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RESEARCH ARTICLE

Some cardinal functions in lexicographic products of LOTS

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Abstract The Lindelöf degree and extent were studied in Buhagiar et al. (Cent Eur J Math 12(8):1249–1264, 2014) in relation to lexicographic products of linearly ordered spaces. In this paper we consider the behaviour of other important cardinal functions, such as spread, density, weight and character, in such lexicographic products. Namely, we study the relation between a particular cardinal function on a lexicographic product of linearly ordered spaces and that cardinal function on each factor of the product.

Keywords Linearly ordered topological space · Lexicographic product · Cardinal functions

Mathematics Subject Classification 54F05 · 54A25 · 06F30

1 Introduction

The *lexicographic product* $X = \mathbb{L}_{\alpha<\mu} X_\alpha$ of a collection $\{X_\alpha : \alpha < \mu\}$, $\mu > 1$, of linearly ordered sets is an unexpectedly useful and ubiquitous set theoretic structure. When each linearly ordered set is given the order topology, it is important to study

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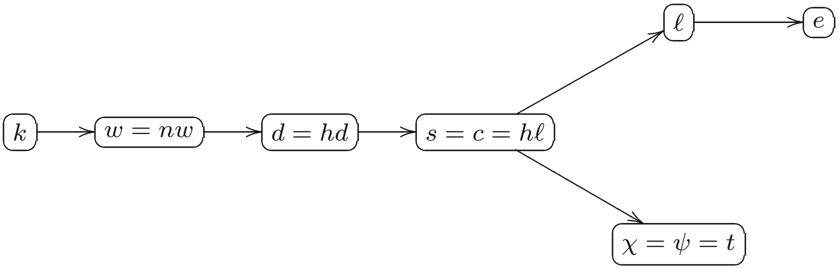

Fig. 1 Cardinal functions in LOTS

the behaviour of topological properties when lexicographic products are taken. Such investigations were done, for example, by Faber [3]. In this paper, the behaviour of some cardinal functions is investigated when lexicographic products of linearly ordered spaces are taken.

A *linearly ordered topological space* (abbreviated LOTS) is a triple $(X, \lambda(\leqslant), \leqslant)$, where $(X, \leqslant)$ is a linearly ordered set (abbreviated LOS) and $\lambda(\leqslant)$ is the usual interval topology defined by $\leqslant$ [7,8].

An ordered pair (A, B) of disjoint subsets of a LOS X is said to be a *jump* if (i) $X = A \cup B$, (ii) $x < y$ whenever $x \in A$ and $y \in B$, and (iii) A has a maximal element a, and B has a minimal element b. Thus $a < b$ and $]a, b[= \varnothing$. For a LOTS X we will let

$$\mathbb{J}_X = |\text{set of } \textit{jumps} \text{ in } X| + \aleph_0.$$

We use standard notation for the cardinal functions on a topological space X. Namely,

$k(X)$	cardinality,	$\ell(X)$	Lindelöf degree,
$w(X)$	weight,	$h\ell(X)$	hereditary Lindelöf degree,
$nw(X)$	net weight,	$e(X)$	extent,
$d(X)$	density,	$\chi(X)$	character,
$hd(X)$	hereditary density,	$\psi(X)$	pseudo-character,
$s(X)$	spread,	$t(X)$	tightness.
$c(X)$	cellularity,		

As usual, we require all cardinal functions to take only infinite cardinals as values. The reader is referred to [4–6] for a thorough study of cardinal functions. Figure 1 shows the relations between the above cardinal functions in LOTS (see [2, p. 222]).

The behaviour of the cardinal functions ℓ and e in lexicographic products of LOTS was studied in [1], in this paper we study the behaviour of the other cardinal functions.

If X is a LOS and for some $x \in X$ we write $x = 0$, then this would mean that X has a minimal element denoted by 0. Analogously, $x = 1$ means that x is the maximal element of X denoted by 1. In particular, when we consider the two-point LOS or LOTS $\{0, 1\}$ we understand that $0 < 1$. In the rest of the paper we assume that $|X| \geqslant 2$ for any LOS X.

2 Spread, density, weight and character of the lexicographic product of two LOTS

In this section we consider the lexicographic product $X \cdot Y$ of two LOTS X and Y.

