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RESEARCH ARTICLE

Exponentially harmonic maps, Morse index and Liouville type theorems

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Abstract
We obtain a result on the Morse index of an exponentially harmonic map from a Riemannian manifold into the unit n-sphere. Next, we prove a Liouville type 1 theorem for exponentially harmonic maps between two Riemannian manifolds. Finally, let (M, g_0) be a complete Riemannian manifold with a pole x_0 and (N, h) a Riemannian manifold, under certain conditions we establish a Liouville type 2 theorem for exponentially harmonic maps $f\colon (M, \rho^2 g_0) \to N$, $0 < \rho \in C^\infty(M)$.

Keywords Exponentially harmonic map · Morse index · Liouville type theorems

Mathematics Subject Classification 58E20 · 58G11 · 35J20

1 Introduction

Exponentially harmonic maps between two Riemannian manifolds were first considered by Eells and Lemaire in [12]. A map $f\colon (M, g) \to (N, h)$ between two Riemannian manifolds is called exponentially harmonic if it is a critical point of the exponential energy functional $E_{\mathrm{e}}(f) = \int_M \exp\bigl(\frac{1}{2}|df|^2\bigr)\, dv_g$. In terms of the Euler–Lagrange equation, f is exponentially harmonic if it satisfies the following second order nonlinear PDE:

$$\exp\biggl(\frac{1}{2}\,|df|^2\biggr)\biggl[\tau(f) + \biggl(\nabla\biggl(\frac{|df|^2}{2}\biggr), df\biggr)\biggr] = 0,$$

where $\tau(f)$ is the tension field of f, and ∇ is the connection on $T^*(M) \otimes f^{-1}TN$ induced by the Levi–Civita connections on M and N, respectively. In the recent three decades, exponentially harmonic maps have been extensively investigated by Duc and Eells [11], Hong et al. [17], Hong and Yang [16], Chiang [3–6], Chiang and Pan [7],

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Chiang and Wolak [8], Chiang and Yang [9], Cheung and Leung [2], Zhang et al. [25], Liu [20,21], and others.

In [16], Hong and Yang showed that there are harmonic maps which are not exponentially harmonic, and conversely there are exponentially harmonic maps which are not harmonic. It is interesting that Kanfon et al. [18] found applications of exponential harmonic maps in the Friedmann–Lemaître universe, and considered some new models of exponentially harmonic maps which are coupled with gravity based on a generalization of Lagrangian for bosonic strings coupled with diatonic field. Moreover, Omori [22,23] recently obtained some results about Eells–Sampson's existence theorem [13] and Sacks–Uhlenbeck's existence theorem [24] for harmonic maps via exponentially harmonic maps.

In [9], Chiang and Yang proved that if f is an exponentially harmonic map from a Riemannian manifold into another Riemannian manifold with non-positive sectional curvature, then f is stable. Chiang [5] also showed that if f is an exponentially harmonic map from a compact Riemannian manifold into the unit n-sphere S^n, $n \geqslant 3$, with $|df|^2 < n-2$, then f is unstable. The degree of instability of a map f is measured by the Morse index. In this paper, we estimate the Morse index of an exponentially harmonic map f from a compact Riemannian manifold into the unit n-sphere S^n, see Theorem 2.4. Next, we obtain a Liouville type 1 theorem for exponentially harmonic maps between two Riemannian manifolds, see Sect. 3. Finally, let (M, g_0) be a complete Riemannian manifold with a pole x_0 and (N, h) a Riemannian manifold, in Sect. 4 we establish a Liouville type 2 theorem for exponentially harmonic maps $f\colon (M, \rho^2 g_0) \to N$, $0 < \rho \in C^\infty(M)$, under certain conditions.

2 Exponentially harmonic maps and Morse index

A map $f\colon (M, g) \to (N, h)$ between two Riemannian manifolds is called *exponentially harmonic* if it is a critical point of the exponential energy functional $E_{\mathrm{e}}(f) = \int_M \exp\left(\frac{1}{2}|df|^2\right) dv_g$. More precisely, a C^2-map $f\colon (M, g) \to (N, h)$ is exponentially harmonic if it satisfies

$$\frac{d}{dt} E_{\mathrm{e}}(f_t)\Big|_{t=0} = 0,$$

for any compactly supported variations $f_t\colon M \to N$ with $f_0 = f$. In terms of the Euler–Lagrange equation, we arrive at the following definition.

Definition 2.1 ([9]) A map $f\colon (M, g_{ij}) \to (N, h_{\alpha\beta})$ from an m-dimensional Riemannian manifold (M^m, g_{ij}) into an n-dimensional Riemannian manifold $(N^n, h_{\alpha\beta})$ is called *exponentially harmonic* if its associated exponential tension field is zero, i.e.,

$$\tau_{\mathrm{e}}(f) = \tau(f) + \left(\nabla\left(\frac{|df|^2}{2}\right), df\right) = 0,$$

where the tension field $\tau^\alpha(f) = g^{ij} f^\alpha_{i|j} = g^{ij}(f^\alpha_{ij} - \Gamma^k_{ij} f^\alpha_k + \Gamma'^\alpha_{\beta\gamma} f^\beta_i f^\gamma_j)$. In terms of local coordinates, f satisfies

