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Foundations of singular Finsler geometry

Pavel Andreev[1]

Abstract We present an approach leading to Finsler geometry without differential calculus of tensors. Several natural examples of such singular Finsler spaces are studied. One class of such examples contains Busemann G-spaces with non-positive curvature. Starting with a singular version of the axiomatics, some simplest properties known in the smooth Finsler geometry are interpreted.

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1 Introduction

The preface of Busemann's *Geometry of Geodesics* [6] contains the following sentence: "… the term Finsler Space means to many not only a type of space, but also a definite approach: The space is considered as a set of linear elements to which euclidean metrics are attached. … In spite of the great success of Finsler's thesis, the later development of this aspect lacks simple geometric facts to the extent that their existence in non-Riemannian geometry has been doubted."

✉ Pavel Andreev
   pdandreev@mail.ru

[1]   Northern (Arctic) Federal University named after M.V. Lomonosov, Severnaya Dvina Emb., 17, 163002 Arkhangelsk, Russia

Busemann considers Finsler geometry as a geometry of G-spaces which satisfy appropriate axioms. The first Busemann version of axiomatics leading to G-spaces was made in [7]. Unfortunately, the topology of general G-spaces is still mysterious: it is unknown whether every G-space is a topological manifold and even whether it is finite-dimensional. Positive answers to these questions exist only in special cases. Consequently, speaking about axioms of generalized non-smooth Finsler geometry, one should adjust Busemann's axioms and take into account the current realities.

The object of the present paper is Finsler spaces in standard and non-standard singular approaches. First let us recall the standard definition. Here standard means the modern definition in terms of differential geometry of manifolds (see, for example, [10] or [9]).

Let M^n be a C^∞-manifold and TM^n its tangent bundle. A function $\mathcal{F}: TM \to [0, +\infty)$ is called *Finsler metric* if it has the following properties:

(a) $\mathcal{F}$ is C^∞ on $TM \setminus \{0\}$ where $\{0\}$ is the null-section of TM;
(b) $\mathcal{F}$ is positively homogeneous on vectors: $\mathcal{F}(x, \lambda\mathbf{y}) = \lambda\mathcal{F}(x, \mathbf{y})$ for all $\lambda > 0$ and all $\mathbf{y} \in T_x M$;
(c) in every point $x \in M^n$ and for every vector $y \in T_x M \setminus \{0\}$ the quadratic form

$$g_y(\mathbf{u}, \mathbf{v}) = \frac{1}{2} \frac{\partial^2}{\partial s \partial t} (\mathcal{F}^2(x, \mathbf{y}+s\mathbf{u}+t\mathbf{v}))|_{s=t=0} \tag{1}$$

is positive definite.

Condition (c) means that $\mathcal{F}^2(x, \cdot)$ is a *strongly convex* function of the variable $\mathbf{y}$ for any fixed $x \in M^n$.

We want to get rid of the C^∞-smoothness condition here. In particular, we speak about manifolds with non-necessarily smooth norms in the tangent spaces. Such a weakening of the definition leads to obstructions for the application of the differential calculus of tensors. Our purpose is to show that Finsler geometry in general has a right to existence without differentiation of tensors.

In particular, we present a version of the axiomatics for such non-smooth Finsler geometry. It is preceded by the introduction of necessary notions and by several examples to illustrate the possibility of such a geometry and its methods.

The work was inspired by the author's recent results on the geometry of Busemann G-spaces with non-positive curvature, showing that such a space is a topological manifold having a tangent space at every point. It occurs that the tangent space has a structure of a normed space with strictly convex norm. The metric of the space is the length metric and one can pick a class of curves with velocity. The velocity of the curve $\gamma: [a, b] \to X$ in the space X at the point $\gamma(t_0)$, $t_0 \in [a, b]$, is a tangent vector $\mathbf{v} = d\gamma(t_0) \in T_{\gamma(t_0)} X$ which can be obtained as a limit of secants to γ. If γ has velocity $\mathbf{v}(t)$ at all points with continuous norm $\|\mathbf{v}(t)\|$, then its length is

$$L(\gamma) = \int_a^b \|\mathbf{v}(t)\| \, dt, \tag{2}$$

In particular, if γ is a naturally parameterized geodesic segment, then it has unit velocity at all points and

$$L(\gamma) = \int_a^b dt = b - a.$$

2 Directed segments in metric spaces

Let (X, d) be a metric space. The closed ball with center $x_0 \in X$ and radius $r > 0$ is denoted by $B(x_0, r)$, the corresponding open ball $U(p, r)$ and the sphere $S(p, r)$.

We also consider non-reversible metrics for which the symmetry property $d(y, x) = d(x, y)$ may fail. The triangle inequality for the the non-reversible metric d must be written strongly in the form

$$d(x, y) \leqslant d(x, z) + d(z, y)$$

for all points $x, y, z \in X$. Hence the closed ball with center $x_0 \in X$ and radius r has two different interpretations:

$$B_+(x_0, r) = \{x \in X : d(x_0, x) \leqslant r\},$$

and

$$B_-(x_0, r) = \{x \in X : d(x, x_0) \leqslant r\}.$$

Corresponding open balls are

$$U_+(x_0, r) = \{x \in X : d(x_0, x) < r\}$$

and

$$U_-(x_0, r) = \{x \in X : d(x, x_0) < r\}.$$

If the metric is reversible, both balls $B_+(x_0, r)$ and $B_-(x_0, r)$ coincide with $B(x_0, r)$ at all points and for all $r > 0$.

Both systems of open balls U_+ and U_- generate topologies on the set X as follows. We say that a subset $U \subset X$ is positively (correspondingly, negatively) open if for any point $x \in U$ there exists $r > 0$ such that $U_+(x, r) \subset U$ (correspondingly $U_-(x, r) \subset U$). The verification of the topology axioms for the family of positively (negatively) open subsets is standard (it is based on the evident argument: if $r_1 < r_2$ then $U_\pm(x, r_1) \subset U_\pm(x, r_2)$). We will denote the two topologies by τ_+ and τ_-. In general such topologies may be different.