Lemma 2.1 (a) *$s(Y) \leqslant s(X \cdot Y)$, $d(Y) \leqslant d(X \cdot Y)$, $w(Y) \leqslant w(X \cdot Y)$ and $\chi(Y) \leqslant \chi(X \cdot Y)$.*
(b) *If $|Y| > 2$, then $k(X) \leqslant s(X \cdot Y) \leqslant d(X \cdot Y) \leqslant w(X \cdot Y)$.*

Proof (a) For $x \in X$, the subspace ${}_xL := \{x\} \cdot Y$ of $X \cdot Y$ is order-isomorphic to Y, the order topology of $X \cdot Y$ induces on ${}_xL$ the order topology of ${}_xL$. The inequalities $s(Y) = s({}_xL) \leqslant s(X \cdot Y)$, $w(Y) = w({}_xL) \leqslant w(X \cdot Y)$ and $\chi(Y) = \chi({}_xL) \leqslant \chi(X \cdot Y)$ thus follow. The inequality $d(Y) = d({}_xL) \leqslant d(X \cdot Y)$ follows from the equality $d = hd$.

(b) Since $L_y := X \cdot \{y\}$, for a non endpoint $y \in Y$, is a discrete subspace of $X \cdot Y$ we get $|X| = |L_y| \leqslant s(X \cdot Y)$. This implies $k(X) \leqslant s(X \cdot Y) \leqslant d(X \cdot Y) \leqslant w(X \cdot Y)$. □

Theorem 2.2 $s(X \cdot Y) = \begin{cases} s(X) \cdot \mathbb{J}_X & \textit{if } Y = \{0, 1\}, \\ k(X) \cdot s(Y) & \textit{if } |Y| > 2. \end{cases}$

Proof Suppose that $Y = \{0, 1\}$. If $]a, b[= \varnothing$, for $a, b \in X$ with $a < b$, then $](a, 1), (b, 1)[= \{(b, 0)\}$, hence $s(X \cdot Y) \geqslant \mathbb{J}_X$. Moreover, if $D \subseteq X$ is discrete, then the set $D' = \{(a, 0) : a \in D,\ 0 \neq a \neq 1\}$ is a discrete set in $X \cdot Y$ and $s(X \cdot Y) \geqslant s(X)$ follows. Consequently, $s(X \cdot Y) \geqslant \max\{s(X), \mathbb{J}_X\}$. Conversely, consider a cellular family $\mathcal{C}$ in $X \cdot Y$, one can assume that each $I \in \mathcal{C}$ is an open interval of the form $](x, y), (x', y')[$ with $x < x'$. The image of $\mathcal{C}$ under the mapping $](x, y), (x', y')[\mapsto]x, x'[$ is a collection of pairwise disjoint open intervals in X. Noting that $]x, x'[= \varnothing$ defines a jump in X, one concludes that $s(X \cdot Y) = c(X \cdot Y) \leqslant \max\{c(X), \mathbb{J}_X\} = \max\{s(X), \mathbb{J}_X\}$.

If $|Y| > 2$, the inequality $s(X \cdot Y) \geqslant k(X) \cdot s(Y)$ follows from Lemma 2.1. Conversely, if $D \subseteq X \cdot Y$ is discrete, then for any $x \in X$, $D_x = \{y : (x, y) \in D\}$ is discrete in Y, so that $|D_x| \leqslant s(Y)$. Consequently, $|D| \leqslant k(X) \cdot s(Y)$ and therefore, $s(X \cdot Y) \leqslant k(X) \cdot s(Y)$. □

An analogous formula holds for the density.

Theorem 2.3 $d(X \cdot Y) = \begin{cases} d(X) \cdot \mathbb{J}_X & \textit{if } Y = \{0, 1\}, \\ k(X) \cdot d(Y) & \textit{if } |Y| > 2. \end{cases}$

Proof If $Y = \{0, 1\}$, then $d(X \cdot Y) \geqslant s(X \cdot Y) \geqslant \mathbb{J}_X$ follows from Theorem 2.2. Suppose $D \subseteq X \cdot Y$ is dense and $|D| \leqslant d(X \cdot Y)$. Then the set $D_X = \{x : (x, 0) \text{ or } (x, 1) \in D\}$ is a dense set in X. Consequently, $d(X \cdot Y) \geqslant d(X)$ and therefore, $d(X \cdot Y) \geqslant \max\{d(X), \mathbb{J}_X\}$. Conversely, suppose $D \subseteq X$ is a dense subset of X satisfying $|D| \leqslant d(X)$. Let $S = \{(x, 0) : x \in D\}$. Then $P = S \cup \{(a_{\mathrm{l}}, 1), (a_{\mathrm{r}}, 0) : a_{\mathrm{l}} < a_{\mathrm{r}} \text{ in } X,]a_{\mathrm{l}}, a_{\mathrm{r}}[= \varnothing\}$ is dense in $X \cdot Y$, adding to P any possible isolated end points. Note that $|P| \leqslant d(X) \cdot \mathbb{J}_X$ so that $d(X \cdot Y) \leqslant d(X) \cdot \mathbb{J}_X$ follows.

