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RESEARCH ARTICLE

Some remarks on Humbert–Edge's curves

Juan B. Frías-Medina[1,2] · **Alexis G. Zamora**[2,3]

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Abstract We discuss William L. Edge's approach to Humbert's curves as a canonical genus 5 curve that is a complete intersection of diagonal quadrics. Moreover, the contribution of Edge to the study of projective curves $X \subset \mathbb{P}^n$ that are complete intersections of $n-1$ quadrics is explained and some results, complementary to Edge's exposition, are proved.

Keywords Curves with automorphisms · Jacobian varieties · Intersection of quadrics

Mathematics Subject Classification 14H37 · 14H40 · 14H45

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✉ Alexis G. Zamora
alexiszamora06@gmail.com

Juan B. Frías-Medina
bfrias99@gmail.com

1 Instituto de Matemáticas, Universidad Nacional Autónoma de México, Área de la Investigación Científica, Ciudad Universitaria, C.P. 04510 Mexico City, Mexico

2 Unidad Académica de Matemáticas, Universidad Autónoma de Zacatecas, Calzada Solidaridad entronque Paseo a la Bufa, C.P. 98000 Zacatecas, Zac., Mexico

3 Instituto de Física y Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo, Edificio C3, Ciudad Universitaria, C.P. 58040 Morelia, Mich., Mexico

1 Introduction

Let $\{p_1, \ldots, p_5\}$ be a set of five points in general position in $\mathbb{P}^3$, and $\{\mathcal{C}_t\}$ be the family of twisted cubic curves through the fixed five points. Consider another point $p \in \mathbb{P}^3$ general with respect to $p_1, \ldots, p_5$ and define

$$X := \left\{ q \in \mathbb{P}^3 : \text{there exists } t \text{ such that } q \in \mathcal{C}_t \text{ and } \overline{pq} \text{ is tangent to } \mathcal{C}_t \text{ at } q \right\},$$

$\overline{pq}$ stands for the line joining p and q. This variety turns out to be an irreducible curve, and was studied by Humbert in [9]. In [1] Baker considered the curve of contact of a Weddle surface with one of its tangent cones at a node (for the definition of Weddle curve and its basic properties see [14], see also [5], specially Remark 8.6.2).

The aim of this note is to present several results obtained by William L. Edge regarding the geometry of Humbert's curves and certain generalization that he introduced and to prove some related results. We work over an algebraically closed field of characteristic different form 2. The material discussed here by no means covers the total contribution of Edge to the study of Humbert's curves and its generalizations, it was a constant topic in his work and even one of his last articles [8], written at his retirement in Nazareth House, was devoted to the study of a particular case of Humbert's curve. Here we focus our attention on two papers [6] and [7].

In the first of these papers Edge proved the equivalence of the constructions by Humbert and Baker and introduced another equivalent construction that was the core of his treatment: the several geometric realizations of Humbert's curve can be obtained by projecting a canonical genus 5 curve that is a complete intersection of three diagonal quadrics. In this sense, the Edge approach to Humbert's curve can be considered as a part of the theory of intersection of quadrics as further developed in [2,12,13]. He proved many interesting properties of these curves, including a characterization of its Weierstrass points and a natural isogenous decomposition of its Jacobian as a sum of five elliptic curves (this being noticed before by Humbert and Baker). In Sect. 2 we present these results, the section is principally of an expository nature and the statement and the idea of the proof of Theorem 2.2 is taken from [14], where some of Edge's results are also reproduced.

In Sect. 3 we focus on the paper [7], in which Edge introduced a generalization of Humbert's curve by considering curves $X \subset \mathbb{P}^n$ that can be expressed as complete intersections of diagonal quadrics, we propose to call these curves Humbert–Edge's curves of type n. We prove that the diagonality of the quadrics equations defining X is equivalent to X admitting a certain abelian group of order 2^n. Strangely enough, in his discussion, Edge did not include a proof that having this group action implies the diagonality of the quadrics. We also provide in Proposition 3.5 a partial equivariant decomposition of the Jacobian variety of X with respect to the action of the above mentioned group. Finally, in Sect. 4 we slightly generalize an Edge's construction providing a family of special Humbert–Edge's curves with a larger, non-abelian group of automorphisms.

