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RESEARCH ARTICLE

Homology, lower central series, and hyperplane arrangements

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To the memory of Ştefan Papadima, 1953–2018

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Abstract
We explore finitely generated groups by studying the nilpotent towers and the various Lie algebras attached to such groups. Our main goal is to relate an isomorphism extension problem in the Postnikov tower to the existence of certain commuting diagrams. This recasts a result of Grigory Rybnikov in a more general framework and leads to an application to hyperplane arrangements, whereby we show that all the nilpotent quotients of a decomposable arrangement group are combinatorially determined.

Keywords Lower central series · Tower of nilpotent quotients · Cohomology ring · Associated graded Lie algebra · Holonomy Lie algebra · Malcev Lie algebra · Hyperplane arrangement · Decomposable arrangement

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1 Introduction

1.1 Motivation

The motivation for this paper comes from an effort to understand Rybnikov's invariant used in [36–38] to distinguish between the fundamental groups of complements

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of two hyperplane arrangements with the same incidence structure. Work of Arnol'd, Brieskorn, and Orlik–Solomon insures that an arrangement complement, $M(\mathcal{A})$, is rationally formal, and that the cohomology ring $H^*(M(\mathcal{A}))$ is determined solely by the intersection lattice, $L(\mathcal{A})$. Thus, the complements of the Rybnikov pair of arrangements share the same rational homotopy type; in particular, the respective fundamental groups share the same rational associated graded Lie algebras and second nilpotent quotients. Nevertheless, the third nilpotent quotients of those two groups are not isomorphic, for reasons that are to this date somewhat mysterious, despite repeated attempts to elucidate this phenomenon, see e.g. [1,2,26,27].

We take here a different approach, closely modeled on Rybnikov's original approach from [36,37], yet from a more general point of view. In the process, we develop a machinery for determining when a given isomorphism between the n-th nilpotent quotients of two groups satisfying certain mild finiteness and homological assumptions extends to an isomorphism between the $(n+1)$-st stages of the respective nilpotent towers.

1.2 The holonomy map

Let X be a connected CW-complex. We will assume throughout that the first homology group $H_1(X)$ is finitely generated and torsion-free. Let $G = \pi_1(X)$ be the fundamental group of X, and let $\Gamma_n(G)$ denote its lower central series subgroups. Finally, let $X \to K(G_{\text{ab}}, 1)$ be a classifying map corresponding to the abelianization homomorphism $G \to G_{\text{ab}}$. The induced homomorphism of second homology groups, $h\colon H_2(X) \to H_2(G_{\text{ab}})$, is called the *holonomy map* of X.

Particularly interesting is the situation when the holonomy map is injective; this happens, for instance, when X is the complement of a complex hyperplane arrangement, or of a 'rigid' link, or of an arrangement of transverse planes in $\mathbb{R}^4$, [27]. Under this injectivity assumption, we show in Theorem 3.1 that there is a split exact sequence

$$0 \longrightarrow \Gamma_n(G)/\Gamma_{n+1}(G) \longrightarrow H_2(G/\Gamma_n(G)) \longrightarrow H_2(X) \longrightarrow 0. \tag{1.1}$$

Many properties of a finitely generated group G are reflected in the Lie algebras associated to it. One of those is the *associated graded Lie algebra*, $\operatorname{gr}(G)$, whose graded pieces are defined as $\operatorname{gr}_n(G) = \Gamma_n(G)/\Gamma_{n+1}(G)$, and whose Lie bracket is induced from the group commutator. The study of the associated graded Lie algebra was initiated in work of Magnus [21], Witt [47], Hall [18], and Lazard [20]. Much of the power of this approach comes from the various connections between the lower central series, nilpotent quotients, and group homology, as evidenced in the work of Stallings [42], Quillen [34], Dwyer [13], and many others.

Another Lie algebra associated to a group G is the *holonomy Lie algebra*, $\mathfrak{h}(G)$, which was introduced in work of Chen [8], Kohno [19], and Markl–Papadima [24], and studied more recently by Papadima–Suciu [30] and Suciu–Wang [44,45]. This Lie algebra depends only on data extracted from the cohomology of G in low degrees. In more detail, assuming G_{ab} is torsion-free, $\mathfrak{h}(G)$ is defined as the quotient of the free

Lie algebra on $G_{\rm ab}$ modulo the ideal generated by the image of the holonomy map, $H_2(G) \to H_2(G_{\rm ab})$.

The holonomy Lie algebra $\mathfrak{h}(G)$ may be viewed as a quadratic approximation of the associated graded Lie algebra $\mathrm{gr}(G)$. More precisely, there is a canonical epimorphism of graded Lie algebras, $\mathfrak{h}(G) \twoheadrightarrow \mathrm{gr}(G)$, which is an isomorphism in degrees $n \leq 2$, but is not necessarily injective in higher degrees (see, for instance, the examples in [45] of groups that are not graded-formal). Nevertheless, we show in Theorem 4.3 that the map $\mathfrak{h}_3(G) \to \mathrm{gr}_3(G)$ is an isomorphism under the aforementioned injectivity assumption for the holonomy map.

1.3 The main result

Let X_a and X_b be two path-connected spaces as above. From [42] it follows that if a map $f\colon X_a \to X_b$ induces an isomorphism of first homology groups and an epimorphism of second homology groups, then f induces an isomorphism $G_a/\Gamma_n(G_a) \xrightarrow{\simeq} G_b/\Gamma_n(G_b)$ for $n \geq 2$, where G_a and G_b denote the fundamental groups of X_a and X_b; respectively.

The main result in this paper gives a necessary and sufficient condition for a given map of coalgebras,

$$H_{\leq 2}(X_a) \xrightarrow{\overline{g}} H_{\leq 2}(X_b),$$

with $\overline{g}_1$ an isomorphism and $\overline{g}_2$ an epimorphism, to *extend* to an isomorphism of nilpotent quotients for a given value of n; more precisely, that there be an isomorphism $f_n\colon G_a/\Gamma_n(G_a) \to G_b/\Gamma_n(G_b)$ such that the diagram

$$\begin{array}{ccc} H_2(G_a/\Gamma_n(G_a)) & \xrightarrow{(f_n)_*} & H_2(G_b/\Gamma_n(G_b)) \\ \uparrow & & \uparrow \\ H_2(X_a) & \xrightarrow{\overline{g}_2} & H_2(X_b) \end{array}$$

commutes. To state the result, let N be a nilpotent group with $N \cong N/\Gamma_n(N)$. Assume that the holonomy maps of X_a and X_b are injective, and there is a map $\ell_b\colon X_b \to K(N,1)$ inducing an isomorphism $G_b/\Gamma_n(G_b) \xrightarrow{\simeq} N$. We then show in Theorems 6.1 and 6.2 that there is an isomorphism

$$f_{n+1}\colon G_a/\Gamma_{n+1}(G_a) \xrightarrow{\cong} G_b/\Gamma_{n+1}(G_b)$$

extending $\overline{g}_2$ if and only if there is a map $\ell_a\colon X_a \to K(N,1)$ inducing an isomorphism $G_a/\Gamma_n(G_a) \xrightarrow{\simeq} N$, and a splitting σ of the exact sequence (1.1) such that the following diagram commutes:

$$\begin{array}{ccccc} & & \mathrm{gr}_n(N) & & \\ & \nearrow^{\mu_a} & \uparrow \sigma & \nwarrow^{\mu_b} & \\ & & H_2(N) & & \\ & \nearrow^{(\ell_a)_*} & & \nwarrow^{(\ell_b)_*} & \\ H_2(X_a) & & \xrightarrow[\cong]{\overline{g}_2} & & H_2(X_b). \end{array}$$

Analogous results in characteristic p are proved in Theorems 6.4 and 6.5. In the case $n = 3$ the obstruction to extending the map $\overline{g}_2$ to an isomorphism $G_a/\Gamma_4(G_a) \xrightarrow{\cong} G_b/\Gamma_4(G_b)$ is computed by generalized Massey triple products. This result along with further results and applications will be given in a subsequent paper.

1.4 Hyperplane arrangements

Returning now to the setting of hyperplane arrangements, let $\mathcal{A}$ be a finite set of hyperplanes in some finite-dimensional complex vector space. The complement $M(\mathcal{A})$, then, has the homotopy type of a connected, finite CW-complex. Moreover, the cohomology ring $H^*(M(\mathcal{A}))$ is torsion-free and generated in degree 1, and so the holonomy map of $M(\mathcal{A})$ is injective. Consequently, if $G = G(\mathcal{A})$ is the fundamental group of the complement, then $\mathrm{gr}_3(G) \cong \mathfrak{h}_3(G)$.

The second nilpotent quotient of an arrangement group is combinatorially determined; that is, if $\mathcal{A}$ and $\mathcal{B}$ are two arrangements such that $L_{\leq 2}(\mathcal{A}) \cong L_{\leq 2}(\mathcal{B})$, then $G(\mathcal{A})/\Gamma_3(G(\mathcal{A})) \cong G(\mathcal{B})/\Gamma_3(G(\mathcal{B}))$. On the other hand, as previously mentioned, Rybnikov showed that the next nilpotent quotient, $G(\mathcal{A})/\Gamma_4(G(\mathcal{A}))$ is not always determined by $L_{\leq 2}(\mathcal{A})$.

The invariant that Rybnikov defined in [36–38] to prove this result comes from the case $n = 3$ of the main result in this paper, as follows. In [36,37] it is further assumed that $\mathfrak{h}_3(G)$ is torsion-free. Replacing then the modules and maps in Theorem 6.1 with their Hom duals gives the result corresponding to [36, Theorem 2.2]. These replacements in Theorem 6.2 yield [37, item 2 of Theorem 12].

Particularly interesting is the class of "decomposable" hyperplane arrangements. Building on work of Papadima and Suciu [31] and applying Theorem 6.2, we prove in Theorem 8.8 that, for such an arrangement $\mathcal{A}$, the tower of nilpotent quotients of $G(\mathcal{A})$ is fully determined by the truncated intersection lattice $L_{\leq 2}(\mathcal{A})$. Our result leaves open the question whether the group $G(\mathcal{A})$ itself is combinatorially determined when $\mathcal{A}$ is decomposable.

1.5 Organization of the paper

The paper is divided into three parts, of roughly equal length.

The first part deals with the nilpotent quotients and Lie algebras associated to a finitely generated group G. In Sect. 2 we describe the tower of nilpotent quotients $\{G/\Gamma_n(G)\}_{n\geq 1}$, while in Sect. 3 we review the associated graded Lie algebra $\mathrm{gr}(G)$ and the Malcev Lie algebra $\mathfrak{m}(G)$. Finally, in Sect. 4 we discuss the holonomy Lie algebra $\mathfrak{h}(G)$ and relate it to $\mathrm{gr}(G)$.

In the second part we reprove and extend Rybnikov's theorem. We start in Sect. 5 with some preparatory material on group extensions, splittings, and k-invariants. The main results, including an extension in characteristic p, are stated and proved in Sect. 6.

In the third part we apply our machinery to the theory of hyperplane arrangements. We start in Sect. 7 with a review of the relevant material on the topology and combinatorics of arrangements, and give a quick application to Lie algebras associated to arrangement groups. Finally, in Sect. 8 we show that the nilpotent quotients of decomposable arrangement groups are combinatorially determined.

2 Lower central series and Postnikov towers

In this section we discuss the lower central series and the tower of nilpotent quotients of a group. General references include the works of Hall [18], Magnus [22], Stallings [42], and Dwyer [13].

2.1 Lower central series

Let G be a group. The *lower central series* (LCS) is the sequence of subgroups $\{\Gamma_n(G)\}_{n\geq 1}$ defined inductively by $\Gamma_1(G) = G$ and

$$\Gamma_{n+1}(G) = [G, \Gamma_n(G)]$$

for $n \geq 1$. Here, if H and K are subgroups of G, then $[H, K]$ denotes the subgroup of G generated by all elements of the form $[a, b] = aba^{-1}b^{-1}$ for $a \in H$ and $b \in K$. If both H and K are normal subgroups, then their commutator $[H, K]$ is again a normal subgroup.

In our situation, the subgroups $\Gamma_n(G)$ are, in fact, characteristic subgroups of G. Moreover, the LCS filtration is *multiplicative*, in the sense that, for all m, n,

$$[\Gamma_n(G), \Gamma_m(G)] \subseteq \Gamma_{m+n}(G).$$

Note that $\Gamma_2(G) = [G, G]$ is the derived subgroup of G, and so $G/\Gamma_2(G) = G_{\mathrm{ab}}$, the abelianization of G. Furthermore, each term $\Gamma_{n+1}(G)$ contains $[\Gamma_n(G), \Gamma_n(G)]$, and thus the quotient group

$$\mathrm{gr}_n(G) := \Gamma_n(G)/\Gamma_{n+1}(G) \tag{2.1}$$

is abelian.

Now let $G = F/R$ be a presentation for our group, with F a free group and R a normal subgroup. Then $\Gamma_n(G) = \Gamma_n(F)/\Gamma_n(F) \cap R$. Moreover, if G is finitely generated, then so are the LCS quotients from (2.1). We will write $\phi_n(G) = \operatorname{rank} \operatorname{gr}_n(G)$ for the ranks of these groups.