If $|Y| > 2$, the inequality $d(X \cdot Y) \geqslant k(X) \cdot d(Y)$ follows from Lemma 2.1. Suppose $D \subseteq Y$ is dense in Y with $|D| \leqslant d(Y)$, one can assume that $0, 1 \in D$ if they exist in Y. Then $D' = \{(x, y) : x \in X,\ y \in D\}$ is dense in $X \cdot Y$ and $d(X \cdot Y) \leqslant k(X) \cdot d(Y)$ follows. □

We next consider the weight.

Theorem 2.4 $w(X \cdot Y) = k(X) \cdot w(Y)$.

Proof Suppose $Y = \{0, 1\}$ and let $\mathcal{B}$ be a base of open intervals in $X \cdot Y$ with $|\mathcal{B}| \leqslant w(X \cdot Y)$. For any $x \in X$ choose $I(x) \in \mathcal{B}$ with $(x, 0) \in I(x)$ and $(x, 1) \notin I(x)$. Since $X \ni x \mapsto I(x)$ is injective, the inequality $w(X \cdot Y) \geqslant k(X)$ follows. On the other hand $w(X \cdot Y) \leqslant k(X \cdot Y) = k(X)$ and therefore, $w(X \cdot Y) = k(X) = k(X) \cdot w(Y)$.

If $|Y| > 2$, the inequality $w(X \cdot Y) \geqslant k(X) \cdot w(Y)$ follows from Lemma 2.1. The inequality $w(X \cdot Y) \leqslant k(X) \cdot w(Y)$ easily follows if Y does not have endpoints. Now suppose $0 \in Y$. Given $x \in X$, a neighbourhood base for $(x, 0)$ must have cardinality $\leqslant w(X) \cdot w(Y) \leqslant k(X) \cdot w(Y)$ so that the collection of neighbourhood bases for $(x, 0)$, for all $x \in X$, must have cardinality $\leqslant k(X) \cdot w(Y)$. Similarly for the case that $1 \in Y$. Consequently, after considering a neighbourhood base for points (x, y) with $y \notin \{0, 1\}$, it follows that $w(X \cdot Y) \leqslant k(X) \cdot w(Y)$. □

We end this section by considering $\chi(X \cdot Y)$. For a LOS X, we will denote the topology generated by sets of the form $[a, b[$, $a, b \in X$, and $\{1\}$ (if $1 \in X$) by τ_{r}, and similarly, the topology generated by sets of the form $]a, b]$, $a, b \in X$, and $\{0\}$ (if $0 \in X$) by τ_{l}. We will write $\chi_{\mathrm{r}}(x)$ to mean the character of $x \in (X, \tau_{\mathrm{r}})$ and $\chi_{\mathrm{r}}(X)$ for the character of (X, τ_{r}). Similarly for $\chi_{\mathrm{l}}(x)$ and $\chi_{\mathrm{l}}(X)$. If one needs to specify that the character of x is taken in X, then one writes $\chi(x, X)$ (similarly $\chi_{\mathrm{r}}(x, X)$ and $\chi_{\mathrm{l}}(x, X)$). Recall that as shown in Fig. 1, the character of a LOTS is equal to its pseudo-character. One can also note that for any $x \in X$, $\chi(x, X) = \max\{\chi_{\mathrm{r}}(x, X), \chi_{\mathrm{l}}(x, X)\}$.

Proposition 2.5 *For two LOTS X and Y,*

$$\chi_{\mathrm{r}}(X \cdot Y) = \begin{cases} \chi_{\mathrm{r}}(Y) & \textit{if } 1 \notin Y, \\ \chi_{\mathrm{r}}(X) \cdot \chi_{\mathrm{r}}(Y) & \textit{if } 1 \in Y, \end{cases}$$

$$\chi_{\mathrm{l}}(X \cdot Y) = \begin{cases} \chi_{\mathrm{l}}(Y) & \textit{if } 0 \notin Y, \\ \chi_{\mathrm{l}}(X) \cdot \chi_{\mathrm{l}}(Y) & \textit{if } 0 \in Y, \end{cases}$$

$$\chi(X \cdot Y) = \begin{cases} \chi(Y) & \textit{if } 0, 1 \notin Y, \\ \chi_{\mathrm{r}}(X) \cdot \chi(Y) & \textit{if } 0 \notin Y,\ 1 \in Y, \\ \chi_{\mathrm{l}}(X) \cdot \chi(Y) & \textit{if } 0 \in Y,\ 1 \notin Y, \\ \chi(X) \cdot \chi(Y) & \textit{if } 0, 1 \in Y. \end{cases}$$

Proof For $(x, y) \in X \cdot Y$ we have

$$\chi_{\mathrm{r}}((x, y), X \cdot Y) = \begin{cases} \chi_{\mathrm{r}}(y, Y) & \text{if } y \neq 1, \\ \chi_{\mathrm{r}}(x, X) & \text{if } y = 1. \end{cases}$$