It should be noted that in [3] the authors rediscovered these curves and some of its basic properties and called them generalized Humbert's curves, they seemed not to be aware of the Edge's work [7].

2 The classical Humbert's curve

Let C be a plane, irreducible, degree 6 curve with five nodes, say $n_1, \ldots, n_5$, in general position. Thus, C has geometric genus 5. Let Γ be the conic passing through the five nodes. Thus, Γ must intersect C in two remaining points p, q. The linear system $3L - \sum_{i=1}^{5} n_i$ of cubics passing through the five nodes defines an immersion ϕ of $\mathbb{P}^2$ into $\mathbb{P}^4$, whose image is a degree 4 Del Pezzo surface. This can be defined as a complete intersection of two diagonal quadrics [5, Theorem 8.6.2] and the image X of C is cut out on S by a further quadric hypersurface. In this way, we obtain the canonical curve X as a complete intersection of three quadrics in $\mathbb{P}^4$. The birational inverse from X to C is obtained by projecting X from the chord $\overline{p'q'}$, with p', q' denoting, respectively, the images of p and q under ϕ.

In [6] Edge identified Humbert's curve as one of these projections in the particular case when the canonical curve X is the intersection of three diagonal quadrics.

We briefly recall the basic facts concerning intersections of quadrics (see [2,12, 13]). Let $X \subset \mathbb{P}^n$ be an irreducible, non-singular complete intersection of quadrics $Q_0, \ldots, Q_{n-2}$. Thus, X can be interpreted as the base locus of the $(n-2)$-dimensional linear system $\lambda_0 Q_0 + \cdots + \lambda_{n-2} Q_{n-2} = 0$. Denote by $\Pi \subset \mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2))$ the corresponding $(n-2)$-subspace. The discriminant hypersurface $\Delta \subset \Pi$ is defined by

$$\Delta := \left\{ \det \left(\sum_{l=0}^{n-2} \lambda_l a_{ij}^{(l)} \right) = 0 \right\},$$

if $Q_l = \sum_{i \leqslant j} a_{ij}^{(l)} x_i x_j$. This is a degree $n+1$ hypersurface parameterizing the singular quadrics in Π.

We have a filtration

$$\Pi \supset \Delta = \Delta^{(1)} \supseteq \cdots \supseteq \Delta^{(n-2)},$$

where $\Delta^{(l)} = \mathrm{Sing}(\Delta^{(l-1)})$, $l = 2, \ldots, n-2$. Moreover,

$$\Delta^{(l)} - \Delta^{(l-1)} = \{Q \in \Pi : \mathrm{rank}\, Q = n + 1 - l\},$$

see [13] for a proof.

In general Δ could be reducible. A hyperplane appears as an irreducible component of Δ if and only if an $(n-1)$-dimensional linear system $\Pi' \subset \Pi$ exists formed by singular quadrics.

Definition 2.1 (Edge, [6]) Let X be a non-hyperelliptic curve of genus 5 (identified with its canonical model $X \subset \mathbb{P}^4$). We say that X is a *Humbert's curve* if there exists a projective system of coordinates in $\mathbb{P}^4$ and diagonal quadrics

$$Q_i = \sum_{j=0}^{4} a_{ij} x_j^2, \quad i = 0, 1, 2,$$

such that X is the base locus of the net $\Pi = \{\lambda_0 Q_0 + \lambda_1 Q_1 + \lambda_2 Q_2 = 0\}$.

In [6] Edge proved that the nodal Humbert's plane sextic can be obtained as an image of the projection of X from a chord and the cuspidal one as an image of the projection of X from a tangent. Edge's study is based on the fact that X admits an automorphism subgroup generated by the involutions:

$$\sigma_i(x_0 : \ldots : x_i : \ldots : x_4) = (x_0 : \ldots : -x_i : \ldots : x_4), \quad i = 0, \ldots, 4.$$

The fundamental properties of X are summarized in the following theorem.