For instance, if F_k is the free group on k generators, then all its LCS quotients are torsion-free, with ranks $\phi_n = \phi_n(F_k)$ given by $\prod_{n=1}^{\infty}(1-t^n)^{\phi_n} = 1 - kt$, or, equivalently, $\phi_n = \frac{1}{k}\sum_{d|k} \mu(d) n^{k/d}$, where μ denotes the Möbius function.

2.2 Nilpotent quotients

It is readily seen that $G/\Gamma_{n+1}(G)$ is a nilpotent group, and in fact, the maximal n-step nilpotent quotient of G. Letting $q_n\colon G/\Gamma_{n+1}(G) \to G/\Gamma_n(G)$ be the projection map, we obtain a tower of nilpotent groups,

$$\cdots \longrightarrow G/\Gamma_4(G) \xrightarrow{q_3} G/\Gamma_3(G) \xrightarrow{q_2} G/\Gamma_2(G).$$

For each $n \geq 1$, we have a central extension,

$$0 \longrightarrow \operatorname{gr}_n(G) \longrightarrow G/\Gamma_{n+1}(G) \xrightarrow{q_n} G/\Gamma_n(G) \longrightarrow 0. \tag{2.2}$$

Passing to classifying spaces, we obtain a commutative diagram

$$\begin{array}{ccc} & & K(G/\Gamma_{n+1}(G),1) \\ & \overset{\psi_{n+1}}{\nearrow} & \big\downarrow \pi_n \\ K(G,1) & \xrightarrow{\psi_n} & K(G/\Gamma_n(G),1), \end{array} \tag{2.3}$$

where ψ_n corresponds to the projection $G \to G/\Gamma_n(G)$ and π_n corresponds to the projection q_n. Note that π_n may be viewed as the fibration with fiber $K(\operatorname{gr}_n(G), 1)$ obtained as the pullback of the pathspace fibration with base $K(\operatorname{gr}_n(G), 2)$ via a k-invariant

$$\chi_n\colon K(G/\Gamma_n(G), 1) \to K(\operatorname{gr}_n(G), 2). \tag{2.4}$$

2.3 Postnikov tower and the holonomy map

Now let X be a connected CW-complex, and let $G = \pi_1(X)$ be its fundamental group. An Eilenberg–MacLane space $K(G, 1)$ can be constructed by adding to X cells of dimension three or more; let $\iota\colon X \to K(G, 1)$ be the inclusion map. For each $n \geq 1$, let $h_n\colon X \to K(G/\Gamma_n(G), 1)$ be the composite $\psi_n \circ \iota$. This gives the following *Postnikov tower* of fibrations:

$$\begin{array}{ccc} & & \vdots \\ & & \downarrow \\ & & K(G/\Gamma_4(G),1) \\ & \overset{h_4}{\nearrow} & \downarrow{\scriptstyle \pi_3} \\ & & K(G/\Gamma_3(G),1) \\ & \underset{h_3}{\nearrow} & \downarrow{\scriptstyle \pi_2} \\ X & \xrightarrow[h_2]{} & K(G/\Gamma_2(G),1). \end{array} \tag{2.5}$$

We take now homology with coefficients in $\mathbb{Z}$. From the above discussion, we deduce the well-known fact that the map $\iota\colon X \to K(G,1)$ induces an isomorphism $\iota_*\colon H_1(X) \xrightarrow{\simeq} H_1(G)$ and an epimorphism $\iota_*\colon H_2(X) \twoheadrightarrow H_2(G)$.

Consider now the Lyndon–Hochschild–Serre spectral sequence defined in [4],

$$H_p(G/N; H_q(N; M)) \Rightarrow H_n(G; M),$$

where G is a group, N is a normal subgroup of G, and M is a G-module. For the central central extension (2.2), the 5-term exact sequence arising from the terms of low degree (see e.g. [42, Theorem 2.1]) reduces to a short exact sequence

$$H_2(G) \xrightarrow{(q_n)_*} H_2(G/\Gamma_n(G)) \xrightarrow{\chi_n} \mathrm{gr}_n(G) \longrightarrow 0, \tag{2.6}$$

where the map χ_n corresponds to the k-invariant from (2.4) via the Universal Coefficient Theorem. Using now the surjectivity of the map $\iota_*\colon H_2(X) \to H_2(G)$ we obtain an exact sequence

$$H_2(X) \xrightarrow{(h_n)_*} H_2(G/\Gamma_n(G)) \xrightarrow{\chi_n} \mathrm{gr}_n(G) \longrightarrow 0. \tag{2.7}$$

In general, the sequence in (2.7) is natural but not split exact. We call the homomorphism

$$(h_2)_*\colon H_2(X) \to H_2(G/\Gamma_2(G)) \cong H_1(X) \wedge H_1(X) \tag{2.8}$$

the *holonomy map* of X. The following lemma easily follows from the definitions and the Universal Coefficient Theorem.

Lemma 2.1 *Suppose $H_1(X)$ is finitely generated and torsion-free. Then the holonomy map of X is dual to the cup-product map*

$$\cup\colon H^1(X) \wedge H^1(X) \to H^2(X). \tag{2.9}$$

Consequently, if the cup-product map from (2.9) *is surjective, the holonomy map from* (2.8) *is injective.*

As the next example shows, the converse of the last statement does not hold.

Example 2.2 Let X be the connected, 2-dimensional CW-complex associated to the group G with presentation $G = \langle x, y \mid x^2yx^{-2}y^{-1} = 1\rangle$. Clearly, $H_1(X) = \mathbb{Z}^2$ and $H_2(X) = \mathbb{Z}$. With these identifications, the holonomy map $(h_2)_*\colon \mathbb{Z} \to \mathbb{Z}$ is multiplication by 2, and thus injective, while the cup-product map $\cup\colon \mathbb{Z} \to \mathbb{Z}$ is also multiplication by 2, and thus not surjective.

3 Associated graded Lie algebras and Malcev Lie algebras

3.1 The associated graded Lie algebra of a group

Given a group G, we let $\operatorname{gr}(G)$ be the direct sum of the successive quotients of the lower central series of G; that is,

$$\operatorname{gr}(G) = \bigoplus_{n\geq 1} \Gamma_n(G)/\Gamma_{n+1}(G).$$

The map $a\otimes b \mapsto aba^{-1}b^{-1}$ induces homomorphisms $[\cdot,\cdot]\colon \operatorname{gr}_m(G)\otimes \operatorname{gr}_n(G) \to \operatorname{gr}_{m+n}(G)$. It is readily seen that the following "Witt–Hall identities" hold in G:

$$[ab, c] = {}^a[b, c]\cdot[a, c], \quad [{}^ba, [c, b]]\cdot[{}^cb, [a, c]]\cdot[{}^ac, [b, a]] = 1,$$

where ${}^ab = aba^{-1}$. It follows that $\operatorname{gr}(G)$, endowed with the aforementioned bracket, has the structure of a graded Lie algebra, see for instance [22,41]. The construction is functorial: every group homomorphism $f\colon G \to H$ induces a morphism of graded Lie algebras, $\operatorname{gr}(f)\colon \operatorname{gr}(G) \to \operatorname{gr}(H)$.

If F is a free group, then, as shown by Magnus and Witt, $\operatorname{gr}(F)$ is the free Lie algebra on the same set of generators as F; in particular, if $F = F_\ell$, then $\operatorname{gr}(F) = \operatorname{Lie}(\mathbb{Z}^\ell)$, the free Lie algebra of rank ℓ.

3.2 Injective holonomy map and an exact sequence

Once again, let X be a path-connected space, with fundamental group $G = \pi_1(X)$.

Theorem 3.1 *Assume that the group $H_1(X)$ is finitely generated, torsion-free, and the holonomy map $(h_2)_*\colon H_2(X) \to H_1(X)\wedge H_1(X)$ from* (2.8) *is injective. For each $n \geq 2$, there is then a natural, split exact sequence*

$$0 \longrightarrow \operatorname{gr}_{n+1}(G) \xrightarrow{\ i\ } H_2(G/\Gamma_{n+1}(G)) \xrightarrow{\ \pi\ } H_2(X) \longrightarrow 0. \tag{3.1}$$

Proof Recall that given a fibration of CW-complexes with base B, fiber F, and total space E, the filtration of $C_*(E)$ by the inverse images of the skeleta of B gives a homology Serre spectral sequence with differentials

$$E^r_{s,t} \xrightarrow{d^r} E^r_{s-r,t+r-1}.$$

Furthermore, if the fundamental group of the base acts trivially on the fibers, then

$$E^2_{s,t} = H_s(B) \otimes H_t(F).$$

For the fibration (2.3), the kernel of the differential $d^2 \colon H_2(B) \to H_1(F)$ is by equation (2.6) the image of $H_2(X)$ in $H_2(G/\Gamma_n(G))$. In turn, this image can be identified with $H_2(X)$, since by assumption the holonomy map is a monomorphism. This gives an exact sequence

$$0 \longrightarrow F^1 \longrightarrow H_2(G/\Gamma_{n+1}(G)) \longrightarrow H_2(X) \longrightarrow 0, \tag{3.2}$$

where F^1 denotes the image in $H_2(G/\Gamma_{n+1}(G))$ of the inverse image of the 1-skeleton in $K(G/\Gamma_n(G), 1)$. It follows that a map $(h_{n+1})_*$ gives a right splitting of the exact sequence (3.2). The result now follows from (2.7). □

In particular, under the above hypothesis there is a natural exact sequence

$$0 \longrightarrow \mathrm{gr}_3(G) \longrightarrow H_2(G/\Gamma_3(G)) \longrightarrow H_2(X) \longrightarrow 0. \tag{3.3}$$

Furthermore, this sequence is split exact. We do not claim that there is a natural splitting of the exact sequence (3.3), or of the other exact sequences from (3.1).

3.3 Malcev completion and the Malcev Lie algebra

In [23], Malcev established a one-to-one correspondence between certain nilpotent Lie algebras over $\mathbb{Q}$, and nilpotent groups over $\mathbb{Q}$, leading to the Malcev Lie algebra of a group. This was extended by Lazard [20] to groups with enough divisibility in central series subgroups to establish a one-to-one correspondence between a wider class of nilpotent groups and Lie algebras. An important next step was taken by Quillen, who established in [35] an equivalence between rational homotopy theory and the homotopy theory of reduced differential graded Lie algebras over $\mathbb{Q}$ with Malcev Lie algebras as the equivalent of the rational fundamental group. This was extended by Dwyer in [14] to an equivalence between the tame homotopy theory of 2-connected spaces and differential graded Lazard Lie algebras.

In more detail, assume that G is a finitely generated group. It is then possible to replace each nilpotent quotient $N_n = G/\Gamma_n(G)$ by $N_n \otimes \mathbb{Q}$, the (rationally defined) nilpotent Lie group associated to the discrete, torsion-free nilpotent group $N_n/\mathrm{tors}(N_n)$. The corresponding inverse limit,

$$\mathfrak{M}(G) = \varprojlim_n \, (G/\Gamma_n(G) \otimes \mathbb{Q}),$$

is a prounipotent, filtered Lie group over $\mathbb{Q}$, which is called the *prounipotent completion*, or *Malcev completion* of G.

Let us denote by $\mathfrak{L}(K)$ the Lie algebra of a Lie group K. The pronilpotent Lie algebra

$$\mathfrak{m}(G) = \varprojlim_n \mathfrak{L}(G/\Gamma_n(G) \otimes \mathbb{Q}),$$

endowed with the inverse limit filtration, is called the *Malcev Lie algebra* of G. By construction, $\mathfrak{m}(\cdot)$ is a functor from the category of finitely generated groups to the category of complete, separated, filtered Lie algebras over $\mathbb{Q}$.

In [34,35], Quillen showed that $\mathfrak{m}(G)$ is the set of all primitive elements in $\widehat{\mathbb{Q}G}$, the completion of the group algebra of G with respect to the filtration by powers of the augmentation ideal, and that the associated graded Lie algebra of $\mathfrak{m}(G)$ with respect to the inverse limit filtration is isomorphic to $\operatorname{gr}(G; \mathbb{Q})$. Furthermore, the set of all group-like elements in $\widehat{\mathbb{Q}G}$, with multiplication and filtration inherited from $\widehat{\mathbb{Q}G}$, forms a complete, filtered group isomorphic to $\mathfrak{M}(G)$.

3.4 The Sullivan minimal model

In a seminal paper [46], Sullivan showed that commutative differential graded algebras (cdgas) of differential forms over $\mathbb{Q}$ can be used to model rational homotopy theory. From this perspective the commutative differential graded algebra corresponding to the Malcev Lie algebra of a group is obtained by taking the free commutative differential graded algebra Hom dual to nilpotent quotients of the Lie algebra and passing to the limit.

More precisely, Sullivan associated to each space X a cdga over the rationals, denoted $A^*_{\mathrm{PL}}(X)$, for which there is an isomorphism $H^*(A_{\mathrm{PL}}(X)) \cong H^*(X, \mathbb{Q})$ under which induced homomorphisms in cohomology correspond. A space X is said to be *formal* if $A^*_{\mathrm{PL}}(X) \cong (H^*(X; \mathbb{Q}), d = 0)$, i.e., Sullivan's algebra can be connected by a zig–zag of quasi-isomorphisms to the rational cohomology ring of X, endowed with the zero differential.