Taking the supremum over all $(x, y) \in X \cdot Y$ we get the formula for $\chi_{\mathrm{r}}(X \cdot Y)$. Dually we get the formula for $\chi_{\mathrm{l}}(X \cdot Y)$, while $\chi(X \cdot Y) = \max\{\chi_{\mathrm{r}}(X \cdot Y), \chi_{\mathrm{l}}(X \cdot Y)\}$. □

Example 2.6 Let us look at the following examples:

1. Consider $Z = \mathbb{R} \cdot \{0, 1\}$. Then $s(Z) = \aleph_0$ (since $s(\mathbb{R}) = \mathbb{J}_{\mathbb{R}} = \aleph_0$). But $s(Z \cdot \{0, 1\}) = \max\{s(Z), \mathbb{J}_Z\} = \max\{\aleph_0, 2^{\aleph_0}\} = 2^{\aleph_0} = \mathfrak{c}$. Analogously, $d(Z) = \aleph_0$ and $d(Z \cdot \{0, 1\}) = \max\{d(Z), \mathbb{J}_Z\} = \max\{\aleph_0, 2^{\aleph_0}\} = 2^{\aleph_0} = \mathfrak{c}$. On the other hand, if one takes $Z = \{0, 1\} \cdot [0, \omega_1[$, then $s(Z) = d(Z) = \aleph_0 \cdot \aleph_1 = \aleph_1$.
2. Let $X = \mathbb{R}$ and $Y = \{0, 1\}$. Then $w(\mathbb{R} \cdot \{0, 1\}) = |\mathbb{R}| \cdot \aleph_0 = 2^{\aleph_0} = \mathfrak{c}$.
3. If $X = [0, \omega_2[$ and $Y = [0, 1[$, then $\chi(X \cdot Y) = \aleph_1$. However, if we take $X = [0, \omega_2[$ and $Y =]0, 1]$, then $\chi(X \cdot Y) = \aleph_0$.

3 Spread, density, weight and character of the lexicographic product of LOTS

In what follows, $X = \mathbb{L}_{\alpha<\mu} X_\alpha$ is the lexicographic product of X_α, where X_α is a LOS for every $\alpha < \mu$ and μ is a limit ordinal.

Theorem 3.1 *If $X = \mathbb{L}_{\alpha<\mu} X_\alpha$, then*

$$s(X) = d(X) = w(X) = \sup_{\alpha<\mu} \Big| \prod_{\gamma<\alpha} X_\gamma \Big|.$$

Proof For every $0 < \alpha < \mu$ we have $X = \mathbb{L}_{\gamma<\alpha} X_\gamma \cdot \mathbb{L}_{\alpha \leqslant \gamma < \mu} X_\gamma$, so that, by Theorem 2.2, $s(X) = |\mathbb{L}_{\gamma<\alpha} X_\gamma| \cdot s(\mathbb{L}_{\alpha \leqslant \gamma<\mu} X_\gamma) \geqslant |\mathbb{L}_{\gamma<\alpha} X_\gamma|$. Hence $s(X) \geqslant \sup_{\alpha<\mu} \big|\prod_{\gamma<\alpha} X_\gamma\big|$.

Now let us look at the weight of X. We show that $w(X) \leqslant \sup_{\alpha<\mu} \big|\prod_{\gamma<\alpha} X_\gamma\big|$ from which the result follows. Let $z = (z_\alpha)_{\alpha<\mu} \in X$ be defined as follows:

$$z_\alpha = \begin{cases} \text{chosen arbitrarily} \in X_\alpha & \text{if } X_\alpha \text{ does not have } 1, \\ 1 & \text{otherwise.} \end{cases}$$

For every $\gamma < \mu$ let $Z_\gamma = \mathbb{L}_{\alpha \leqslant \gamma} X_\alpha \cdot \mathbb{L}_{\gamma<\alpha<\mu} \{z_\alpha\}$ and let $D = \bigcup_{\gamma<\mu} Z_\gamma$. Then $|Z_\gamma| = \big|\prod_{\alpha \leqslant \gamma} X_\alpha\big|$ and therefore,

$$|D| = \Big|\bigcup_{\gamma<\mu} Z_\gamma\Big| \leqslant \mu \cdot \sup_{\gamma<\mu} |Z_\gamma| = \sup_{\gamma<\mu} |Z_\gamma| = \sup_{\gamma<\mu} \Big|\prod_{\alpha \leqslant \gamma} X_\alpha\Big| = \sup_{\gamma<\mu} \Big|\prod_{\alpha<\gamma} X_\alpha\Big|.$$