Theorem 2.2 *Let $X \subset \mathbb{P}^4$ be a canonical curve. The following statements are equivalent:*

(i) *X is a Humbert's curve.*
(ii) *X admits five involutions $\sigma_0, \ldots, \sigma_4$ such that $[\sigma_i, \sigma_j] = 1$, $\sigma_0 \cdots \sigma_4 = 1$, and $E_i := X/\langle\sigma_i\rangle$ are elliptic curves.*
(iii) *X admits 10 even and effective theta-characteristics.*
(iv) *The discriminant curve of Π, $\Delta \subset \mathbb{P}^2$, is a product of five lines.*

That (i) implies (ii) and (iii) was proved (in a rather scattered way) in [6]. Even though the other implications were not explicitly formulated, it is possible that Edge was aware of these equivalences. As formulated here this result and an indication of its proof appear in [14]. We include a proof for the reader's convenience.

Proof We start with a non-trigonal canonical curve $X \subset \mathbb{P}^4$ which is a complete intersection of three diagonal quadrics Q_0, Q_1, Q_2,

$$Q_j = \sum_{i=0}^{4} a_{ij} x_i^2.$$

Note that in order for X to be non-degenerated and non-singular the determinant of any 3×3 minor of the matrix (a_{ij}) must be different from 0. Denote by $\{e_0, \ldots, e_4\}$ the standard frame of reference in $\mathbb{P}^4$, i.e., $e_i = (0 : \ldots : 1 : \ldots : 0)$ with 1 on the i-th position. Note that $e_i \notin X$ for $i = 0, \ldots, 4$.

Let $H_i := Z(x_i)$ and denote $X.H_i = p_{i1} + \cdots + p_{i8}$, i.e., p_{ij}, $j = 1, \ldots, 8$, are the points of intersection of X with the hyperplane $x_i = 0$. Note that, on X, $p_{i1} + \cdots + p_{i8} \sim K_X$.

(i)$\Rightarrow$(ii). Fix, for instance $i = 0$. Let $p \in X$ and consider the line l_p joining e_0 and p. An easy computation shows that there exists another point $q \in X \cap l_p$. Explicitly in coordinates, if $p = (b_0 : \ldots : b_4)$, then $q = (-b_0 : \ldots : b_4)$. Let us call this involution σ_0 and consider the quotient $E_0 = X/\langle\sigma_0\rangle$. The ramification points of this quotient are precisely $\{p_{01}, \ldots, p_{08}\}$. Thus, it follows from the Riemann–Hurwitz formula that E_0 is an elliptic curve. Analogous considerations apply for $i = 1, \ldots, 4$.

(ii)$\Rightarrow$(iii). Consider the subgroup generated by two involutions, say $\langle\sigma_0, \sigma_1\rangle$, we obtain

$$f : X \xrightarrow{4:1} X/\langle\sigma_0, \sigma_1\rangle \simeq \mathbb{P}^1.$$

This covering is simply ramified in the 16 points $\{p_{01}, \ldots, p_{08}, p_{11}, \ldots, p_{18}\}$. Thus,

$$K_X \sim f^*K_{\mathbb{P}^1} + p_{01} + \cdots + p_{08} + p_{11} + \cdots + p_{18} \sim f^*K_{\mathbb{P}^1} + 2K_X.$$

Therefore, $K_X \sim -f^*K_{\mathbb{P}^1}$ and we obtain that $f^*\mathcal{O}_{\mathbb{P}^1}(1)$ is a theta-characteristic of X. Since $h^0(X, f^*\mathcal{O}_{\mathbb{P}^1}(1)) = 2$ we obtain the desired theta-characteristics.

(iii) ⇒ (iv). Theta-characteristics of dimension at least 2 are in a 1 : 1 correspondence with rank 3 quadrics containing X (see, for instance [13, Section 4]). Taking into account our previous remark on $\Delta^{(l)}$ we see that the existence of the 10 theta-characteristics implies that the plane quintic Δ has 10 singular points. This implies that Δ is a product of five lines.

(iv) ⇒ (i). As X is a genus 5 canonical curve it must be contained in the base locus of a net Π of quadrics. We are assuming that the discriminant Δ of this net is a product of five lines. We can choose coordinates in such a way that $e_i \notin X$ and the pencil of quadrics in Π containing e_i is formed by rank 4 quadrics, from this the diagonal form for the Q_i is deduced. □

The following corollary was also central in Edge's investigation of Humbert's curves.

