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RESEARCH ARTICLE

Attractors for classes of iterated function systems

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Abstract
We consider various classes of iterated function systems, such as those comprised of similarities and those satisfying the open set condition, and weakly contractive systems. Necessary properties are developed for a compact set to be an attractor for the various classes of iterated function systems. Since the Hausdorff metric space is complete, we are able to develop several results concerning typical compact sets and their invariance with respect to iterated function systems.

Keywords Iterated function system · Contractive system · Invariant set · Attractor · Hausdorff measure

Mathematics Subject Classification 28A80 · 54E40 · 54C50 · 26A21

1 Introduction

Let $X = (X, d)$ be a complete metric space, with $\mathcal{S} = \{S_1, \dots, S_m\}$ a finite set of contraction maps $S_i : X \to X$. We call a set E an *attractor* (*non-empty invariant set*) of the iterated function system (IFS) $\mathcal{S}$ if $E = \mathcal{S}(E) = \bigcup_{i=1}^{m} S_i(E)$; that is, E is invariant with respect to $\mathcal{S}$. For any given finite set of contraction maps $\mathcal{S}$, there is a unique non-empty compact set E in X such that $E = \mathcal{S}(E)$. In what follows, we frequently take $|\mathcal{S}|$ to represent the necessarily unique attractor of the IFS $\mathcal{S}$. This and other results in [10] are recorded below.

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Theorem 1.1 *Let X be a complete metric space with $\mathcal{S} = \{S_1, \ldots, S_m\}$ a finite set of contraction maps $S_i : X \to X$.*

(1) *There exists a unique non-empty compact set $E \subseteq X$ such that $E = \mathcal{S}(E)$.*
(2) *The set E is the closure of the set of fixed points of the finite compositions $S_{i_1} \circ \cdots \circ S_{i_p}$ of elements of $\mathcal{S}$.*
(3) *If A is any non-empty compact set in X, then $\lim_{p\to\infty} \mathcal{S}^p(A) = E$ in the Hausdorff metric.*

Now, suppose that the contraction maps found in $\mathcal{S}$ are similarities. That is, $d(S_i(x), S_i(y)) = r_i d(x, y)$ for all x, y in X, and $0 < r_i < 1$. Each S_i takes subsets of X and sends them into geometrically similar sets. This produces attractors that are self-similar. Should the images of the sets $S_i(F)$ not overlap "too much", the self-similar attractor $E = \bigcup_{i=1}^m S_i(E)$ has Hausdorff dimension equal to the value of s satisfying $\sum_{i=1}^m {r_i}^s = 1$.

The notion of insignificant overlap comes from the open set condition (OSC). Suppose $\mathcal{S} = \{S_1, \ldots, S_m\}$ is a finite set of contraction maps $S_i : X \to X$. The contractive system $\mathcal{S}$ satisfies the OSC if there exists a non-empty open set V for which $\bigcup_{i=1}^m S_i(V) \subseteq V$ and $S_i(V) \cap S_j(V) = \varnothing$ whenever $i \neq j$ [10,15].

If each of the S_i is a similarity, and $d(S_i(x), S_i(y)) = r_i d(x, y)$ for all x, y in X, where $0 < r_i < 1$, then the Hausdorff dimension, $\dim_{\mathcal{H}}(E)$, is equal to s, and $0 < \mathcal{H}^s(E) < \infty$, where $E = \mathcal{S}(E)$ is the attractor for $\mathcal{S}$, and $\sum_{i=1}^m {r_i}^s = 1$.

As proved as an exercise in [8], if the unique non-empty invariant set E satisfies $S_i(E) \cap S_j(E) = \varnothing$ whenever $i \neq j$, then the system $\mathcal{S} = \{S_1, \ldots, S_m\}$ of contraction maps $S_i : X \to X$ satisfies the OSC.

The purpose of this paper is to study the attractors of the IFS that are comprised of similarities, that satisfy the OSC, or both. Let X be a complete metric space, take $\mathcal{K}(X)$ to be the collection of non-empty compact subsets of X, with $\mathcal{S} = \{S_1, \ldots, S_m\}$ a finite set of contraction maps $S_i : X \to X$. We set

$$\mathcal{I}_1 = \big\{E \in \mathcal{K}(X) : E = \mathcal{S}(E),\ \text{each } S_i \text{ is a similarity and } \mathcal{S} \text{ satisfies the OSC}\big\},$$
$$\mathcal{I}_2 = \big\{E \in \mathcal{K}(X) : E = \mathcal{S}(E) \text{ and each } S_i \text{ is a similarity}\big\},$$
$$\mathcal{I}_3 = \big\{E \in \mathcal{K}(X) : E = \mathcal{S}(E) \text{ and } \mathcal{S} \text{ satisfies the OSC}\big\}.$$

Finally, let $\mathcal{I} = \{E \in \mathcal{K}(X) : E = \mathcal{S}(E)\}$ be the set of all attractors in $\mathcal{K}(X)$ generated by some $\mathcal{S} = \{S_1, \ldots, S_m\}$, a finite collection of contraction maps $S_i : X \to X$. We call a finite collection of contraction maps a *contractive system* and a finite collection of weak contractions a *weakly contractive system*.

We proceed through several sections. Definitions, notation and some previously known results are presented in Sect. 2. There, we also record some observations concerning the sets $\mathcal{I}_1$ and $\mathcal{I}_2$. Section 3 is dedicated to the study of countable attractors. If $E \in \mathcal{K}([0, 1])$ is homeomorphic to an element of $\mathcal{I}_1$ or $\mathcal{I}_2$, then E must be a singleton, or a Cantor set, or the closure of the union of countably many disjoint non-trivial closed intervals. Any nowhere dense element $E \subseteq [0, 1]$ of $\mathcal{I}$ has a homeomorphic copy E' in $\mathcal{I}_3$. If E is a countable attractor, then it has a homeomorphic copy F such that $F = \mathcal{S}(F)$ for some $\mathcal{S} = \{S_1, S_2\}$.

In Sect. 4, one studies weakly contractive systems. From [4], we know that the set of attractors in $\mathcal{K}([0,1]^n)$ generated by contractive systems is a first category F_σ. This is also true for IFS composed of weak contractions. One also sees that the set of attractors generated by IFS of the form $\{(f_1,\ldots,f_L) : f_i$ is a weak contraction, $L \leqslant M\}$ is nowhere dense in $\mathcal{K}([0,1])$. Since the set of attractors is dense in $\mathcal{K}([0,1])$, one concludes that most attractors can only be generated by IFS $\mathcal{S} = \{S_1,\ldots,S_L\}$, where $L > M$.

Not only is the typical element of $\mathcal{K}([0,1]^n)$ not an attractor for any IFS, but one also can take that dense G_δ subset of $\mathcal{K}([0,1]^n)$ so that $\mathcal{H}^\phi(E) = 0$ for each of its elements E, where ϕ is some fixed gauge function. In Sect. 5, one sees that attractors with "large" measure—that is, $\mathcal{H}^\phi_m(E) \geqslant c$—are also exceptional.

Section 6 is dedicated to attractors with non-empty interior. There, we develop a necessary condition for a compact set with non-empty interior to be an attractor. This is similar to that developed by Nowak for countable sets [1,12].

2 Preliminaries

We take (X,d) to be a compact metric space. For A subset of X, we denote by $|A|$ its diameter, by $\overline{A}$ its closure, by $\mathrm{int}(A)$ its interior, by $\overline{\mathrm{conv}}(A)$ its convex closure (meaning the closure of its convex hull), and by $\mathrm{card}(A)$ its cardinality when A is finite.

Much of the analysis takes place in the Hausdorff metric space $\mathcal{K}(X) = (\mathcal{K}(X),\mathcal{H})$. Take $\mathcal{K}(X)$ to be the collection of non-empty compact subsets of X. Endow $\mathcal{K}(X)$ with the Hausdorff metric $\mathcal{H}$ given by

$$\mathcal{H}(E,F) = \inf\left\{\delta > 0 : E \subset B_\delta(F),\, F \subset B_\delta(E)\right\},$$

where the δ-neighbourhood or δ-parallel body, $B_\delta(A)$, of a set A is the set of points within distance δ of A. Thus, $B_\delta(A) = \{x : d(x,y) < \delta \text{ for some } y \text{ in } A\} = \bigcup_{x\in A} B_\delta(x)$, where $B_\delta(x) = \{y \in X : d(y,x) < \delta\}$ is the open ball in X centered at x of radius δ. The Hausdorff metric space $(\mathcal{K}(X),\mathcal{H})$ is also compact.

Since $(\mathcal{K}(X),\mathcal{H})$ is complete, one can make good use of the Baire category theorem. A set is of the first category in the complete space (X,d) whenever it can be written as a countable union of nowhere dense sets; otherwise, the set is of the second category. A set is residual if it is the complement of a first category set, and an element of a residual set is called either typical or generic.

Theorem 2.1 (Baire category theorem) *Let (X,d) be a complete metric space with B a first category subset of X. Then $X \setminus B$ is dense in X.*

Let Φ denote the set of functions ϕ that are continuous and increasing on $I = [0,1]$, with $\phi(0) = 0$. For $\phi \in \Phi$ and $s \in \mathbb{N}$, set

$$\mathcal{H}^\phi_s(E) = \inf\left\{\sum \phi(|I_j|) : E \subseteq \bigcup I_j,\; I_j \subseteq [0,1]^n \text{ an open set of diameter } |I_j| \leqslant \frac{1}{s}\right\}.$$

Then $\mathcal{H}^{\phi} = \lim_{s\to\infty} \mathcal{H}^{\phi}_s$ defines a measure on the Borel sets in $[0, 1]^n$. In what follows, our concern will be primarily with closed sets. In the case that $\phi(x) = x^s$, one gets the usual s-dimensional Hausdorff measure [2,4,8,9]. Define the Hausdorff dimension of a set E such that

$$\dim_{\mathcal{H}}(E) = \inf\{s : \mathcal{H}^s(E) = 0\} = \sup\{s : \mathcal{H}^s(E) = \infty\}.$$

In Sect. 3, we use λ^n to refer to the Lebesgue n-dimensional measure.

Since our interest lies with contractions and weak contractions, the functions considered are all continuous. Let $C(X, X)$ to be the set of continuous functions $f: X \to X$. Within $C(X, X)$ we use the uniform metric: $\|f - g\| = \sup\{d(f(x), g(x)) : x \in X\}$. Since X is compact, $\|f - g\| = \max\{d(f(x), g(x)) : x \in X\}$.

We recall that a topological space is a Cantor space if it is non-empty, perfect, compact, totally disconnected, and metrizable. In particular, a topological space is a Cantor space if it is homeomorphic to the Cantor ternary set.