Example 2.1 Consider $X = \mathbb{R}$ with the metric

$$
d(s, t) = \begin{cases} s - t & \text{if } s \geqslant t, \\ 1 + t - s & \text{if } s < t. \end{cases}
$$

It is easy to check that d is a metric. The open balls of this metric are the following:

$$
U_+(x_0, r) = \begin{cases} (x_0 - r, x_0] & \text{if } r \leqslant 1, \\ (x_0 - r, x_0 + r - 1) & \text{otherwise,} \end{cases}
$$

and

$$
U_-(x_0, r) = \begin{cases} [x_0, x_0 + r) & \text{if } r \leqslant 1 \\ (x_0 - r + 1, x_0 + r) & \text{otherwise,} \end{cases}
$$

Evidently, the topologies τ_+ and τ_- are different: semiintervals of type $(s, t]$ are open in the first one and they are not open in the second one.

The map $\gamma : [a, b] \to X$ from the numerical segment $[a, b] \subset \mathbb{R}$ to the metric space X is called *positively isometric* if for any $s, t \in [a, b]$ such that $s < t$ the following equality holds:

$$
d(\gamma(s), \gamma(t)) = t - s.
$$

The map γ is called *isometric* if for any $s, t \in [a, b]$

$$
d(\gamma(s), \gamma(t)) = |t - s|.
$$

The *directed segment* $[xy\rangle$ in the space (X, d) is the image of the numerical segment $[a, b] \subset \mathbb{R}$ under the positively isometric map $\gamma : [a, b] \to X$ such that $\gamma(a) = x$ and $\gamma(b) = y$. The point x is the *beginning* of the directed segment $[xy\rangle$, the point y is its *endpoint*. The directed segment $[xy\rangle$ is called *segment* and is denoted $[xy]$ if γ is isometric. The points x and y are called *ends* of the segment $[xy]$. Every segment $[xy] \subset X$ leads to a pair of directed segments $[xy\rangle$ and $[yx\rangle$. On the other hand, the directed segments $[xy\rangle$ and $[yx\rangle$ may be essentially different as point sets when the metric d is asymmetric and $d(x, y) \neq d(y, x)$.

An equivalent definition of the directed segment $[xy\rangle \subset X$ and the segment $[xy]$ uses the notion of length. Let $\gamma : [a, b] \to X$ be a map. Given a subdivision

$$
a = t_0 < t_1 < \cdots < t_n = b
$$

of the numerical segment $[a, b]$, the corresponding *positive variation of the length* is

$$
V_+(\gamma, T) = \sum_{i=1}^{n} d(\gamma(t_{i-1}), \gamma(t_i))
$$

and its *negative variation* is

$$V_-(\gamma, T) = \sum_{i=0}^{n-1} d(\gamma(t_{i+1}), \gamma(t_i)).$$

We say that γ represents a *positively rectifiable curve* in X if the set

$$\mathcal{V}_+(\gamma) = \{V_+(\gamma, T) : T \text{ is a subdivision of } [a, b]\} \qquad (3)$$

is bounded. The *positive length* of a positively rectifiable curve γ is by definition

$$L_+(\gamma) = \sup_T V_+(\gamma, T).$$

The map γ represents a *rectifiable curve* with length $L(\gamma)$ if it is both positively rectifiable and metrically invertible:

$$d(\gamma(s), \gamma(t)) = d(\gamma(t), \gamma(s))$$

for all pairs $s, t \in [a, b]$. The image $\gamma([a, b])$ is a *directed segment* $[xy\rangle$ where $x = \gamma(a)$ and $y = \gamma(b)$ if and only if γ represents a positively rectifiable curve and $L_+(\gamma) = d(x, y)$. Correspondingly, the segment $[xy]$ is an image of the rectifiable curve γ with length $L(\gamma) = d(x, y) = d(y, x)$.

Definition 2.2 The space (X, d) is called *geodesic* if for any ordered pair of different points $(x, y) \in X \times X$ there exists a directed segment $[xy\rangle$.

For a better understanding of the above definitions, let us consider some examples.

Example 2.3 Let V^n be a linear space. We consider its elements as vectors and as points at the same time. The *linear segment* $[xy]$ is a part of the straight line bounded by x and y, the *endpoints* of $[xy]$. The *Minkowski space* is a finite-dimensional linear space V^n equipped with a norm. In general, the norm $\|x\|$ is not necessarily symmetric, strictly convex or smooth. Identifying every vector $\mathbf{x}$ with its endpoint in the corresponding affine space A^n when the origin $o \in A^n$ is fixed and identified with the null-vector $\boldsymbol{\theta}$, one gets a metric d on A^n with

$$d(x, y) = \|y - x\|.$$

The possible asymmetry of the norm leads to metric asymmetry: the equality $d(x, y) = d(y, x)$ fails in general. Modern Finsler geometry allows non-symmetric metrics, so asymmetry is admissible.

The possible non-smoothness of the norm contradicts in general the standard smoothness assumption in modern Finsler geometry. The possible non-strongness of the convexity also contradicts it: the positive definiteness of the form (1) fails. This condition may be false even if the norm is smooth and strictly convex: when the Gauss–Kroneker curvature (calculated in the sense of the Euclidean metric in the

space A^n) of the boundary sphere $S_+(o, 1)$ of the ball $B_+(o, 1)$ vanishes in some point x with $\|\mathbf{x}\| = 1$.

If the norm in Minkowski space is non-strictly convex, we have non-uniqueness of directed segments. Precisely, if $S_+(o, 1)$ contains a pair of different points $x, y \in S_+(o, 1)$ with $\|\mathbf{x} + \mathbf{y}\| = 2$, then any two points $u, v \in A^n$ such that

$$\overrightarrow{uv} = (\mathbf{v} - \mathbf{u}) \uparrow\uparrow (\mathbf{x} + \mathbf{y})$$

are connected by infinitely many non-linear directed segments $[uv\rangle$.

Example 2.4 (*Funk metrics*) Let Ω be a bounded convex domain in the affine space A^n. Given two distinct points $x, y \in \Omega$, the point $z \in \partial\Omega$ for which y lies between x and z is uniquely defined. The Funk distance $d(x, y)$ is

$$d_{\mathrm{F}}(x, y) = \ln \frac{|x - z|}{|y - z|}. \tag{4}$$

The linear segment $[xy]$ is the directed segment in the sense of the Funk metric d_{F}. Such a directed segment is unique for any two $x, y \in \Omega$ if Ω is strictly convex. If not, the boundary $\partial\Omega$ contains linear segments, and the directed segment $[xy\rangle$ is not unique if z is an interior point of the linear segment lying in $\partial\Omega$.