We show that for every $x \in X$ and every $c < x$, there exists $a \in D$ such that $c \leqslant a < x$. Let $x = (x_\alpha)_{\alpha<\mu}$ and $c = (c_\alpha)_{\alpha<\mu}$. Let α_0 be the first index such that $c_{\alpha_0} < x_{\alpha_0}$. If $c_\alpha = 1$ for all $\alpha > \alpha_0$, then let $a = c \in D$, otherwise there exists some $\delta > \alpha_0$ such that $c_\delta < y$ for some $y \in X_\delta$, and we define $a = (a_\alpha)_{\alpha<\mu} \in D$ by

$$a_\alpha = \begin{cases} c_\alpha & \text{if } \alpha < \delta, \\ y & \text{if } \alpha = \delta, \\ z_\alpha & \text{if } \alpha > \delta. \end{cases}$$

Dually, there exists a subset $D' \subseteq X$ with $|D'| \leqslant \sup_{\gamma<\mu}\left|\prod_{\alpha<\gamma} X_\alpha\right|$ such that for every $x \in X$ and every $d > x$, there exists $b \in D'$ such that $d \geqslant b > x$. Consequently, there exists a base of cardinality $\sup_{\gamma<\mu}\left|\prod_{\alpha<\gamma} X_\alpha\right|$ as required to show. □

We are left with the calculation of the character $\chi(X)$ of a lexicographic product $X = \mathbb{L}_{\alpha<\mu} X_\alpha$, where μ is a limit ordinal. We first calculate $\chi_{\mathrm{r}}(x, X)$ for $x \in X$ and then $\chi_{\mathrm{r}}(X)$. By duality we obtain $\chi_{\mathrm{l}}(x, X)$ and $\chi_{\mathrm{l}}(X)$. Finally, we get $\chi(x, X)$ and $\chi(X)$ as a maximum of χ_{r} and χ_{l}.

For $x = (x_\alpha)_{\alpha<\mu} \in X$, let

$$\begin{aligned} A_1(x) &= \{\alpha < \mu : x_\alpha \neq 1\}, \\ \mu_1(x) &= \inf\{\gamma < \mu : x_\alpha = 1 \text{ for all } \gamma \leqslant \alpha < \mu\}. \end{aligned}$$

[We use the following convention: $\mu_1(x) = \mu$ when $\{\gamma < \mu : x_\alpha = 1 \text{ for all } \gamma \leqslant \alpha < \mu\} = \varnothing$. In other words, $\inf \varnothing = \mu$, which is true when $A_1(x)$ is cofinal in μ. If $A_1(x)$ is not cofinal in μ then $\mu_1(x) = \min\{\gamma < \mu : x_\alpha = 1 \text{ for all } \gamma \leqslant \alpha < \mu\}$.]

Lemma 3.2 *For $x = (x_\alpha)_{\alpha<\mu} \in X$ we have*

$$\chi_{\mathrm{r}}(x, X) = \begin{cases} \mathrm{cf}\,(\mu_1(x)) & \textit{if } \mu_1(x) \textit{ is a limit ordinal (or } 0), \\ \chi_{\mathrm{r}}(x_\beta, X_\beta) & \textit{if } \mu_1(x) = \beta + 1. \end{cases}$$

Proof Case I: $A_1(x)$ is cofinal in μ. In this case we show that $\chi_{\mathrm{r}}(x, X) = \mathrm{cf}\,(\mu)$. Choose $z_\alpha > x_\alpha$ for every $\alpha \in A_1(x)$ and let $y^\gamma = (y^\gamma_\alpha)_{\alpha<\mu} \in X$, for every $\gamma \in A_1(x)$, be defined by:

$$y^\gamma_\alpha = \begin{cases} x_\alpha & \text{if } \alpha \neq \gamma, \\ z_\gamma & \text{if } \alpha = \gamma. \end{cases}$$

Take any $A \subseteq A_1(x)$ with $|A| = \mathrm{cf}\,(\mu)$, then $y^\gamma > x$ for every $\gamma \in A$ and $\inf_{\gamma\in A} y^\gamma = x$. Hence $\chi_{\mathrm{r}}(x, X) \leqslant |A| = \mathrm{cf}\,(\mu)$. To prove the converse, suppose $\chi_{\mathrm{r}}(x, X) < \mathrm{cf}\,(\mu)$. There exists a set B with $|B| < \mathrm{cf}\,(\mu)$ and elements $b^\beta = (b^\beta_\alpha)_{\alpha<\mu} \in X$ with $b^\beta > x$ for all $\beta \in B$ such that $\inf_{\beta\in B} b^\beta = x$. Let $\alpha_\beta = \min\{\alpha < \mu : b^\beta_\alpha \neq x_\alpha\}$, then $|\{\alpha_\beta : \beta \in B\}| \leqslant |B| < \mathrm{cf}\,(\mu)$. Therefore, there exists $\xi < \mu$ such that $\alpha_\beta < \xi$ for all $\beta \in B$. Take any $\gamma \in A_1(x)$ with $\gamma > \xi$. Then $x < y^\gamma < b^\beta$ for all $\beta \in B$, so that $\inf_{\beta\in B} b^\beta \geqslant y^\gamma > x$, a contradiction.