$$g^{ij}\left(\frac{\partial^2 f^\alpha}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial f^\alpha}{\partial x^k} + \Gamma'^\alpha_{\beta\gamma} \frac{\partial f^\beta}{\partial x^i} \frac{\partial f^\gamma}{\partial x^j}\right)$$
$$+ g^{il} g^{jm} h_{\beta\gamma} \frac{\partial f^\alpha}{\partial x^l} \frac{\partial f^\gamma}{\partial x^m} \frac{\partial^2 f^\beta}{\partial x^i \partial x^j}$$
$$- g^{il} g^{jm} h_{\beta\gamma} \Gamma_{ij}^k \frac{\partial f^\alpha}{\partial x^l} \frac{\partial f^\beta}{\partial x^m} \frac{\partial f^\gamma}{\partial x^k}$$
$$+ g^{ij} g^{lm} h_{\beta\gamma} \Gamma'^\beta_{\mu\nu} \frac{\partial f^\mu}{\partial x^i} \frac{\partial f^\nu}{\partial x^l} \frac{\partial f^\gamma}{\partial x^m} \frac{\partial f^\alpha}{\partial x^j} = 0,$$

where Γ_{ij}^k and $\Gamma'^\alpha_{\beta\gamma}$ are the Christoffel symbols of the Levi–Civita connections on M and N, respectively.

Theorem 2.2 ([9]) *Let $f: M \to N$ be an exponentially harmonic map.*

(a) *If N has non-positive sectional curvature (i.e., $\mathrm{R}^N_{\alpha\beta\gamma\mu}\lambda^\alpha \eta^\beta \lambda^\gamma \eta^\mu \leqslant 0$ for vector fields λ, η), then f is stable.*
(b) *If $\dot{f}$ is a Jacobi field, then f is stable.*

Theorem 2.3 ([5]) *If $f: M^m \to S^n$ is a non-constant exponentially harmonic map from a compact Riemannian manifold M into the n-dimensional sphere S^n, $n \geqslant 3$, with $|df|^2 < n-2$, then f is unstable.*

Let $f: (M^m, g) \to (N^n, h)$ be a differentiable map from an m-dimensional Riemannian manifold M into an n-dimensional Riemannian manifold N. Let v be a vector field on N, and (f_t^v) be the flows of diffeomorphisms induced by v on N, i.e., $f_0^v = f$, $\frac{d}{dt} f_t^v|_{t=0} = v$. Recall that the first variation of the exponential energy functional is

$$\frac{d}{dt} E_{\mathrm{e}}(f_t)\Big|_{t=0} = \int_M e^{|df|^2/2} (\nabla_{\partial t} df_t, df_t)_{t=0}\, dv$$
$$= -\int_M \big(\mathrm{trace}_g \nabla(e^{|df|^2/2} df), v\big)\, dv.$$

Then the second variation of the exponential energy functional is

$$\frac{d^2}{dt^2} E_{\mathrm{e}}(f_t)\Big|_{t=0} = \int_M e^{|df|^2/2} \big[(\nabla v, df_t)^2 + |\nabla v|^2\big]\, dv$$
$$- \int_M \left(\nabla_{\partial t} \frac{\partial f}{\partial t}\Big|_{t=0}, \ \mathrm{trace}_g \nabla(e^{|df|^2/2} df)\right) dv$$
$$- \int_M e^{|df|^2/2} \sum_{i=1}^m \big(\mathrm{R}^N(v, df(e_i)) df(e_i), v\big)\, dv,$$

where $\{e_i\}$ is a local orthonormal frame at a point in M and R^N is the Riemannian curvature of N.

We now consider a differentiable map $f: (M, g) \to (S^n, \mathrm{stn})$ from a Riemannian manifold into the unit n-sphere, where stn is the standard metric on S^n. Let $f^{-1} T S^n$

be the pull-back vector field bundle of TS^n, $\Gamma(f^{-1}TS^n)$ be the space of sections on $f^{-1}TS^n$, and denote by ∇^M, ∇^{S^n} and $\widetilde{\nabla}$ the Levi–Civita connections on TM, TS^n and $f^{-1}TS^n$, respectively. Then $\widetilde{\nabla}$ is given by $\widetilde{\nabla}_X Y = \nabla^{S^n}_{f_*X} Y$, where $X \in TM$ and $Y \in \Gamma(f^{-1}TS^n)$. The variation in the directions of the vector fields of the subspace $\mathcal{L}(f)$ of $\Gamma(f^{-1}TS^n)$ is defined by

$$\mathcal{L}(f) = \left\{\overline{v} \circ f : v \in \mathbb{R}^{n+1}\right\},$$

where $\overline{v}$ is a vector field on S^n given by $\overline{v}(y) = v - (v, y)y$ for any $y \in S^n$. It is known that $\overline{v}$ is a conformal vector field on S^n. Clearly, if f is not constant, $\mathcal{L}(f)$ is of dimension $n + 1$.

For any vector field v on S^n along an exponentially harmonic map $f : (M, g) \to (S^n, \mathrm{stn})$, we associate the quadratic form

$$Q_f(v) = \frac{d^2 E_{\mathrm{e}}(f_t)}{dt^2}\Big|_{t=0}.$$

The *Morse index* of f is defined as the positive integer

$$\mathrm{Ind}(f) = \sup\left\{\dim W : W \subset \Gamma(f) \text{ such that } Q_f(v) \text{ is negative defined on } W\right\},$$

where W is the subspace of $\Gamma(f)$. The Morse index measures the degree of the instability of f. A map f is called *stable* if $\mathrm{Ind}(f) = 0$. In view of Theorems 2.2 and 2.3, we shall estimate the Morse index of an exponentially harmonic map into the unit n-sphere. We define the (*modified*) *exponential stress energy* of f as

$$S_{\mathrm{e}}(f) = e^{|df|^2/2}|df|^2 g - 2e^{|df|^2/2}\left(1 + \frac{|df|^2}{2}\right) f^*\mathrm{stn}$$

(for the definition of the exponential stress energy, see [5,12]). For $x \in M$, we set

$$S^{\mathrm{o}}_{\mathrm{e}}(f) = \inf\left\{S_{\mathrm{e}}(f)(X, X) : X \in T_x(M) \text{ such that } g(X, X) = 1\right\}.$$

The tensor $S_{\mathrm{e}}(f)$ is called *positive* (resp. *semi-positive*) if $S^{\mathrm{o}}_{\mathrm{e}}(f) > 0$ (resp. $S^{\mathrm{o}}_{\mathrm{e}}(f) \geqslant 0$).