A *Hirsch extension* (of degree i) is a cdga inclusion $(A, d) \hookrightarrow (A \otimes \bigwedge(V), d)$, where V is a $\mathbb{Q}$-vector space concentrated in degree i, while $\bigwedge(V)$ is the free graded-commutative algebra generated by V, and d sends V into A^{i+1}. A cdga (A, d) is called *minimal* if A is connected (i.e., $A^0 = \mathbb{Q}$), and the following two conditions are satisfied: (1) $A = \bigcup_{j \geq 0} A_j$, where $A_0 = \mathbb{Q}$ and each A_j is a Hirsch extension of A_{j-1}; (2) the differential is decomposable, i.e., $dA \subset A^+ \wedge A^+$, where $A^+ = \bigoplus_{i \geq 1} A^i$. A basic result of Sullivan [46] and Morgan [28] asserts the following: Each connected cdga (A, d) has a minimal model $\mathcal{M}(A)$, unique up to isomorphism.

Suppose now that X is a connected CW-complex with finitely many 1-cells. Then the Lie algebra dual to the first stage of the minimal model associated to $A^*_{\mathrm{PL}}(X)$ is isomorphic to the Malcev Lie algebra $\mathfrak{m}(\pi_1(X))$. A finitely generated group G is said to be 1-*formal* (over $\mathbb{Q}$) if it has a classifying space $K(G, 1)$ which is 1-formal, or, equivalently, if the Malcev Lie algebra $\mathfrak{m}(G)$ is the completion of a quadratic Lie algebra. For a comprehensive discussion of all these notions and more we refer to the monographs [16,17] and to the papers [32,45].

The next step is to look for invariants beyond rational homotopy theory. Chen, Fox, and Lyndon [9] gave examples using Fox derivatives to find groups whose successive quotients in the lower central series have torsion. Stallings [42] related homological properties of a group to successive quotients in the lower central series and also to successive quotients in a mod p descending series. Building on this work, Dwyer [13] related Massey products in the cohomology of a group to properties of the quotients in the lower central series and also to a mod p central series different than the one used by Stallings. In [5–7], Cenkl and Porter used a commutative algebra of differential forms to model tame homotopy theory and the Lazard Lie algebra completion of the fundamental group.

Massey products were defined in [25] and applied to prove the Jacobi identity for Whitehead products. Porter [33] gave a general formula for Massey products in a commutator relators group in terms of coefficients in the Magnus expansions of the relators and provided applications to links. In [26], Matei gave examples of complements of hyperplane arrangements with nonzero mod p Massey products. This shows that while arrangement complements are formal over the rationals—and hence all Massey products with rational coefficients contain zero—they are not necessarily formal over the integers. Recently, Salvatore [39] gave examples of configuration spaces that are not formal over the integers.

4 Holonomy Lie algebras

Among all the Lie algebras one can associate to a group, the simplest is the holonomy Lie algebra, which only depends on data extracted from cohomology in low degrees. In this section we shed new light on the relationship between the holonomy Lie algebra and the associated graded Lie algebra of a group.

4.1 The holonomy Lie algebra of a group

Let G be a group, and fix a coefficient ring $\Bbbk$, which we will take to be either a field or the integers. We will assume throughout that $H = H_1(G, \Bbbk)$ is a finitely generated $\Bbbk$-module; moreover, when $\Bbbk = \mathbb{Z}$, we will assume for simplicity that H is torsion-free. We let $\mathrm{Lie}(H)$ denote the free Lie algebra on the free $\Bbbk$-module H; note that $\mathrm{Lie}_1(H) = H$ and $\mathrm{Lie}_2(H) = H \wedge H$.

Following [8,19,24,30,44,45], we define $\mathfrak{h}(G, \Bbbk)$, the *holonomy Lie algebra* of G, as the quotient of the free Lie algebra on $H_1(G, \Bbbk)$ by the Lie ideal generated by the image of the holonomy map, $(h_2)_*\colon H_2(G, \Bbbk) \to H_1(G, \Bbbk) \wedge H_1(G, \Bbbk)$:

$$\mathfrak{h}(G, \Bbbk) = \mathrm{Lie}(H_1(G, \Bbbk))/\mathrm{ideal}(\mathrm{im}((h_2)_*)).$$

The holonomy Lie algebra of G is a quadratic Lie algebra: it is generated in degree 1 by $\mathfrak{h}_1(G, \Bbbk) = H_1(G, \Bbbk)$, and all the relations are in degree 2. For $\Bbbk = \mathbb{Z}$, we simply write $\mathfrak{h}(G) = \mathfrak{h}(G, \mathbb{Z})$. Clearly, the construction is functorial: every group homomorphism $f\colon G \to H$ induces a morphism of graded Lie algebras, $\mathfrak{h}(f)\colon \mathfrak{h}(G, \Bbbk) \to \mathfrak{h}(H, \Bbbk)$.

As noted in [44], the projection map $\psi_n\colon G \twoheadrightarrow G/\Gamma_n(G)$ induces an isomorphism $\mathfrak{h}(\psi_n)\colon \mathfrak{h}(G) \xrightarrow{\simeq} \mathfrak{h}(G/\Gamma_n(G))$ for all $n \geq 3$. In particular, the holonomy Lie algebra of G depends only on its second nilpotent quotient, $G/\Gamma_3(G)$.

In a completely analogous fashion, one may define the holonomy Lie algebra $\mathfrak{h}(A)$ of a graded, graded-commutative $\Bbbk$-algebra A, provided that $A^0 = \Bbbk$ and A^1 is finite-dimensional (and torsion-free if $\Bbbk = \mathbb{Z}$). It is readily seen that $\mathfrak{h}(A) = \mathfrak{h}(A^{\leq 2})$. Moreover, if G is a group as above, $\mathfrak{h}(G) = \mathfrak{h}(H^*(G;\Bbbk))$. In fact, if X is any path-connected space with $G = \pi_1(X)$, then we may define $\mathfrak{h}(X) := \mathfrak{h}(H^*(X;\Bbbk))$, after which it is easily verified that $\mathfrak{h}(X) \cong \mathfrak{h}(G)$.

On a historical note, the holonomy Lie algebra of a group G was first defined (over $\Bbbk = \mathbb{Q}$) by Chen in [8], and later considered by Kohno in [19] in the case when G is the fundamental group of the complement of a complex projective hypersurface. In [24], Markl and Papadima extended the definition of the holonomy Lie algebra to integral coefficients. Further in-depth studies were done by Papadima–Suciu [30] and Suciu–Wang [44,45]; in particular, the more general case when the group $H = H_1(G,\mathbb{Z})$ is allowed to have torsion is treated in [44].

4.2 A comparison map

Now set $\operatorname{gr}_n(G,\Bbbk) = \operatorname{gr}_n(G)\otimes\Bbbk$, and let $\operatorname{gr}(G,\Bbbk) = \bigoplus_{n\geq 1}\operatorname{gr}_n(G,\Bbbk)$ be the associated graded Lie algebra of G over $\Bbbk$. As shown in [24,30,44,45], there is a (functorially defined) surjective morphism of graded Lie algebras,

$$\mathfrak{h}(G,\Bbbk) \twoheadrightarrow \operatorname{gr}(G;\Bbbk),$$

which restricts to isomorphisms $\mathfrak{h}_n(G,\Bbbk) \to \operatorname{gr}_n(G;\Bbbk)$ for $n \leq 2$.

The above map is an isomorphism if the group G is 1-formal over a field $\Bbbk$ of characteristic 0, but in general it fails to be injective in degrees $n \geq 3$. Nevertheless, for a large class of (not necessarily 1-formal) groups, the map $\mathfrak{h}_3(G,\Bbbk) \to \operatorname{gr}_3(G;\Bbbk)$ is an isomorphism, even for $\Bbbk = \mathbb{Z}$. This will be made more precise in Theorem 4.3 below.

4.3 Another exact sequence

As before, let $h_2\colon X \to K(G/\Gamma_2(G),1)$ be the continuous map induced by the projection of G to $G/\Gamma_2(G)$, and let $(h_2)_*\colon H_2(X) \to H_2(G/\Gamma_2(G))$ be the corresponding holonomy map.

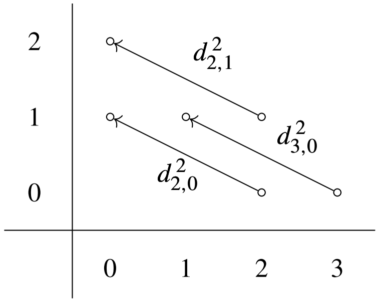
Theorem 4.1 *If $H_1(X)$ is finitely generated and torsion-free, then there is an exact sequence*

$$0 \longrightarrow \mathfrak{h}_3(G) \longrightarrow H_2(G/\Gamma_3(G)) \longrightarrow H_2(X)/(\ker(h_2)_*) \longrightarrow 0. \qquad (4.1)$$

Proof We shall make use of the homology Serre spectral sequence associated to the extension

$$0 \longrightarrow \mathrm{gr}_2(G) \longrightarrow G/\Gamma_3(G) \longrightarrow G/\Gamma_2(G) \longrightarrow 0. \tag{4.2}$$

The E^2 page of this spectral sequence is depicted in diagram (4.3) below.



(4.3)

Assume first that $\mathfrak{h}_2$ is torsion-free, where $\mathfrak{h} = \mathfrak{h}(G)$. Our hypotheses on the abelian groups $G/\Gamma_2(G) = \mathfrak{h}_1 = H_1(X)$ and $\mathrm{gr}_2(G) = \mathfrak{h}_2$ imply that all the terms $E^2_{p,q} = H_p(G/\Gamma_2(G), H_q(\mathfrak{h}_2))$ are finitely generated and torsion-free, and hence, Hom dual to the E_2 terms and differentials d_2 in the cohomology spectral sequence associated to extension (4.2). Since the E_2 terms and differentials form a commutative differential graded algebra, the differentials d_2 are determined by the differential $d_2^{0,1}$ dual to $d^2_{2,0}\colon H_2(G/\Gamma_2(G)) \to H_1(\mathfrak{h}_2)$, which is given by

$$d^2_{2,0}(x_a \wedge x_b) = -[x_a, x_b], \tag{4.4}$$

where $[\cdot,\cdot]$ denotes the bracket map from $\mathfrak{h}_1 \wedge \mathfrak{h}_1$ to $\mathfrak{h}_2$. Computing the differential $d_2^{1,1}$ and $d_2^{0,2}$ and then taking the dual maps gives the following:

$$d^2_{3,0}(x_a \wedge x_b \wedge x_c) = x_a \wedge [x_b, x_c] - x_b \wedge [x_a, x_c] + x_c \wedge [x_a, x_b] \tag{4.5}$$

for $x_a, x_b, x_c \in \mathfrak{h}_1$ and

$$d^2_{2,1}(x_a \wedge x_b \wedge x_d) = -[x_a, x_b] \wedge x_d \tag{4.6}$$

for $x_a, x_b \in \mathfrak{h}_1$ and $x_d \in \mathfrak{h}_2$.

From (4.4) and (4.6) it follows that $E^3_{0,2} = 0$, while from (4.5) it follows that $E^3_{1,1} = \mathfrak{h}_3$. Now note that $E^3_{2,0}$ is the kernel of the map $d^2_{2,0}$ in equation (4.4). From the formula for $d^2_{2,0}$ in (4.4) and the exact sequence (2.6), it follows that $E^3_{2,0}$ is the image of $H_2(X)$ in $H_2(G/\Gamma_2(G))$, which is $H_2(X)/(\ker(h_2)_*)$.

Looking at the domains and ranges of the higher-order differentials in the spectral sequence, we see that since $E^3_{0,2} = 0$, it follows that $E^3_{p,q} = E^\infty_{p,q}$ for $p+q \leq 2$. We conclude that

$$E^{\infty}_{0,2} = 0, \quad E^{\infty}_{1,1} = \mathfrak{h}_3, \quad \text{and} \quad E^{\infty}_{2,0} = H_2(X)/(\ker(h_2)_*). \tag{4.7}$$

Equation (4.1) now follows, and the proof of the lemma is complete in the case where $\mathfrak{h}_2(G)$ is torsion-free.