Example 2.5 Let $f : [0, +\infty) \to [0, +\infty)$ be a non-constant concave monotone non-decreasing function with $f(0) = 0$. Concavity means that

$$f\left(\frac{s + t}{2}\right) \geqslant \frac{f(s) + f(t)}{2}$$

for all pairs $s, t \in [0, +\infty)$ and monotone non-decreasing means that the inequality $0 \leqslant s < t$ implies $f(s) \leqslant f(t)$. It is well known that the function $d_f(s, t) = f(|s - t|)$ is a reversible metric on $\mathbb{R}$. It follows that the function

$$d_{f_-}(s, t) = \begin{cases} t - s, & \text{if } s \leqslant t, \\ d_f(s, t) & \text{otherwise} \end{cases}$$

is also a metric on $\mathbb{R}$ but it is not reversible if f is not the identity function $f(t) = t$. All translations $T_h : \mathbb{R} \to \mathbb{R}$ of the type $T_h(x) = x + h$ are isometries of the metric space $(\mathbb{R}, d_{f_-})$ independently on the choice of the function f. If $s < t$ then the point s may be connected with t by a directed segment with length $L_+ = t - s$. The reverse directed segment is possible only if f is linear ($f(x) = k \cdot x$ for some $k > 0$) on the numerical segment $[0, t - s]$. In the last case, the length L_- of the reverse segment is $L_- = k \cdot (t - s)$.

If $f(x) \geqslant C \cdot x^p$ in the interval $[0, \varepsilon)$ for some $\varepsilon, C > 0$ and $0 < p < 1$ then for any $s, t \in \mathbb{R}$, $s < t$, there is no rectifiable path from t to s.

Consider the unit circle $\mathbb{S}^1 \subset \mathbb{C}$ as a factor group of $(\mathbb{R}, +)$ by the action of $\mathbb{Z}$ on $\mathbb{R}$ by $(2\pi k)$-translations: $x \mapsto T_{2\pi k}(x) = x + 2\pi k$. The metric d_{f_-} generates a new metric $d_{f,\mathbb{S}}$ on $\mathbb{S}^1$. Denote

$$[t] = \{t + 2\pi k : k \in \mathbb{Z}\}$$

and consider the class $[t]$ as a point $e^{(t+2\pi k) \cdot i} \in \mathbb{S}^1$. Then we have for $[s]$ and $[t]$ two quantities:

$$d_+([s], [t]) = \min\{t - s + 2\pi k : k \in \mathbb{Z}, \, t - s + 2\pi k \geqslant 0\}$$

and

$$d_-([s], [t]) = \min\{f(s - t - 2\pi k) : k \in \mathbb{Z}, \, t - s + 2\pi k < 0\}.$$

The metric $d_{f,\mathbb{S}}$ is defined as the minimum

$$d_{f,\mathbb{S}}([s], [t]) = \min\{d_+([s], [t]), d_-([s], [t])\}.$$

For example, consider the case $f(x) = \lambda x$ where $\lambda > 1$. Assume for convenience that $s < t < s + 2\pi$. Then

$$d_{f,\mathbb{S}}([s], [t]) = \begin{cases} t - s, & \text{if } t - s \leqslant \lambda \cdot (s - t + 2\pi), \\ \lambda \cdot (s - t + 2\pi) & \text{otherwise.} \end{cases}$$

The point $[s]$ is connected to $[t]$ by a directed segment $[[s][t]\rangle$. Namely, if

$$t - s < \lambda \cdot (s - t + 2\pi)$$

and consequently

$$t - s < \frac{2\lambda\pi}{1 + \lambda},$$

then $[[s][t]\rangle$ is the arc of the circle starting from $[s]$ to $[t]$ counterclockwise. If

$$\frac{2\lambda\pi}{1 + \lambda} < t - s < 2\pi,$$

then $[[s][t]\rangle$ is the clockwise arc $[s]$ to $[t]$. If the equality

$$t - s = \frac{2\lambda\pi}{1 + \lambda}$$

holds then both these arcs are directed segments. Speaking about the inverse directed segment $[[t][s]\rangle$ we also have several essentially different situations.

- If $t - s < 2\pi/(1 + \lambda)$, then the counterclockwise arc from $[s]$ to $[t]$ is a directed segment $[[s][t]\rangle$ and its inverse arc is an inverse segment $[[t][s]\rangle$.
- If $t - s = 2\pi/(1 + \lambda)$, then there is one more directed segment from $[t]$ to $[s]$: now both arcs of $\mathbb{S}^1$ from $[t]$ to $[s]$ are directed segments.

- If $2\pi/(1 + \lambda) < t - s < 2\pi\lambda/(1 + \lambda)$, then the directed segments $[[s][t])$ and $[[t][s])$ are unique and different. Both are directed counterclockwise.

The two cases

$$t - s \geqslant \frac{2\pi\lambda}{1 + \lambda}$$

reduce to the previous ones after redenoting $[s] \leftrightarrow [t]$.

Remark 2.6 It is not difficult to extend the metric $d_{f,\mathbb{S}}$ considered here to the metric on $\mathbb{C}$ with prescribed properties of directed segments connecting points on $\mathbb{S}^1$.

The following example constructed by the same scheme as the previous one presents a geodesic space with different topologies τ_+ and τ_-

Example 2.7 Let d be a metric on $\mathbb{R}$ from Example 2.1. Then the factorisation of $\mathbb{R}$ by the action of $\mathbb{Z}$ by k-translations $t \mapsto t + k$, $k \in \mathbb{Z}$ leads to a geodesic space with different topologies τ_+ and τ_-. Neither $(\mathbb{R}/\mathbb{Z}, \tau_+)$ nor $(\mathbb{R}/\mathbb{Z}, \tau_+)$ is homeomorphic to $\mathbb{S}^1$. The map $[t] \rightarrow [-t]$ is a homeomorphism $(\mathbb{R}/\mathbb{Z}, \tau_+) \rightarrow (\mathbb{R}/\mathbb{Z}, \tau_-)$.