Case II: $A_1(x)$ is not cofinal in μ. If $x_\alpha = 1$ for all $\alpha < \mu$ then x is the greatest element of X, hence $\chi_{\mathrm{r}}(x, X) = 1$. Otherwise, $\chi_{\mathrm{r}}(x, X) = \chi_{\mathrm{r}}((x_\alpha)_{\alpha<\mu_1(x)}, \mathbb{L}_{\alpha<\mu_1(x)} X_\alpha)$, see the proof of Proposition 2.5 for $X = \mathbb{L}_{\alpha<\mu_1(x)} X_\alpha \cdot \mathbb{L}_{\mu_1(x)\leqslant\alpha<\mu} X_\alpha$. If $\mu_1(x)$ is a limit ordinal, then $\chi_{\mathrm{r}}((x_\alpha)_{\alpha<\mu_1(x)}, \mathbb{L}_{\alpha<\mu_1(x)} X_\alpha) = \mathrm{cf}\,(\mu_1(x))$ by Case I above.

Otherwise, let $\beta < \mu$ satisfy $\beta + 1 = \mu_1(x)$. Then $\mathbb{L}_{\alpha<\mu_1(x)} X_\alpha = \mathbb{L}_{\alpha<\beta} X_\alpha \cdot X_\beta$ and by Proposition 2.5 one obtains

$$\chi_{\mathrm{r}}\big((x_\alpha)_{\alpha<\mu_1(x)}, \mathbb{L}_{\alpha<\mu_1(x)} X_\alpha\big) = \chi_{\mathrm{r}}(x_\beta, X_\beta). \qquad \square$$

For our next corollary we let

$$\mu_1 = \inf\{\gamma < \mu : 1 \in X_\alpha \text{ for all } \gamma \leqslant \alpha < \mu\}.$$

Hence, using the above convention, $\mu_1 = \mu$ if and only if $A_1 = \{\alpha < \mu : 1 \notin X_\alpha\}$ is cofinal in μ. For an ordinal α, let us denote by α^- the immediate predecessor of α if it exists; otherwise, if α is a limit ordinal (or 0) we let $\alpha^- = \alpha$. Moreover, in line with our interest in infinite cardinals, we let $\sup \varnothing = \aleph_0$ (instead of $\sup \varnothing = 0$).

Corollary 3.3 *For $X = \mathbb{L}_{\alpha<\mu} X_\alpha$ we have*

$$\chi_{\mathrm{r}}(X) = \sup_{\mu_1^- \leqslant \sigma < \mu} \chi_{\mathrm{r}}(X_\sigma) \cdot \sup_{\mu_1 \leqslant \sigma \leqslant \mu} \mathrm{cf}(\sigma).$$

To calculate $\chi_{\mathrm{l}}(x, X)$ and $\chi_{\mathrm{l}}(X)$ for $x = (x_\alpha)_{\alpha<\mu} \in X$, let

$$\mu_0(x) = \inf\{\gamma < \mu : x_\alpha = 0 \text{ for all } \gamma \leqslant \alpha < \mu\}.$$

As above we use the convention: $\mu_0(x) = \mu$ when $\{\gamma < \mu : x_\alpha = 0 \text{ for all } \gamma \leqslant \alpha < \mu\} = \varnothing$. By duality we have the following two results:

Lemma 3.4 *For $x = (x_\alpha)_{\alpha<\mu} \in X$ we have*

$$\chi_{\mathrm{l}}(x, X) = \begin{cases} \mathrm{cf}(\mu_0(x)) & \textit{if } \mu_0(x) \textit{ is a limit ordinal (or } 0), \\ \chi_{\mathrm{l}}(x_\beta, X_\beta) & \textit{if } \mu_0(x) = \beta + 1. \end{cases}$$

Also, if we let $\mu_0 = \inf\{\gamma < \mu : 0 \in X_\alpha \text{ for all } \gamma \leqslant \alpha < \mu\}$, we have the following corollary.

Corollary 3.5 *For $X = \mathbb{L}_{\alpha<\mu} X_\alpha$ we have*

$$\chi_{\mathrm{l}}(X) = \sup_{\mu_0^- \leqslant \sigma < \mu} \chi_{\mathrm{l}}(X_\sigma) \cdot \sup_{\mu_0 \leqslant \sigma \leqslant \mu} \mathrm{cf}(\sigma).$$

Using Lemmas 3.2 and 3.4 we can calculate $\chi(x, X)$ by the formula

$$\chi(x, X) = \max\{\chi_{\mathrm{l}}(x, X), \chi_{\mathrm{r}}(x, X)\}.$$

Finally, by Corollaries 3.3 and 3.5 and the formula $\chi(X) = \max\{\chi_{\mathrm{l}}(X), \chi_{\mathrm{r}}(X)\}$, we have the following result for the character of a lexicographic product.