In the case where $\mathfrak{h}_2$ has torsion, let $x_1, \dots, x_\ell$ be elements in G that project to a basis for $G/\Gamma_2(G)$. Set $\mathcal{F}$ equal to the free group on the generators x_i, and note that $\mathfrak{h}_1(\mathcal{F}) = \mathfrak{h}_1$ and $\mathfrak{h}_2(\mathcal{F}) = \mathfrak{h}_1 \wedge \mathfrak{h}_1$. The identity map of generators gives a map of central extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{h}_1 \wedge \mathfrak{h}_1 & \longrightarrow & \mathcal{F}/\Gamma_3(\mathcal{F}) & \longrightarrow & \mathcal{F}/\Gamma_2(\mathcal{F}) & \longrightarrow & 0 \\ & & \Big\downarrow {\scriptstyle [\cdot,\cdot]} & & \Big\downarrow & & \Big\downarrow & & \\ 0 & \longrightarrow & \mathfrak{h}_2 & \longrightarrow & G/\Gamma_3(G) & \longrightarrow & G/\Gamma_2(G) & \longrightarrow & 0, \end{array}$$

and hence a map of the respective homology spectral sequences. By the argument above, equations (4.4), (4.5), and (4.6) hold in the spectral sequence for $\mathcal{F}$ and hence in the spectral sequence for G. Moreover, each of the maps of the E^2 terms involved in these equations is onto, so it follows that the equations in (4.7) hold for G as well. This completes the proof. □

Remark 4.2 If the group $\mathfrak{h}_2 = \mathfrak{h}_2(G)$ is torsion-free, then the commutative differential graded algebra (E_2, d_2) is the Chevalley–Eilenberg cochain complex [4] of the Lie algebra $\mathfrak{h}/\Gamma_3(\mathfrak{h})$. Since $E^3_{p,q} = E^{\infty}_{p,q}$ for $p+q=2$, it follows that the Lie algebra homology group $H_2(\mathfrak{h}/\Gamma_3(\mathfrak{h}))$ is isomorphic to $H_2(G/\Gamma_3(G))$.

4.4 Identifying $\mathfrak{h}_3(G)$ with $\mathrm{gr}_3(G)$

We are now ready to state and prove the main result of this section. A proof of this theorem was first sketched by Rybnikov in [37, Section 3]; we provide here an alternate proof, with full details.

Theorem 4.3 *Suppose $H = H_1(G;\mathbb{Z})$ is a finitely-generated, free abelian group, and the holonomy map $(h_2)_*\colon H_2(G) \to H \wedge H$ is injective. Then the canonical projection $\mathfrak{h}_3(G) \to \mathrm{gr}_3(G)$ is an isomorphism.*

Proof Consider the homology spectral sequence of the exact sequence from (4.2), whose E^2 page is pictured in diagram (4.3). As in the proof of Theorem 3.1, let F^1 denote the image in $H_2(G/\Gamma_3(G))$ of the inverse image of the 1-skeleton of $K(G/\Gamma_2(G), 1)$.

The proof of Theorem 4.1 shows that if H is torsion-free, then $F^1 \cong \mathfrak{h}_3(G)$.

The proof of Theorem 3.1 shows that if H is torsion-free and the holonomy map $(h_2)_*$ is injective, then $F^1 \cong \mathrm{gr}_3(G)$, and the result follows. □

5 Second cohomology of nilpotent groups and associated *k*-invariants

The purpose of this section is to use the exact sequence in equation (3.1) to relate homomorphisms from $H_2(X)$ to $\mathrm{gr}_n(G)$ to the possible k-invariants of the central extension of $G/\Gamma_n(G)$ to $G/\Gamma_{n+1}(G)$ from (2.2). Throughout this section, homology will be taken with coefficients in $\mathbb{Z}$.

In general, given an exact sequence of abelian groups

$$0 \longrightarrow A \xrightarrow{\ i\ } B \xrightarrow{\ j\ } C \longrightarrow 0, \tag{5.1}$$

a map $\sigma\colon B \to A$ with $\sigma \circ i = \mathrm{id}_A$ is called a *left splitting* and a map $h\colon C \to B$ with $j \circ h = \mathrm{id}_C$ is called a *right splitting*. Recall the following well-known fact: The exact sequence in (5.1) splits (either on the left or the right) if and only if $B \cong A \oplus C$. Furthermore, the direct sum decompositions of this sort are in one-to-one correspondence with splitting maps $B \to A$ (or $C \to B$). Moreover, as shown in the proof of Lemma 5.1 below, a splitting yields a bijection between all splittings and the set of homomorphisms from C to A.

Once again, let X be a path-connected space such that $H_1(X)$ is finitely generated and torsion-free, and such that the holonomy map $(h_2)_*\colon H_2(X) \to H_1(X) \wedge H_1(X)$ is injective. Set $G = \pi_1(X)$. Recall from Theorem 3.1 that for $n \geq 3$ there is a split exact sequence

$$0 \longrightarrow \mathrm{gr}_n(G) \xrightarrow{\ i\ } H_2(G/\Gamma_n(G)) \xrightarrow{\ \pi\ } H_2(X) \longrightarrow 0 \tag{5.2}$$

and from (2.6), the k-invariant χ_n gives a splitting; that is, in the diagram

$$0 \longrightarrow \mathrm{gr}_n(G) \underset{\chi_n}{\overset{i}{\rightleftarrows}} H_2(G/\Gamma_n(G)) \underset{(h_n)_*}{\overset{\pi}{\rightleftarrows}} H_2(X) \longrightarrow 0 \tag{5.3}$$

the map $\chi_n \circ i$ is the identity on $\mathrm{gr}_n(G)$, while $\pi \circ (h_n)_*$ is the identity on $H_2(X)$ and $\ker(\chi_n) = \mathrm{im}(h_n)_*$.

Lemma 5.1 *For $n \geq 3$, any homomorphism $\sigma\colon H_2(G/\Gamma_n(G)) \to \mathrm{gr}_n(G)$ with $\sigma \circ i$ equal to the identity on $\mathrm{gr}_n(G)$ yields a bijection between splittings of the exact sequence* (5.2) *and elements in* $\mathrm{Hom}(H_2(X), \mathrm{gr}_n(G))$.

Proof The map σ gives an isomorphism between $H_2(G/\Gamma_n(G))$ and $\mathrm{gr}_n(G) \oplus H_2(X)$. Without loss of generality, we can assume that via this isomorphism the inclusion i and the projection π in (5.3) correspond to the maps $\widetilde{i}$ and $\widetilde{\pi}$ in the diagram below

$$0 \longrightarrow \mathrm{gr}_n(G) \xrightarrow{\ \widetilde{i}\ } \mathrm{gr}_n(G) \oplus H_2(X) \xrightarrow{\ \widetilde{\pi}\ } H_2(X) \longrightarrow 0, \tag{5.4}$$

where $\widetilde{i}$ is the inclusion into the first coordinate and $\widetilde{\pi}$ is the projection onto the second coordinate.

An element $\lambda \in \mathrm{Hom}(H_2(X), \mathrm{gr}_n(G))$ determines a splitting of (5.4) as follows. Given λ, define a map $h\colon H_2(X) \to \mathrm{gr}_n(G) \oplus H_2(X)$ by $h(c) = (\lambda(c), c)$, and define $\chi\colon \mathrm{gr}_n(G) \oplus H_2(X) \to \mathrm{gr}_n(G)$ by $\chi(x, c) = x - \lambda(c)$.

It is straightforward to check that in the diagram

$$0 \longrightarrow \mathrm{gr}_n(G) \underset{\chi}{\overset{\widetilde{i}}{\rightleftarrows}} \mathrm{gr}_n(G) \oplus H_2(X) \underset{h}{\overset{\widetilde{\pi}}{\rightleftarrows}} H_2(X) \longrightarrow 0 \tag{5.5}$$

the map $\chi \circ \widetilde{i}$ is the identity on $\mathrm{gr}_n(G)$, while $\widetilde{\pi} \circ h$ is the identity on $H_2(X)$ and $\ker(\chi) = \mathrm{im}(h)$. Every homomorphism $h\colon H_2(X) \to \mathrm{gr}_n(G) \oplus H_2(X)$ with $\widetilde{\pi} \circ h$ equal to the identity on $H_2(X)$ has the form $h(c) = (\lambda(c), c)$ and the lemma follows. □

In the context of the Postnikov tower (2.5) and the exact sequence in (5.2), this leads to a formula for the k-invariant of the extension (2.2) from $G/\Gamma_n(G)$ to $G/\Gamma_{n+1}(G)$ for a fixed $n \geq 3$, in terms of a splitting map $\sigma\colon H_2(G/\Gamma_n(G)) \to \mathrm{gr}_n(G)$ and a map $h_n\colon X \to K(G/\Gamma_n(G), 1)$ corresponding to the projection of G to $G/\Gamma_n(G)$.

Corollary 5.2 *With assumptions and notation as above, the k-invariant of the extension* $0 \to \mathrm{gr}_2(G) \to G/\Gamma_3(G) \to G/\Gamma_2(G) \to 0$ *with respect to the direct sum decomposition given by the splitting* $\sigma\colon H_2(G/\Gamma_n(G)) \to \mathrm{gr}_n(G)$ *is the element*

$$\chi_n \in \mathrm{Hom}(H_2(G/\Gamma_n(G)), \mathrm{gr}_n(G)) \cong H^2(G/\Gamma_n(G); \mathrm{gr}_n(G))$$

given by the homomorphism $\chi_n(x, c) = x - \lambda(c)$*, where* $\lambda = \sigma \circ (h_n)_*\colon H_2(X) \to \mathrm{gr}_n(G)$.

Proof The claim follows from Lemma 5.1 and the observation that for the map h in (5.5), we have that $\lambda = \sigma \circ h$. □

Example 5.3 We illustrate the above corollary with a simple example (for a more general context, see Proposition 7.2 below). Let X be a wedge of ℓ circles, so that $G = \pi_1(X)$ is isomorphic to F_ℓ, the free group of rank ℓ. Identifying $G/\Gamma_2(G) = \mathbb{Z}^\ell$ and $\mathrm{gr}_2(G) = \bigwedge^2 \mathbb{Z}^\ell$, the second nilpotent quotient $N = G/\Gamma_3(G)$ fits into a central extension

$$0 \longrightarrow \textstyle\bigwedge^2 \mathbb{Z}^\ell \longrightarrow N \xrightarrow{q_2} \mathbb{Z}^\ell \longrightarrow 0. \tag{5.6}$$

Note that $H_2(X) = 0$, and so the homomorphism $\lambda\colon H_2(X) \to \mathbb{Z}^\ell$ is the zero map. Hence, by Corollary 5.2, the extension (5.6) is classified by the k-invariant $\chi_2 = \mathrm{id} \in \mathrm{Hom}(\bigwedge^2 \mathbb{Z}^\ell, \bigwedge^2 \mathbb{Z}^\ell)$.

6 Generalizations of Rybnikov's theorem

6.1 The setup

Let X be a connected CW-complex. We will assume throughout that the homology group $H_1(X)$ is finitely generated and torsion-free, and that the holonomy map $(h_2)_*\colon H_2(X) \to H_1(X) \wedge H_1(X)$ is injective.

Let $G = \pi_1(X)$, and fix an integer $n \geq 2$. Recall from (2.7) the exact sequence

$$H_2(X) \xrightarrow{(h_n)_*} H_2(G/\Gamma_n(G)) \longrightarrow \mathrm{gr}_n(G) \longrightarrow 0,$$

where the map $h_n\colon X \to K(G/\Gamma_n(G), 1)$ is induced by the projection of $G \twoheadrightarrow G/\Gamma_n(G)$. If N is a nilpotent group with $N \cong N/\Gamma_n(N) \cong G/\Gamma_n(G)$, then Theorem 3.1 gives a split exact sequence

$$0 \longrightarrow \mathrm{gr}_n(G) \longrightarrow H_2(N) \longrightarrow H_2(X) \longrightarrow 0. \tag{6.1}$$

Now let X_a and X_b be two spaces as above and let G_a, and G_b be the respective fundamental groups. Suppose there is a map $g\colon H^{\leq 2}(X_b) \to H^{\leq 2}(X_a)$ which is an isomorphism of graded rings. Set $\overline{g}\colon H_{\leq 2}(X_a) \to H_{\leq 2}(X_b)$ equal to the dual to g. Then

- There is an isomorphism $G_a/\Gamma_3(G_a) \xrightarrow{\simeq} G_b/\Gamma_3(G_b)$.
- The isomorphism $\overline{g}_1\colon H_1(X_a) \to H_1(X_b)$ induces an isomorphism $\overline{g}_\#\colon \mathfrak{h}_3(G_a) \to \mathfrak{h}_3(G_b)$.

If $f\colon G_a \to G_b$ is a group homomorphism, we will denote by $f_n\colon G_a/\Gamma_n(G_a) \to G_b/\Gamma_n(G_b)$ the induced homomorphisms between the respective nilpotent quotients.

6.2 Statement and proof of the theorem

We are ready now to state and proof our generalization of [37, Theorem 12].

Theorem 6.1 *With the assumptions above, fix $n \geq 3$, let $\sigma_a\colon H_2(G_a/\Gamma_n(G_a)) \to \mathrm{gr}_n(G_a)$ be any left splitting of the exact sequence (6.1), and let $f_n\colon G_a/\Gamma_n(G_a) \to G_b/\Gamma_n(G_b)$ be any isomorphism that extends the map $\overline{g}_1\colon G_a/\Gamma_2(G_a) \to G_b/\Gamma_2(G_b)$. The following conditions are then equivalent:*

1. *The map $\overline{g}_1$ extends to an isomorphism $f_{n+1}\colon G_a/\Gamma_{n+1}(G_a) \xrightarrow{\simeq} G_b/\Gamma_{n+1}(G_b)$.*

2. *There are liftings $h_n^c\colon X_c \to K(G_c/\Gamma_n(G_c), 1)$ for $c = a$ and b such that the following diagram commutes:*

$$
\begin{array}{ccccc}
 & \mathrm{gr}_n(G_a) & \xrightarrow[\cong]{\overline{g}_\#} & \mathrm{gr}_n(G_b) & \\
 & \uparrow{\scriptstyle \sigma_a} & & \uparrow{\scriptstyle \sigma_b} & \\
\lambda_b & H_2(G_a/\Gamma_n(G_a)) & \xrightarrow{(f_n)_*} & H_2(G_b/\Gamma_n(G_b)) & \lambda_b \\
 & \uparrow{\scriptstyle (h_n^a)_*} & & \uparrow{\scriptstyle (h_n^b)_*} & \\
 & H_2(X_a) & \xrightarrow[\cong]{\overline{g}_2} & H_2(X_b). &
\end{array}
\tag{6.2}
$$

In the above diagram, the map $\overline{g}_\#$ is the restriction of the map $(f_n)_*$ between the respective extensions of type (6.1).