3 The spaces of directions and the tangent cones in a geodesic space

Let (X, d) be a (not necessarily reversible) geodesic metric space. We do not suppose uniqueness of $[xy)$ nor coincidence of $[xy)$ with $[yx)$ as point sets here. Let $o \in X$ be an arbitrary point and $[ox)$ and $[oy)$ two directed segments starting from o.

Definition 3.1 We say that the directed segments $[ox)$ and $[oy)$ *have the same joint direction* at o if the following holds. Let $\gamma_x : [0, d(o, x)] \rightarrow X$ be a positively isometric map with image $[ox)$ (natural parameterization of $[ox)$) and $\gamma_y : [0, d(o, y)] \rightarrow X$ the corresponding positively isometric map for $[oy)$. Then

$$\limsup_{t \rightarrow 0} \frac{d(\gamma_x(t), \gamma_y(t))}{t} = \limsup_{t \rightarrow 0} \frac{d(\gamma_y(t), \gamma_x(t))}{t} = 0.$$

The binary relation "to have the same joint direction" is an equivalence relation on the set of all directed segments starting at o. From now on, we omit the word "joint" and use the term "the same direction" meaning namely this definition. Each equivalence class is called a *direction* at o.

Assume that the set of segments starting at the point $o \in X$ satisfies the following condition: if

$$\limsup_{t \rightarrow 0} \frac{d(\gamma_x(t), \gamma_y(t))}{t} = 0,$$

then

$$\limsup_{t \rightarrow 0} \frac{d(\gamma_y(t), \gamma_x(t))}{t} = 0$$

for any two segments $[ox\rangle$ and $[oy\rangle$ with corresponding natural parameterizations γ_x and γ_y. Then we say that the directions at the point o are defined *independently of the asymmetry*. The geodesic space X is called a *space with directions independent of the asymmetry* if this property holds at all points of X.

The following example explains the necessity of the independence condition.

Example 3.2 Denote by X the closed subset $\eta \geqslant \sqrt{\xi}$ of the upper half-plane $\eta > 0$ of the plane with coordinates (ξ, η) bounded by the η-axis γ_1 from the left and by the semiparabola γ_2 with equation $\eta = \sqrt{\xi}$ from the right. Consider the function $\delta \colon X \times X \to \mathbb{R}_+ \cup \{+\infty\}$ defined as

$$\delta((\xi_1, \eta_1), (\xi_2, \eta_2)) = \begin{cases} \delta_+((\xi_1, \eta_1), (\xi_2, \eta_2)) & \text{if } \xi_1 \leqslant \xi_2, \\ \delta_-((\xi_1, \eta_1), (\xi_2, \eta_2)) & \text{otherwise.} \end{cases}$$

Here $\delta_+((\xi_1, \eta_1), (\xi_2, \eta_2))$ is the length distance between the considered points generated in the set X from the standard Euclidean metric and

$$\delta_-((\xi_1, \eta_1), (\xi_2, \eta_2)) = \eta_1 - \sqrt{\xi_1} + \eta_2 - \sqrt{\xi_2} + \int_{\xi_2}^{\xi_1} \sqrt{1 + \frac{1}{4\xi}} \, d\xi.$$

Note that δ_- becomes also the length metric symmetrized as $\delta_-(u, v)$ when it is defined or $\delta_-(v, u)$ otherwise. In such symmetrization it represents the metric of a non-separable $\mathbb{R}$-tree. Hence δ is the length metric on the set X and (X, δ) is a geodesic space. Both boundary lines: the positive ray γ_1 of the η-axis and the semiparabola γ_2 from the right side of X are geodesic rays starting from the origin o. The metric δ restricted to the boundary rays is reversible on each of them. We have

$$\limsup_{t \to 0} \frac{d(\gamma_1(t), \gamma_2(t))}{t} = 0$$

and

$$\limsup_{t \to 0} \frac{d(\gamma_2(t), \gamma_1(t))}{t} = 2.$$

The equivalence class of the segment $[ox\rangle$ defined as an image of the isometry γ_x will be denoted by $[\gamma_x]$. The following statement is well known for reversible metrics. In fact, the non-reversibility has no influence on its proof.

Lemma 3.3 *Let $o \in X$ be a point with directions defined independently of the asymmetry. Then the function $\overline{\angle}([\gamma_x], [\gamma_y])$ defined as*

$$\overline{\angle}([\gamma_x], [\gamma_y]) = 2 \limsup_{t \to 0} \arcsin \frac{d(\gamma_x(t), \gamma_y(t))}{2t}$$

is well defined and it is a metric on the set of directions at the point o.

The distance $\overline{\angle}$ is called the *upper isosceles angle* between the directions $[\gamma_x]$ and $[\gamma_y]$. The metric $\overline{\angle}$ is not reversible in general. The upper angle between directions can be defined in a more general way but we will not generalize the definition of upper isosceles angle here because in most situations both quantities do not work as effective tools for calculations.

The *lower isosceles angle* $\underline{\angle}$ is also well defined in a similar way, but it does not lead to a metric in general.

Example 3.4 Consider 2-dimensional Minkowski space V^2 with the Randers norm $\mathcal{F} = \alpha + \beta$ where

$$\alpha(dx^1, dx^2) = \sqrt{(dx^1)^2 + (dx^2)^2}$$

is the Euclidean norm and

$$\beta(dx^1, dx^2) = b_1 dx^1 + b_2 dx^2$$

is a linear form such that

$$\sqrt{(b_1)^2 + (b_2)^2} < 1.$$

After rotation of coordinates one can assume that for $0 < b < 1$ the norm $\mathcal{F}$ gets the form

$$\mathcal{F}(dx^1, dx^2) = \sqrt{(dx^1)^2 + (dx^2)^2} + b \cdot dx^1.$$

The unit sphere $\mathcal{S}(o, 1)$ given by

$$\mathcal{F}(x^1, x^2) = 1$$

after a transformation of its equation to the form

$$\left(\sqrt{1 - b^2} \cdot x^1 + \frac{b}{\sqrt{1 - b^2}} \right)^2 + (x^2)^2 = \frac{1}{1 - b^2}$$

can be parameterized by the angular parameter ϕ:

$$x^1 = \frac{\cos \phi - b}{1 - b^2}, \qquad x^2 = \frac{\sin \phi}{\sqrt{1 - b^2}}. \tag{5}$$