Corollary 2.3 *Let X be a Humbert's curve. The set of Weierstrass points of X is formed by the* 40 *intersections of X with the hyperplanes H_i, that is the set* $\{p_{ij}\}_{0 \leqslant i \leqslant 4,\, 1 \leqslant j \leqslant 8}$ *defined in the proof of Theorem 2.2. Each of these points has weight* 3 *and gap sequence* $\{1, 2, 3, 5, 7\}$.

Proof Recall that given a projective curve X of genus g, a general point $p \in X$ satisfies $h^0(X, \mathcal{O}_X(lp)) = 1$ for $l = 1, \ldots, g$. Points failing to fulfil this condition are the Weierstrass points. For any point $p \in X$ the gap sequence is defined by $\{l_1, \ldots, l_g\}$ such that $h^0(X, \mathcal{O}_X(l_i p)) = h^0(X, \mathcal{O}_X((l_i - 1)p))$. Thus, p is a Weierstrass point if and only if its gap sequence is not $\{1, 2, \ldots, g\}$. The weight of a Weierstrass point is defined as the sum $\sum_{i=1}^{g}(l_i - i)$ and the sum of the weights of all the Weierstrass points equals $g(g-1)(g+1)$ (for a complete discussion, see for instance [11]).

Returning to our particular case, consider the quotient $\sigma_i : X \to E_i$. Let $p_{ij} \in X$ be as defined in the proof of Theorem 2.2. Since in the elliptic curve E_i,

$$h^0(E_i, \mathcal{O}_{E_i}(l\sigma_i(p_{ij}))) = l$$

for all $l \geqslant 1$, we have a non-constant meromorphic function on E_i having exactly a pole of order 2 at $\sigma_i(p_{ij})$ and two non-constant linearly independent meromorphic functions having exactly a pole of order 3 at $\sigma_i(p_{ij})$. Composing these functions with σ_i, we deduce that

$$h^0(X, \mathcal{O}_X(4p_{ij})) = 2 \quad \text{and} \quad h^0(X, \mathcal{O}_X(6p_{ij})) = 3.$$

From this we obtain the desired gap sequence. Since each of these 40 points has weight 3, they form the totality of Weierstrass points. □

Another important feature of Humbert's curve is that its Jacobian is isogenous to a product of the five elliptic curves given by the pull-back of the quotients $X \to E_i$:

Proposition 2.4 *Let X be a Humbert's curve and $\pi_i : X \to E_i$ be the quotient by the involution σ_i. Then,*

$$JX = \pi_0^* E_0 + \cdots + \pi_4^* E_4.$$

Proof The proof is elementary. Given $L \in JX$, we can write

$$L = \sigma_0^* \cdots \sigma_4^* L \otimes (\sigma_1^* \cdots \sigma_4^* L) \otimes (\sigma_1^* \cdots \sigma_4^* L)^{-1} \otimes (\sigma_2^* \sigma_3^* \sigma_4^* L^{-1}) \otimes \cdots \otimes L^{-1}.$$

In this way, for any $L \in JX$, $L^2 \in \pi_0^* E_0 + \cdots + \pi_4^* E_4$. Since JX is a divisible group the result follows. □

3 Intersection of quadrics and Humbert–Edge's curves

In [7] Edge considered the following situation: let $X_n \subset \mathbb{P}^n$ be a non-singular, irreducible complete intersection of $n-1$ diagonal quadrics $Q_0, \ldots, Q_{n-2}$:

$$Q_i = \sum_{j=0}^{n} a_{ij} x_j^2, \quad i = 0, \ldots, n-2.$$

In [3] such a curve is called a generalized Humbert's curve. Taking into account the quoted paper [7] we think it is more appropriate to call it a Humbert–Edge's curve.

Definition 3.1 An irreducible, non-degenerated and non-singular curve $X_n \subset \mathbb{P}^n$, $n \geqslant 2$, is a *Humbert–Edge's curve* (HE curve for short) *of type n* if it is the complete intersection of $n-1$ diagonal quadrics $Q_0, \ldots, Q_{n-2}$.