Theorem 3.6 *If $X = \mathbb{L}_{\alpha<\mu} X_\alpha$, then*

$$\chi(X) = \sup_{\mu_0^- \leqslant \sigma < \mu} \chi_{\mathrm{l}}(X_\sigma) \cdot \sup_{\mu_1^- \leqslant \sigma < \mu} \chi_{\mathrm{r}}(X_\sigma) \cdot \sup_{\delta \leqslant \sigma \leqslant \mu} \mathrm{cf}(\sigma),$$

where $\delta = \min\{\mu_0, \mu_1\}$.

Remark 3.7 A particular case of Theorem 3.6 is when $\delta = \mu_0 = \mu_1$, then

$$\chi(X) = \sup_{\delta^- \leqslant \sigma < \mu} \chi(X_\sigma) \cdot \sup_{\delta \leqslant \sigma \leqslant \mu} \mathrm{cf}(\sigma).$$

Example 3.8 Let us look at the following examples.

1. Suppose $X_\alpha = [0, \omega_3[$ for all $\alpha < \omega_1$ and let $X = \mathbb{L}_{\alpha<\omega_1} X_\alpha$. Then $\chi(X) = \aleph_2$.
2. Suppose $X_\alpha =]\omega_0, 0] + [0, \omega_3]$ for all $\alpha < \omega_1$ and let $X = \mathbb{L}_{\alpha<\omega_1} X_\alpha$. Then $\chi(X) = \aleph_1$.
3. Let $Y =]\omega_3, 0] + [0, \omega_3[$ and $Z = [0, \omega_2[$. If $X = \mathbb{L}_{\alpha<\omega_1} X_\alpha$, where $X_\alpha = Y$ for all $\alpha < \omega_0$ and $X_\alpha = Z$ for all $\omega_0 \leqslant \alpha < \omega_1$, then $\chi(X) = \aleph_1$. However if, $X = \mathbb{L}_{\alpha<\omega_1} X_\alpha$, where $X_\alpha = Y$ for all $\alpha \leqslant \omega_0$ and $X_\alpha = Z$ for all $\omega_0 < \alpha < \omega_1$, then $\chi(X) = \aleph_2$.
4. Let $X_\alpha = [0, \omega_3[$ for all $\alpha < \omega_\omega$, $\alpha \neq \omega_n$ for all $n < \omega_0$, and $X_\alpha =]\omega_0, 0] + [0, \omega_3]$ for all $\alpha < \omega_\omega$, $\alpha = \omega_n$ for some $n < \omega_0$. If $X = \mathbb{L}_{\alpha<\omega_\omega} X_\alpha$, then $\chi(X) = \mathrm{cf}(\omega_\omega) = \aleph_0$.

We end this section by noting that the results of Sect. 2 and of this section give us a way of calculating $\phi(X)$, where $X = \mathbb{L}_{\alpha<\mu} X_\alpha$, $\mu = \lambda + 1$ is a successor ordinal and ϕ is any of the considered cardinal functions.

One needs to look into the following four cases to calculate the spread and use Theorems 2.2 and 3.1:

(1) $|X_\lambda| > 2$: $s(X) = \left|\prod_{\alpha<\lambda} X_\alpha\right| \cdot s(X_\lambda)$;
(2a) $|X_\lambda| = 2$, λ is a limit ordinal: $s(X) = \sup_{\alpha<\lambda} \left|\prod_{\gamma<\alpha} X_\gamma\right| \cdot \mathbb{J}_{\mathbb{L}_{\alpha<\lambda} X_\alpha} = \sup_{\alpha<\lambda} \left|\prod_{\gamma<\alpha} X_\gamma\right|$, because
$\mathbb{J}_{\mathbb{L}_{\alpha<\lambda} X_\alpha} \leqslant \sum_{\alpha<\lambda} \left|\prod_{\gamma<\alpha} X_\gamma\right| \leqslant \sup_{\alpha<\lambda} \left|\prod_{\gamma<\alpha} X_\gamma\right|$;
(2b) (i) $|X_\lambda| = 2$, $\lambda = \nu + 1$ with $|X_\nu| > 2$;
(2b) (ii) $|X_\lambda| = 2$, $\lambda = \nu + 1$ with $|X_\nu| = 2$:
In both cases $s(X) = \left|\prod_{\alpha<\nu} X_\alpha\right| \cdot s(X_\nu \cdot X_\lambda) = \left|\prod_{\alpha<\nu} X_\alpha\right| \cdot s(X_\nu) \cdot \mathbb{J}_{X_\nu}$.
In (2b) (ii), in general if $|X_\nu| \leqslant \aleph_0$, we can simplify to obtain $s(X) = k(X) = \left|\prod_{\alpha<\mu} X_\alpha\right|$.