A topological space X is said to be scattered if every non-empty subspace M has an isolated point in M. Every compact scattered Hausdorff space has a base consisting of clopen sets (is zero-dimensional), and a compact metric space is scattered if and only if it is countable. Let X be a compact scattered space. We let

$$X' = \{x \in X : x \text{ is an accumulation point of } X\}$$

be the *Cantor–Bendixon derived set* of X, and define inductively

$$X^{(\alpha+1)} = \left(X^{(\alpha)}\right)'; \qquad X^{\alpha} = \bigcap_{\beta<\alpha} X^{(\beta)} \quad \text{for a limit ordinal } \alpha.$$

The *height* of X is

$$\mathrm{ht}(X) = \min\{\alpha : X^{(\alpha)} \text{ is discrete}\}.$$

By the Mazurkiewicz–Sierpiński Theorem [11], every countable compact scattered space X is homeomorphic to the space $\omega^{\beta}\cdot n + 1$, where $\beta = \mathrm{ht}(X)$ and n is the number of elements of $X^{(\beta)}$.

Next we present some simple conclusions concerning elements in $\mathcal{I}_1$ and $\mathcal{I}_2$.

Proposition 2.2 *Let $\mathcal{S} = \{S_1, \ldots, S_N\}$ be a finite set of similarities from $\mathbb{R}^n$ to $\mathbb{R}^n$ satisfying the OSC. Then the following are equivalent:*

(1) *$F = \mathcal{S}(F)$ is nowhere dense and perfect,*
(2) *$N \geqslant 2$ and $\sum_{i=1}^{N} {r_i}^n < 1$.*

Proof (1)$\Rightarrow$(2): Suppose $F = \mathcal{S}(F)$ is nowhere dense and perfect. Then, $N \geqslant 2$. Suppose $\mathcal{S}$ satisfies the OSC with V open. Since F is nowhere dense, V must contain some G, an open ball contained in the complement of F. Since $S_i(V) \cap S_j(V) = \varnothing$ whenever $i \neq j$, it follows that $S_{i_1\ldots i_k}(V) \cap S_{j_1\ldots j_k}(V) = \varnothing$ whenever $i_1 \ldots i_k \neq j_1 \ldots j_k$. Let $x \in F$. For each $k \in \mathbb{N}$, there exists a string of length k, say $i_1 i_2 \ldots i_k$

such that $x \in S_{i_1 i_2 \ldots i_k}(F)$. As $G \subseteq V$, $S_{i_1 i_2 \ldots i_k}(G) \subseteq S_{i_1 i_2 \ldots i_k}(V)$, $S_{i_1 i_2 \ldots i_k}(G)$ is complementary to F [10, p. 736], and the porosity

$$p(x, F) \geqslant \frac{r_{i_1} \ldots r_{i_k} \operatorname{diam}(G)}{r_{i_1} \ldots r_{i_k} \operatorname{diam}(F)}.$$

Thus, $\dim_{\mathcal{H}}(F) < n$, and $\sum_{i=1}^{N} r_i{}^n < 1$ ([9, Theorem 9.3], [17, Section 4F]).
(2)$\Rightarrow$(1): Suppose $F = \mathcal{S}(F)$ with $N \geqslant 2$ and $\sum_{i=1}^{N} r_i{}^n < 1$. We show that it has no isolated point. Let $x \in F$ with $\epsilon > 0$. Take $i_1', \ldots, i_k'$ so that $S_{i_1' \ldots i_k'}(F) \subseteq B_\epsilon(x)$. Since F is not a singleton, there exists $y \in S_{i_1' \ldots i_k'}(F)$ distinct from x. In particular, x is a limit point of F. So, F is perfect. Since $\dim_{\mathcal{H}}(|\mathcal{S}|) < n$, F contains no ball. □

Proposition 2.3 *Let $\mathcal{S} = \{S_1, \ldots, S_N\}$ be a finite set of similarities from $\mathbb{R}^n$ to $\mathbb{R}^n$ satisfying the OSC. If $F = \mathcal{S}(F)$, then the following are equivalent:*

(1) *F contains a ball in $\mathbb{R}^n$.*
(2) *F is the closure of a countable union of non-degenerate closed balls in $\mathbb{R}^n$.*
(3) *$\sum_{i=1}^{N} r_i{}^n = 1$ and $N \geqslant 2$.*

Proof (1)$\Rightarrow$(2): Let $x \in F$. It suffices to show that there is a sequence of balls converging to x. Let $\epsilon > 0$, and take $i_1 i_2 \ldots i_k$ so that $S_{i_1 \ldots i_k}(F) \subseteq B_\epsilon(x)$. Since F contains a ball and $S_{i_1 \ldots i_k}$ is a composition of similarities, $S_{i_1 \ldots i_k}(F)$ also contains a ball.
(2)$\Rightarrow$(3): Since F has non-empty interior, $N \geqslant 2$. Moreover, F is not nowhere dense, so $\sum_{i=1}^{n} r_i{}^n = 1$ by Proposition 2.2.
(3)$\Rightarrow$(1): Since $\sum_{i=1}^{N} r_i{}^n = 1$, F is not nowhere dense by Proposition 2.2, so it must necessarily contain a ball in $\mathbb{R}^n$. □

Proposition 2.4 *Let $\mathcal{S} = \{S_1, S_2\}$ where each $S_i : \mathbb{R} \to \mathbb{R}$ is a similarity. If $r_1 + r_2 \leqslant 1$, and $\operatorname{Fix}(S_1) \neq \operatorname{Fix}(S_2)$, then $\mathcal{S}$ satisfies the OSC with $V = \operatorname{int}(\overline{\operatorname{conv}}(|\mathcal{S}|))$.*

This allows us to characterize, up to homeomorphism, those elements of $\mathcal{I}$ generated by a pair of similarities.

Proposition 2.5 *Let $\mathcal{S} = \{S_1, S_2\}$ where each $S_i : \mathbb{R} \to \mathbb{R}$ is a similarity. Necessarily, $\mathcal{S}$ satisfies the OSC so long as $\operatorname{Fix}(S_1) \neq \operatorname{Fix}(S_2)$ and $r_1 + r_2 \leqslant 1$. Set*

$$\mathcal{T} = \left\{ F \in \mathcal{K}([0, 1]) : F = \mathcal{S}(F) \text{ where } \mathcal{S} = \{S_1, S_2\}, S_i \text{ is a similarity} \right\}.$$

Then $E \in \mathcal{K}([0, 1])$ is homeomorphic to an element of $\mathcal{T}$ if and only if one of the following occurs:

(1) *E is a singleton,*
(2) *E is a Cantor set, or*
(3) *E is an interval.*

3 Countable attractors

In [12], Nowak completely characterizes, up to homeomorphism, countable attractors for iterated function systems. Nowak shows that every countable compact set of successor Cantor–Bendixon height with a single point of maximal rank has a homeomorphic copy in [0, 1] that is an attractor of an IFS consisting of two contractions. This IFS satisfies the OSC, and the Lipschitz constants of its two components can be chosen to be as small as one wishes. Moreover, Nowak shows that if a countable compact metric space is an IFS-attractor, then its Cantor–Bendixson height cannot be a limit ordinal.

Theorem 3.1 ([12, Theorem 3]) *A compact scattered metric space of limit Cantor–Bendixson height is not homeomorphic to any IFS-attractor consisting of weak contractions.*

Theorem 3.2 ([12, Theorem 4]) *For every $\epsilon > 0$ and every countable ordinal δ the scattered space $\omega^{\delta+1} + 1$ is homeomorphic to the attractor of an iterated function system consisting of two contractions $\{\phi, \phi_{\delta+1}\}$ in the unit interval $I = [0, 1]$, such that*

$$\max\{\mathrm{Lip}(\phi), \mathrm{Lip}(\phi_{\delta+1})\} < \epsilon.$$

The main result of this section is the following theorem.

Theorem 3.3 *Let $E \subseteq [0, 1]$ be countable and compact. Then E is homeomorphic to an attractor E' for some $\mathcal{S} = \{S_1, S_2\}$ which satisfies the OSC if and only if the height of E is a successor ordinal.*

Theorem 3.3 is an immediate consequence of Theorem 3.2, and the following proposition. Frequently, we make use of the convex closure of a set E, and write $\overline{\mathrm{conv}}(E)$. We use this notation even in the case that E is closed, in an effort to somewhat limit the various notations used.

Proposition 3.4 *Suppose $E \in \mathcal{I}$, and $E = \mathcal{S}(E)$ where $\mathcal{S} = \{S_1, \ldots, S_m\}$. Then, for any $n \geqslant m$, there exists $F \in \mathcal{I}$ and $\mathcal{S}' = \{T_1, T_2\}$ which satisfies the OSC such that*

(1) $F = \mathcal{S}'(F)$,
(2) $F = \dot{\bigcup}_{i=1}^{n} E_i$, *where each E_i is similar to E ($\dot{\bigcup}$ means disjoint union), and*
(3) $\overline{\mathrm{conv}}(E_i) \cap \overline{\mathrm{conv}}(E_j) = \varnothing$ *whenever $i \neq j$.*

Proof We give a proof in the case that $m = 2$. The general case then follows easily. Suppose $E \in \mathcal{I}$, and for convenience and without loss of generality, $\overline{\mathrm{conv}}(E) = [0, 1]$, and $E = \mathcal{S}(E)$ for some $\mathcal{S} = \{S_1, S_2\}$. Take $E_1, E_2, \ldots, E_n$ such that

(i) each is similar to E,
(ii) E_j lies to the right of E_l whenever $j < l$,
(iii) $\mathrm{diam}(E_j) = \delta\,\mathrm{diam}(E_{j-1})$, where $\mathrm{Lip}(S_2) = r_2 < \delta < 1$, for any $1 < j \leqslant n$, and
(iv) $\min\{\mathrm{diam}(G_i) : 1 \leqslant i \leqslant n-1\} > 2\max\{\mathrm{diam}(E_i) : 1 \leqslant i \leqslant n\}$, where G_i is the open interval lying between E_i and E_{i+1}, that is $G_i = (\max E_{i+1}, \min E_i)$.

Moreover, take $\operatorname{diam}(G_{i+1}) < \operatorname{diam}(G_i)$. Define T_1 so that it is linear on each of the intervals $\overline{\operatorname{conv}}(E_i)$ and G_i. In particular, take T_1 so that

(v) $\overline{\operatorname{conv}}(E_i)$ is mapped onto $\overline{\operatorname{conv}}(E_{i+1})$ with $T_1(E_i) = E_{i+1}$ for $i = 1, 2, \ldots, n-1$,
(vi) $T_1(\overline{\operatorname{conv}}(E_n)) = \min E_n$,
(vii) G_i is mapped onto G_{i+1} for $i = 1, 2, \ldots, n-2$, and
(viii) $T_1(G_{n-1}) = \min E_n$, too.

Then $T_1 \colon [0, 1] \to [0, 1]$ is a contraction given the diameters of the sets E_i and G_i, and $T_1\big(\bigcup_{i=1}^{n} E_i\big) = \bigcup_{i=2}^{n} E_i$.