The basic properties of an HE curve X_n are as follows.

Lemma 3.2 *Let X_n be an HE curve. Then:*

(i) $\deg X_n = 2^{n-1}$.
(ii) *The canonical sheaf of X_n is*

$$\mathcal{O}_{X_n}(K_{X_n}) \simeq \mathcal{O}_{X_n}(n-3),$$

in particular $g_{X_n} - 1 = 2^{n-2}(n-3)$.
(iii) *Every $(n-3)$-minor of the matrix (a_{ij}) is non-degenerated.*
(iv) *The linear involutions of $\mathbb{P}^n$ sending x_i into $-x_i$ induce involutions σ_i on X_n. These together generate a subgroup of* $\mathrm{Aut}(X_n)$ *isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$ with relation $\sigma_0 \cdots \sigma_n = 1$.*
(v) *The quotient of X_n by each involution σ_i is geometrically interpreted as projection with center e_i onto the hyperplane $H_i = Z(x_i)$ and this quotient is an HE curve of type $n-1$.*

All these properties are stated in [7] and rediscovered in [3]. The cases $n = 2, 3$ correspond to a plane conic and an elliptic space quartic curve, respectively, the case $n = 4$ is the classical Humbert's curve, as presented by Edge.

Definition 3.3 Let X_n be a non-singular, irreducible, projective curve of genus $2^{n-2} \cdot (n-3) + 1$. We say that a group E acting on X_n is an *HE group* if it is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$ and generators $\sigma_0, \ldots, \sigma_n$ of E exist such that $\sigma_0 \cdots \sigma_n = 1$ and the fixed set of each σ_i is of cardinality 2^{n-1} and disjoint from the fixed set of σ_j if $i \neq j$.

The following theorem is not explicitly stated in [7], but probably was implicitly assumed.

Theorem 3.4 *An irreducible and non-singular curve of genus $2^{n-2}(n-3)+1$, $X \subset \mathbb{P}^n$ is an HE curve of type n if and only if it admits an action of an HE group. Moreover, in this case the generators σ_i are the restriction to X_n of the involutions $x_i \mapsto -x_i$.*

This fact was proved over $\mathbb{C}$ in [3], using the theory of uniformization and Klenian groups. We give here an algebraic proof valid over any algebraically closed field of characteristic different from 2.

Proof We make an induction on n (the result being trivial for $n = 2$ and previously proved for $n = 4$). Assume that a non-singular projective curve X_n of genus $2^{n-2} \cdot (n-3)+1$ admits an HE group as a subgroup of $\mathrm{Aut}(X_n)$. Then, the quotient $X_{n-1} := X_n/\langle \sigma_n \rangle$ admits a group of automorphisms generated by $\overline{\sigma}_0, \ldots, \overline{\sigma}_{n-1}$ satisfying $\overline{\sigma}_0 \cdots \overline{\sigma}_{n-1} = 1$. Thus, by induction X_{n-1} is an HE curve of type $n-1$. Thus, it is expressed as the complete intersection of $n-2$ diagonal quadrics $\{\overline{Q}_1, \ldots, \overline{Q}_{n-2}\}$ in $\mathbb{P}^{n-1}$.

In order to recover the 2:1 covering $\pi_n : X_n \to X_{n-1}$ we must consider the ramified covering associated with the divisor $E = q_1 + \cdots + q_{2^{n-1}}$, where q_i are the ramification values of π_n. We have

$$\pi_{n*}\mathcal{O}_{X_n} = \mathcal{O}_{X_{n-1}} \oplus \mathcal{O}_{X_{n-1}}(-\eta),$$

with $2\eta = E$. Let $D_i = \sum_{j=1}^{2^{n-2}} p_{ij}$ be the divisor given by $X_{n-1}.H_i$ (equivalently $\{p_{ij}\}_j$ is the set of fixed points of $\overline{\sigma}_i$). Then, $D_i \sim \eta$ and thus

$$\pi_{n*}\mathcal{O}_{X_n}(\pi_n^* D_i) = \mathcal{O}_{X_{n-1}}(D_i) \oplus \mathcal{O}_{X_{n-1}}.$$

In this way $|\pi_n^* D_i|$ defines a diagram

$$\begin{array}{ccc} X_n & \xrightarrow{|\pi_n^* D_i|} & \mathbb{P}^n \\ \downarrow{\scriptstyle \pi_n} & & \downarrow{\scriptstyle p_n} \\ X_{n-1} & \xrightarrow{|D_i|} & \mathbb{P}^{n-1}, \end{array}$$

with p_n the projection of X_n with center e_n onto H_n.