Proof First we show that if there is a commutative diagram such as the one above, then the isomorphism $f_n\colon G_a/\Gamma_n(G_a) \xrightarrow{\cong} G_b/\Gamma_n(G_b)$ extends to an isomorphism $G_a/\Gamma_{n+1}(G_a) \xrightarrow{\cong} G_b/\Gamma_{n+1}(G_b)$.

From the commutativity of diagram (6.2), it follows that σ_b is a left splitting. Using the direct sum decompositions given by the left splittings, we may define maps

$$\kappa_c\colon H_2(G_c/\Gamma_n(G_c)) \cong \mathfrak{h}_n(G_c) \oplus H_2(X_c) \to \mathfrak{h}_n(G_c)$$

for $c = a$ or b by

$$\kappa_c(x, y) = x - \lambda_c(y).$$

Consider now the homology spectral sequences associated to the extensions (2.2) for $G = G_a$ and $G = G_b$, respectively. From the naturality of the Serre spectral sequence and the commutativity of the aforementioned diagram, it follows that, with respect to the direct sum decompositions, the map $(f_n)_*$ corresponds to the map $(x, y) \to (g_\#(y), \overline{g}_2(y))$. Thus, the following diagram is commutative:

$$
\begin{array}{ccc}
\mathfrak{h}_n(G_a) & \xrightarrow{\overline{g}_\#} & \mathfrak{h}_n(G_b) \\
\uparrow{\scriptstyle \kappa_a} & & \uparrow{\scriptstyle \kappa_b} \\
\mathfrak{h}_n(G_a) \oplus H_2(X_a) & \xrightarrow{(f_n)_*} & \mathfrak{h}_n(G_b) \oplus H_2(X_b).
\end{array}
\tag{6.3}
$$

Let $E(\kappa_a)$ and $E(\kappa_b)$ be the central extensions with k-invariants κ_a and κ_b, respectively. Then from the commutativity of the diagram in (6.3) it follows that f_n lifts to an isomorphism $\widetilde{f}_n\colon E(\kappa_a) \to E(\kappa_b)$. On the other hand, Corollary 5.2 implies that $E(\kappa_a) = K(G_a/\Gamma_{n+1}(G_a), 1)$ and $E(\kappa_b) = K(G_b/\Gamma_{n+1}(G_b), 1)$, and this completes the proof of the first part of the theorem.

To prove the reverse implication, assume that a left splitting $\sigma_a\colon H_2(G_a/\Gamma_n(G_a)\to \mathfrak{h}_n(G_a)$ and an isomorphism $f_n\colon G_a/\Gamma_n(G_a)\to G_b/\Gamma_n(G_b)$ are given; we will then show that there is a commutative diagram of the form (6.2).

Let $e_{n+1}\colon G_a/\Gamma_{n+1}(G_a)\to G_b/\Gamma_{n+1}(G_b)$ be an isomorphism. The first step is to prove that there is a commutative diagram

$$\begin{array}{ccc} H_2(G_a/\Gamma_n(G_a)) & \xrightarrow{(f_n)_*} & H_2(G_b/\Gamma_n(G_b)) \\ \Big\uparrow{\scriptstyle (h_n^a)_*} & & \Big\uparrow{\scriptstyle (h_n^b)_*} \\ H_2(X_a) & \xrightarrow[\cong]{\overline{g}_2} & H_2(X_b). \end{array} \tag{6.4}$$

Let e_n be the isomorphism from $G_a/\Gamma_n(G_a)$ to $G_b/\Gamma_n(G_b)$ induced by e_{n+1}. Then e_n gives rise to the following commutative diagram in the tower of nilpotent quotients:

$$\begin{array}{ccccccc} & \overset{\widetilde{h}_n^a}{\nearrow} & K(G_a/\Gamma_n(G_a),1) & \xrightarrow{e_n} & K(G_b/\Gamma_n(G_b),1) & \overset{h_n^b}{\nwarrow} & \\ & & \Big\downarrow{\scriptstyle \pi_n^a} & & \Big\downarrow{\scriptstyle \pi_n^b} & & \\ X_a & \xrightarrow{h_2^a} & K(G_a/\Gamma_2(G_a),1) & \xrightarrow{e_2} & K(G_b/\Gamma_2(G_b),1) & \xleftarrow{h_2^b} & X_b. \end{array} \tag{6.5}$$

Since e_n and f_n are both extensions of $\overline{g}$, it follows that the automorphism $e_n^{-1}\circ f_n\colon G_a/\Gamma_n(G_a)\to G_a/\Gamma_n(G_a)$ is an extension of the identity map. This gives the following commutative diagram:

$$\begin{array}{ccccccc} & \overset{h_n^a}{\nearrow} & K(G_a/\Gamma_n(G_a),1) & \xrightarrow{e_n^{-1}\circ f_n} & K(G_a/\Gamma_n(G_a),1) & \overset{\widetilde{h}_n^a}{\nwarrow} & \\ & & \Big\downarrow{\scriptstyle \pi_n^a} & & \Big\downarrow{\scriptstyle \pi_n^a} & & \\ X_a & \xrightarrow[h_2^a]{} & K(G_a/\Gamma_2(G_a),1) & \xrightarrow{\mathrm{id}} & K(G_a/\Gamma_2(G_a),1) & \xleftarrow[h_2^a]{} & X_a. \end{array} \tag{6.6}$$

Putting diagrams (6.5) and (6.6) together and passing to homology gives the commutative diagram (6.4). Then the left splitting σ_a determines a left splitting σ_b such that the upper rectangle in the diagram (6.2) commutes, and the proof of the theorem is complete. □

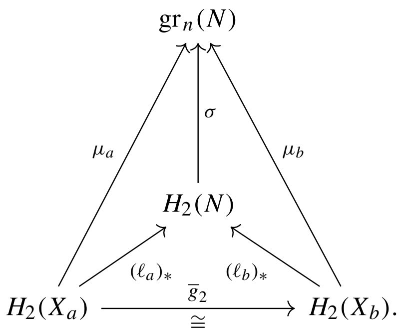
6.3 Further refinements

The above proof shows the following: if the map $h_2^b\colon X_b\to K(G_b/\Gamma_2(G_b),1)$ is given, and if $f_n\colon G_a/\Gamma_n(G_a)\xrightarrow{\simeq} G_b/\Gamma_n(G_b)$ is an isomorphism, then there is an extension of f_n to an isomorphism $f_{n+1}\colon G_a/\Gamma_{n+1}(G_a)\xrightarrow{\simeq} G_b/\Gamma_{n+1}(G_b)$ if and only if there is a lifting $h_n^a\colon X\to K(G_a/\Gamma_n(G_a),1)$ such that diagram (6.2) commutes. The next theorem recasts this result in a more compact fashion.

Theorem 6.2 *With notation and assumptions as above, suppose N is a nilpotent group with $N \cong N/\Gamma_n(N)$ and that the map $\ell_b\colon X_b \to K(N,1)$ induces an isomorphism $G_b/\Gamma_n(G_b) \xrightarrow{\cong} N$. Let $\sigma\colon H_2(N) \to \mathrm{gr}_n(N)$ be a splitting of the exact sequence (6.1) and $f_n\colon G_a/\Gamma_n(G_a) \to G_b/\Gamma_n(G_b)$ an isomorphism. Then there is an isomorphism*

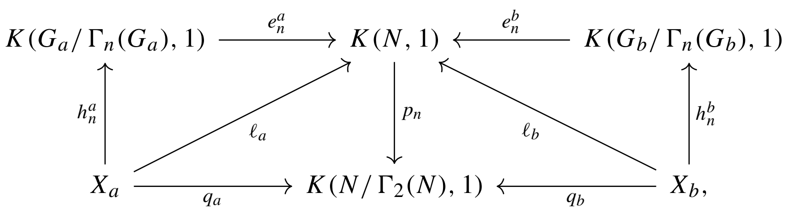
$$f_{n+1}\colon G_a/\Gamma_{n+1}(G_a) \xrightarrow{\cong} G_b/\Gamma_{n+1}(G_b)$$

extending f_n if and only if there is a map $\ell_a\colon X_a \to K(N,1)$ inducing an isomorphism $G_a/\Gamma_n(G_a) \xrightarrow{\cong} N$ such that the following diagram commutes:



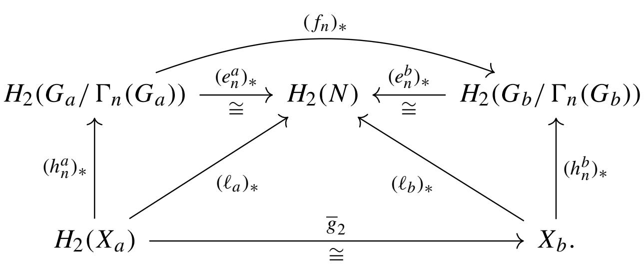
(6.7)

Proof Let e_n^b be a isomorphism from $K(G_b/\Gamma_n(G_b),1)$ to $K(N,1)$ and consider the following diagram:



(6.8)

where $e_n^a = e_n^b \circ f_n$ and q_a is determined by the condition that on the first homology groups $\overline{q}_a = \overline{q}_b \circ \overline{g}_1$. The corresponding diagram of homology groups and maps is



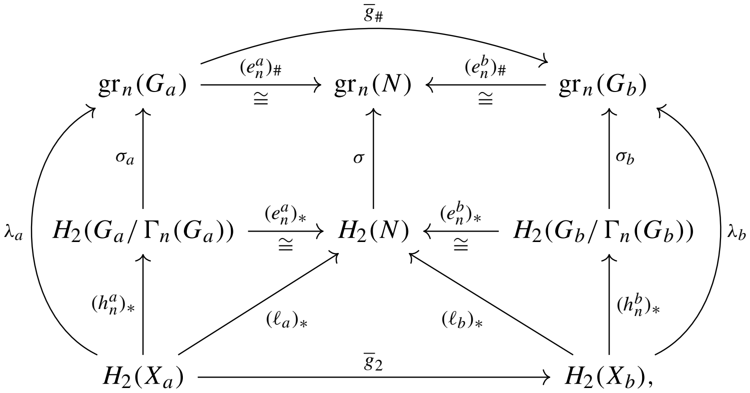
(6.9)

Since the maps e_n^a and e_n^b in (6.8) are isomorphisms, it follows that there is a bijection between liftings ℓ_a and h_n^a in (6.8), and also a bijection between liftings ℓ_b and h_n^b.

Moreover, in (6.9)

$$(\ell_a)_* = (\ell_b)_* \circ \overline{g}_2 \quad \Longleftrightarrow \quad (h_n^b)_* \circ \overline{g}_2 = (f_n)_* \circ (h_n^a)_*. \tag{6.10}$$

Consider now the diagram



(6.11)

where the maps $(e_n^a)_\#$ and $(e_n^b)_\#$ are induced by the corresponding isomorphisms of groups e_n^a and e_n^b, and the splittings σ_a and σ_b are defined by requiring that the top two rows in (6.11) form a commutative diagram.

From (6.10) and a diagram chase, it follows that (6.11) commutes if and only if the corresponding diagram (6.2) commutes, and also that (6.11) commutes if and only if diagram (6.7) commutes with $\mu_a = (e_n^a)_\# \circ \sigma_a \circ (h_n^a)_*$ and $\mu_b = (e_n^b)_\# \circ \sigma_b \circ (h_n^b)_*$. The desired conclusion follows. □

Remark 6.3 In the work of Rybnikov [36,37], it is assumed that the groups $\mathfrak{h}_2$ and $\mathfrak{h}_3$ are torsion-free. Then replacing the modules and maps in Theorem 6.1 for $n = 3$ with their Hom duals yields [37, item 2 of Theorem 12]. The result in Theorem 6.2 for $n = 3$ corresponds to [36, Theorem 2.2].