Let us now define T_2. Recall that E_1 and E_2 are both similar to E and $\operatorname{diam}(E_2) = \delta \operatorname{diam}(E_1)$. Set $\operatorname{conv}(E_i) = [a_i, b_i]$ and take $\phi_i \colon E \to E_i$ to be the line such that $\phi_i(0) = \min E_i$ and $\phi_i(1) = \max E_i$. Define T_2 on $[a_2, b_2]$ so that $T_2 = \phi_1 \circ S_2 \circ (\phi_2)^{-1}$, and on $[a_1, b_1]$, define T_2 so that $T_2 = \phi_1 \circ S_1 \circ (\phi_1)^{-1}$. Set $T_2(x) = T_2(a_2)$ for all $x \leqslant a_2$, and extend T_2 linearly on G_1. Then

$$T_2(F) = T_2(E_1 \cup E_2) = T_2(E_1) \cup T_2(E_2) = E_1,$$
$$\operatorname{Lip}\big(T_{2|[0,a_2]}\big) = 0,$$

and

$$\operatorname{Lip}\big(T_{2|[a_1,b_1=1]}\big) = r_1 = \operatorname{Lip}(S_1),$$
$$\operatorname{Lip}\big(T_{2|[a_2,b_2]}\big) = \frac{\operatorname{diam}(E)}{\operatorname{diam}(E_2)} r_2 \frac{\operatorname{diam}(E_1)}{\operatorname{diam}(E)} = \frac{\operatorname{diam}(E_1)}{\operatorname{diam}(E_2)} r_2 = \frac{r_2}{\delta} < 1.$$

It follows, then, that $T_2 \colon [0, 1] \to [0, 1]$ is a contraction map, and

$$F = \mathcal{S}'(F) = \bigcup_{i=1}^{2} T_i(F) = T_1(F) \cup T_2(F)$$
$$= \left(\bigcup_{i=2}^{n} E_i\right) \cup E_1 = \bigcup_{i=1}^{n} E_i = F. \qquad \square$$

Remark 3.5 From [6, Theorem 5.4] and the construction found in its proof, if $F \in \mathcal{K}([0, 1])$ is nowhere dense and uncountable, then there exist F', a homeomorphic copy of F, and an IFS $\mathcal{S}$ satisfying the OSC such that $F' = \mathcal{S}(F')$. This observation, coupled with Theorem 3.3, allows one to conclude that any nowhere dense attractor is homeomorphic to an element of $\mathcal{I}_3$.

4 Systems of weak contractions

In this section we investigate IFS consisting of weak contractions. First, we observe that the typical element of $\mathcal{K}([0, 1]^n)$ is not an attractor for any system of weak contractions. Then, we consider the function $\mathcal{T}\colon (\text{wLip } 1)^m \to (\mathcal{K}([0,1]), \mathcal{H})$ given by $\mathcal{S} \mapsto |\mathcal{S}|$, and investigate the images of $\left(\text{Lip } \frac{l-1}{l}\right)^m$, $l \in \mathbb{N}\setminus\{1\}$, and of $(\text{wLip } 1)^m$. There, we focus on the case when $X = [0,1]$.

In [4], we construct a residual subset $\mathcal{K}_2$ of $\mathcal{K}([0,1]^n)$ with the property that if $E \in \mathcal{K}_2$, then $E \neq \mathcal{S}(E)$ for any finite set of contraction maps $\mathcal{S}$ defined on $[0,1]^n$. The analysis developed for contractions is valid for weak contractions. This is recorded in the next result.

Theorem 4.1 *There is a residual subset $\mathcal{K}_2$ of $\mathcal{K}([0,1]^n)$ such that $E \neq \mathcal{S}(E)$ for any finite set of weak contraction maps $\mathcal{S}$ defined on $[0,1]^n$, for any $E \in \mathcal{K}_2$.*

For $s > 0$, set

$$\text{Lip } s = \left\{ f \in C(X,X) : d(f(x), f(y)) \leqslant s d(x,y) \text{ for all } x, y \text{ in } X \right\}$$

and, for $m \in \mathbb{N}$, set

$$(\text{Lip } s)^m = \left\{ (f_1, \ldots, f_m) : f_i \in \text{Lip } s,\ 1 \leqslant i \leqslant m \right\}.$$

As the following result in [6] shows, should we limit the number of contractions in our IFS to no more than N, and uniformly bound the Lipschitz constants of the contractions, then the set of attractors generated by such systems is closed.

Lemma 4.2 ([6, Lemma 4.3]) *Let (X,d) be a compact metric space. Let $N \in \mathbb{N}$ and $0 < m < 1$. The set*

$$\mathcal{L}_{N,m} = \Big\{ F \in \mathcal{K}(X) : F = \mathcal{S}(F) \text{ with } \mathcal{S} = \{S_1, \ldots, S_L\}, \\ \text{for } L \leqslant N \text{ and } \text{Lip}(S_i) \leqslant m \Big\}$$

is closed.

The following straightforward lemma shows not only that $\mathcal{I}_1$ is dense in the Hausdorff metric space $(\mathcal{K}([0,1]^n), \mathcal{H})$, but also that the attractors for iterated function systems generated by contractions with arbitrarily small Lipschitz constants are dense in $(\mathcal{K}([0,1]^n), \mathcal{H})$.

Lemma 4.3 *The set of attractors $\mathcal{I}_1$ is dense in $(\mathcal{K}([0,1]^n), \mathcal{H})$.*

Proof Let $E \in \mathcal{K}([0,1]^n)$, and take $\epsilon > 0$. We use the Euclidean metric within $[0,1]^n$. Let $F \subseteq E$ be a finite set so that $\mathcal{H}(E,F) < \epsilon/2$; say $F = \{x_1, x_2, \ldots, x_n\}$. To each x_i associate a non-degenerate closed ball I_i and a similarity S_i so that $x_i \in I_i$, $I_i \cap I_j = \varnothing$ whenever $i \neq j$, $I_i \subseteq B_{\epsilon/2}(F)$, $S_i([0,1]^n) \subset I_i$ and $\text{Fix } S_i = x_i$. It follows that $F \subseteq |\{S_1, S_2, \ldots, S_n\}| \subset B_{\epsilon/2}(F)$, and $\mathcal{H}(F, |\{S_1, S_2, \ldots, S_n\}|) < \epsilon/2$. Now

$$\mathcal{H}(E, |\{S_1, S_2, \ldots, S_n\}|) \leqslant \mathcal{H}(E,F) + \mathcal{H}(F, |\{S_1, S_2, \ldots, S_n\}|) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

The conclusion follows. □

Set

$$\text{wLip}\,1 = \left\{ f \in C(X, X) : d(f(x), f(y)) < d(x, y) \text{ for all } x \neq y \text{ in } X \right\}$$

and, for $m \in \mathbb{N}$, set

$$(\text{wLip}\,1)^m = \left\{ (f_1, \ldots, f_m) : f_i \in \text{wLip}\,1,\ 1 \leqslant i \leqslant m \right\}.$$

Call $f \in \text{wLip}\,1$ a weak contraction. We recall some simple observations.

Remark 4.4 Let $X = [0, 1]$.

(1) The set Lip 1 is closed.
(2) The set Lip s is a closed and nowhere dense subset of Lip 1, for all $s \in (0, 1)$.
(3) Let

$$\text{Iso}_{[a,b]} = \left\{ f \in \text{Lip}\,1 : |f(x) - f(y)| = |x - y| \text{ for all } x \neq y \text{ in } [a, b] \right\}.$$

Then $\text{Iso}_{[a,b]}$ is closed in Lip 1.

See also [7] for (2) in the case of maps on closed, convex and bounded subsets of a Hilbert space.

Should $\mathcal{S} = \{S_1, \ldots, S_m\}$ and $\mathcal{T} = \{T_1, \ldots, T_m\}$ be elements of $(\text{wLip}\,1)^m$, set $\|\mathcal{S} - \mathcal{T}\| = \max\{\|S_i - T_i\| : 1 \leqslant i \leqslant m\}$, where $\|S_i - T_i\| = \max\{|S_i(x) - T_i(x)| : x \in I\}$. Take a and b in $\mathbb{R}$. By $I\{a, b\}$ we denote the closed interval with endpoints a and b. That is, either $I\{a, b\} = [a, b]$ or $I\{a, b\} = [b, a]$, should $a < b$ or $b < a$.

The next lemma shows that, while Lip s is a very small subset of Lip 1 for every $s \in (0, 1)$, wLip 1 is considerably larger. This is similar to a result found in [13] for maps defined on closed, convex and bounded subsets of a Banach space.

Lemma 4.5 *Let* $X = [0, 1]$. *The set of weak contractions* wLip 1, *defined on* X, *is a dense* G_δ *subset of* Lip 1.

Proof Let $\epsilon > 0$. Suppose that $f \in \text{Lip}\,1 \setminus \text{wLip}\,1$. Then there exist points $x \neq y$ in $[0, 1]$ such that $|f(x) - f(y)| = |x - y|$. Should $|f(x) - f(y)| = |x - y|$, then it follows that $|f(a) - f(b)| = |a - b|$ for all points a and b in $I\{x, y\}$, as $f \in \text{Lip}\,1$. Let $\{x_n\}$ be an enumeration of $\mathbb{Q} \cap [0, 1]$ and set $G_{n,m} = \text{Lip}\,1 \setminus \text{Iso}_{I\{x_n, x_m\}}$. From Remark 4.4 it follows that $G_{n,m}$ is open in Lip 1. Take $f \in \text{Iso}_{[a,b]}$ such that $f(b) - f(a) = b - a$. Consider $g\colon [0, 1] \to [0, 1]$ such that

$$\begin{cases} g(x) = f(x) & \text{for } 0 \leqslant x \leqslant a, \\ g(x) = f(x) - \epsilon/2 & \text{for } b \leqslant x \leqslant 1 \end{cases}$$

and g is extended linearly on (a, b). Then $\|f - g\| < \epsilon$, and $g \in \text{Lip}\,1 \setminus \text{Iso}_{[a,b]}$. The construction of g is similar, should $f(b) - f(a) = a - b$. We conclude that $\text{Lip}\,1 \setminus \text{Iso}_{[a,b]}$ is dense. Since $G_{n,m}$ is both dense and open, one concludes that $\text{wLip}\,1 = \bigcap_{n,m} G_{n,m}$ is a dense G_δ subset of Lip 1. □

Let us now turn our attention to attractors generated by sets of m-many weak contractions.

Lemma 4.6 *Let X be a compact metric space. The function*

$$\mathcal{T}\colon (\mathrm{wLip}\,1)^m \to (\mathcal{K}(X), \mathcal{H}),$$

given by $\mathcal{S} \mapsto |\mathcal{S}|$, *is continuous.*

Proof Let $\{\mathcal{S}_k\} \subseteq (\mathrm{wLip}\,1)^m$ be such that $\mathcal{S}_k \to \mathcal{S}$, or equivalently, $\|\mathcal{S}_k - \mathcal{S}\| \to 0$. Say $\mathcal{S}_k(E_k) = E_k$. Since $(\mathcal{K}(X), \mathcal{H})$ is compact, we may assume that $E_k \to E$ in the Hausdorff metric.