The upper isosceles angle $\overline{\angle}$ is equal to the lower isosceles angle $\underline{\angle}$ and

$$\overline{\angle}(\phi, \psi) = \underline{\angle}(\phi, \psi)$$
$$= 2\arcsin\left(\frac{1}{2}\mathcal{F}\left(\frac{\cos\psi - \cos\phi}{1 - b^2}, \frac{\sin\psi - \sin\phi}{\sqrt{1 - b^2}}\right)\right)$$
$$= 2\arcsin\left(\frac{\sqrt{(\cos\psi - \cos\phi)^2 + (\sin\psi - \sin\phi)^2(1 - b^2)}}{2(1 - b^2)} + \frac{b(\cos\psi - \cos\phi)}{2(1 - b^2)}\right)$$

in the parametrization (5). The difference of the signs in the final items in the numerator leads to the irreversibility of the metric $\underline{\angle}(\phi, \psi)$.

Speaking about the tangent cone $K_p X$ to the geodesic space (X, d) one should take into the account that its definition has the following alternative form. If X is an Alexandrov space with bounded (from above or from below) curvature, then its space of directions is a well-defined metric space whose metric is a length metric generated from the angle metric (see [4] for precise definitions and details). In this case the natural meaning of the tangent cone is the Euclidean cone over the space of directions. Such an approach does not work for geodesic spaces in general. Fortunately, there exists another meaning of the tangent cone which we use here.

From now on, all the metric spaces are topologically symmetric: the topologies τ_+ and τ_- coincide. Moreover, speaking about directions we mean that all directions are defined independently of the asymmetry.

Definition 3.5 Let (X, d) be a metric space and $A, B \subset X$ be two closed sets. The *Hausdorff distance* $\mathrm{Hd}(A, B)$ in X is

$$\mathrm{Hd}(A, B) = \inf\{\varepsilon > 0 : A \subset \mathcal{N}_\varepsilon(B) \text{ and } B \subset \mathcal{N}_\varepsilon(A)\}$$

where

$$\mathcal{N}_\varepsilon(C) = \{x \in X : d(x, y), d(y, x) < \varepsilon \text{ for some } y \in C\}$$

denotes the symmetric ε-neighborhood of the set C.

Remark 3.6 Since the neighborhood $\mathcal{N}_\varepsilon(C)$ is the intersection

$$\mathcal{N}_{+,\varepsilon} \cap \mathcal{N}_{-,\varepsilon}(C)$$

where

$$\mathcal{N}_{+,\varepsilon}(C) = \{x \in X : d(x, y) < \varepsilon \text{ for some } y \in C\}$$

is open in the topology τ_+ and

$$\mathcal{N}_{-,\varepsilon(C)} = \{x \in X : d(y, x) < \varepsilon \text{ for some } y \in C\}$$

is open in τ_-, it is an open set in $\tau = \tau_\pm$.

Remark 3.7 We allow here the value $\mathrm{Hd}(A, B) = +\infty$.

Remark 3.8 Note that the metric Hd on the family of closed subsets in X is reversible.

Definition 3.9 Let (X, d_X), (Y, d_Y) be two metric spaces. The *Gromov–Hausdorff* distance $\mathrm{GHd}(X, Y)$ between them is by definition

$$\mathrm{GHd}(X, Y) = \inf_{Z, f, g} \{\mathrm{Hd}(f(X), g(Y))\}$$

where the infimum is taken among all metric spaces (Z, d_Z) and all isometric embeddings $f : X \to Z$ and $g : Y \to Z$.

Definition 3.10 Let (X_t, d_t) be a one-parameter family of compact metric spaces. We say that it *converges* when $t \to t_0$ *in the sense of Gromov–Hausdorff* to a compact space (X, d), if $\mathrm{GHd}(X, X_t) \to 0$. The value t_0 is a real number or $\pm\infty$ here.

The version of Definition 3.10 in the non-compact proper case deals with pointed spaces.

Definition 3.11 Let (X_t, d_t, o_t) be a one-parameter family of proper metric spaces with marked points o_t correspondingly. We say that the space (X, d, o) is a *Gromov–Hausdorff limit* of this family when $t \to t_0$ if for every $r > 0$ the ball $B_+(o, r)$ is a Gromov–Hausdorff limit of d_t-balls $B_+(o_t, r)$.

In order to estimate the distances d_{GH} it is convenient to use [4, Theorem 7.3.25].

Definition 3.12 ([4, Definition 7.3.17]) A *correspondence* φ between sets A and B is by definition a subset of the Cartesian product $A \times B$ such that for every $a \in A$ there exists $b \in B$ for which $(a, b) \in \varphi$, and conversely, for every $b \in B$ there exists $a \in A$ for which $(a, b) \in \varphi$.

Theorem 3.13 *For any two metric spaces X and Y*

$$\mathrm{GHd}(X, Y) = \inf_{\varphi} \mathrm{dis}(\varphi)$$

where

$$\mathrm{dis}(\varphi) = \sup\big\{|d_X(x, x') - d_Y(y, y')| : (x, y), (x', y') \in \varphi\big\}$$

is the distortion of the correspondence φ between X and Y and the infimum is taken among all such correspondences.

The proof of this theorem for the spaces with non-reversible metrics does not differ from the symmetric case presented in [4].

Definition 3.14 Let (X, d) be a proper geodesic space, $p \in X$ an arbitrary point. Consider a family of spaces (X, d_t), $t > 0$, where $d_t = t \cdot d$. The space $K_p X = (X_*, d_*)$ is called the *tangent cone* to (X, d) at the point p if the pointed space (X_*, d_*, p_*) with some marked point $p_* \in K_p X$ is a Gromov–Hausdorff limit of the family (X, d_t, p) when $t \to +\infty$. The point p_* is a *vertex* of the cone $K_p X$. It is convenient to identify p_* with p and to consider p as a vertex of $K_p X$.

The following natural example and the consideration after it partially explain the motivation of the paper.