Theorem 2.4 *Let* $f : (M^m, g) \to (S^n, \mathrm{stn})$ *be an exponentially harmonic map from a compact m-dimensional Riemannian manifold, $m \geqslant 2$, into the unit n-sphere, $n \geqslant 2$. Suppose that the exponential stress energy tensor $S_{\mathrm{e}}(f)$ is positive. Then* $\mathrm{Ind}(f) \geqslant n + 1$.

Proof Let $u = \overline{v} \circ f \in \mathcal{L}(f)$ and set $(v, f) = f_v$. For any point $x \in M$, we denote by u^{T} and u^{N} the tangential and normal components of the vector $u(x)$ on the spaces $df(T_xM)$ and $df(T_xM)^{\perp}$, respectively. Let $\{e_1, \ldots, e_m\}$ be an orthonormal basis of T_xM which diagonalizes $f^*\mathrm{stn}$ so that $\{df(e_1), \ldots, df(e_k)\}$ forms a basis of $df(T_xM)$. Since $e^{|df|^2/2}(1 + |df|^2/2) \neq 0$ at the point $x \in M$,

$$|\overline{v}^{\mathrm{T}}(x)|^2 = \sum_{i=1}^{k} \frac{(\overline{v}(x), df(e_i))^2}{|df(e_i)|^2}.$$

For any $i \leqslant k$ we have

$$\begin{aligned} 2e^{|df|^2/2}\left(1+\frac{|df|^2}{2}\right)|df(e_i)|^2 &= e^{|df|^2/2}|df|^2 - S_{\mathrm{e}}(f)(x)(e_i, e_i) \\ &\leqslant e^{|df|^2/2}|df|^2 - S_{\mathrm{e}}^{\mathrm{o}}(f)(x). \end{aligned} \tag{2.1}$$

This implies

$$2e^{|df|^2/2}\left(1+\frac{|df|^2}{2}\right)\sum_{i=1}^{k}(\overline{v}(x), df(e_i))^2 \leqslant \left(e^{|df|^2/2}|df|^2 - S_{\mathrm{e}}^{\mathrm{o}}(f)(x)\right)|\overline{v}^{\mathrm{T}}(x)|^2.$$

Since

$$(\overline{v}(x), df(e_i))^2 = (v-(v, f)f, df(e_i))^2 = (v, df(e_i))^2 = |df_v(e_i)|^2,$$

we deduce

$$\left(e^{|df|^2/2}|df|^2 - S_{\mathrm{e}}^{\mathrm{o}}(f)(x)\right)|u^{\mathrm{T}}(x)|^2 \geqslant 2e^{|df|^2/2}\left(1+\frac{|df|^2}{2}\right)|df_v(x)|^2. \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$\begin{aligned} &2e^{|df|^2/2}\left(1+\frac{|df|^2}{2}\right)|df_v(x)|^2 - e^{|df|^2/2}|df|^2|\overline{v}|^2 \\ &\leqslant -|df|^2 e^{|df|^2/2}|\overline{v}^{\mathrm{N}}(x)|^2 - S_{\mathrm{e}}^{\mathrm{o}}(f)(x)|\overline{v}^{\mathrm{T}}(x)|^2 \leqslant -S_{\mathrm{e}}^{\mathrm{o}}(f)(x)|\overline{v}(x)|^2. \end{aligned} \tag{2.3}$$

The second variation of the exponential energy can be expressed as

$$\frac{d^2E_{\mathrm{e}}(f_t)}{dt^2}\Big|_{t=0} = \int_M e^{|df|^2/2}\left((\nabla\overline{v}, df)^2 + |\nabla\overline{v}|^2 - |df|^2|\overline{v}|^2 + |df_v|^2\right)dv.$$

Therefore, we obtain

$$Q_f(v) = \int_M \left(2e^{|df|^2/2}\left(1+\frac{|df|^2}{2}\right)|df_v|^2 - e^{|df|^2/2}|df|^2|\overline{v}|^2\right)dv.$$

Hence, (2.3) implies

$$Q_f(v) \leqslant -\int_M S_{\mathrm{e}}^{\mathrm{o}}(f)|\overline{v}|^2 dv.$$

Since $S_e(f)$ is positive, $Q_f(v)$ is negative defined on $\mathfrak{L}(f)$. Consequently, the Morse index $\mathrm{Ind}(f) \geqslant n+1$. □

Example 2.5 Consider a homothetic map $f\colon (M,g) \to (S^n, \mathrm{stn})$, i.e., $f^*\mathrm{stn} = k^2 g$, $k \in \mathbb{R}$. Then $|df|^2 = mk^2$ with $m = \dim(M)$. The (modified) exponential stress energy equals

$$\begin{aligned} S_e(f) &= e^{|df|^2/2}|df|^2 g - 2e^{|df|^2/2}\left(1+\frac{|df|^2}{2}\right)k^2 g \\ &= e^{|df|^2/2}\left(1-\frac{2}{m}-\frac{|df|^2}{m}\right)|df|^2 g. \end{aligned}$$

If $f\colon (M,g) \to (S^n, \mathrm{stn})$ is homothetic exponentially harmonic with $|df|^2 < m-2$, then $S_e(f)$ is positive defined. Consequently, it follows from Theorem 2.4 that the Morse index $\mathrm{Ind}(f) \geqslant n+1$.