Say $\mathcal{S}_k = \{S_1^k, \ldots, S_m^k\}$ and $\mathcal{S} = \{S_1, \ldots, S_m\}$. Since $\mathcal{S}_k \to \mathcal{S}$, it follows that $S_i^k \to S_i$ uniformly for each $1 \leqslant i \leqslant m$. Moreover, $\mathcal{S}_k(E_k) = \bigcup_{i=1}^m S_i^k(E_k) = E_k \to E$ and $\bigcup_{i=1}^m S_i^k(E_k) \to \bigcup_{i=1}^m S_i(E)$, so that $\bigcup_{i=1}^m S_i(E) = E$. Thus, if $\mathcal{S}_k \to \mathcal{S}$, then $|\mathcal{S}_k| \to |\mathcal{S}|$, and our conclusion follows. □

Lemma 4.7 *For any $s \in (0, 1)$, the set $\mathcal{T}((\mathrm{Lip}\,s)^m)$ is closed and nowhere dense in* $(\mathcal{K}([0, 1]), \mathcal{H})$.

Proof Let $s \in (0, 1)$. Since $(\mathrm{Lip}\,s)^m$ is compact, and $\mathcal{T}$ is continuous, it follows that $\mathcal{T}((\mathrm{Lip}\,s)^m)$ is closed. Since $\mathcal{T}((\mathrm{Lip}\,s)^m) \subseteq (\mathcal{K}_2)^c$, $\mathcal{T}((\mathrm{Lip}\,s)^m)$ is nowhere dense. □

It is easy to determine the size of $\mathcal{T}((\mathrm{Lip}\,s)^m)$ since $(\mathrm{Lip}\,s)^m$ is compact in $(\mathrm{Lip}\,1)^m$. The proof of the following is considerably more involved, as $(\mathrm{wLip}\,1)^m$ is a G_δ set.

The following observations are fundamental to the construction found in the proof of Theorem 4.8.

(1) Let $E = \bigcup_{i=1}^N E_i$. If for any i, j and k in $\{1, 2, \ldots, N\}$ with $i \neq j$, one has $\mathrm{diam}(E_k) \leqslant \mathrm{dist}(E_i, E_j)$, then for every weak contraction $f\colon E \to E$ and $1 \leqslant k \leqslant N$, it follows that $f(E_k) \subseteq E_l$, $1 \leqslant l \leqslant N$.
(2) Let $E = \{x_1, x_2, \ldots, x_n\}$ and $F = \{y_1, y_2, \ldots, y_m\}$. Suppose that $|x_i - x_j| < \epsilon$ for all $i \neq j$ in $\{1, 2, \ldots, n\}$, and $|y_l - y_k| > \epsilon$ for all $l \neq k$ in $\{1, 2, \ldots, m\}$. If $f\colon E \to F$ is a weak contraction, then $f(E) = y_i$, for some $1 \leqslant i \leqslant m$.

Theorem 4.8 *The set $\mathcal{T}((\mathrm{wLip}\,1)^m)$ is nowhere dense in* $(\mathcal{K}([0, 1]), \mathcal{H})$.

Proof We can assume $m > 1$ since the result is clear for $m = 1$. Fix $\delta > 0$. We begin by developing a set E.

(1) $E = \bigcup_{i=0}^k E_i$, with $E_i \cap E_j = \varnothing$ for each $i \neq j$.
(2) Each E_i contains 2^{2^i} points that are distributed uniformly throughout the interval $[\min E_i, \max E_i]$. Set $E_i = \bigcup_{j=1}^{2^{2^i}} \{x_{i,j}\}$, with $x_{i,j} < x_{i,j+1}$, $i \in \{0, \ldots, k\}$ and $j \in \{1, \ldots, 2^{2^i}\}$.
(3) E_j lies to the right of E_l whenever $j < l$ and $\mathrm{dist}(E_i, E_{i+1}) = (i + 1)\delta$.
(4) $\mathrm{diam}(E_i) = d$ for all i, where $d < \delta/4$.

The idea, now, is to associate to each point $x_{i,j}$ a non-empty compact neighborhood $E'_{i,j} \subseteq B_r(x_{i,j})$, where $r < \min\{\delta/8, |x_{k,1} - x_{k,2}|/8\}$. More precisely, take $E' \in \mathcal{K}([0,1])$ such that $\mathcal{H}\big(E', \bigcup_{i=0}^{k} \bigcup_{j=1}^{2^{2^i}} \{x_{i,j}\}\big) < r$, and set $E'_{i,j} = B_r(x_{i,j}) \cap E'$, for $i \in \{0, 1, \ldots, k\}$ and $j \in \{1, 2, \ldots, 2^{2^i}\}$, and $E'_i = \bigcup_{j=1}^{2^{2^i}} E'_{i,j}$.

Suppose $S\colon E' \to E'$ is a weak contraction. Then

(i) $S(E'_j) \subseteq E'_l$; that is each E'_j can map into only one E'_l as $\operatorname{diam}(E'{}_i) \leqslant \delta/4 + \delta/8 + \delta/8 = \delta/2$, and the smallest interval in $\overline{\operatorname{conv}}(E)$ complementary to the sets $\overline{\operatorname{conv}}(E'_j)$ is of length at least $\delta - \delta/8 - \delta/8 = 3\delta/4$.

(ii) $S(E'_{k,j}) \subseteq E'_{l,p}$, so that each $E'_{k,j}$ can map into only one $E'_{l,p}$ as $\operatorname{diam}(E'_{k,j}) \leqslant |x_{k,1} - x_{k,2}|/4$ and the smallest interval complementary to the sets $\overline{\operatorname{conv}}(E'_{l,p})$ is at least of length $|x_{k,1} - x_{k,2}| - 2|x_{k,1} - x_{k,2}|/8 = 3|x_{k,1} - x_{k,2}|/4$.

(iii) If $l > j$ and $S(E'_l) \subseteq E'_j$, then $S(E'_l) \subseteq E'_{j,s}$, for some s as $\operatorname{dist}(E'_{l,i}, E'_{l,i+1}) < \operatorname{dist}(E'_{j,p}, E'_{j,p+1})$ for any i and p.

(iv) If $S(E'_l) \subseteq E'_l$, then $S(E'_j) \subseteq E'_l$, whenever $j < l$ as $S(E'_l) \subseteq E'_l$ implies that $S(\overline{\operatorname{conv}}(E'_l)) \subseteq \overline{\operatorname{conv}}(E'_l)$, so that $\operatorname{Fix}(S) \in \overline{\operatorname{conv}}(E'_l)$. Now, note that $S(E') \subseteq E'$, and S is a weak contraction.

Let $\mathcal{S} = \{S_1, S_2, \ldots, S_m\}$, where each S_i is a weak contraction, m is now a fixed number, and take $\epsilon > 0$. Let $F_1 \in \mathcal{K}([0,1])$. Since the collection of finite sets is dense in $\mathcal{K}([0,1])$, there exists F_2 finite such that $\mathcal{H}(F_1, F_2) < \epsilon/2$. Say $F_2 = \{x_1, x_2, \ldots, x_n\}$. Take F_3 finite in $\mathcal{K}([0,1])$ such that

(v) $\mathcal{H}(F_2, F_3) < \gamma = \min\{\epsilon/8, z/8\}$, where $z = \min\{|x_i - x_j| : x_i, x_j \in F_2, i \neq j\}$,

(vi) $F_2 \backslash \{x_1\} \subseteq F_3$, and

(vii) there is a similar copy of E in $B_\gamma(x_1)$. We take $k > m \geqslant 2$, so that

  a. $\sum_{i=0}^{k-m} \operatorname{card}(E_i) = \sum_{i=0}^{k-m} 2^{2^i} > m(m+n)$, and
  b. $\operatorname{card}(E_j) = 2^{2^j} > m\big[\sum_{i=0}^{j-1} \operatorname{card}(E_i) + (n-1) + k\big]$ whenever $k \geqslant j \geqslant k-m$.

Suppose that $\mathcal{H}(F_4, F_3) < \sigma$, where $\sigma < \min\{\delta'/8, z/8\}$, $z = \min\{|x_i - x_j|, x_i \neq x_j \text{ in } F_3\}$, and $\delta' = |x_{0,2} - x_{1,1}|$, the distance between E_0 and E_1, the first two components of the similar copy of E found in $B_\gamma(x_1)$. Thus $3\delta' < 2\gamma < \epsilon/4$ and $\delta' < \epsilon/12$. We continue to use the notation of F_3, remembering that each point x_i in F_3 now represents $F_4 \cap B_\sigma(x_i)$, a subset of F_4 both open and closed in the relative topology of F_4. Suppose $\mathcal{S}(F_4) = F_4$. By our choice of k (see (vii) b), there are so many components of E'_k that it is not possible to cover E'_k with m weak contractions defined on $F_4 \setminus E'_k$. Thus, there must exist some S_i—say S_1—such that $S_1(E'_k) \subseteq E'_k$. This implies $S_1(E') \subseteq E'_k$. Similarly, there exists some S_j—say S_2—such that $S_2(E'_{k-1}) \subseteq E'_{k-1}$, which implies that $S_2(E' \setminus E'_k) \subseteq E'_{k-1}$. (It is also possible that $S_2(E'_k) \subseteq E'_{k-1}$.) Continuing, for each $0 \leqslant i \leqslant m-1$, we have, with possible renumbering of the elements of $\mathcal{S}$, that $S_{i+1}(E'_{k-i}) \subseteq E'_{k-i}$, so that $S_{i+1}\big(E' \setminus \bigcup_{l=k-i+1}^{k} E'_l\big) \subseteq E'_{k-i}$. (Analogously to what we noted with S_2, it is also possible that there exists $j \in \{k-i+1, \ldots, k\}$ such that $S_{i+1}(E'_l) \subseteq E'_{k-i}$.) Note that $S_{i+1}\big(\bigcup_{l=k-i+1}^{k} E'_l\big)$ can intersect at most i components of $\bigcup_{l=0}^{k-i} E'_l$. This follows from (iii). The m contractions $\{S_1, \ldots, S_m\}$

acting on the $n-1$ components of $F_4 \setminus E'$ can intersect at most $m(n-1) < mn$ components of F_4. We conclude that $\bigcup_{i=0}^{m-1} S_{i+1}\big(\bigcup_{l=k-i+1}^{k} E'_l\big)$ and $\bigcup_{i=0}^{m-1} S_{i+1}(F_4 \setminus E')$ can intersect at most $\sum_{i=1}^{m} i + mn < m(m+n)$ components of $\bigcup_{i=0}^{k-m} E'_i$. Since $\sum_{i=0}^{k-m} \operatorname{card}(E_i) > m(m+n)$, we conclude that $F_4 \nsubseteq \mathcal{S}(F_4)$. □

As a corollary, we have that the set of attractors generated by those IFS with m many contractions is a nowhere dense F_σ set in $\mathcal{K}([0,1])$.