Example 3.15 Let X be a smooth surface in the affine space A^n equipped with a strictly convex norm. We do not suppose the norm to be reversible, smooth or strongly convex. By its own smoothness, the surface X has a well-defined family of smooth curves and a tangent space T_pX at every point p. This space inherits the norm from the ambient space A^n. The length $L(\gamma)$ of a smooth curve γ in X calculated by the formula (2) generates a length metric d on X. The topologies τ_+ and τ_- with respect to the metric d coincide, the space (X, d) is proper in these topologies and it is a geodesic space. It is obvious that the metric space (T_pX, d_p) is a tangent cone K_pX to the space (X, d) with vertex p, where the metric d_p in T_pX is induced from A^n.

Definition 3.16 We say that the family d_t, $t \in [a, +\infty)$, of metrics on the set A *uniformly converges* to the function $d_*\colon A \times A \to \mathbb{R}$ if for any $\varepsilon > 0$ there exists $T > 0$ such that

$$|d_t(x, y) - d_*(x, y)| < \varepsilon \tag{6}$$

for all $x, y \in A$ and all $t > T$. The function d_* is automatically non-negative and satisfies the triangle inequality

$$d_*(x, y) \leqslant d_*(x, z) + d_*(z, y).$$

It is called a *pseudometric* if the equality $d_*(x, y) = 0$ implies $d_*(y, x) = 0$. Analogously to the symmetric case, the pseudometric d_* generates a metric space on the set A/θ where θ is an equivalence relation on A such that $a\theta b$ if and only if $d_*(a, b) = 0$.

Definition 3.17 We say that the metric space (X, d) satisfies the *property of segments prolongability* at the point $p \in X$ if there exists $r > 0$ such that every point $x \in U_+(p, r)$, $x \neq p$, is an interior point of a directed segment $[py)$ with $d(p, y) = r$. If such r is detected, we say that (X, d) satisfies the *r-prolongability* of segments at p.

Let (X, d) be a proper geodesic space with $\tau_+ = \tau_-$ and with the property of local uniqueness of directed segments: for every point $p \in X$ there exists a ball $B_+(p, r)$ where any ordered pair of points $x, y \in B_+(p, r)$ defines a unique directed segment $[xy)$. In particular, for every $x \in B_+(p, r)$ the directed segment $[px)$ is defined uniquely. For $x \in B_+(p, r)$ and $t \geqslant 1$, denote by x_t the point of the segment $[px)$ such that

$$d(p, x) = t \cdot d(p, x_t).$$

Define the family of metrics $d_t(x, y) = t \cdot d(x_t, y_t)$ on the ball $B_+(p, r)$.

Lemma 3.18 *Let the space (X, d) satisfy the property of segments r-prolongability at $p \in X$. Assume that the metrics d_t are well defined on $B_+(p, r)$ and the family d_t converges uniformly to a pseudometric*

$$d_*(x, y) = \lim_{t \to +\infty} d_t(x, y).$$

Then the one-parameter family of pointed spaces $(X, t \cdot d, p)$ has a limit in the sense of Gromov–Hausdorff when $t \to +\infty$.

Proof First, let us define a metric space (Y, d_Y) as follows. It follows from the condition, that the function d_* is a pseudometric on the set $B_+(p, r)$. We will denote by B_* the metric space generated from the pseudometric space $(B_+(p, r), d_*)$ as a factor-space by the equivalence relation θ as in Definition 3.16. Consider the family of spaces

$$t \cdot B_* = (B(p, r)/\theta, t \cdot d_*),$$

$t > 0$. It easily follows from the definition of the metrics d_t that every space $s \cdot B_*$ is isometric to the ball $B_+(p, sr)$ in $t \cdot B_*$ when $s \leqslant t$. Consequently, there is a well-defined inductive limit

$$Y = \varinjlim (t \cdot B_*).$$

We will denote its metric by d_Y. Note that the segments r-prolongability at p implies that the ball B_* is foliated by directed segments $[py\rangle$ with $d_*(p, y) = r$. It follows that Y consists of the rays which are infinite continuations of these segments and the r-ball $B_+(p, r)$ in Y coincides with B_*. We claim that the pointed space (Y, d_Y, p) is exactly the Gromov–Hausdorff limit of the family $(X, t \cdot d, p)$.

Note that there is a similarity of balls: the ball $B_+(p, t \cdot r)$ of the space $(X, t \cdot d)$ is similar with the coefficient t to the ball $B_+(p, r)$ in (X, d). Consequently, to prove $(X, t \cdot d, p) \xrightarrow{GH} (Y, d_Y, p)$, it is sufficient to prove the convergence of r-balls $(X, t \cdot d) \supset B_+(p, r) \xrightarrow{GH} B_*$.

Given any $\varepsilon > 0$ denote $T(\varepsilon)$ a number such such that inequality (6) holds for any $t \geqslant T(\varepsilon)$ and all pairs $x, y \in B(p, r)$.

The composition of pseudometric projection $\mathrm{pr} \colon (B_+(p, r), d_*) \to (B_*, d_*)$ with the identity map $\mathrm{Id} \colon (B_+(p, r), d_t) \to (B_+(p, r), d_*)$ has the distortion estimated by

$$\mathrm{dis}(\mathrm{pr} \circ \mathrm{Id}) \leqslant \varepsilon$$

for every $t \geqslant T(\varepsilon)$. Consequently, it follows from Theorem 3.13 that $(B_+(p, r), d_t) \xrightarrow{GH} B_*$.

Finally, it is clear that

$$\mathrm{Hd}(B_*, B_+(p, r)) = \varepsilon$$

and consequently,

$$\mathrm{GHd}((B_*, d_*), (B_+(p, r), d_t)) \leqslant \varepsilon.$$

This proves the lemma. $\qquad\square$

The following example demonstrates the essentiality of the condition of segments prolongability at p.

Example 3.19 Consider a subset A of the plane with coordinates (ρ, ϕ) which we assume to be Cartesian, bounded by the lines $\phi \geqslant 0$, $\rho \geqslant 0$ and $\rho \leqslant 1/(\phi + 1)$. Define a pseudometric d_A on A by

$$d_A((\rho_1, \phi_1), (\rho_2, \phi_2)) = \begin{cases} \sqrt{\rho_1^2 + \rho_2^2 - 2\rho_1\rho_2 \cos(\phi_2 - \phi_1)} & \text{if } |\phi_2 - \phi_1| \leqslant \pi, \\ \rho_1 + \rho_2 & \text{otherwise} \end{cases}$$

and the metric space $(A/\theta, d_A)$. Here θ is a pseudometric equivalence with the unique non-trivial class defined by the equality $\rho = 0$.