Hence, the equations of $X_n \subset \mathbb{P}^n$ are given by $Q_i := p_n^* \overline{Q}_i$ and the extra equation defining the 2:1 covering, namely:

$$x_n^2 = p_n^* s,$$

$s \in \mathbb{P}H^0(\mathcal{O}_{X_{n-1}}(E)) = \mathbb{P}H^0(\mathcal{O}_{X_{n-1}}(2D))$. As s is invariant under the action of the HE group induced on X_{n-1}, we see that s can be written as a diagonal quadric in the projective coordinates of $\mathbb{P}^{n-1}$. □

It is not clear to us if the Jacobian of an HE curve X can be written as a sum of elliptic curves. However, a partial decomposition exists: let $\pi_i := X \to X/\langle\sigma_i\rangle$ be the quotient and $A \subset JX$ be the abelian subvariety defined as

$$A := \sum_{i=0}^{n} \pi_i^* J(X/\langle\sigma_i\rangle).$$

Moreover, let JX^- be the abelian subvariety

$$\{L \in JX : \sigma_i L = L^{-1},\ i = 0, \dots, n\}.$$

Then we have:

Proposition 3.5 *Let JX be the Jacobian variety of an HE curve. Then*

(i) *$JX = A + JX^-$ and A and JX^- are complementary, in the sense that its intersection is finite.*
(ii) *$JX^- = \{L : \sigma_i \sigma_j L = L$ for all $i, j \in \{0, \dots, n\}\}$.*

Proof Given an abelian subvariety $B \subset JX$, define

$$B_i := \{L \in B : \sigma_i L = L\},$$

and, for any $I \subset \{0, \dots, n\}$

$$B^I := \{L \in B : \sigma_i L = L \text{ for all } i \in I\}.$$

We clearly have

$$JX = JX_0 + JX_1^{\{0\}} + JX_2^{\{0,1\}} + \cdots + JX_n^{\{0,1,\dots,n-1\}} + JX^-.$$

Next, look at the representation of the HE group E in T_0X. As E is abelian a basis exists such that all the matrices representing elements of E are diagonal and with eigenvalues 1 or -1. Write, for each $\sigma \in E$

$$T_0X = V_\sigma^+ \oplus V_\sigma^-,$$

with V_σ^+ (respectively V_σ^-) denoting the eigenspace associated with 1 (resp. -1).

Then, we see that $T_0 JX^- = \bigcap V^-_{\sigma_i}$ and $T_0 A = \sum V^+_{\sigma_i}$. Moreover,

$$V^+_{\sigma_i \sigma_j} = V^-_{\sigma_i} \cap V^-_{\sigma_j}.$$

The proposition follows from these considerations. □

Remark 3.6 The dimension of some of the spaces appearing in the previous decomposition can be explicitly computed. For instance, if n is even, then the argument in the proof of Proposition 2.4 provides $JX^- = 0$. It must also be noticed that the obtained decomposition gives a partial solution to the computation of the E-equivariant decomposition in the sense of [10]. Since E induces on JX^- the group $\{\pm 1\}$ the E-simplicity of JX^- is equivalent to its simplicity as abelian variety.

4 Specializations with larger automorphism group

Recall that by definition, if X_n is an HE curve of type n there exist $n-1$ diagonal quadrics,

$$Q_i = \sum_{j=0}^{n} a_{ij} x_j^2, \quad i = 0, \ldots, n-2,$$

such that X_n is the complete intersection of these quadrics. Also, an HE curve admits naturally $n+1$ involutions $\sigma_i : x_i \mapsto -x_i$ for $i = 0, \ldots, n$. Denote the HE group generated by the involutions σ_i by $E = \langle \sigma_0, \ldots, \sigma_n \rangle$.