Proposition 2.6 *If $f\colon (M,g) \to (N,h)$ is an exponentially harmonic and homothetic map between two Riemannian manifolds, then* $\mathrm{Ind}(f) \geqslant \mathrm{Ind}(\iota)$, *where ι is the identity map of M.*

Proof Let $f\colon (M,g) \to (N,h)$ be a homothetic map, i.e., $f^*h = \lambda^2 g$, $\lambda \in \mathbb{R}$. In this case, the exponential tension field $\tau_e(f)$ is proportional to the mean curvature of f, and so f is exponentially harmonic if and only if f is minimal immersion.

Since $f\colon (M,g) \to (N,h)$ is exponentially harmonic, the second variation in the direction of a vector field v reduces to

$$\begin{aligned} &Q_f(v) \\ &= e^{m\lambda^2/2}\int_M \left[(\nabla v, df)^2_{f^{-1}TN} + |\nabla v|^2 - \sum_{i=1}^m \left(\mathrm{R}^N(v, df(e_i)df(e_i), v\right)\right]dv, \end{aligned} \tag{2.4}$$

where $\{e_i\}_{i=1}^m$ is an orthonormal basis on M.

Let $\Gamma^T(f)$ be the subspace of $\Gamma(f^{-1}TN)$ containing the vector fields on N of the form $df(X)$ where X is a vector field on M. The restriction of Q_f^ι to $\Gamma^T(f)$ can be written as (cf. [14])

$$Q_f^\iota(df(X)) = \lambda^2 Q_\iota^\iota(X). \tag{2.5}$$

Since ∇df takes its values in the normal fiber bundle of N, we have

$$\begin{aligned} (\nabla_X df(Y), df(Z)) &= ((\nabla df)(X,Y), Z) + (df(\nabla_X Y), df(Z)) \\ &= \lambda^2(\nabla_X Y, Z). \end{aligned} \tag{2.6}$$

Substituting (2.6) and (2.5) into (2.4), we obtain

$$Q_f(df(X)) = e^{m\lambda^2/2}\lambda^2\int_M (\nabla_{e_i}X, e_i)^2 dv + e^{m\lambda^2/2}\lambda^2 Q_\iota^\iota(X) = \lambda^2 Q_\iota(X),$$

and the result follows. □

3 Liouville type 1 theorem

We establish a Liouville type 1 theorem for exponentially harmonic maps between two Riemannian manifolds. What we present here is very different from Liu's result in [21] involving the sectional curvature of the source manifold M under certain condition. We derive the Bochner formula for exponential energy density in the following lemma. Then we can apply it to prove Theorem 3.2.

Lemma 3.1 *Let $f: M \to N$ be a differentiable map between two Riemannian manifolds. Then*

$$\begin{aligned} &\triangle e^{|df|^2/2} \\ &= e^{|df|^2/2}\Big[|\nabla df|^2 - (\triangle_{\mathrm{H}} df, df) - \sum_{i,j}\big(\mathrm{R}^N(f_*e_i, f_*e_j)f_*e_j, f_*e_i\big) \\ &\qquad + \sum_i (f_*\mathrm{Ric}^M e_i, f_*e_i) + |df|^2 \cdot |\nabla|df||^2\Big], \end{aligned} \tag{3.1}$$

where $\triangle$ is the Laplacian–Beltram operator, $\triangle_{\mathrm{H}}$ is the Hodge–Laplace operator, R^N is the Riemannian curvature of N and Ric^M is the Ricci curvature of M.

Proof Let $\{e_i\}_{i=1,\ldots,m}$ be a local orthonormal frame at a point in M. We compute

$$\begin{aligned} \triangle e^{|df|^2/2} &= e^{|df|^2/2}\big[(\nabla df, df)^2 + (\triangle df, df) + |\nabla df|^2\big] \\ &= e^{|df|^2/2}\big[|df|^2 \cdot |\nabla|df||^2 - (\triangle_{\mathrm{H}} df, df) + |\nabla df|^2 \\ &\quad - \sum_{i,j}\big(\mathrm{R}^N(f_*e_i, f_*e_j)f_*e_j, f_*e_i\big) + \sum_i (f_*\mathrm{Ric}^M e_i, f_*e_i)\big]. \end{aligned}$$ □

Theorem 3.2 *Let $f: M \to N$ be a non-constant exponentially harmonic map between two Riemannian manifolds. Suppose that the Ricci curvature of M is non-negative and the Riemannian curvature of N is non-positive. Then f is totally geodesic. Moreover, if $\mathrm{Ric}^M > 0$ at some point, then f is constant. If $\mathrm{R}^N < 0$, then f is either constant or a map of rank one (i.e., whose image is a closed geodesic).*