Geometrically, the space A is isometric to a subspace of the Riemannian surface of branching covering over the Euclidean plane with one singular point in the origin. Its boundary projects to the union of the spiral $\rho = 1/(\phi + 1)$ in the polar coordinates with the segment $0 \leqslant \rho \leqslant 1$ of the axis $\phi = 0$.

All the spaces (A, d_t) are isometric to each other, but the spaces $(A, t \cdot d)$ do not converge in the sense of Gromov–Hausdorff because the family of balls $B(p, r)$ in these spaces has no Gromov–Hausdorff limit.

4 Definition and examples

Let (X, d) be a metric space such that

- X is a geodesic space;
- the topologies τ_+ and τ_- coincide;
- the space (X, τ) is a topological manifold;
- for any point $p \in X$ the space of directions is defined independently of the asymmetry, the tangent cone $K_p X$ is well defied and it has the structure of a normed linear space with strictly convex norm.

Then the space X is called a *singular Finsler space*.

Example 4.1 Let (X, d) be a Busemann G-space with non-positive curvature. It is a symmetric geodesic space by definition. Then it is proved in [1] that X is a topological manifold and the tangent cone $K_p X$ is well defined at every point $p \in X$. Moreover, it is shown in [2] that $K_p X$ has the structure of a normed space with strictly convex norm. It follows that every Busemann G-space with non-positive curvature is a singular Finsler space.

Example 4.2 The Funk metric (4) leads to a regular Finsler space only if the boundary $\partial\Omega$ is smooth and strongly convex. The Funk space defined with strictly convex boundary surface $\partial\Omega$ is a singular Finsler space which can be non-smooth.

The following example should be viewed as a counterexample because not every quasihyperbolic plane satisfies the definition of singular Finsler space.

Example 4.3 Quasihyperbolic planes which were studied in [5] and [8] are deeply connected with singular Finsler spaces: they lead to a new understanding of the definition of singular Finsler spaces and to some problems. First, we recall the definition and properties of a quasihyperbolic plane.

Remark 4.4 One should not confuse this notion of quasihyperbolic plane with another object introduced by B.A. Rozenfeld also known as a quasi-hyperbolic space.

Let

$$X = \mathbb{R} \times \mathbb{R}_+ = \{(x, y) : y > 0\}$$

be the upper Euclidean half-plane and Γ a group which acts on X by the transformations $(x, y) \mapsto (\alpha x + \beta, \alpha y)$, where $\alpha > 0$, $\beta \in \mathbb{R}$. The group Γ is generated by positive homotheties $(x, y) \to (\alpha x, \alpha y)$ and x-parallel translations $(x, y) \to (x + \beta, y)$.

The metric space (X, d) with a Γ-invariant metric d is called a *quasihyperbolic plane* if it is a Busemann G-space.

The following theorem of Gribanova gives a characterization of quasihyperbolic planes in terms of the linear element. The overall formulation is taken from [3].

Theorem 4.5 *Let (X, d) be a quasihyperbolic plane. Then the linear element in X has the form*

$$ds = \frac{F(dx, dy)}{y}, \tag{7}$$

where the function $F(u, v)$ satisfies the conditions

- $F(u, v) > 0$ *when* $(u, v) \neq (0, 0)$;
- $F(ku, kv) = k \cdot F(u, v)$ *for all* $k \geqslant 0$;
- F *is convex;*
- F *is smooth in all points* (u, v) *except* $(0, 0)$;
- *tangents for the curve* $F(u, v) = 1$ *parallel to the line* $v = 0$ *touch this curve in a unique point.*

Conversely, each line element of the form (7) *with F satisfying these properties defines a quasihyperbolic plane. The geodesics of the space (X, d) are the intersections of X with*

- *the curves* $F^*(x - a, y) = k$, $k > 0$, $a \in \mathbb{R}$, *where*

$$F^*(x, y) = \max_{F(u,v) \leqslant 1} (xv - yu)$$

  is a function dual to F in coordinates turned over $\pi/2$, and
- *the tangents to these curves in points of the x-axis.*

For two distinct points there exists exactly one geodesic passing through them.

In some sense, the above example presents the class of metrics which is dual to the singular Finsler metrics admitting the action of Γ on X. In general, a quasihyperbolic plane is not necessarily a singular Finsler space and vice-versa. The stadium space

described in [3] is a counterexample: the norm in its tangent space is not strictly convex. Conversely, it follows from the smoothness condition that if the linear element of the space X is strictly convex but not smooth, it is a singular Finsler space but not a quasihyperbolic plane. In that case it admits branching of geodesics.

More examples of singular Finsler spaces arise as smooth submanifolds of a strictly convex normed space as in Example 3.15.

5 Curves with velocity

Let (X, d) be a singular Finsler space, $c: [a, b] \to X$ a parameterization of a rectifiable curve $\gamma \subset X$, $a < s_0 < b$ and $p = c(s_0)$. Let $t > 0$ and $\omega_t: [0, 1] \to X$ be a parameterization of the directed segment from p to $c(s_0 + t)$ proportional to the natural parameterization. It defines a vector $\mathbf{w}_t$ in the tangent space $T_p X$ with length $\|\mathbf{w}_t\| = d(p, c(s_0 + t))$. The vector

$$\mathbf{v}_t = \frac{1}{t} \mathbf{w}_t$$

is called the *secant-directing* vector of γ at the point p corresponding to the parameter t. If the limit

$$\mathbf{v}_+ = \lim_{t \to +0} \mathbf{v}_t \in T_p X \tag{8}$$

exists, then we say that the curve γ *has an upcoming velocity in the positive direction* at the point p. The vector $\mathbf{v}_+$ represents this upcoming velocity. Similarly, for $t < 0$ the vector $\mathbf{v}_t$ is defined by

$$\mathbf{v}_t = \frac{d(c(s_0 + t), p)}{|t|} \mathbf{w}_t$$

where $\mathbf{w}_t \in T_p X$ is the unit vector in the direction of the continuation of the directed segment $[c(s_0 + t)p\rangle$ after the point p, and

$$\mathbf{v}_- = \lim_{t \to -0} \mathbf{v}_t \in T_p X \tag{9}$$

represents the *coming velocity from the negative direction* at the point p if it exists.