Corollary 4.9 *The set* $\bigcup_{l=1}^{\infty} \mathcal{T}\big(\big(\mathrm{Lip}\,\frac{l-1}{l}\big)^m\big) = \mathcal{T}\big(\bigcup_{l=1}^{\infty} \big(\mathrm{Lip}\,\frac{l-1}{l}\big)^m\big)$ *is a nowhere dense* F_σ *set in* $(\mathcal{K}([0,1]), \mathcal{H})$.

Proof We note that $\bigcup_{l=1}^{\infty} \big(\mathrm{Lip}\,\frac{l-1}{l}\big)^m \subseteq (\mathrm{wLip}\,1)^m$. □

Remark 4.10 Let l and m be elements of $\mathbb{N}$. Then

$$(\mathcal{K}([0,1]), \mathcal{H}) \supseteq (\mathcal{K}_2)^{\mathrm{c}} \supseteq \mathcal{T}((\mathrm{wLip}\,1)^m) \supseteq \mathcal{T}\left(\bigcup_{k=1}^{\infty} \left(\mathrm{Lip}\,\frac{k-1}{k}\right)^m\right) \supseteq \mathcal{T}\left(\left(\mathrm{Lip}\,\frac{l-1}{l}\right)^m\right),$$

where

(1) $(\mathcal{K}([0,1]), \mathcal{H})$ is compact,
(2) $(\mathcal{K}_2)^{\mathrm{c}}$ is a first category and dense F_σ subset of $(\mathcal{K}([0,1]), \mathcal{H})$,
(3) $\mathcal{T}\big(\bigcup_{l=1}^{\infty} \big(\mathrm{Lip}\,\frac{l-1}{l}\big)^m\big)$ is a nowhere dense F_σ subset of $(\mathcal{K}([0,1]), \mathcal{H})$, and
(4) $\mathcal{T}\big(\big(\mathrm{Lip}\,\frac{l-1}{l}\big)^m\big)$ is nowhere dense and closed.

In particular, the set of all attractors $\mathcal{I}$ is dense in $(\mathcal{K}([0,1]), \mathcal{H})$, but, when we limit ourselves to only m maps, we are able to generate as attractors only a very small subset of $(\mathcal{K}([0,1]), \mathcal{H})$.

5 The set of attractors with large measure

As in [4], we denote by $\mathcal{I}R$ the collection of all points in $[0,1]^n$ with all of the coordinates irrational. Set

$$\mathcal{K}_1 = \big\{E \in \mathcal{K}([0,1]^n) : E \subseteq \mathcal{I}R \text{ is a nowhere dense and perfect set}\big\}.$$

We call *a rational open interval* in $[0,1]^n$ each set $J = (a_1, b_1) \times \cdots \times (a_n, b_n)$, where a_i and b_i are in $\mathbb{Q} \cap [0,1]$ for all i. We say that $E \cap J$ is *a rational portion* of $E \in \mathcal{K}([0,1]^n)$ if $E \cap J$ is non-empty and J a rational open interval. If $E \in \mathcal{K}_1$, then there exists some $\phi \in \Phi$ such that $\mathcal{H}^\phi(E \cap J) > 0$ for every rational portion $E \cap J$ of E. Moreover, the set $\{E \in \mathcal{K}([0,1]^n) : \mathcal{H}^\phi(E) = 0\}$ is a dense G_δ subset of $\mathcal{K}([0,1]^n)$ [2,4]. One concludes, then, that the typical element of $\mathcal{K}([0,1]^n)$ has very small measure, and that the elements of $\mathcal{K}([0,1]^n)$ which comprise the elements of $\mathcal{K}_2$ in Proposition 4.1 can be taken to have arbitrarily small measure. In what follows it will be convenient to use the following results. We let λ be the Lebesgue measure in $\mathbb{R}$, and denote by Φ_{c} the family of concave functions from Φ.

If f is a finite function of a real variable defined in the neighbourhood of a point x_0, we denote the lower limit of $(f(x) - f(x_0))/(x - x_0)$ as x tends to x_0 by values $x > x_0$, that is the *lower right Dini derivate* of the function f at the point x_0, as $D_+ f(x_0)$ [14].

Proposition 5.1 ([16, Lemmas 3.3, 3.5, Proposition 3.2]) *Let* $\phi \in \Phi$.

(1) *If the lower right Dini derivate of* ϕ *at* 0 *is finite, for instance* $D_+\phi(0) = s < \infty$, *and* $E \in \mathcal{K}([0, 1])$ *for which* $\lambda(E) = c$, *then* $\mathcal{H}^\phi(E) = sc$.
(2) *If* $D_+\phi(0) = +\infty$, *then there exists* $\tilde{\phi} \in \Phi_c$ *such that* $\mathcal{H}^\phi(E) = \mathcal{H}^{\tilde{\phi}}(E)$, *for all* $E \in \mathcal{K}([0, 1])$.
(3) *If* $\phi \in \Phi_c$ *and* $f \in \operatorname{Lip} M$, *then* $L = \lim_{\delta \to 0} \phi(M\delta)/\phi(\delta) \neq 0$ *exists, and* $\mathcal{H}^\phi(f(E)) \leqslant L\mathcal{H}^\phi(E)$. *Moreover, should* $|f(x) - f(y)| = M|x - y|$, *then* $\mathcal{H}^\phi(f(E)) = M\mathcal{H}^\phi(E)$.

For the remainder of this section, we presume that each ϕ taken is concave, so that we can apply Proposition 5.1 to our measures. The next result shows that the mass distribution within elements of $\mathcal{I}_2$ is relatively uniform.

Lemma 5.2 *Let* $X = \mathbb{R}$. *Let* $F \in \mathcal{I}_2$, *and* $\phi \in \Phi_c$. *Then one of the following is true:*

(1) *For every portion* P *of* F, $\mathcal{H}^\phi(P) = 0$.
(2) *For every portion* P *of* F, $\mathcal{H}^\phi(P) = \infty$.
(3) *For every portion* P *of* F, $0 < \mathcal{H}^\phi(P) < \infty$.

Proof Let P, J be portions of F, where $F = \mathcal{S}(F)$ and $\mathcal{S} = \{S_1, \ldots, S_N\}$ and $\phi \in \Phi_c$. There exists $i_1 i_2 \ldots i_n$ such that $S_{i_1 i_2 \ldots i_n}(P) \subset J$. Thus, there is some $r_{P,J} \neq 0$ for which $\mathcal{H}^\phi(J) \geqslant r_{P,J}\mathcal{H}^\phi(P)$. There exists $j_1 j_2 \ldots j_m$ such that $S_{j_1 j_2 \ldots j_m}(J) \subset P$, which ensures the existence of $r_{J,P} > 0$ for which $\mathcal{H}^\phi(P) \geqslant r_{J,P}\mathcal{H}^\phi(J)$. It follows that $\mathcal{H}^\phi(J) \geqslant r_{P,J}\mathcal{H}^\phi(P) \geqslant r_{P,J} r_{J,P}\mathcal{H}^\phi(J)$ and $\mathcal{H}^\phi(P) \geqslant r_{J,P}\mathcal{H}^\phi(J) \geqslant r_{P,J} r_{J,P}\mathcal{H}^\phi(P)$. □

In [3,5] sets of arbitrary Hausdorff dimension are constructed which are not attractors of any finite family of weak contractions in [0, 1] and, more generally, in $[0, 1]^N$. Let $\phi \in \Phi_c$. In [4] and Theorem 4.1, one sees that the typical element of $(\mathcal{K}([0, 1]^n), \mathcal{H})$ has both $\mathcal{H}^\phi$-measure zero, and is not an attractor for any IFS. Working on the unit interval, we now show that the typical element of $(\mathcal{K}([0, 1]), \mathcal{H})$ for which $\mathcal{H}^\phi_m$ is bounded away from zero also is not an attractor for any IFS. That is, the remainder of this section is dedicated to showing that while Lemma 5.2 provides a necessary condition for some compact set to be an element of $\mathcal{I}_2$, that condition is far from sufficient. As with Theorem 4.1, the typical "large" set that satisfies the necessary condition found in Lemma 5.2 is not an attractor for any IFS.

We begin by recalling the following result from [2], which describes a typical element of $\mathcal{K}([0, 1])$.

Proposition 5.3 ([2, Proposition 2.2 (1)]) *Let*

$$\Sigma_0 = \left\{ E \in \mathcal{K}([0, 1]) : E \text{ is nowhere dense, perfect and } E \subseteq [0, 1] \backslash \mathbb{Q} \right\}.$$

Then Σ_0 *is a dense* G_δ *subset of* $\mathcal{K}([0, 1])$, *hence topologically complete.*

Lemma 5.4 *Let $\gamma > 0$. Then*

$$\Sigma_1 = \left\{ E \in \mathcal{K}([0,1]) : \mathcal{H}_m^\phi(E) < \gamma \right\}$$

is dense and open in the complete metric space $\mathcal{K}([0,1])$.

Proof The density of Σ_1 follows from the fact that each compact set can be approximated by finite sets in the Hausdorff metric. Let $E \in \Sigma_1$. To show that Σ_1 is open, take $\{U_i\}_{i=1}^\infty$ such that

(1) $E \subseteq \bigcup_{i=1}^\infty U_i$,
(2) $|U_i| \leqslant 1/m$ for all i, and
(3) $\sum_{i=1}^\infty \phi(|U_i|) < \gamma$.