Proof Integrating (3.1) and using the exponential harmonicity of f,

$$\int_M e^{|df|^2/2}(\triangle_{\mathrm{H}} df, df)\, dv = \int_M \big(\delta df, \delta(e^{|df|^2/2} df)\big)\, dv = 0,$$

we have

$$
\begin{aligned}
0 &\leqslant \int_M e^{|df|^2/2} |\nabla df|^2 dv \\
&= \int_M e^{|df|^2/2} \Big[\big(\mathrm{R}^N(f_* e_i, f_* e_j) f_* e_j, f_* e_i \big) \\
&\qquad - \sum_i (f_* \mathrm{Ric}^M e_i, e_i) - |df|^2 |\nabla |df||^2 \Big] dv \leqslant 0,
\end{aligned}
$$

since $\mathrm{Ric}^M \geqslant 0$ and $\mathrm{R}^N \leqslant 0$. It follows that $\nabla df = 0$. Hence, f is totally geodesic. Moreover, if $\mathrm{Ric}^M > 0$ at some point, then $df = 0$ and so f is constant. If $\mathrm{R}^N < 0$, then $(\mathrm{R}^N(f_* e_i, f_* e_j) f_* e_j, f_* e_i) = 0$, and the rank of f is either zero (i.e. f is constant), or one (i.e. the image of a totally geodesic is a closed geodesic). □

We are interested in exponentially harmonic maps to manifolds which admit convex functions (cf. [15,19]), and the following lemma is important for Proposition 3.4 and Theorem 3.5.

Lemma 3.3 *Let $f : M \to N$ be a C^1-map between Riemannian manifolds and ϕ be a real-valued C^2-function on N. Then for every C^1-function ψ on M we have*

$$
\begin{aligned}
\big(e^{|df|^2/2} d(\phi \circ f), d\psi\big) = &- e^{|df|^2/2} \operatorname{trace} (\nabla d\phi)(df, df) \psi \\
&+ \big(\nabla(\psi \cdot (\operatorname{grad} \phi) \circ f), e^{|df|^2/2} df\big).
\end{aligned}
$$

Proof Let $\{e_i\}$ be an orthonormal frame around a point in M such that $\nabla e_i = 0$ at that point. We calculate

$$
\begin{aligned}
&\big(\nabla(\psi \cdot (\operatorname{grad} \phi) \circ f), e^{|df|^2/2} df\big) \\
&\quad = \sum_i \big(\nabla_{e_i}(\psi \cdot (\operatorname{grad} \phi) \circ f), e^{|df|^2/2} df(e_i)\big) \\
&\quad = \sum_i \big(d\psi(e_i)((\operatorname{grad} \phi) \circ f), e^{|df|^2/2} df(e_i)\big) \\
&\qquad\qquad + \sum_i \psi e^{|df|^2/2} \big(\nabla_{df(e_i)}((\operatorname{grad} \phi) \circ f), df(e_i)\big) \\
&\quad = \big(e^{|df|^2/2} d(\phi \circ f), d\psi\big) + \psi e^{|df|^2/2} \operatorname{trace} (\nabla d\phi)(df, df),
\end{aligned}
$$

and the result follows. □

Proposition 3.4 *Let M be a compact connected Riemannian manifold and N be a Riemannian manifold admitting a convex function on N. Then every exponentially harmonic map $f : M \to N$ is constant.*

Proof Let ϕ be a real-valued convex function on N. Taking $\psi = 1$ in the above lemma and integrating on M, via the first variational formula for an exponentially harmonic map, we obtain

$$
\int_M e^{|df|^2/2} \operatorname{trace} (\nabla d\phi)(df, df)\, dv = 0.
$$

This implies that $df = 0$ everywhere on M, and concludes the result. □

Theorem 3.5 *Let M be a complete and non-compact connected Riemannian manifold and N be a Riemannian manifold admitting a convex function ϕ on N such that the uniform norm $\|d\phi\|_\infty$ is bounded. Then every exponentially harmonic map $f\colon M \to N$ with finite $\int_M e^{|df|^2/2}|df|\,dv$ is constant.*

Proof For each $\sigma > 0$ we can find a Lipschitz continuous function ψ on M such that $\psi(x) = 1$ for $x \in B_\sigma$, $\psi(x) = 0$ for $x \in M - B_{2\sigma}$, $0 \leqslant \psi \leqslant 1$, and $|d\psi| \leqslant C/\sigma$ with $C > 0$ independent of σ, where B_σ is a geodesic ball with radius σ about a fixed point x_0. Applying Lemma 3.3, we obtain

$$\begin{aligned}\int_M e^{|df|^2/2}\,\mathrm{trace}\,(\nabla d\phi)(df, df)\,df\,\psi\,dv &= -\int_M e^{|df|^2/2}(d(\phi\circ f), d\psi)\,dv\\ &\leqslant \int_M e^{|df|^2/2}\cdot\|d\phi\|_\infty\cdot|df|\cdot|d\psi|\,dv.\end{aligned}$$

Since $\|d\phi\|_\infty$ is bounded and $\int_M e^{|df|^2/2}|df|dv < \infty$, we have

$$\int_M e^{|df|^2/2}\,\mathrm{trace}\,(\nabla d\phi)(df, df)\,dv \leqslant \frac{C}{\sigma}\int_M e^{|df|^2/2}|df|\,dv.$$

As $\sigma \to \infty$, this implies $df = 0$ and the result follows. □

We can construct a smooth and convex function whose uniform norm is bounded on a simply connected manifold with non-positive sectional curvature (cf. [19]). Indeed, let M be a complete and non-compact connected Riemannian manifold and N be a simply connected Riemannian manifold with non-positive sectional curvature. Then every exponentially harmonic map $f\colon M \to N$, with finite $\int_M e^{|df|^2/2}|df|\,dv$, is constant. In particular, when $N = \mathbb{R}$, we deal with exponentially subharmonic functions. A function f on M is exponentially subharmonic iff $\mathrm{trace}\,\nabla(e^{|df|^2/2}df) \geqslant 0$. Let M be a complete and non-compact connected Riemannian manifold. Then every exponentially subharmonic function f on M, with finite $\int_M e^{|df|^2/2}|df|\,dv$, is constant, since there is a non-decreasing convex function ϕ with bounded derivative on the real line. Thus we have

$$\int_M e^{|df|^2/2}\,\mathrm{trace}\,(\nabla d\phi)(df, df)\psi\,dv \leqslant -\int_M e^{|df|^2/2}(d(\phi\circ f), d\psi)\,dv$$

for every non-negative function ψ with compact support. It follows from a similar argument as in Theorem 3.5.