Analogous formulas apply for $d(X)$. If we want to calculate the weight of X we use Theorem 2.4 to conclude

$$w(X) = \left|\prod_{\alpha<\lambda} X_\alpha\right| \cdot w(X_\lambda).$$

Finally, one uses Proposition 2.5 to calculate the character of X. As above, let $\mu = \lambda + 1 = \lambda_0 + n$, $1 \leqslant n < \omega_0$, be a successor ordinal, where λ_0 is a limit ordinal. Let $N_1 = \{k \in \{0, \ldots, n-1\} : 1 \notin X_{\lambda_0+k}\}$ and let

$$n_1 = \begin{cases} \max N_1 & \text{if } N_1 \neq \varnothing, \\ -1 & \text{otherwise.} \end{cases}$$

Analogously define $n_0 \in \{-1, 0, \ldots, n-1\}$. Put $Z_{-1} = \mathbb{L}_{\alpha<\lambda_0} X_\alpha$ and $Z_k = X_{\lambda_0+k}$ for $k \in \{0, \ldots, n-1\}$. Note that $\chi_r(Z_{-1})$, $\chi_l(Z_{-1})$ and $\chi(Z_{-1})$ can be calculated by Corollaries 3.3, 3.5 and Theorem 3.6, respectively. Then

$$\chi(X) = \begin{cases} \prod_{k=n_1}^{n-1} \chi(Z_k) & \text{if } n_0 = n_1, \\ \prod_{k=n_0}^{n_1-1} \chi_l(Z_k) \cdot \prod_{k=n_1}^{n-1} \chi(Z_k) & \text{if } n_0 < n_1, \\ \prod_{k=n_1}^{n_0-1} \chi_r(Z_k) \cdot \prod_{k=n_0}^{n-1} \chi(Z_k) & \text{if } n_0 > n_1. \end{cases}$$

Example 3.9 Let us look at a simple example. Suppose $X_\alpha = [0, \omega_3]$ for all $\alpha < \omega_1$ and let $X = \mathbb{L}_{\alpha<\omega_1} X_\alpha$.

1. Let $X_{\omega_1} = [0, \omega_3]$ and let $Y = X \cdot X_{\omega_1} = \mathbb{L}_{\alpha\leqslant\omega_1} X_\alpha$. Then

$$s(Y) = k(X) \cdot s([0, \omega_3]) = \aleph_3^{\aleph_1} \cdot \aleph_3 = \aleph_3^{\aleph_1} = 2^{\aleph_1} \cdot \aleph_3$$
$$d(Y) = k(X) \cdot d([0, \omega_3]) = \aleph_3^{\aleph_1} \cdot \aleph_3 = \aleph_3^{\aleph_1} = 2^{\aleph_1} \cdot \aleph_3$$
$$w(Y) = k(X) \cdot w([0, \omega_3]) = \aleph_3^{\aleph_1} \cdot \aleph_3 = \aleph_3^{\aleph_1} = 2^{\aleph_1} \cdot \aleph_3$$
$$\chi(Y) = \chi(X) \cdot \chi([0, \omega_3]) = \aleph_3 \cdot \aleph_3 = \aleph_3.$$

2. Let $X_{\omega_1} = \{0, 1\}$ and let $Y = X \cdot X_{\omega_1} = \mathbb{L}_{\alpha<\omega_1} X_\alpha \cdot \{0, 1\}$. Then

$$s(Y) = s(X) \cdot \mathbb{J}_X = \sup_{\alpha<\omega_1} \left| \prod_{\gamma<\alpha} X_\gamma \right| \cdot \mathbb{J}_X = \aleph_3^{\aleph_0} = 2^{\aleph_0} \cdot \aleph_3,$$
$$d(Y) = d(X) \cdot \mathbb{J}_X = \sup_{\alpha<\omega_1} \left| \prod_{\gamma<\alpha} X_\gamma \right| \cdot \mathbb{J}_X = 2^{\aleph_0} \cdot \aleph_3,$$
$$w(Y) = k(X) \cdot w(\{0, 1\}) = \aleph_3^{\aleph_1} \cdot \aleph_0 = \aleph_3^{\aleph_1} = 2^{\aleph_1} \cdot \aleph_3,$$
$$\chi(Y) = \chi(X) \cdot \chi(\{0, 1\}) = \aleph_3 \cdot \aleph_0 = \aleph_3.$$

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