6.4 The Stallings mod *p* lower central series

Let G be a group, and let $p = 0$ or a prime. Following Stallings [42], define subgroups $\Gamma_n^p(G) < G$ as follows:

$$\Gamma_1^p(G) = G,$$
$$\Gamma_{n+1}^p(G) = \langle gug^{-1}u^{-1}v^p \mid g \in G, u, v \in \Gamma_n^p(G) \rangle,$$

where $\langle U \rangle$ denotes the subgroup generated by a subset $U \subset G$. Then $\{\Gamma_n^p(G)\}_{n\geq 1}$ is a descending central series of normal subgroups. For $p = 0$ it is the lower central series; for $p \neq 0$ it is the most rapidly descending central series whose successive quotients are vector spaces over the field of p elements. If we set $\mathrm{gr}_n^p(G) = \Gamma_n^p(G)/\Gamma_{n+1}^p(G)$, then $\mathrm{gr}^p(G) := \bigoplus_{n\geq 1} \mathrm{gr}_n^p(G)$ is a graded Lie algebra over $\mathbb{Z}_p$ in a natural way.

Now let X be a path-connected space, and $G = \pi_1(X)$. For the remainder of this section all homology groups are with $\mathbb{Z}_p$ coefficients, where $\mathbb{Z}_0$ denotes the integers. As shown in [42], there is an exact sequence

$$H_2(X) \xrightarrow{(h_n)_*} H_2(G/\Gamma_n^p(G)) \longrightarrow \mathrm{gr}_n^p(G) \longrightarrow 0, \tag{6.12}$$

where the map $h_n\colon X \to G/\Gamma_n^p(G)$ is induced by the projection of $G = \pi_1(X)$ to $G/\Gamma_n^p(G)$. The proof of Theorem 3.1 extends to show that if $(h_2)_*$ is a monomorphism and N is a nilpotent group with $N \cong N/\Gamma_n^p(N) \cong G/\Gamma_n^p(G)$, then there is a split exact sequence

$$0 \longrightarrow \mathrm{gr}_n^p(G) \longrightarrow H_2(N) \longrightarrow H_2(X) \longrightarrow 0 \tag{6.13}$$

for all $n \geq 3$.

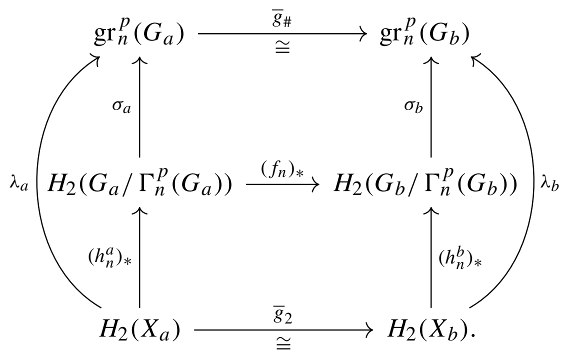
6.5 An extension of Rybnikov's theorem in characteristic *p*

Let X_a and X_b be path-connected spaces with $G_a = \pi_1(X_a)$ and $G_b = \pi_1(X_b)$. Assume $p = 0$ or p a prime has been chosen; all homology groups in the following theorem are with $\mathbb{Z}_p$ coefficients. Assume also that $H_1(X_a)$ and $H_1(X_b)$ are finitely generated, and the respective maps $(h_2)_*$ are monomorphisms.

Suppose we are given an isomorphism $g\colon H^{\leq 2}(X_b) \to H^{\leq 2}(X_a)$ of graded algebras. Set $\overline{g}\colon H_{\leq 2}(X_a) \to H_{\leq 2}(X_b)$ equal to the dual to g. Then given the exact sequences from (6.12) and (6.13), the steps in the proof of Theorem 6.1 apply to prove the following.

Theorem 6.4 *With the assumptions above, fix $n \geq 3$, let $\sigma_a\colon H_2(G_a/\Gamma_n^p(G)) \to \mathrm{gr}_n^p(G_a)$ be any left splitting of the exact sequence (6.13), and let $f_n\colon G_a/\Gamma_n^p(G_a) \to G_b/\Gamma_n^p(G_b)$ be any isomorphism that extends the map $\overline{g}_1\colon G_a/\Gamma_2^p(G_a) \to G_b/\Gamma_2^p(G_b)$. The following conditions are then equivalent:*

1. *The map $\overline{g}_1$ extends to an isomorphism $f_{n+1}\colon G_a/\Gamma_{n+1}^p(G_a) \xrightarrow{\simeq} G_b/\Gamma_{n+1}^p(G_b)$.*
2. *There are liftings $h_n^c\colon X_c \to K(G_c/\Gamma_n^p(G_c), 1)$ for $c = a$ and b such that the following diagram commutes:*

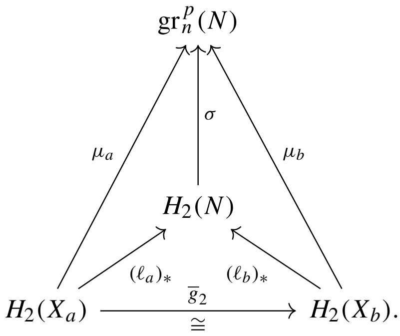


In the above diagram, the map $\overline{g}_\#$ is the restriction of the map $(f_n)_*$ between the respective extensions of type (6.13). The reasoning from Theorem 6.2 generalizes to show that Theorem 6.4 implies the following result.

Theorem 6.5 *With the assumptions as in Theorem 6.4, assume N is a nilpotent group with $N \cong N/\Gamma_n^p(N)$ and that $\ell_b\colon X_b \to K(N,1)$ induces an isomorphism $G_b/\Gamma_n^p(G_b) \xrightarrow{\simeq} N$. Let σ be a splitting of the exact sequence (6.13). Then there is an isomorphism*

$$f_{n+1}\colon G_a/\Gamma_{n+1}^p(G_a) \xrightarrow{\cong} G_b/\Gamma_{n+1}^p(G_b)$$

extending $\overline{g}_2$ if and only if there is a map $\ell_a\colon X_a \to K(N,1)$ inducing an isomorphism of $G_a/\Gamma_n^p(G_a) \to N$ such that the following diagram commutes:



7 Hyperplane arrangements

We now apply the tools developed in the previous sections to a class of spaces that arise in a combinatorial context. These spaces—complements of complex hyperplane arrangements—have motivated to a large extent the approach taken here, and provide a blueprint for further applications.

7.1 Complement and intersection lattice

We start with a brief review of arrangement theory; for details and references, we refer to the monograph of Orlik and Terao [29].

An *arrangement of hyperplanes* is a finite set $\mathcal{A}$ of codimension-1 linear subspaces in a finite-dimensional, complex vector space $\mathbb{C}^n$. The combinatorics of the arrangement is encoded in its *intersection lattice*, $L(\mathcal{A})$, that is, the poset of all intersections of hyperplanes in $\mathcal{A}$ (also known as flats), ordered by reverse inclusion, and ranked by codimension. For a flat $Y = \bigcap_{H \in \mathcal{B}} H$ defined by a sub-arrangement $\mathcal{B} \subset \mathcal{A}$, we let rank $Y = \operatorname{codim} Y$; we also write $L_k(\mathcal{A}) = \{Y \in L(\mathcal{A}) \mid \operatorname{rank} Y = k\}$.

The main topological invariant associated to such an arrangement $\mathcal{A}$ is its *complement*, $M(\mathcal{A}) = \mathbb{C}^n \setminus \bigcup_{H \in \mathcal{A}} H$. This is a connected, smooth complex quasi-projective variety. Moreover, $M(\mathcal{A})$ is a Stein manifold, and thus has the homotopy type of a finite CW-complex of dimension at most n.

Probably the best-known example is the braid arrangement $\mathcal{A}_n$, consisting of the diagonal hyperplanes in $\mathbb{C}^n$. It is readily seen that $L(\mathcal{A}_n)$ is the lattice of partitions of $[n] = \{1, \dots, n\}$, ordered by refinement, while $M(\mathcal{A}_n)$ is the configuration space $F(\mathbb{C}, n)$ of n ordered points in $\mathbb{C}$. In the early 1960s, Fox, Neuwirth, and Fadell showed that $M(\mathcal{A}_n)$ is a classifying space for P_n, the pure braid group on n strings.

For a general arrangement $\mathcal{A}$, the cohomology ring $H^*(M(\mathcal{A}), \mathbb{Z})$ was computed by Brieskorn in the early 1970s, building on pioneering work of Arnol'd on the cohomology ring of the braid arrangement. It follows from Brieskorn's work that the space $M(\mathcal{A})$ is formal over $\mathbb{Q}$. Consequently, the fundamental group of the complement, $G(\mathcal{A}) = \pi_1(M(\mathcal{A}), x_0)$, is 1-formal over $\mathbb{Q}$.

In 1980, Orlik and Solomon gave a simple combinatorial description of the ring $H^*(M(\mathcal{A}), \mathbb{Z})$: it is the quotient $E(\mathcal{A})/I(\mathcal{A})$ of the exterior algebra $E(\mathcal{A})$ on classes dual to the meridians around the hyperplanes, modulo a certain ideal $I(\mathcal{A})$ defined in terms of the intersection lattice of $\mathcal{A}$. In particular, the cohomology ring of the complement is *combinatorially determined*; that is to say, if $\mathcal{A}$ and $\mathcal{B}$ are arrangements with $L(\mathcal{A}) \cong L(\mathcal{B})$, then $H^*(M(\mathcal{A}), \mathbb{Z}) \cong H^*(M(\mathcal{B}), \mathbb{Z})$.

7.2 Localized sub-arrangements

The *localization* of an arrangement $\mathcal{A}$ at a flat $Y \in L(\mathcal{A})$ is defined as the sub-arrangement

$$\mathcal{A}_Y = \{H \in \mathcal{A} \mid H \supset Y\}.$$

The inclusion $\mathcal{A}_Y \subset \mathcal{A}$ gives rise to an inclusion of complements, $j_Y \colon M(\mathcal{A}) \hookrightarrow M(\mathcal{A}_Y)$. Choosing a point x_0 sufficiently close to $0 \in \mathbb{C}^n$, we can make it a common basepoint for both $M(\mathcal{A})$ and all local complements $M(\mathcal{A}_Y)$.

Lemma 7.1 [12] *There exist basepoint-preserving maps $r_Y \colon M(\mathcal{A}_Y) \to M(\mathcal{A})$ such that $j_Y \circ r_Y \simeq \mathrm{id}$ relative to x_0. Moreover, if $H \in \mathcal{A}$ and $H \not\supset Y$, then the composite $r_Y \circ j_Y \circ r_H$ is null-homotopic.*

In particular, if we set $G(\mathcal{A}_Y) = \pi_1(M(\mathcal{A}_Y), x_0)$, then the induced homomorphisms $(r_Y)_\# \colon G(\mathcal{A}_Y) \to G(\mathcal{A})$ are all injective.

The inclusions $\{j_Y\}_{Y \in L(\mathcal{A})}$ assemble into a map

$$j \colon M(\mathcal{A}) \to \prod_{Y \in L(\mathcal{A})} M(\mathcal{A}_Y). \tag{7.1}$$

The classical Brieskorn Lemma insures that the induced homomorphism in cohomology is an isomorphism in each degree $k \geq 1$. By the Künneth formula, then, we have that

$$H^k(M(\mathcal{A}), \mathbb{Z}) \cong \bigoplus_{Y \in L_k(\mathcal{A})} H^k(M(\mathcal{A}_Y), \mathbb{Z})$$

for all $k \geq 1$. Likewise, the Orlik–Solomon ideal decomposes in each degree as

$$I^k(\mathcal{A}) \cong \bigoplus_{Y \in L_k(\mathcal{A})} I^k(\mathcal{A}_Y).$$

It follows that the homology groups of the complement of $\mathcal{A}$ are torsion-free, with ranks given by

$$b_k(M(\mathcal{A})) = \sum_{Y \in L_k(\mathcal{A})} (-1)^k \mu(Y),$$

where $\mu\colon L(\mathcal{A}) \to \mathbb{Z}$ is the Möbius function of the intersection lattice, defined inductively by $\mu(\mathbb{C}^n) = 1$ and $\mu(Y) = -\sum_{Z \supsetneq Y} \mu(Z)$. In particular, $H_1(M(\mathcal{A}), \mathbb{Z})$ is free abelian of rank equal to the cardinality of the arrangement, $|\mathcal{A}|$.

Of particular interest to us is what happens in degree $k = 2$. For a 2-flat Y, the localized sub-arrangement $\mathcal{A}_Y$ is a pencil of $|Y| = \mu(Y) + 1$ hyperplanes. Consequently, $M(\mathcal{A}_Y)$ is homeomorphic to $(\mathbb{C} \setminus \{\mu(Y) \text{ points}\}) \times \mathbb{C}^* \times \mathbb{C}^{n-2}$, and so $M(\mathcal{A}_Y)$ is a classifying space for the group $G(\mathcal{A}_Y) = F_{\mu(Y)} \times \mathbb{Z}$.

7.3 The second nilpotent quotient of an arrangement group

Let $G = G(\mathcal{A})$ be an arrangement group. Then G admits a commutator-relators presentation of the form $G = F/R$, where F is the free group on generators $\{x_H\}_{H \in \mathcal{A}}$ corresponding to meridians about the hyperplanes, and $R \subset [F, F]$ (see for instance [10] as well as [43] and references therein).

Plainly, the abelianization $G_{\mathrm{ab}} = H_1(M(\mathcal{A}))$ is the free abelian group on $\{x_H\}_{H \in \mathcal{A}}$. On the other hand, as noted for instance in [27], the abelian group $\mathrm{gr}_2(G)$ is the $\mathbb{Z}$-dual of $I^2(\mathcal{A})$; in particular, $\mathrm{gr}_2(G)$ is also torsion-free. The central extension (2.2) with $n = 2$ takes a very explicit form, detailed in the next result.