4 Liouville type 2 theorem

Let M be a Riemannian manifold. For a 2-tensor $K \in \Gamma(T^*M \otimes T^*M)$, its divergence $\mathrm{div}\,K \in \Gamma(T^*M)$ is defined as

$$\operatorname{div} K(X) = \sum_{i=1}^{m} (\nabla_{e_i} K)(e_i, X),$$

where X is any smooth vector field on M. For two 2-tensors K_1, K_2, their inner product is defined as

$$\langle K_1, K_2 \rangle = \sum_{i,j=1}^{m} K_1(e_i, e_j) K_2(e_i, e_j),$$

where $\{e_i\}$ is an orthonormal frame on M with respect to g. For a vector field $X \in \Gamma(TM)$, let θ_X be its dual one form, i.e., $\theta_X(Y) = (X, Y)_g$ with $Y \in \Gamma(TM)$. The covariant derivative of θ_X gives a 2-tensor field $\nabla\theta_X$:

$$\nabla\theta_X(Y, Z) = \nabla_Y \theta_X(Z) = (\nabla_Y X, Z)_g.$$

If $X = \nabla\rho$ is the gradient field of a C^2-function ρ on M, then $\theta_X = d\rho$ and $\nabla\theta_X = \operatorname{Hess} \rho$.

Lemma 4.1 (cf. [1,10]) *Let K be a symmetric $(0, 2)$-type tensor field and X be a vector field. Then*

$$\operatorname{div}(i_X K) = (\operatorname{div} K)(X) + \langle K, \nabla\theta_X \rangle = (\operatorname{div} K)(X) + \frac{1}{2} \langle K, L_X g \rangle,$$

where L_X is the Lie derivative of the metric g in the direction of X. Let $\{e_1, \ldots, e_m\}$ be a local orthonormal frame on M. Then

$$\begin{aligned} \frac{1}{2} \langle K, L_X g \rangle &= \sum_{i,j=1}^{m} \frac{1}{2} \langle K(e_i, e_j), L_X g(e_i, e_j) \rangle \\ &= \sum_{i,j=1}^{m} K(e_i, e_j)(\nabla e_i X, e_j)_g = \langle K, \nabla\theta_X \rangle. \end{aligned}$$

Let D be a bounded domain of M with C^1-boundary. Applying the Stokes theorem, we have

$$\int_{\partial D} K(X, n)\, ds_g = \int_D \left((\operatorname{div} K)(X) + \left\langle K, \frac{1}{2} L_X g \right\rangle \right) dv_g, \tag{4.1}$$

where n is the unit outward normal vector field along ∂D.

The exponential stress energy tensor of a differentiable map $f : M \to N$ between Riemannian manifolds is defined by

$$S_e(f) = e^{|df|^2/2} \left(\frac{|df|^2}{2} g - f^* h \right).$$

The exponential stress energy tensor of f is conserved if $\operatorname{div} S_e(f) = 0$.

Lemma 4.2 ([5,12]) *If $f : (M, g) \to (N, h)$ is an exponentially harmonic map, then*

$$\operatorname{div} S_e(f) = -(\tau_e(f), df(X)) = 0,$$

where X is a vector field on M. Hence, the associated exponential stress energy tensor of f is conserved.

If f is an exponentially harmonic map, then we arrive at

$$\int_{\partial D} S_e(f)(X, n)\, ds_g = \int_D \left\langle S_e(f), \frac{1}{2} L_X g \right\rangle dv_g, \tag{4.2}$$

using Lemma 4.2 and letting $K = S_e(f)$ in (4.1).

Now, let (M, g_0) be a complete m-dimensional Riemannian manifold with a pole x_0 and (N, h) be an n-dimensional Riemannian manifold. Set $r(x) = \operatorname{dist}_{g_0}(x, x_0)$ the g_0-distance function with respect to the pole x_0. Put $B(r) = \{x \in M : r(x) \leqslant r\}$. It is well known that $\frac{\partial}{\partial r}$ is an eigenvector of $\operatorname{Hess}_{g_0}(r^2)$ associated with the eigenvalue 2. Denote by $\mu_{\max}$ (resp. $\mu_{\min}$) the maximum (resp. minimal) eigenvalues of $\operatorname{Hess}_{g_0}(r^2) - 2dr \otimes dr$ at each point of $M - \{x_0\}$. Suppose that $f : (M, g) \to (N, h)$ is a stationary map (via exponential energy) with $g = \rho^2 g_0$, $0 < \rho \in C^\infty(M)$. It is clear that the vector field $n = \rho^{-1} \frac{\partial}{\partial r}$ is an outer normal vector field along $\partial B(r) \subset (M, g)$. Under certain conditions we establish the following Liouville type 2 theorem for exponentially harmonic maps $f : (M, \rho^2 g_0) \to (N, h)$.

