, z) \in V \times \mathbb{C}_z. \tag{16}$$

We may extend the action of Γ to $V \times \mathbb{P}^1 = V \times \overline{\mathbb{C}}_z$ by the same formula (16) and get $\overline{Y} = (V \times \overline{\mathbb{C}}_z)/\Gamma$.

Let us consider the following three holomorphic automorphisms of $\overline{Y}$.

(i) The line bundles $\mathcal{L}$ and $\mathcal{L}^{-1}$ are isomorphic. Hence, there is a holomorphic *involution map* $I_L : \overline{Y} \to \overline{Y}$ such that $I_L(D_0) = D_\infty$ and $I_L \circ I_L = \text{id}$. The automorphism I_L may be included into the commutative diagram

$$\begin{array}{ccc} \overline{Y} & \xrightarrow{I_L} & \overline{Y} \\ \downarrow \overline{q} & & \downarrow \overline{q} \\ S & \xrightarrow{\text{id}} & S. \end{array}$$

In order to define I_L explicitly, let us consider a holomorphic involution

$$\widetilde{I}_L : V \times \overline{\mathbb{C}}_z \to V \times \overline{\mathbb{C}}_z, \quad (v, z) \mapsto \left(v, \frac{1}{z}\right).$$

We have for all $\gamma \in \Gamma$,

$$g_\gamma \circ \widetilde{I}_L(v, z) = \left(v + \gamma, \xi(\gamma) \cdot \frac{1}{z}\right),$$
$$\widetilde{I}_L \circ g_\gamma(v, z) = \left(v + \gamma, \frac{1}{\xi(\gamma)z}\right) = g_\gamma \circ \widetilde{I}_L(v, z),$$

since $\xi(\gamma)^2 = 1$. In other words, $\widetilde{I}_L$ commutes with the action of Γ and therefore descends to the holomorphic involution of $(V \times \overline{\mathbb{C}}_z)/\Gamma = \overline{Y}$ and this involution is our I_L.

(ii) Let us choose $\gamma_0 \in \Gamma$ such that

$$\gamma_0 \notin 2\Gamma, \quad \xi(\gamma_0) = 1.$$

(Such a γ_0 does exist, since the rank of Γ is greater than 1.) Let us put

$$v_0 := \frac{\gamma_0}{2} \in \frac{1}{2}\Gamma \subset V$$

and consider an order 2 point $P := v_0 + \Gamma \in V/\Gamma = S$. Then the translation map

$$\mathbf{T}_P : S \to S, \quad s \mapsto s + P$$

is a holomorphic involution on S: $\mathbf{T}_P^2 = \mathrm{id}$. Since $\mathcal{L} \in \mathrm{Pic}^0(S)$, the translation $\mathbf{T}_P$ induces a holomorphic involution $I_P : \overline{Y} \to \overline{Y}$ [37] that "lifts" $\mathbf{T}_P$ and leaves D_0 and D_∞ invariant. The automorphism I_P may be included in the commutative diagram

$$\begin{array}{ccc} \overline{Y} & \xrightarrow{I_P} & \overline{Y} \\ \downarrow{\scriptstyle \overline{q}} & & \downarrow{\scriptstyle \overline{q}} \\ S & \xrightarrow{\mathbf{T}_P} & S. \end{array}$$

In order to describe I_P explicitly, let us consider a holomorphic automorphism

$$\widetilde{I}_P : V \times \overline{\mathbb{C}}_z \to V \times \overline{\mathbb{C}}_z, \quad (v, z) \mapsto (v + v_0, z).$$

Clearly, $\widetilde{I}_P^2 = g_{\gamma_0}$ (recall that $\xi(\gamma_0) = 1$). For all $\gamma \in \Gamma$

$$g_\gamma \circ \widetilde{I}_P(v, z) = (v + v_0 + \gamma, \xi(\gamma)z),$$
$$\widetilde{I}_P \circ g_\gamma(v, z) = (v + \gamma + v_0, \xi(\gamma)z) = (v + v_0 + \gamma, \xi(\gamma)z),$$

i.e., $\widetilde{I}_P$ and g_γ do commute. This implies that $\widetilde{I}_P$ descends to the holomorphic involution of $(V \times \overline{\mathbb{C}}_z)/\Gamma = \overline{Y}$, and this involution is our I_P.

(iii) Let $\overline{h} \in \mathrm{Aut}(\overline{Y})$ be the holomorphic involution that acts as multiplication by -1 in every fiber of $\mathcal{L}$. (In notation of [37], $\overline{h} = \mathrm{mult}(-1)$.) In $\overline{Y} = (V \times \overline{\mathbb{C}}_z)/\Gamma$ the map $\overline{h}$ is induced by the holomorphic involution

$$\tilde{h} : V \times \overline{\mathbb{C}}_z \to V \times \overline{\mathbb{C}}_z, \quad (v, z) \mapsto (v, -z),$$

which commutes with all g_γ. Indeed, for all $\gamma \in \Gamma$,

$$g_\gamma \circ \tilde{h}(v, z) = (v + \gamma, \xi(\gamma)(-z)) = (v + \gamma, -\xi(\gamma)z),$$
$$\tilde{h} \circ g_\gamma(v, z) = (v + \gamma, -\xi(\gamma)z) = g_\gamma \circ \tilde{h}(v, z).$$