Proposition 7.2 ([27]) *For any arrangement $\mathcal{A}$, the second nilpotent quotient of $G(\mathcal{A})$ fits into a central extension of the form*

$$0 \longrightarrow (I^2(\mathcal{A}))^* \longrightarrow G(\mathcal{A})/\Gamma_3(G(\mathcal{A})) \longrightarrow H_1(M(\mathcal{A})) \longrightarrow 0.$$

Furthermore, the k-invariant of this extension, $\chi_2\colon H_2(G_{\mathrm{ab}}) \to \mathrm{gr}_2(G)$, is the dual of the inclusion map $I^2(\mathcal{A}) \hookrightarrow E^2(\mathcal{A}) = \bigwedge^2 G_{\mathrm{ab}}$.

It follows that $G/\Gamma_3(G)$ is the quotient of the free, 2-step nilpotent group $F/\Gamma_3(F)$ by all commutation relations of the form

$$\Big[x_H, \prod_{\substack{K \in \mathcal{A} \\ K \supset Y}} x_K\Big],$$

indexed by pairs of hyperplanes $H \in \mathcal{A}$ and flats $Y \in L_2(\mathcal{A})$ such that $H \supset Y$ (see [27,36]). From this description it is apparent that the second nilpotent quotient of an arrangement group is combinatorially determined. More precisely, if $\mathcal{A}$ and $\mathcal{B}$ are two arrangements such that $L_{\le 2}(\mathcal{A}) \cong L_{\le 2}(\mathcal{B})$, there is then an induced isomorphism, $G(\mathcal{A})/\Gamma_3(G(\mathcal{A})) \cong G(\mathcal{B})/\Gamma_3(G(\mathcal{B}))$.

7.4 Holonomy Lie algebra

The holonomy Lie algebra of an arrangement $\mathcal{A}$ is defined as $\mathfrak{h}(\mathcal{A}) = \mathfrak{h}(G(\mathcal{A}))$. Using the Orlik–Solomon description of the cohomology ring of $M(\mathcal{A})$, it is readily seen that $\mathfrak{h}(\mathcal{A})$ is the quotient of $\mathbf{L}(\mathcal{A})$, the free Lie algebra on variables $\{x_H\}_{H\in\mathcal{A}}$, modulo the ideal generated by relations arising from the rank 2 flats:

$$\mathfrak{h}(\mathcal{A}) = \mathbf{L}(\mathcal{A}) \Big/ \text{ideal}\left\{ \Big[x_H, \sum_{\substack{K\in\mathcal{A} \\ K\supset Y}} x_K \Big] \;\Big|\; H \in \mathcal{A}, Y \in L_2(\mathcal{A}), \text{ and } H \supset Y \right\}. \tag{7.2}$$

By construction, this is a quadratic Lie algebra which depends solely on the ranked poset $L_{\le 2}(\mathcal{A})$. More precisely, if $\mathcal{A}$ and $\mathcal{B}$ are two arrangements such that $L_{\le 2}(\mathcal{A}) \cong L_{\le 2}(\mathcal{B})$, there is then an induced isomorphism $\mathfrak{h}(\mathcal{A}) \cong \mathfrak{h}(\mathcal{B})$.

As shown by Kohno [19] (based on foundational work by Sullivan [46] and Morgan [28]), the associated graded Lie algebra $\operatorname{gr}(G(\mathcal{A}))$ and the holonomy Lie algebra $\mathfrak{h}(\mathcal{A})$ are rationally isomorphic:

$$\mathfrak{h}(\mathcal{A}) \otimes \mathbb{Q} \cong \operatorname{gr}(G(\mathcal{A})) \otimes \mathbb{Q}.$$

In [15], Falk sketched the construction of a 1-minimal model for $M(\mathcal{A})$ and used this to show that the rank of $\operatorname{gr}_3(G(\mathcal{A}))$—now sometimes known as the “Falk invariant" of the arrangement—is equal to the nullity of the multiplication map $E^1(\mathcal{A}) \otimes I^2(\mathcal{A}) \to E^3(\mathcal{A})$ over $\mathbb{Q}$. Further information on the ranks of the LCS quotients $\operatorname{gr}_n(G(\mathcal{A}))$ can be found in [40].

At the integral level, there is a surjective Lie algebra map, $\Psi\colon \mathfrak{h}(\mathcal{A}) \twoheadrightarrow \operatorname{gr}(G(\mathcal{A}))$, such that $\Psi \otimes \mathbb{Q}$ is an isomorphism. In general, there exist arrangements for which the map Ψ is not injective. Nevertheless, as a consequence of Theorem 4.3 and the preceding discussion, we have the following result.

Theorem 7.3 *For any arrangement $\mathcal{A}$, the map $\Psi_3\colon \mathfrak{h}_3(\mathcal{A}) \to \operatorname{gr}_3(G(\mathcal{A}))$ is an isomorphism.*

Consequently, the group $\operatorname{gr}_3(G(\mathcal{A}))$ is combinatorially determined; that is, if $\mathcal{A}$ and $\mathcal{B}$ are two arrangements such that $L_{\le 2}(\mathcal{A}) \cong L_{\le 2}(\mathcal{B})$, then $\operatorname{gr}_3(G(\mathcal{A})) \cong \operatorname{gr}_3(G(\mathcal{B}))$.

On the other hand, as first noted in [43], there exist arrangements $\mathcal{A}$ for which $\operatorname{gr}_k(G(\mathcal{A}))$ has non-zero torsion for some $k > 3$. This naturally raised the question whether such torsion in the LCS quotients of arrangement groups is combinatorially determined. The question was recently answered by Artal Bartolo, Guerville-Ballé, and Viu-Sos [3], who produced a pair of arrangements $\mathcal{A}$ and $\mathcal{B}$ with $L_{\le 2}(\mathcal{A}) \cong L_{\le 2}(\mathcal{B})$, yet $\operatorname{gr}_4(G(\mathcal{A})) \not\cong \operatorname{gr}_4(G(\mathcal{B}))$; the difference (detected by computer-aided computation) lies in the 2-torsion of the respective groups.

8 Decomposable arrangements and nilpotent quotients

We conclude with an in-depth study of a particularly nice class of hyperplane arrangements. Building on work of Papadima and Suciu [31], we show that the tower of nilpotent quotients of the fundamental group of the complement of a decomposable arrangement is fully determined by the intersection lattice.

8.1 Decomposable arrangements

Let $\mathcal{A}$ be an arrangement. As we saw in Sect. 7.2, for each 2-flat $Y \in L_2(\mathcal{A})$, the group $G(\mathcal{A}_Y)$ is isomorphic to $F_{\mu(Y)} \times \mathbb{Z}$; hence, $\operatorname{gr}(G(\mathcal{A}_Y)) \cong \mathbf{L}_{\mu(Y)} \times \mathbf{L}_1$. Furthermore, from the defining relations (7.2), we infer that $\mathfrak{h}(\mathcal{A}_Y) \cong \operatorname{gr}(G(\mathcal{A}_Y))$.

Let j be the map from (7.1). Projecting onto the factors corresponding to rank 2 flats we obtain a map

$$j\colon M(\mathcal{A}) \to \prod_{Y\in L_2(\mathcal{A})} M(\mathcal{A}_Y).$$

The induced homomorphism on fundamental groups

$$j_\#\colon G(\mathcal{A}) \to \prod_{Y\in L_2(\mathcal{A})} G(\mathcal{A}_Y)$$

defines a morphism of graded Lie algebras

$$\mathfrak{h}(j_\#)\colon \mathfrak{h}(\mathcal{A}) \to \prod_{Y\in L_2(\mathcal{A})} \mathfrak{h}(\mathcal{A}_Y).$$

Proposition 8.1 ([31]) *For any arrangement $\mathcal{A}$, the homomorphism*

$$\mathfrak{h}_n(j_\#)\colon \mathfrak{h}_n(\mathcal{A}) \to \prod_{Y\in L_2(\mathcal{A})} \mathfrak{h}_n(\mathcal{A}_Y)$$

is a surjection for $n \geq 3$ and an isomorphism for $n = 2$.

Following [31], we say that the arrangement $\mathcal{A}$ is *decomposable* if the map $\mathfrak{h}_3(j_\#)$ is an isomorphism (for related notions of decomposability, see also [11,40]). The following theorem completely describes the structure of the associated graded and holonomy Lie algebras of a decomposable arrangement.

Theorem 8.2 ([31]) *If $\mathcal{A}$ is a decomposable arrangement, then the following hold:*

1. *The map $\mathfrak{h}'(j_\#)\colon \mathfrak{h}'(\mathcal{A}) \to \prod_{Y\in L_2(\mathcal{A})} \mathfrak{h}'(\mathcal{A}_Y)$ is an isomorphism of graded Lie algebras.*
2. *The map $\Psi_{\mathcal{A}}\colon \mathfrak{h}(\mathcal{A}) \twoheadrightarrow \operatorname{gr}(G(\mathcal{A}))$ is an isomorphism.*

It follows from this theorem that the groups $\mathfrak{h}_n(\mathcal{A}) = \mathrm{gr}_n(G(\mathcal{A}))$ are torsion-free, with ranks $\phi_n = \phi_n(G(\mathcal{A}))$ given by

$$\prod_{n=1}^{\infty}(1-t^n)^{\phi_n} = (1-t)^{|\mathcal{A}|-\sum_{Y\in L_2(\mathcal{A})}\mu(Y)} \prod_{Y\in L_2(\mathcal{A})}(1-\mu(Y)t).$$

Moreover, since the holonomy Lie algebra of any arrangement is combinatorially determined, we have the following immediate corollary.

Corollary 8.3 *If $\mathcal{A}$ and $\mathcal{B}$ are decomposable arrangements with $L_{\le 2}(\mathcal{A}) \cong L_{\le 2}(\mathcal{B})$, then* $\mathrm{gr}(G(\mathcal{A})) \cong \mathrm{gr}(G(\mathcal{B}))$.

8.2 Nilpotent quotients and localized arrangements

Our goal now is to strengthen Corollary 8.3 from the level of the LCS quotients $\mathrm{gr}_n(G(\mathcal{A}))$ to the level of the nilpotent quotients $G(\mathcal{A})/\Gamma_n(G(\mathcal{A}))$. We start with some preparatory results on the second homology of these nilpotent groups.

Lemma 8.4 *Let $\mathcal{A}$ be an arrangement and set $G = G(\mathcal{A})$. There is then a natural, split exact sequence*

$$0 \longrightarrow \mathfrak{h}_3(\mathcal{A}) \longrightarrow H_2(G/\Gamma_3(G)) \longrightarrow H_2(M(\mathcal{A})) \longrightarrow 0.$$

Moreover, if $\mathcal{A}$ is decomposable, then for every $n \ge 3$ there is a natural, split exact sequence

$$0 \longrightarrow \mathfrak{h}_n(\mathcal{A}) \longrightarrow H_2(G/\Gamma_n(G)) \longrightarrow H_2(M(\mathcal{A})) \longrightarrow 0. \tag{8.1}$$

Proof The first assertion follows from Theorems 3.1 and 7.3, while the second assertion follows from Theorems 3.1 and 8.2. □

For an arbitrary arrangement $\mathcal{A}$ and for a 2-flat $Y \in L_2(\mathcal{A})$, we let $\mathcal{A}_Y$ be the corresponding localized arrangement, and write $G_Y = G(\mathcal{A}_Y)$. The inclusion map $j_Y : M(\mathcal{A}) \to M(\mathcal{A}_Y)$ induces a homomorphism $(j_Y)_\# : G \to G_Y$ on fundamental groups, which in turn induces homomorphisms

$$N_n(j_Y) : G/\Gamma_n(G) \to G_Y/\Gamma_n(G_Y)$$

on the respective nilpotent quotients. Assembling these maps, we obtain a homomorphism

$$N_n(j) : G/\Gamma_n(G) \to \prod_{Y\in L_2(\mathcal{A})} G_Y/\Gamma_n(G_Y).$$

Proposition 8.5 *For any arrangement $\mathcal{A}$, and for each $n \geq 3$, the map $N_n(j)$ induces a surjection in second homology,*

$$N_n(j)_*\colon H_2(G/\Gamma_n(G)) \twoheadrightarrow \bigoplus_{Y \in L_2(\mathcal{A})} H_2(G_Y/\Gamma_n(G_Y)).$$

Moreover, if $\mathcal{A}$ is decomposable, then the maps $N_n(j)_*$ are isomorphisms, for all $n \geq 3$.