If $F \subseteq \bigcup_{i=1}^\infty U_i$, then $\mathcal{H}_m^\phi(F) < \gamma$, too. □

Proposition 5.5 *Let $\gamma > 0$. The collection $\Sigma_2 = \{E \in \Sigma_0 : \mathcal{H}_m^\phi(E) \geqslant \gamma\}$ is topologically complete with respect to the Hausdorff metric $\mathcal{H}$.*

Proof Let $\gamma > 0$. Then $\Sigma_2 = \Sigma_0 \setminus \Sigma_1$, so that Σ_2 is closed in the topologically complete metric space Σ_0. □

Proposition 5.6 *Fix $F \in \Sigma_2$. The set $\Sigma_3 = \{E \in \Sigma_2 : f(E) \supseteq F$ for some $f \in$ Lip $\rho\}$ is closed in Σ_2.*

Proof Let $\{E_n\} \subseteq \Sigma_3$ with $\{f_n\} \subseteq \operatorname{Lip} m$ such that $f_n(E_n) \supseteq F$. Since $(\mathcal{K}([0,1]), \mathcal{H})$ is compact and $\{f_n\}$ is uniformly bounded and equicontinuous, there exist $\{n_k\}$ in $\mathbb{N}$ and $E \in \mathcal{K}([0,1])$ such that $\lim_{k\to\infty} \mathcal{H}(E_{n_k}, E) = 0$ and f_{n_k} converges uniformly to some f. Then

$$\lim_{k\to\infty} \mathcal{H}(f_{n_k}(E_{n_k}), f(E)) = 0.$$

Since $F \subseteq f_{n_k}(E_{n_k})$, it follows that $F \subseteq f(E)$. □

Theorem 5.7 *Let $X = [0,1]$. The typical $E \in \Sigma_2$ is not an attractor of any system of weak contractions.*

Proof Suppose E and F are nowhere dense subsets of $[0,1]$, and $f \in \operatorname{Lip} \rho$ is such that $f(E) = F$. If $(c,d) \subseteq \overline{\operatorname{conv}}(F)$ is complementary to F, then there exists $(a,b) \subseteq \overline{\operatorname{conv}}(E)$ complementary to E such that $f((a,b)) \supseteq (c,d)$ and $(d-c)/(b-a) \leqslant \rho$. Let $\{J_k\}$ be an enumeration of the intervals complementary to F such that $|J_k| \geqslant |J_{k+1}|$ for all k. Similarly, let $\{I_k\}$ be an enumeration of the intervals complementary to E. There exists a subsequence $\{I_{k_s}\}$ such that $|J_s|/|I_{k_s}| \leqslant \rho$ for all i [16, Lemma 3.6]. This observation provides a necessary condition on the sets E and F in order for there to exist some $f \in \operatorname{Lip} \rho$ with the property that $f(E) \supseteq F$. We use this observation below. Let I and J be disjoint rational intervals. Our goal is to show that the typical element of Σ_2 is not an attractor of any system of weak contractions. A critical step in this direction is establishing that given some fixed $\rho > 0$, for the typical $E \in \Sigma_2$ and disjoint clopen portions $E \cap I$ and $E \cap J$, one has that $f(E \cap I) \not\supseteq E \cap J$ whenever f is in $\mathrm{Lip}\,\rho$. We restrict our attention to disjoint portions as the identity function i_d is in $\mathrm{Lip}\,1$, and $i_d(E \cap I) = E \cap I$ for all E in Σ_2 and rational intervals I. Set $\Sigma_4^{I,J} = \{E \in \Sigma_2 : f(I \cap E) \supseteq J \cap E$ for some $f \in \mathrm{Lip}\,\rho\}$. Then $\Sigma_4^{I,J}$ is closed. This can be proved as for Σ_3 in Proposition 5.6. We wish to show that $\Sigma_4^{I,J}$ is nowhere dense. Since $\Sigma_4^{I,J}$ is closed, it is sufficient to show that its complement, necessarily open in Σ_2, is also dense. Let $E \in \Sigma_4^{I,J}$ and $\epsilon > 0$. We modify E to get E' in Σ_0 such that

(1) $E' \setminus I = E \setminus I$, so that E and E' are the same outside of I;
(2) Take E' on I so that

   a. $E \cap I \subseteq E' \cap I$ and $E' \subseteq B_\epsilon(E)$. Thus, $\mathcal{H}(E, E') < \epsilon$. Since $E \subseteq E'$, it follows that $E' \in \Sigma_2$, too.
   b. Consider $\{J_k\}$ for $J \cap E$ and $\{I_k\}$ for $I \cap E$. Suppose $I_k = (c, d)$ is a complementary interval of $E \cap I$. Then c and d are elements of E and if $d - c < \epsilon$, then $(c, d) \subseteq B_\epsilon(c) \subseteq B_\epsilon(E)$. It follows that there exists some N such that $|I_k| < \epsilon$ whenever $k > N$, and hence $I_k \subseteq B_\epsilon(E)$. Take $E' \cap I$ such that if $\{I'_k\}$ are the complementary intervals of $E' \cap I$, then $I_i = I'_i$ for $1 \leqslant i \leqslant N$, and $|J_i|/|I'_i| > \rho$ for all $i > N$. Thus $E' \notin \Sigma_4^{I,J}$, so $\Sigma_4^{I,J}$ is nowhere dense.

Set $\Sigma_5 = \bigcup_{I,J} \Sigma_4^{I,J}$. Then Σ_5 is a first category F_σ subset of Σ_2. If $E \in \Sigma_2 \setminus \Sigma_5$, and $f \in \mathrm{Lip}\,\rho$, then $f(E \cap I) \not\supseteq E \cap J$ whenever I and J are any disjoint rational intervals. In particular, $f(E \cap I)$ must be nowhere dense in $E \cap J$ as $f(E \cap I)$ is a continuous image of the compact set $E \cap I$, hence is itself compact. We show that E is not the attractor of any weakly contractive system. Since we are interested in weak contractions, in what follows, we may fix $\rho = 1$ so as to work within $\mathrm{Lip}\,1$. Let $\{T_1, T_2, \dots, T_s\}$ be a set of weak contractions, $T_i : E \to E$. It suffices to show that $T_i(E)$ is nowhere dense in E, for any i. In fact, if it is so, then $\bigcup_{i=1}^s T_i(E)$ also is nowhere dense in E. Thus $\bigcup_{i=1}^s T_i(E)$ is a proper subset of E, and, hence, E is not an attractor of $\{T_1, T_2, \dots, T_s\}$. To this end, fix $T : [0, 1] \to [0, 1]$, a weak contraction, with $x \in [0, 1]$ the unique fixed point of T in $[0, 1]$. Let $y \neq x$, $y \in E$. Since T is a weak contraction,

$$|T(y) - T(x)| = |T(y) - x| < |y - x|.$$

Let $\epsilon = |T(y) - y|$. Since T is continuous, there exists $\delta > 0$ such that $|T(z) - T(y)| < \epsilon/2$ whenever $|z - y| < \delta$. Set $\delta' = \min\{\delta, \epsilon/2\}$. Then $T(B_{\delta'}(y)) \subseteq B_{\epsilon/2}(T(y))$, and $B_{\delta'}(y) \cap B_{\epsilon/2}(T(y)) = \varnothing$. Now, take (a, b) a rational interval of $[0, 1]$ such that $y \in (a, b) \subseteq B_{\delta'}(y)$. Since $E \in \Sigma_2$, it follows that $E \cap \mathbb{Q} = \varnothing$, so that a and b are not in E. Now, $E = ([0, a) \cap E) \cup ((a, b) \cap E) \cup ((b, 1] \cap E)$. Moreover, $T([0, a) \cap E)$ and $T((b, 1] \cap E)$ are both nowhere dense in $(a, b) \cap E$ since $E \in \Sigma_2 \setminus \Sigma_5$ and $T((a, b) \cap E) \cap ((a, b) \cap E) = \varnothing$. We conclude that there exists a neighborhood of y, namely (a, b), such that $T(E)$ is nowhere dense in $(a, b) \cap E$. Since this is true for any $y \neq x$, $T(E)$ is nowhere dense in E. □

6 Attractors with non-empty interior

In Proposition 2.3, one sees that when working within $\mathcal{I}_2$ and $X = \mathbb{R}^n$, an attractor F has non-empty interior if and only if it is the closure of its components with non-empty interior. Simple examples, as our first example below, show that in $\mathcal{I}_2$ we can have attractors composed of an arbitrarily large number of non-trivial components.

Example 6.1 We show that $[0, 1/3] \cup [2/3, 1]$ is an attractor for $\mathcal{S} = \{S_i\}_{i=1}^4$, where each S_i is a similarity and $\mathcal{S}$ satisfies the OSC.

Construction. Let $S_1(x) = x/4$, $S_2(x) = x/4 + 1/12$, $S_3(x) = x/4 + 2/3$, $S_4(x) = x/4 + 3/4$ and set $F = [0, 1/3] \cup [2/3, 1]$. Then

$$S_1(F) = \left[0, \frac{1}{12}\right] \cup \left[\frac{1}{6}, \frac{1}{4}\right],$$
$$S_2(F) = \left[\frac{1}{12}, \frac{1}{6}\right] \cup \left[\frac{1}{4}, \frac{1}{3}\right],$$
$$S_3(F) = \left[\frac{2}{3}, \frac{3}{4}\right] \cup \left[\frac{5}{6}, \frac{11}{12}\right],$$
$$S_4(F) = \left[\frac{3}{4}, \frac{5}{6}\right] \cup \left[\frac{11}{12}, 1\right].$$

This gives $\bigcup_{i=1}^4 S_i(F) = [0, 1/3] \cup [2/3, 1] = F$. Moreover, $\mathcal{S} = \{S_1, S_2, S_3, S_4\}$ satisfies the OSC with $G = (0, 1/3) \cup (2/3, 1)$.

This construction can be generalized to $\mathbb{R}^n$ for any number of components k. That is, for any n and k in $\mathbb{N}$, there exists $F = \bigcup_{i=1}^k F_i$ a disjoint union, where each F_i is an n-cube, $\mathcal{S} = \{S_j\}_{j=1}^{(2k)^n}$ where each S_j is a similarity, $\mathcal{S}$ satisfies the OSC, and $F = \mathcal{S}(F)$.

As we remarked in Proposition 2.5, the only type of attractor with non-empty interior formed by two similarities is an interval. The next example shows that, with three similarities, one can form an attractor that is the closure of the union of countably many disjoint and non-degenerate closed components. In what follows, we use $\dot{\bigcup}$ to indicate the union of pairwise disjoint subsets of $[0, 1]$. Each of the intervals $[a_i, b_i]$ or (a_i, b_i) taken is non-degenerate, so that $b_i > a_i$.

Example 6.2 We develop an attractor of type $\overline{\left(\dot{\bigcup}_{i=1}^\infty [a_i, b_i]\right)}$ for $\mathcal{S} = \{S_1, S_2, S_3\}$ composed of similarities that satisfy the OSC.

Construction. Fix $0 < r < 1$.

1. Set $E = \left(\bigcup_{n=0}^\infty [r^n(\sqrt{r}), r^n]\right) \cup \{0\}$.
2. Set $S_1(x) = rx$. Then $S_1(E) = E \setminus [\sqrt{r}, 1]$.
3. Set $S_2'(x) = \sqrt{r}x$. Then $S_2'(E) = \overline{[0, 1] \setminus E}$.
4. Let $h \colon \mathbb{R} \to \mathbb{R}$ be the linear homeomorphism taking $[0, 1]$ to $[\sqrt{r}, 1]$: $h(0) = \sqrt{r}$ and $h(1) = 1$ so that $h(x) = (1 - \sqrt{r})x + \sqrt{r}$.