Theorem 4.3 (a) *Let $f : (M, \rho^2 g_0) \to (N, h)$ be an exponentially harmonic map. Assume that ρ satisfies condition ($\star$): $\frac{\partial \log \rho}{\partial r} \geqslant 0$ and there is a constant $C > 0$ such that*

$$(m-2) r \frac{\partial \log \rho}{\partial r} + \frac{m-1}{2} \mu_{\min} + 1 - \max\{2, \mu_{\max}\} \geqslant C.$$

Then

$$\sigma_1^{-C} \int_{B(\sigma_1)} e^{|df|^2/2} \frac{|df|^2}{2}\, dv \leqslant \sigma_2^{-C} \int_{B(\sigma_2)} e^{|df|^2/2} \frac{|df|^2}{2}\, dv$$

for any $0 < \sigma_1 \leqslant \sigma_2$.

(b) *If $\int_{B(R)} e^{|df|^2/2} \frac{|df|^2}{2} dv = o(R^C)$, then f is constant.*

Proof In (4.2), take $D = B(r)$ and $X = r \frac{\partial}{\partial r} = \frac{1}{2} \nabla^0 r^2$ (the covariant derivative ∇^0 determined by g_0), we have

$$\int_{B(r)} \left\langle S_e(f), \frac{1}{2} L_X g \right\rangle dv_g = \int_{\partial B(r)} S_e(f) \langle X, n \rangle\, ds_g.$$

We have

$$\begin{aligned} \left\langle S_e(f), \frac{1}{2} L_X g \right\rangle &= \left\langle S_e(f), r \frac{\partial \log \rho}{\partial r} g \right\rangle + \left\langle S_e(f), \frac{1}{2} \rho^2 L_X g_0 \right\rangle \\ &= r \frac{\partial \log \rho}{\partial r} \langle S_e(f), g \rangle + \frac{1}{2} \eta^2 \left\langle S_e(f), \operatorname{Hess}_{g_0}(r^2) \right\rangle. \end{aligned} \tag{4.3}$$

Let $\{e_i\}_{i=1}^m$ be an orthonormal frame with respect to g_0 and $e_m = \frac{\partial}{\partial r}$. We may assume that $\mathrm{Hess}_{g_0}(r^2)$ is a diagonal matrix with respect to $\{e_i\}$. Keep in mind that $\{\widehat{e_i} = \rho^{-1}e_i\}$ is an orthonormal frame with respect to g. We derive the following two inequalities:

$$\begin{aligned}
&\frac{1}{2}\rho^2\big\langle S_{\mathrm{e}}(f), \mathrm{Hess}_{g_0}(r^2)\big\rangle \\
&\quad= \frac{1}{2}\rho^2\sum_{i,j=1}^m S_{\mathrm{e}}(f)(\widehat{e_i},\widehat{e_j})\,\mathrm{Hess}_{g_0}(r^2)(\widehat{e_i},\widehat{e_j}) \\
&\quad= \frac{1}{2}\rho^2\Big[\sum_{i=1}^m e^{|df|^2/2}\frac{|df|^2}{2}\,\mathrm{Hess}_{g_0}(r^2)(\widehat{e_i},\widehat{e_j}) \\
&\qquad\qquad - \sum_{i,j=1}^m e^{|df|^2/2}(df(\widehat{e_i}), df(\widehat{e_j}))\,\mathrm{Hess}_{g_0}(r^2)(\widehat{e_i},\widehat{e_j})\Big] \\
&\quad= \frac{1}{2}e^{|df|^2/2}\frac{|df|^2}{2}\sum_{i=1}^m \mathrm{Hess}_{g_0}(r^2)(e_i,e_i) \\
&\qquad\qquad - \frac{1}{2}e^{|df|^2/2}\sum_{i=1}^m (df(\widehat{e_i}), df(\widehat{e_i}))\,\mathrm{Hess}_{g_0}(r^2)(e_i,e_i) \\
&\quad\geqslant \frac{1}{2}e^{|df|^2/2}\frac{|df|^2}{2}\,[(m-1)\mu_{\min}+2] \\
&\qquad\qquad - \frac{1}{2}\max\{2,\mu_{\max}\}\,e^{|df|^2/2}\sum_{i=1}^m (df(\widehat{e_i}),\widehat{e_i})) \\
&\quad= \frac{1}{2}e^{|df|^2/2}\frac{|df|^2}{2}\,[(m-1)\mu_{\min}+2] \\
&\qquad\qquad - \frac{1}{2}\max\{2,\mu_{\max}\}\,e^{|df|^2/2}|df|^2 \\
&\quad\geqslant \frac{1}{2}\big[(m-1)\mu_{\min}+2-2\max\{2,\mu_{\max}\}\big]e^{|df|^2/2}\frac{|df|^2}{2},
\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}
\langle S_{\mathrm{e}}(f), g\rangle &= me^{|df|^2/2}\frac{|df|^2}{2} - e^{|df|^2/2}(df(\widehat{e_i}), df(\widehat{e_j}))_h(\widehat{e_i},\widehat{e_j})_g \\
&= me^{|df|^2/2}\frac{|df|^2}{2} - e^{|df|^2/2}|df|^2 \geqslant (m-2)\,e^{|df|^2/2}\frac{|df|^2}{2}.
\end{aligned} \tag{4.5}$$