Proof Fix $n \geq 3$, and set $N = G(\mathcal{A})/\Gamma_n(G(\mathcal{A}))$ and $N_Y = G(\mathcal{A}_Y)/\Gamma_n(G(\mathcal{A}_Y))$. Consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{h}_n(\mathcal{A}) & \longrightarrow & H_2(N) & \longrightarrow & H_2(M(\mathcal{A})) & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \mathfrak{h}_n(j_\#)} & & \big\downarrow{\scriptstyle N_n(j)_*} & & \big\downarrow{\scriptstyle j_*} & & \\ 0 & \longrightarrow & \bigoplus_Y \mathfrak{h}_n(\mathcal{A}_Y) & \longrightarrow & \bigoplus_Y H_2(N_Y) & \longrightarrow & \bigoplus_Y H_2(M(\mathcal{A}_Y)) & \longrightarrow & 0. \end{array}$$

It follows from Lemma 8.4 that the top and bottom rows are (split) exact. Furthermore, the naturality of the exact sequence (8.1) implies that the diagram commutes. By Brieskorn's Lemma, the map j_* is an isomorphism. Furthermore, by Proposition 8.1, the map $\mathfrak{h}_n(j_\#)$ is a surjection. The first claim follows at once.

If the arrangement is decomposable, then by Theorem 8.2 the map $\mathfrak{h}_n(j_\#)$ is an isomorphism, whence $N_n(j)_*$ is also an isomorphism. □

8.3 Lifting maps to nilpotent quotients

Let $\mathcal{A}$ be an arrangement and set $G = \pi_1(M(\mathcal{A}))$. Composing a classifying map $M(\mathcal{A}) \to K(G, 1)$ with the map $K(G, 1) \to K(G_{\mathrm{ab}}, 1)$ induced by the abelianization homomorphism $G \to G_{\mathrm{ab}}$, we obtain a map of spaces, $h\colon M(\mathcal{A}) \to K(G_{\mathrm{ab}}, 1)$, uniquely defined up to homotopy. Fix an integer $n \geq 3$, and write $N = G/\Gamma_n(G)$ and $N_Y = G_Y/\Gamma_n(G_Y)$ for $Y \in L_2(\mathcal{A})$.

Lemma 8.6 *Suppose $\ell\colon M(\mathcal{A}) \to K(N, 1)$ is a (homotopy) lifting of h. For each 2-flat $Y \in L_2(\mathcal{A})$, there is then a map $\ell_Y\colon M(\mathcal{A}_Y) \to K(N_Y, 1)$ which lifts the map $h_Y\colon M(\mathcal{A}_Y) \to K((G_Y)_{\mathrm{ab}}, 1)$ and fits in the commuting diagram*

$$\begin{array}{ccc} K(N,1) & \xrightarrow{N_n(j_Y)} & K(N_Y,1) \\ {\scriptstyle \ell}\big\uparrow & & \big\uparrow{\scriptstyle \ell_Y} \\ M(\mathcal{A}) & \xrightarrow{j_Y} & M(\mathcal{A}_Y). \end{array}$$

Proof We define the map ℓ_Y by forming the composite

$$M(\mathcal{A}_Y) \xrightarrow{r_Y} M(\mathcal{A}) \xrightarrow{\ell} K(N,1) \xrightarrow{N_n(j_Y)} K(N_Y,1),$$

where the first map is the one from Lemma 7.1, while the last map is induced by the homomorphism $N_n(j_Y)\colon N \to N_Y$. The two claims follow at once. □

We now prove a converse to Lemma 8.6: given "local lifts" ℓ_Y, there is a way to assemble them into a "global lift," which we will denote by $\widetilde{\ell}$. To state the result more precisely, start by recalling that the map $j\colon M(\mathcal{A}) \to \prod_Y M(\mathcal{A}_Y)$ induces an isomorphism $\mathfrak{h}_2(j_\#)\colon \mathfrak{h}_2(G) \xrightarrow{\simeq} \bigoplus_Y \mathfrak{h}_2(G_Y)$.

Lemma 8.7 *Let $\mathcal{A}$ be an arrangement. Suppose that, for each 2-flat $Y \in L_2(\mathcal{A})$, we are given a lift $\ell_Y\colon M(\mathcal{A}_Y) \to K(N_Y, 1)$ of the map $h_Y\colon M(\mathcal{A}_Y) \to K((G_Y)_{\mathrm{ab}}, 1)$. There is then a map $\widetilde{\ell}\colon M(\mathcal{A}) \to K(N, 1)$ which lifts the map $h\colon M(\mathcal{A}) \to K(G_{\mathrm{ab}}, 1)$, and such that the following diagram commutes:*

$$\begin{array}{ccc} H_2(N) & \xrightarrow{N_n(j)_*} & \bigoplus_Y H_2(N_Y) \\ \uparrow{\scriptstyle \widetilde{\ell}_*} & & \uparrow{\scriptstyle \bigoplus_Y (\ell_Y)_*} \\ H_2(M(\mathcal{A})) & \xrightarrow[\cong]{j_*} & \bigoplus_Y H_2(M(\mathcal{A}_Y)). \end{array} \tag{8.2}$$

Proof Recall that we have a central extension

$$0 \longrightarrow \mathrm{gr}_n(G) \longrightarrow G/\Gamma_n G \longrightarrow G/\Gamma_{n-1} G \longrightarrow 0.$$

Recall also that the group G is generated by meridians x_H about the hyperplanes $H \in \mathcal{A}$, and likewise for G_{ab}. Thus, if $\ell\colon M(\mathcal{A}) \to K(N, 1)$ is any map which lifts $h\colon M(\mathcal{A}) \to K(G_{\mathrm{ab}}, 1)$, the homomorphism $\ell_\#\colon G \to N$ is given on generators by

$$\ell_\#(x_H) = x_H a_2(H) \cdots a_{n-1}(H), \tag{8.3}$$

for some $a_i(H) \in \mathrm{gr}_i(G)$. What we need to do is pick these elements $a_i(H)$ in such a way so that diagram (8.2) commutes.

First note the following consequence of Lemma 7.1: If Z and Y are different 2-flats and $H \supset Z$, then $(j_Y)_\# \circ (r_Z)_\#(x_H)$ is the identity element in N_Y.

The next step is to see that if Z and Y are distinct 2-flats and if $a \in \mathrm{gr}_i(G_Z)$, for $2 \le i \le n-1$, then $(j_Y)_\#(a)$ is the identity element in N_Y. The group $\mathrm{gr}_i(G_Z)$ is generated by iterated brackets of the generators x_H for $H \in \mathcal{A}_Z$. If any one or more of these generators is replaced by the identity, then the resulting bracket equals the identity. Let a be an iterated bracket in $\mathrm{gr}_i(G_Z)$. Since a involves at least one generator $x_{H'}$ with $H' \not\supset Y$, and since $(j_Y)_\# \circ (r_Z)_\#(x_{H'})$ is the identity in N_Y, it follows that $(j_Y)_\# \circ (r_Z)_\#(a)$ is also the identity in N_Y.

By (8.3), the homomorphism $(\ell_Y)_\#\colon G_Y \to N_Y$ is given on generators by

$$(\ell_Y)_\#(x_H) = x_H a_2(H, Y) \cdots a_{n-1}(H, Y),$$

for some $a_i(H, Y) \in \mathfrak{h}_i(G_Y)$, where $H \supset Y$. Define a map $\widetilde{\ell}\colon M(\mathcal{A}) \to K(N, 1)$ by requiring that

$$\widetilde{\ell}_{\#}(x_H) = x_H \prod_{H \supset Y} a_2(H, Y) \cdots \prod_{H \supset Y} a_{n-1}(H, Y).$$

Now let X and Y be different 2-flats in $L_2(\mathcal{A})$ and consider the composition

$$\xi_{XY}\colon M(\mathcal{A}_X) \xrightarrow{r_X} M(\mathcal{A}) \xrightarrow{\widetilde{\ell}} K(N, 1) \xrightarrow{N_n(j_Y)} K(N_Y, 1).$$

From the result above, it follows that $(\xi_{XY})_{\#}(x_H)$ is the identity in N_Y for all hyperplanes $H \in \mathcal{A}$ such that $H \supset X$ but $H \not\supset Y$. Since there is a unique hyperplane K with $K \supset X$ and $K \supset Y$, the image of the homomorphism $(\xi_{XY})_{\#}\colon G_X \to N_Y$ is the (infinite cyclic) subgroup generated by the single element $(\xi_{XY})_{\#}(x_K)$. Hence, the map ξ_{XY} factors through $K(\mathbb{Z}, 1)$. Since $H_2(\mathbb{Z}) = 0$, it follows that the induced homomorphism $(\xi_{XY})_*\colon H_2(M(\mathcal{A}_X)) \to H_2(N_Y)$ is the zero map. The lemma now follows by a diagram chase. □

8.4 The nilpotent quotients of a decomposable arrangement group

In [36,37], Rybnikov showed that, in general, the third nilpotent quotient of an arrangement group is *not* determined by the intersection lattice. Specifically, he produced a pair of arrangements $\mathcal{A}$ and $\mathcal{B}$ of 13 hyperplanes in $\mathbb{C}^3$ such that $L(\mathcal{A}) \cong L(\mathcal{B})$, yet $G(\mathcal{A})/\Gamma_4(G(\mathcal{A})) \ncong G(\mathcal{B})/\Gamma_4(G(\mathcal{B}))$. By contrast, we can use our approach to show that the phenomenon detected by Rybnikov cannot happen among decomposable arrangements. Here, then, is the main result of this section.

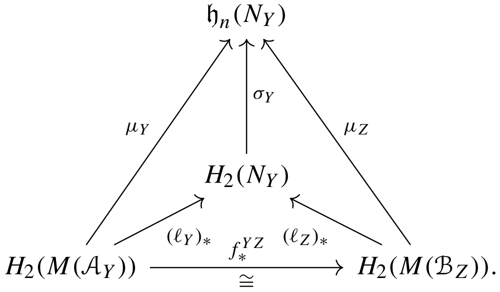
Theorem 8.8 *If $\mathcal{A}$ and $\mathcal{B}$ are decomposable arrangements with $L_{\le 2}(\mathcal{A}) \cong L_{\le 2}(\mathcal{B})$, then, for each $n \ge 2$, there is an isomorphism*

$$G(\mathcal{A})/\Gamma_n(G(\mathcal{A})) \cong G(\mathcal{B})/\Gamma_n(G(\mathcal{B})).$$

Proof Let $G = \pi_1(M(\mathcal{B}))$ and set $N = G/\Gamma_n(G)$. We start by picking a lifting $\ell_B\colon M(\mathcal{B}) \to K(N, 1)$ of the map $M(\mathcal{B}) \to K(G_{\mathrm{ab}}, 1)$. For each 2-flat $Z \in L_2(\mathcal{B})$, we obtain a map $\ell_Z\colon M(\mathcal{B}_Z) \to K(N_Z, 1)$, defined as in Lemma 8.6.

Having an isomorphism of posets $L_{\le 2}(\mathcal{A}) \cong L_{\le 2}(\mathcal{B})$ means we have a bijection $\mathcal{A} \to \mathcal{B}$ which induces a compatible bijection $L_2(\mathcal{A}) \to L_2(\mathcal{B})$. Let $Y \in L_2(\mathcal{A})$ and $Z \in L_2(\mathcal{B})$ be a pair of 2-flats which correspond under the aforementioned bijection. Using the description of localized arrangement complements from (7.2), we obtain a homeomorphism $f^{YZ}\colon M(\mathcal{A}_Y) \to M(\mathcal{B}_Z)$ between the respective complements.

By the forward implication of Theorem 6.2, there is a map $\ell_Y\colon M(\mathcal{A}_Y) \to K(N_Y, 1)$ and a splitting $\sigma_Y\colon H_2(N_Y) \to \mathfrak{h}_n(N_Y)$ such that the following diagram commutes:



Using the bijection $L_2(\mathcal{A}) \to L_2(\mathcal{B})$, the maps f_*^{YZ} assemble to give an isomorphism $\Phi\colon H_2(M(\mathcal{A})) \xrightarrow{\cong} H_2(M(\mathcal{B}))$. The decomposability assumption together with Theorem 8.2 insure that

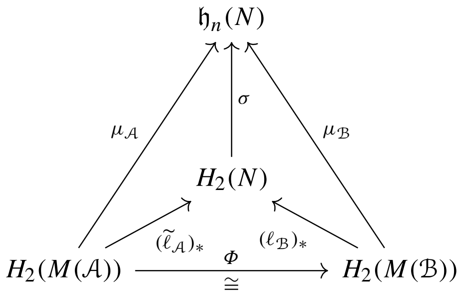
$$\mathfrak{h}_n(G) \cong \bigoplus_{Z\in L_2(\mathcal{B})} \mathfrak{h}_n(G(\mathcal{B}_Z)).$$

Furthermore, by Proposition 8.5, we have that

$$H_2(N) \cong \bigoplus_{Z\in L_2(\mathcal{B})} H_2(N_Z).$$

Consequently, the homomorphisms $\sigma_Z\colon H_2(N_Y) \to \mathfrak{h}_n(N_Y)$ may be assembled into a homomorphism $\sigma\colon H_2(N) \to \mathfrak{h}_n(N)$.

Next, using the maps $\ell_Y\colon M(\mathcal{A}_Y) \to N_Y$, we define a lifting $\widetilde{\ell}_{\mathcal{A}}\colon M(\mathcal{A}) \to K(N,1)$ by the procedure outlined in Lemma 8.7. It is then readily verified that the diagram



commutes. The result follows from the backwards implication of Theorem 6.2. ◻

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