Edge found in [7] a convenient representation for an HE curve given as above. In fact, for each $j = 0, \ldots, n$, regard the coefficients a_{ij} as the homogeneous coordinates of a point $p_j = (a_{0j} : a_{1j} : \ldots : a_{(n-2)j})$ in the projective space $\mathbb{P}^{n-2}$, thus, we have $n+1$ points. Using the fact that there is a unique rational normal curve $C \subset \mathbb{P}^{n-2}$ through the $n+1$ points $p_0, \ldots, p_n$ in general position and changing the coordinates of $\mathbb{P}^{n-2}$, one may assume that C is given in the standard parametric form and consequently, $p_j = (1 : a_j : a_j^2 : \ldots : a_j^{n-2})$ for every $j = 0, \ldots, n$. Therefore, we can assume that an HE curve X_n is defined by diagonal quadrics with the form

$$Q_i = \sum_{j=0}^{n} a_j^i x_j^2, \quad i = 0, \ldots, n-2.$$

A direct consequence of this representation noted by Edge is that X_n has only $n-2$ moduli.

On the other hand, Edge specialized an HE curve (denoted by Γ_n in his paper) to obtain another curve which has a larger automorphism group and fewer moduli. Edge's idea was to modify the diagonal quadrics that define the HE curve in such a way that the curve defined by these new quadrics is preserved under the original automorphism group E and under a specific automorphism of finite order. Indeed, he presented three such curves: $\Gamma'_n \subset \mathbb{P}^{2p+1}$ admitting an automorphism of order 2, $\Gamma''_n \subset \mathbb{P}^{3s-1}$ admitting an automorphism of order 3 and $\Delta_n \subset \mathbb{P}^n$ admitting an automorphism of order $n+1$. The diagonal quadrics that define such curves enabled

Edge to conclude that the curve Γ'_n has $p+1$ moduli, Γ''_n has s moduli and Δ_n has no moduli (see [7, Sections II.8, II.13 and III.17]).

Following the procedure used by Edge, we generalize this process to construct curves admitting the HE group E and other specific automorphism of finite order with fewer moduli. Consider an HE curve $X_{mt-1} \subset \mathbb{P}^{mt-1}$ defined by the diagonal quadrics $Q_i = \sum_{j=0}^{mt-1} a_j^i x_j^2$, where m and t are positive integers. Fix a primitive m^{th} root ξ of unity. Let $H_{m,t}$ be the curve defined as the intersection of the diagonal quadrics

$$\Omega_i = \sum_{j=0}^{t-1} a_j^i \big(x_j^2 + \xi^i x_{j+t}^2 + \xi^{2i} x_{j+2t}^2 + \cdots + \xi^{(m-1)i} x_{j+(m-1)t}^2\big), \quad i = 0, \ldots, mt-3.$$

Definition 4.1 Given an HE curve $X_{mt-1} \subset \mathbb{P}^{mt-1}$, a *specialization* of X_{mt-1} is the curve $H_{m,t}$ given by the intersection of the $mt-2$ diagonal quadrics $\Omega_0, \ldots, \Omega_{mt-3}$.

The next result presents some properties of the specialization $H_{m,t}$ of an HE curve X_{mt-1}.

Proposition 4.2 *The following assertions hold:*

(i) *The genus $g_{H_{m,t}}$ of $H_{m,t}$ satisfies $g_{H_{m,t}} - 1 = 2^{mt-3}(mt-4)$.*
(ii) *$H_{m,t}$ has $t-3$ moduli if $t > 3$ and has no moduli if $t \leqslant 3$.*
(iii) *$H_{m,t}$ admits the action of a group $\widehat{E}$ of order $m2^{mt-1}$ which contains the HE group E, isomorphic to $E \rtimes \langle \tau_{m,t} \rangle$ where $\tau_{m,t}$ is an automorphism of order m.*

Proof (i) holds since $H_{m,t}$ is a specialization of X_{mt-1} and (ii) follows immediately from the equations that define $H_{m,t}$.