Let us show that I_L, I_Y, and $\overline{h}$ commute. It suffices to check that $\widetilde{I}_L$, $\widetilde{I}_Y$ and $\widetilde{h}$ commute, which is an immediate corollary of the following direct computations:

$$\widetilde{I}_L \circ \tilde{h}(v, z) = \left(v, \frac{1}{-z}\right) = \left(v, -\frac{1}{z}\right) = \tilde{h} \circ \widetilde{I}_L(v, z),$$
$$\widetilde{I}_L \circ \widetilde{I}_P(v, z) = \left(v + v_0, \frac{1}{z}\right) = \widetilde{I}_P \circ \widetilde{I}_L(v, z),$$
$$\tilde{h} \circ \widetilde{I}_P(v, z) = (v + v_0, -z) = \widetilde{I}_P(v, -z) = \widetilde{I}_P \circ \tilde{h}(v, z).$$

Let us put now

$$\mathrm{inv} := I_P \circ I_L : \overline{Y} \to \overline{Y}.$$

Then:

(a) $\mathrm{inv}^2 = \mathrm{id}$;
(b) $\mathrm{inv} \circ \overline{h} = \overline{h} \circ \mathrm{inv}$;
(c) $\overline{q} \circ \mathrm{inv} = \mathbf{T}_P \circ \overline{q}$;
(d) $\mathbf{T}_P$ has no fixed points, thus inv has no fixed points;
(e) $\mathrm{inv}(D_0) = D_\infty$;
(f) if $d_1, d_2 \in D_0$ then $\mathrm{inv}(d_1) \neq d_2$.

Let X be the quotient of $\overline{Y}$ by the action of the order 2 group $\{\mathrm{id}, \mathrm{inv}\}$, and $\pi_Y : \overline{Y} \to X$ be the corresponding quotient map. Let T be the quotient of S by the action of the order 2 group $\{\mathrm{id}, \mathbf{T}_P\}$, and $\pi_S : S \to T$ be the corresponding quotient map. Then X and T enjoy the following properties:

- For any $x \in X$ there are precisely two points $y, \mathrm{inv}(y)$ in $\pi_Y^{-1}(x)$.
- For any $t \in T$ there are precisely two points $s, \mathbf{T}_P(s)$ in $\pi_S^{-1}(t)$.
- Both $\pi_Y : \overline{Y} \to X$ and $\pi_S : S \to T$ are double unramified coverings.
- X is a smooth complex manifold (by (d)).
- T is a complex torus with $a(T) = 0$, $\dim(T) = \dim(S) \geqslant 2$.
- It follows from (c) that there is a holomorphic map $p : X \to T$ such that the following diagram commutes:

$$\begin{array}{ccc} \overline{Y} & \xrightarrow{\pi_Y} & X \\ {\scriptstyle \overline{q}}\downarrow & & \downarrow{\scriptstyle p} \\ S & \xrightarrow{\pi_S} & T. \end{array}$$

- If $\pi_S(s) = t \in T$ then $p^{-1}(t) \sim \overline{q}^{-1}(s) \sim \mathbb{P}^1$.
- It follows from (b) that there is a holomorphic map (pushdown) $h : X \to X$ such that the following diagram commutes:

$$\begin{array}{ccc} \overline{Y} & \xrightarrow{\overline{h}} & \overline{Y} \\ {\scriptstyle \pi_Y}\downarrow & & \downarrow{\scriptstyle \pi_Y} \\ X & \xrightarrow{h} & X. \end{array}$$

- Thanks to (e), we have $\pi_Y(D_0) = \pi_Y(D_\infty) := D$.
- Thanks to (f), the restriction $p|_D : D \to T$ is a double covering.

It follows that X is a $\mathbb{P}^1$-bundle over T, D is a bisection of p, and h is a non-trivial automorphism in $\mathrm{Aut}(X)_p$ of order 2, whose set of fixed points coincides with D.

Lemma 6.1 *There is no section of p.*

Proof Assume that p has a section $\sigma : T \to X$. Let $\Sigma := \sigma(T) \subset X$ and $\Delta := \pi_Y^{-1}(\Sigma)$. As we have already seen, both maps π_S and π_Y are double unramified covers. For every point $t \in T$ there are precisely two distinct points s and $\mathrm{inv}(s)$ in $\pi_S^{-1}(t)$, and there are precisely two distinct points in $\pi_Y^{-1}(\sigma(t))$, say, y_t and $\mathrm{inv}(y_t)$. One of them is mapped by $\overline{q}$ to s, another to $\mathrm{inv}(s)$. It follows that for every $s \in S$ there is precisely one point in $\Delta \cap \overline{q}^{-1}(s)$. Hence, Δ is a section of $\overline{q}$. By construction, $\overline{q}$ has no sections except D_0 and D_∞. But Δ cannot coincide with D_0 or D_∞ since $\pi_Y(\Delta) = \Sigma$ is a section of p and D is not. The contradiction shows that section $\sigma : T \to X$ does not exist. □

Hence, p has no sections and, therefore, there are no automorphisms of type **A** and **B** in $\mathrm{Aut}(X)_p$.

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