5. Set $S_2 = h \circ S_2'$ and $S_3 = h$. Then $S_2(E) \cup S_3(E) = [\sqrt{r}, 1]$ as $S_2'(E) = \overline{[0,1] \setminus E}$ and $i_d(E) = E$ imply that $S_2'(E) \cup i_d(E) = [0, 1]$, and $h(S_2'(E) \cup i_d(E)) = h([0, 1]) = [\sqrt{r}, 1]$.
6. We conclude that $E = \mathcal{S}(E)$, with $\mathcal{S} = \{S_1, S_2, S_3\}$, where $S_1(x) = rx$, $S_2(x) = (h \circ S_2')(x)$, with $S_2'(x) = \sqrt{r}x$, and $S_3(x) = h(x) = (1 - \sqrt{r})x + \sqrt{r}$.
7. The collection $\mathcal{S}$ satisfies the OSC with $V = \text{int}(E)$.
8. If we let m_i be the slope of S_i, then $\sum_{i=1}^{3} m_i = r + \sqrt{r}(1 - \sqrt{r}) + (1 - \sqrt{r}) = 1$.

Let $\mathcal{I}$ be the set of attractors contained in $[0, 1]$ and set $\mathcal{L} = \{E \in \mathcal{I} : \text{int}(E) \neq \varnothing\}$. For A and B subsets of $[0, 1]$, we take $d(A, B) = \inf\{|a - b| : a \in A, b \in B\}$. In what remains of this section, we study the elements of $\mathcal{L}$. Definition 6.3 and Lemma 6.4 provide a topological characterization of the elements of $\mathcal{L}$ analogous to that for countable and compact sets described in Theorems 3.1 and 3.2. Definition 6.3 gives us a tool that allows us to relate elements of $\mathcal{L}$ with nowhere dense attractors.

In what follows, let $\{[a_i, b_i]\}_{i=1}^{P}$ be a pairwise disjoint enumeration of the components of E in $\mathcal{L}$ with non-empty interior. Should the collection be finite, then $P \in \mathbb{N}$. Otherwise, $P = \infty$.

Definition 6.3 (*collapse morphism*) Let $E \in \mathcal{L}$, with $\{[a_i, b_i]\}_{i=1}^{P}$ a pairwise disjoint enumeration of the components of E with non-empty interior. We define ϕ as the uniform limit of a sequence of functions. Let $\phi_0(x) = x$ be the identity function. Define ϕ_1 such that

$$\phi_1(x) = \begin{cases} \phi_0(x) & \text{for } x \leqslant a_1, \\ \phi_0(a_1) & \text{for } a_1 \leqslant x \leqslant b_1, \\ \phi_0(x) - (b_1 - a_1) & \text{for } x \geqslant b_1. \end{cases}$$

In general, define ϕ_n such that

$$\phi_n(x) = \begin{cases} \phi_{n-1}(x) & \text{for } x \leqslant a_n, \\ \phi_{n-1}(a_n) & \text{for } a_n \leqslant x \leqslant b_n, \\ \phi_{n-1}(x) - (\phi_{n-1}(b_n) - \phi_{n-1}(a_n)) & \text{for } x \geqslant b_n. \end{cases}$$

Should $P \in \mathbb{N}$, set $\phi = \phi_P$.

By our construction, $\|\phi_n - \phi_{n+1}\| = |b_{n+1} - a_{n+1}|$. Thus, ϕ_n converges uniformly to ϕ on $[0, 1]$. The following are consequences of this uniform convergence:

(1) Since ϕ_n is continuous for all n, ϕ is continuous.
(2) Since $\phi_n \in \text{Lip } 1$ for all n, $\phi \in \text{Lip } 1$.
(3) Since ϕ_n is non-decreasing for all n, ϕ is non-decreasing.
(4) Since ϕ_k is constant on $[a_n, b_n]$ for all $k \geqslant n$, ϕ is constant on $[a_n, b_n]$ for all n.
(5) If $(a, b) \subset [\min E, \max E]$ is complementary to E, then ϕ_n is linear with slope $m = 1$ on (a, b), for all n. Thus, ϕ is linear with slope $m = 1$ on (a, b).
(6) If $A \subseteq [0, 1] \setminus \dot{\bigcup}_{i=1}^{P} [a_i, b_i]$, then $\lambda(A) = \lambda(\phi_n(A))$, for all n. Thus, $\lambda(A) = \lambda(\phi(A))$.

Lemma 6.4 *Suppose that the pairwise disjoint sequence* $\{[a_i, b_i]\}_{i\in\mathbb{N}}$ *is contained in* $E \in \mathcal{L}$ *and* $x \notin \bigcup_{i=1}^{\infty}[a_i, b_i]$. *Then* $d([a_i, b_i], \{x\}) \to 0$ *as* $i \to \infty$ *if and only if* $d(\phi([a_i, b_i]), \phi(x)) \to 0$ *as* $i \to \infty$.

Proof Suppose $\{[a_i, b_i]\}_{i\in\mathbb{N}}$ is contained in E and that $\lim_{i\to\infty} d([a_i, b_i], \{x\}) = 0$. By definition, for any $\epsilon > 0$, there is N a natural number so that $d([a_i, b_i], \{x\}) < \epsilon$ whenever $i > N$. Since $\phi([c, d]) \leqslant d - c$ for any $(c, d) \subseteq [0, 1]$, it follows that $d(\phi([a_i, b_i]), \phi(x)) \leqslant d([a_i, b_i], \{x\}) < \epsilon$ whenever $i > N$. We conclude that $d(\phi([a_i, b_i]), \phi(x)) \to 0$ as $i \to \infty$.

Suppose that $d(\phi([a_i, b_i]), \phi(x)) \to 0$ as $i \to \infty$. Then, for any $\epsilon > 0$, there exists $N(\epsilon)$ such that $d(\phi([a_i, b_i]), \phi(x)) < \epsilon/2$ whenever $i > N(\epsilon)$. Since $\{[a_i, b_i]\}_{i\in\mathbb{N}}$ is a pairwise disjoint sequence contained in $[0, 1]$, the series $\sum_{i=1}^{\infty} |b_i - a_i|$ converges. Take M such that $\sum_{i=M}^{\infty} |b_i - a_i| < \epsilon/2$. Let $\delta_i = d(\phi([a_i, b_i]), \phi(x))$, and set $\delta = \min\{\delta_i : 1 \leqslant i \leqslant M\}/2$. If $i > N(\delta)$ and $[a_l, b_l] \cap \overline{\text{conv}}([a_i, b_i] \cup \{x\}) \neq \varnothing$, then $l > M$. Let $k > \max\{N(\epsilon), N(\delta)\}$. Then

$$d([a_k, b_k], \{x\}) < \sum_{i=M}^{\infty} |b_i - a_i| + d(\phi([a_k, b_k]), \phi(x)) < \frac{\epsilon}{2} + \frac{\epsilon}{2}.$$

This follows from (4), (5) and (6) in the observation following Definition 6.3. □

Example 6.5 Let F be a countable space homeomorphic to the space $\omega^{\alpha} \cdot n + 1$. Suppose $x \in F$ is isolated; say $\epsilon > 0$ such that $B_{\epsilon}(x) \cap F = \{x\}$. To each such isolated point x, associate the interval $[x - \epsilon/2, x + \epsilon/2]$. Let

$$E = \overline{\bigcup_i \left\{ \left[x_i - \frac{\epsilon_i}{2}, x_i + \frac{\epsilon_i}{2} \right] : x_i \in F \text{ is isolated} \right\}}.$$

Then $\phi(E)$ is homeomorphic to the space $\omega^{\alpha} \cdot n + 1$.

Example 6.6 Let Q be the middle thirds Cantor set with C the set of midpoints of the intervals complementary to Q in $[0, 1]$. Let $x \in (c, d)$ be in C, where (c, d) is an interval complementary to Q. Take $[a_x, b_x]$ so that $x \in [a_x, b_x] \subset (c, d)$. Then $E = \overline{\bigcup_{x\in C}[a_x, b_x]} = Q \cup \left(\bigcup_{x\in C}[a_x, b_x]\right)$, and $\phi(E)$ is uncountable.

Let $E \in \mathcal{L}$ with $\{[a_i, b_i]\}_{i=1}^{P}$ an enumeration of the components of E with non-empty interior and ϕ the collapse morphism defined on E. Say $E' = \phi(E)$. In the following proposition, we make use of ϕ^{-1} defined on E', which is intended to reverse the collapse affected by ϕ on $\{[a_i, b_i]\}_{i=1}^{P} \subseteq E$. If $x \in \phi\left(\bigcup_{i=1}^{P}[a_i, b_i]\right)$, then there exists a unique element $[a_k, b_k]$ of $\{[a_i, b_i]\}_{i=1}^{P}$ such that $\phi([a_k, b_k]) = x$. This follows from the observations (3)–(6). If $x \in E'$ and $x \notin \phi\left(\bigcup_{i=1}^{P}[a_i, b_i]\right)$, then there exists a unique element $y \in E$ such that $\phi(y) = x$. This follows from (3), (5) and (6). Let $\mathcal{P}([0, 1]) = \{A : A \subseteq [0, 1]\}$ be the power set of $[0, 1]$. We define $\phi^{-1} : E' \to \mathcal{P}([0, 1])$ such that $\phi^{-1}(x) = [a_k, b_k]$, the unique element of $\{[a_i, b_i]\}_{i=1}^{P}$ such that $\phi([a_k, b_k]) = x$, whenever $x \in \phi\left(\bigcup_{i=1}^{P}[a_i, b_i]\right)$, and $\phi^{-1}(x) = y$, the unique point in E such that $\phi(y) = x$, whenever $x \in E' \setminus \phi\left(\bigcup_{i=1}^{P}[a_i, b_i]\right)$.