(iii) Note that by construction Ω_i is invariant under E for every $i = 0, \ldots, mt-3$. On the other hand, consider the automorphism $\tau_{m,t}$ of $\mathbb{P}^{mt-1}$ induced by the following permutation consisting of t m-cycles:

$$(x_0 x_t x_{2t} \ldots x_{(m-1)t}) \cdots (x_j x_{j+t} \ldots x_{j+(m-1)t}) \cdots (x_{t-1} x_{2t-1} \ldots x_{mt-1}).$$

$H_{m,t}$ is invariant under $\tau_{m,t}$. Indeed, for each $i = 0, \ldots, mt-3$,

$$\begin{aligned}\tau_{m,t}(\Omega_i) &= \sum_{j=0}^{t-1} a_j^i \tau_{m,t} \big(x_j^2 + \xi^i x_{j+t}^2 + \xi^{2i} x_{j+2t}^2 + \cdots + \xi^{(m-1)i} x_{j+(m-1)t}^2\big) \\ &= \sum_{j=0}^{t-1} a_j^i \big(x_{j+t}^2 + \xi^i x_{j+2t}^2 + \xi^{2i} x_{j+3t}^2 \\ &\qquad + \cdots + \xi^{(m-2)i} x_{j+(m-1)t}^2 + \xi^{(m-1)i} x_j^2\big) \\ &= \xi^{(m-1)i} \Omega_i.\end{aligned}$$

Define the group $\widehat{E}$ as the group spanned by E and $\tau_{m,t}$. This group is non-abelian, for example, if $p=(a_0:a_1:\ldots:a_{mt-1})\in H_{m,t}$, then

$$\sigma_0\tau_{m,t}(p)=\big(-a_t:a_{t+1}:\ldots:a_{2t-1}:\ldots:a_{(m-1)t}\\ :a_{(m-1)t+1}:\ldots:a_{mt-1}:a_0:a_1:\ldots:a_{t-1}\big),$$

while

$$\tau_{m,t}\sigma_0(p)=\big(a_t:a_{t+1}:\ldots:a_{2t-1}:\ldots:a_{(m-1)t}\\ :a_{(m-1)t+1}:\ldots:a_{mt-1}:-a_0:a_1:\ldots:a_{t-1}\big).$$

In $\widehat{E}$, we have the original relations between the elements of E and also there are new relations coming from the elements $\tau_{m,t}$ and $\sigma_i\tau_{m,t}$. The relations in $\widehat{E}$ are the following:

$$\begin{aligned}\sigma_i^2&=1,\\ \sigma_i\sigma_j&=\sigma_j\sigma_i,\\ \sigma_0\sigma_1\cdots\sigma_{mt-1}&=1,\\ \tau_{m,t}^m&=1,\\ \sigma_i\tau_{m,t}&=\tau_{m,t}\sigma_{i+t\,(\mathrm{mod}\ mt)}.\end{aligned}\tag{1}$$

The relation (1) implies that E is a normal subgroup of $\widehat{E}$. In addition, the fact that $\widehat{E}$ is the product of the subgroups E and $\langle\tau_{m,t}\rangle$ and $E\cap\langle\tau_{m,t}\rangle=\{1\}$ imply that $\widehat{E}$ is the semidirect product of E by $\langle\tau_{m,t}\rangle$, i.e., $\widehat{E}=E\rtimes\langle\tau_{m,t}\rangle$. So, the order of $\widehat{E}$ is equal to $m2^{mt-1}$. □

Remark 4.3 There exist Humbert's curves admitting larger automorphisms groups than the mentioned before. For example, the automorphism group of the Humbert's curve given by the equations

$$\begin{aligned}x_0^2+x_1^2+x_2^2+x_3^2+x_4^2&=0,\\ x_0^2+\zeta_5x_1^2+\zeta_5^2x_2^2+\zeta_5^3x_3^2+\zeta_5^4x_4^2&=0,\\ \zeta_5^4x_0^2+\zeta_5^3x_1^2+\zeta_5^2x_2^2+\zeta_5x_3^2+x_4^2&=0,\end{aligned}$$

where ζ_5 is a primitive fifth root of unity, has order 160. This curve was recently considered by Cheltsov and Shramov in the context of the study of groups of rigid transformations of $\mathbb{P}^3$, [4, Remark 4.18].

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