We obtain from (4.3), (4.4), (4.5) and condition ($\star$) that

$$
\begin{aligned}
\left\langle S_{\mathrm{e}}(f), \frac{1}{2} L_X g \right\rangle \geqslant \Bigg[& r \frac{\partial \log \rho}{\partial r} (m-2) + \frac{m-1}{2} \mu_{\min} \\
& + 1 - \max\{2, \mu_{\max}\} \Bigg] e^{|df|^2/2} \frac{|df|^2}{2} \\
\geqslant & \; C e^{|df|^2/2} \frac{|df|^2}{2}.
\end{aligned} \tag{4.6}
$$

Using co-area and the following fact:

$$
\begin{aligned}
|\nabla r|_g^2 &= \sum_{i=1}^{m} (\widehat{e_i} r)^2 = \sum_{i=1}^{m-1} \rho^{-2} (e_i r)^2 + \rho^{-2} \\
&= \sum_{i=1}^{m-1} \rho^{-2} \left[\left(e_i, \frac{\partial}{\partial r} \right)_{g_0} \right]^2 + \rho^{-2} = \rho^{-2} \quad (\text{i.e., } |\nabla r|_g = \rho^{-1}),
\end{aligned}
$$

we arrive at

$$
\begin{aligned}
\int_{\partial B(r)} & S_{\mathrm{e}}(f)(X, n) \, ds_g \\
&= \int_{\partial B(r)} e^{|df|^2/2} \left[\frac{|df|^2}{2} (X, n) - (df(X), df(n))_h \right] ds_g \\
&= r \int_{\partial B(r)} e^{|df|^2/2} \frac{|df|^2}{2} \rho \, ds_g \\
&\qquad - \int_{\partial B(r)} e^{|df|^2/2} r \rho^{-1} \left(df\left(\frac{\partial}{\partial r} \right), df\left(\frac{\partial}{\partial r} \right) \right)_h ds_g \\
&\leqslant r \int_{\partial B(r)} e^{|df|^2/2} \frac{|df|^2}{2} \rho \, ds_g \\
&= r \frac{d}{dr} \int_0^r \int_{\partial B(t)} \left[|\nabla r|^{-1} e^{|df|^2/2} \frac{|df|^2}{2} ds_g \right] dt \\
&= r \frac{d}{dr} \int_{B(r)} e^{|df|^2/2} \frac{|df|^2}{2} dv.
\end{aligned} \tag{4.7}
$$

It follows from (4.3), (4.6) and (4.7) that

$$
0 \leqslant C \int_{B(r)} e^{|df|^2/2} \frac{|df|^2}{2} dv \leqslant r \frac{d}{dr} \int_{B(r)} e^{|df|^2/2} \frac{|df|^2}{2} dv,
$$

or equivalently

$$
\frac{d}{dr} \frac{1}{r^C} \int_{B(r)} e^{|df|^2/2} \frac{|df|^2}{2} dv \geqslant 0.
$$

Hence,

$$\frac{1}{\sigma_1^C}\int_{B(\sigma_1)} e^{|df|^2/2}\frac{|df|^2}{2}\,dv \leqslant \frac{1}{\sigma_2^C}\int_{B(\sigma_2)} e^{|df|^2/2}\frac{|df|^2}{2}\,dv$$

for any $0<\sigma_1\leqslant\sigma_2$. □

The energy functional $E_1(f)=\int_M e^{|df|^2/2}\frac{|df|^2}{2}dv$ of a map $f\colon M\to N$ is called *slowly divergent* if there exists a positive function $\phi(r)$ with $\int_{R_0}^{\infty}\frac{dr}{r\phi(r)}=+\infty$, $R_0>0$, such that

$$\lim_{R\to\infty}\int_{B(R)}\frac{e^{|df|^2/2}}{\phi(r(x))}\frac{|df|^2}{2}\,dv<\infty. \tag{4.8}$$

Theorem 4.4 *Let $f\colon(M,\rho^2 g_0)\to(N,h)$ be an exponentially harmonic map. Suppose that ρ satisfies condition $(\star)$ and $E_1(f)$ is slowly divergent, then f is constant.*

Proof From the proof of Theorem 4.3 we have

$$C\int_{B(R)} e^{|df|^2/2}\frac{|df|^2}{2}\,dv_g \leqslant R\frac{d}{dr}\int_{\partial B(R)} e^{|df|^2/2}\frac{|df|^2}{2}\rho\,ds_g \tag{4.9}$$

Assume that f is a non-constant map. Then there exists $R_0>0$ such that for $R\geqslant R_0$,

$$\int_{B(R)} e^{|df|^2/2}\frac{|df|^2}{2}\,dv\geqslant C_1, \tag{4.10}$$

where C_1 is a positive constant. It follows from (4.9) and (4.10) that

$$\int_{\partial B(R)} e^{|df|^2/2}\frac{|df|^2}{2}\rho\,ds_g\geqslant\frac{C_1\cdot C}{R},$$

for $R\geqslant R_0$. Consequently,

$$\begin{aligned}\lim_{R\to\infty}\int_{B(R)}\frac{e^{|df|^2/2}}{\phi(r(x))}\frac{|df|^2}{2}\,dv &= \int_0^{\infty}\frac{dR}{\phi(R)}\int_{\partial B(R)} e^{|df|^2/2}\frac{|df|^2}{2}\rho\,ds_g\\ &\geqslant \int_{R_0}^{\infty}\frac{dR}{\phi(R)}\int_{\partial B(R)} e^{|df|^2/2}\frac{|df|^2}{2}\rho\,ds_g\\ &\geqslant C\cdot C_1\int_{R_0}^{\infty}\frac{dR}{R\phi(R)}=\infty,\end{aligned}$$

which contradicts (4.8). Hence, f must be constant. □

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