Proposition 6.7 *If $E \in \mathcal{L}$ is an attractor for some IFS, then $\phi(E)$ is an attractor, too.*

Proof Suppose $E \in \mathcal{L}$, with $\{[a_i, b_i]\}_{i=1}^P$ an enumeration of the components of E with non-empty interior, and $E = \mathcal{S}(E)$ where $\mathcal{S} = \{S_1, S_2, \ldots, S_N\}$. Let ϕ be the collapse morphism as in Definition 6.3, and set $\phi(E) = E'$. Take $\mathcal{S}' = \{S'_j = \phi \circ S_j \circ \phi^{-1} : 1 \leqslant j \leqslant N\}$. We first show that if $S_j \in \operatorname{Lip} r_j$, then $S'_j \in \operatorname{Lip} r_j$, too. We make frequent reference to the observations following Definition 6.3. Let $[c, d] \subseteq \phi([0, 1])$. Take $\mathcal{A} = \{i : \phi([a_i, b_i]) \subseteq [c, d]\}$. Then $\phi^{-1}([c, d]) = [\min \phi^{-1}(c), \max \phi^{-1}(d)]$ from (3),

$$\phi^{-1}([c, d]) = \left(\bigcup_{i \in \mathcal{A}} [a_i, b_i]\right) \cup \left(\phi^{-1}([c, d]) \setminus \bigcup_{i \in \mathcal{A}} [a_i, b_i]\right),$$

with

$$\lambda\left(\phi^{-1}([c, d]) \setminus \bigcup_{i \in \mathcal{A}} [a_i, b_i]\right) = d - c,$$

and

$$|\phi^{-1}([c, d])| = (d - c) + \sum_{i \in \mathcal{A}} (b_i - a_i)$$

is the length of $\phi^{-1}([c, d])$. This follows from (4), (5) and (6). As $S_j \in \operatorname{Lip} r_j$,

$$\begin{aligned} \left|(S_j \circ \phi^{-1})([c, d])\right| &= \left|S_j(\phi^{-1}([c, d]))\right| \\ &\leqslant r_j |\phi^{-1}([c, d])| = r_j \left[(d - c) + \sum_{i \in \mathcal{A}} (b_i - a_i)\right], \end{aligned}$$

so that

$$\left|(\phi \circ S_j \circ \phi^{-1})([c, d])\right| \leqslant r_j (d - c),$$

which again follows from (4), (5) and (6). Or, alternatively, we can, again, write

$$\phi^{-1}([c, d]) = \left(\phi^{-1}([c, d]) \setminus \bigcup_{i \in \mathcal{A}} [a_i, b_i]\right) \cup \left(\bigcup_{i \in \mathcal{A}} [a_i, b_i]\right),$$

so that

$$\begin{aligned} (S_j \circ \phi^{-1})([c, d]) &= S_j(\phi^{-1}([c, d])) \\ &\subseteq S_j\left(\phi^{-1}([c, d]) \setminus \bigcup_{i \in \mathcal{A}} [a_i, b_i]\right) \cup S_j\left(\bigcup_{i \in \mathcal{A}} [a_i, b_i]\right), \end{aligned}$$

and hence

$$
\begin{aligned}
(\phi \circ S_j \circ \phi^{-1})([c,d]) &= \phi((S_j \circ \phi^{-1})([c,d])) \\
&\subseteq \phi\Big(S_j\Big(\phi^{-1}([c,d]) \setminus \bigcup_{i \in \mathcal{A}} [a_i, b_i] \Big)\Big) \cup \phi\Big(S_j \Big(\bigcup_{i \in \mathcal{A}} [a_i, b_i] \Big)\Big).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\big|(\phi \circ S_j \circ \phi^{-1})([c,d])\big| = \big|\phi((S_j \circ \phi^{-1})([c,d]))\big| \\
&\quad = \lambda\big((\phi \circ S_j \circ \phi^{-1})([c,d])\big) = \lambda\big(\phi((S_j \circ \phi^{-1})([c,d]))\big) \\
&\quad \leqslant \lambda\Big(\phi\Big(S_j\Big(\phi^{-1}([c,d]) \setminus \bigcup_{i \in \mathcal{A}} [a_i, b_i] \Big)\Big)\Big) + \lambda\Big(\phi\Big(S_j \Big(\bigcup_{i \in \mathcal{A}} [a_i, b_i] \Big)\Big)\Big) \\
&\quad = \lambda\Big(S_j\Big(\phi^{-1}([c,d]) \setminus \bigcup_{i \in \mathcal{A}} [a_i, b_i] \Big)\Big) + 0 \leqslant r_j(d-c).
\end{aligned}
$$

From (5) and (6), one sees that ϕ, on $\phi^{-1}([c,d])$, preserves measure off the set $\bigcup_{i \in \mathcal{A}} [a_i, b_i]$. Since $\mathcal{A}$ is countable and $\phi([a_i, b_i])$ is a singleton for each i, we have $\lambda\big(\phi\big(\bigcup_{i \in \mathcal{A}} [a_i, b_i]\big)\big) = 0$. This follows from (6). Thus $S'_j = \phi \circ S_j \circ \phi^{-1} \in \operatorname{Lip} r_j$.

We now show that $E' = \mathcal{S}'(E')$. Let $x \in E'$, and take $y \in \phi^{-1}(x)$. Then $y \in E$, so there exists $1 \leqslant j \leqslant N$ such that $y \in S_j(E)$. Say $z \in E$ for which $S_j(z) = y$. Then $\phi(z) \in E'$. Now, $\phi^{-1}(\phi(z)) = [r,s] \subseteq E$, where $z \in [r,s]$, so that $S_j([r,s]) = [t,u] \subseteq E$, and $S_j(z) = y \in [t,u]$. Finally, $\phi([t,u]) = \phi(y) = x$. We conclude that $S'_j(\phi(z)) = (\phi \circ S_j \circ \phi^{-1})(\phi(z)) = x$. Thus, $E' \subseteq S'(E')$.

We now show that $S'(E') \subseteq E'$. Let $x \in E'$. It suffices to show that $S'_j(x) = (\phi \circ S_j \circ \phi^{-1})(x) \in E'$, for each $1 \leqslant j \leqslant N$. From the construction of ϕ, one sees that $\phi^{-1}(x) = [r,s] \subseteq E$. Since $S_j(E) \subseteq E$ for any j, it follows that $S_j(\phi^{-1}(x)) \subseteq E$, too. Again, by the construction of ϕ, $\phi(E) = E'$, so $S'_j(x) = \phi(S_j(\phi^{-1}(x))) \in E'$. □

The above proposition allows us to show that certain classes of elements of $\mathcal{K}([0,1])$ are not attractors for any IFS, using Theorem 3.1.

Corollary 6.8 *A compact set $E \in \mathcal{L}$ such that $\phi(E)$ is of limit Cantor–Bendixon height is not homeomorphic to any IFS-attractor consisting of weak contractions.*

As the next example shows, and as one would expect, Corollary 6.8 provides a necessary but not a sufficient condition for a set of the form $\overline{\dot{\bigcup}_{i=1}^{\infty} [a_i, b_i]}$ to be an attractor for some IFS.

Example 6.9 A compact set $E \in \mathcal{L}$ with the property that $\phi(E)$ has height of a successor ordinal can be embedded into the real line so that it is not an attractor of any iterated function system consisting of weak contractions defined on E.

Construction. We begin with the example found in [12, Theorem 2]. Nowak proves that a compact scattered metric space with successor height can be embedded topologically in the real line so that it is not an attractor of any iterated function system consisting of weak contractions. Consider the scattered set $X = \{0\} \cup \left(\bigcup_{n=1}^{\infty} X_n\right)$, where $\{0\} \cap X_n = \varnothing$ for all n, each of the X_n is homeomorphic to $\omega^{\alpha} + 1$ and $\overline{\text{conv}}(X_n) \cap \overline{\text{conv}}(X_m) = \varnothing$ whenever $n \neq m$, and $d(\{0\}, X_n) \to 0$ as $n \to \infty$. Since $\{0\} = \overline{\left(\bigcup_{n=1}^{\infty} X_n\right)} \setminus \bigcup_{n=1}^{\infty} X_n$, it follows that X is homeomorphic to $\omega^{\alpha+1}+1$. Let $\{x_i\}_{i=1}^{\infty}$ be the collection of isolated points found in X. Let $E_0 = X$. Take E_1 so that $E_1 \cap [0, x_1] = E_0 \cap [0, x_1]$, $E_1 \cap [x_1+1/2, 1+1/2]$ is a translation $1/2$ units to the right of $E_0 \cap [x_1, 1]$, and $[x_1, x_1+1/2] \subseteq E_1$. That is, we start our construction with E_1 using E_0, replacing x_1 with an interval $[a, b]$ of length $1/2$ so that $a = x_1$, and that part of E_0 to the right of x_1 is translated to the right a distance of $1/2$. In general, we take E_{n+1} so that $E_{n+1} \cap [0, x_{n+1}] = E_n \cap [0, x_{n+1}]$, $[x_{n+1}, x_{n+1} + 1/2^{n+1}] \subseteq E_{n+1}$, and $E_{n+1} \cap [x_{n+1} + 1/2^{n+1}, \max E_n + 1/2^{n+1}]$ is a translation $1/2^{n+1}$ units to the right of $E_n \cap [x_{n+1}, \max E_n]$.

Since $\{E_n\}_{n=0}^{\infty}$ is Cauchy in $(\mathcal{K}(\mathbb{R}), \mathcal{H})$, $\lim_{n\to\infty} E_n = E$ exists. Since $\overline{\text{conv}}(E_0) = [0, 1]$, and $\sum_{i=1}^{\infty} 1/2^i = 1$, we see that $\overline{\text{conv}}(E) = [0, 2]$. Moreover, $\phi(E) = X$, where ϕ is the collapse morphism of E. Now, suppose $E = \mathcal{S}(E)$ for some collection $\mathcal{S} = \{S_1, S_2, \ldots, S_N\}$ of weak contractions. From Proposition 6.7, it follows that $\phi(E) = X$ is the attractor of the IFS $\mathcal{S}' = \{S_j' = \phi \circ S_j \circ \phi^{-1} : 1 \leqslant j \leqslant N\}$. This contradicts the fact that X is not an attractor for any IFS defined on $[0, 1]$. We conclude that E is not an attractor for any IFS.

The next example constructs an attractor E of the form $\overline{\dot{\bigcup}_{i=1}^{\infty}[a_i, b_i]}$ such that $\phi(E)$ is homeomorphic to $Q \cup C$, where Q is the middle thirds Cantor set, and C is the set of midpoints of the intervals complementary to Q in $[0, 1]$.

Example 6.10 There exists an IFS $\mathcal{S} = \{S_1, S_2, S_3, S_4\}$ satisfying the OSC for which $E = |\mathcal{S}|$, and $\phi(E)$ is homeomorphic to $Q \cup C$, where Q is the middle thirds Cantor set, and C is the set of midpoints of the intervals complementary to Q in $[0, 1]$.

Construction. Let $\mathcal{S} = \{S_1, S_2, S_3, S_4\}$, where $S_1(x) = 1/3x$, $S_2(x) = x/3 + 2/3$, and S_3 and S_4 are piecewise linear Lip $\frac{1}{2}$ functions such that $[5/12, 7/12] = K = \bigcup_{i=3}^{4} S_i(K) = \bigcup_{i=3}^{4} S_i([0, 1])$. Let $E_n = \mathcal{S}^n(K)$. Then $E_0 = K$, and $E_1 = K \cup S_1(K) \cup S_2(K)$ is composed of three intervals, where K is centered at $1/2$ of length $1/6$, $S_1(K)$ is centered at $1/6$ of length $1/18$, and $S_2(K)$ is centered at $5/6$, also of length $1/18$. Moreover, $E_0 \subsetneq E_1$. As one can easily verify, $E_0 \subsetneq E_1 \subsetneq E_2 \subsetneq \cdots$, each E_k is composed of $\sum_{i=0}^{k} 2^i$ disjoint closed intervals centered in intervals complementary to Q, the middle thirds Cantor set, and $E = \lim E_k = \text{cl}\left(\bigcup_{k=0}^{\infty} E_k\right) = Q \cup \left(\bigcup_{k=0}^{\infty} E_k\right)$ in $\mathcal{K}([0, 1], \mathcal{H})$.

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