If the vectors $\mathbf{v}_+$ and $\mathbf{v}_-$ coincide, we say that γ *has velocity* presented by the *velocity vector* $\mathbf{v} = \mathbf{v}_+ = \mathbf{v}_-$ at p. The proof of the following theorem is standard but we review it here to avoid questions.

Theorem 5.1 *Let γ be a curve in a singular Finsler space (X, d) represented by a parameterization $c: [a, b] \to X$ such that the velocity vector $v(s)$ exists for all parameters s in $[a, b]$ and such that the function $\|v(s)\|$ is continuous. Then γ is positively rectifiable with length*

$$L_+(\gamma) = \int_a^b \|v(s)\| \, ds. \tag{10}$$

Proof First, we show that the curve γ is positively rectifiable, that is, the length variations (3) are bounded from above. Denote by

$$M = \max_{s \in [a,b]} \|\mathbf{v}(s)\|$$

the maximal value of the norm of the vector $\mathbf{v}$, which is reached at some $t_M \in [a, b]$ since $\|\mathbf{v}(s)\|$ is continuous. It follows from the definition of the vector $\mathbf{v}(s)$ as a common limit in (8) and (9) that for any $s \in [a, b]$ there exists $\delta(s) > 0$ such that if $0 < t < \delta(s)$ then

$$d(c(s), c(s + t)) < 2M \cdot t$$

and

$$d(c(s - t), c(s)) < 2M \cdot t.$$

Consequently, applying the triangle inequality, we have

$$d(c(\sigma), c(\tau)) < 4M \cdot (\tau - \sigma)$$

for any $\sigma, \tau \in (s - \delta(s), s + \delta(s))$ where $\sigma < \tau$.

Picking a finite subcovering

$$[a, b] \subset \bigcup_{i=1}^{n} (s_j - \delta(s_j), s_j + \delta(s_j)),$$

we find the estimate

$$V_+(\gamma, T) \leqslant 4M \cdot (b - a)$$

of the length variation of γ for the subdivision T of the segment $[a, b]$ such that $t_i, t_{i+1} \in (s_j - \delta(s_j), s + \delta(s_i))$ for any two neighbour elements t_i, t_{i+1} of T and some j depending on i. An arbitrary subdivision T can be accomplished by additional points to get the same estimate of the variation $V_+(\gamma, T)$ (adding points can only enlarge the value of $V_+(\gamma, T)$). Hence, the set of positive length variations is bounded and γ is positively rectifiable.

Denote by $L_+(\gamma)$ the positive length of γ. Next we show that it is exactly of the form (10). First, for any $\varepsilon > 0$ there exists $\delta_1 > 0$ such that for any subdivision T of $[a, b]$ with diameter $\Delta < \delta_1$ the estimate holds

$$L_+(\gamma) - V_+(\gamma, T) < \frac{\varepsilon}{3}.$$

Next, consider the function $f(s, \tau)$ defined for $s, s + \tau \in [a, b]$ by

$$f(s, \tau) = \begin{cases} \dfrac{d(c(s), c(s + \tau))}{\tau} & \text{if } \tau \neq 0, \\[2ex] \lim\limits_{t \to 0} \dfrac{d(c(s), c(s + t))}{t} & \text{if } \tau = 0. \end{cases}$$

It is continuous by the condition of Theorem 5.1 and by the definition of $\mathbf{v}$. Hence it is uniformly continuous and there exists δ_2 such that if $|\tau| < \delta_2$ then

$$\left| f(c(s), c(s + \tau)) - \|\mathbf{v}(s)\| \right| < \frac{\varepsilon}{3(b - a)}$$

or

$$\left| d(c(s), c(s + \tau)) - \|\mathbf{v}(s)\| \cdot \tau \right| < \tau \frac{\varepsilon}{3(b - a)} \tag{11}$$

for all $s \in [a, b]$. For a subdivision T of $[a, b]$ presented by numbers $a = s_0 < s_1 < \cdots < s_{n-1} < s_n$, if the diameter $\Delta < \delta_2$, the summing inequalities (11) by $i = 0, n - 1$ taking $\tau_i = \Delta s_i = s_{i+1} - s_i$ gives

$$\left| V_+(\gamma, T) - \sum_{i=0}^{n-1} \|\mathbf{v}(s_i)\| \cdot \Delta s_i \right| < \frac{\varepsilon}{3}.$$

Note that the second item under the modulus here is in fact the integral sum for the integral $\int_a^b \|\mathbf{v}(t)\| \, dt$. The integral exists since the norm $\|\mathbf{v}(t)\|$ is continuous. Finally, there exists $\delta_3 > 0$ such that

$$\left| \sum_{i=0}^{n-1} \|\mathbf{v}(s_i)\| \cdot \Delta s_i - \int_a^b \|\mathbf{v}(t)\| \, dt \right| < \frac{\varepsilon}{3}$$

for any subdivision T with diameter $\Delta(T) < \delta_3$. Taking $\delta = \min\{\delta_1, \delta_2, \delta_3\}$ and an arbitrary subdivision T with $\Delta(T) < \delta$ we obtain that

$$\left| L_+(\gamma) - \int_a^b \|\mathbf{v}\| \, dt \right| < \varepsilon$$

proving the theorem because of arbitrariness of $\varepsilon > 0$. □

6 Some questions and problems

We conclude by a short list of questions which are close to the theory developed or may have independent interest.

First, since the singular Finsler geometry is an extension of smooth Finsler geometry, it is natural to ask about the circle of its objects: what is kept in such an extension.

Question 6.1 What objects of the smooth Finsler geometry admit a description without differentiation and hence belong to the singular extension of Finsler geometry?

The first pretenders for singular Finsler geometry are standard measures: different versions of angular measure, areas and volumes. One can speak about different versions of perpendicularity, of course, without symmetry. Note that Busemann studied these values in the symmetric case of G-spaces. The class of singular Finsler spaces has some differences from the class of Busemann G-spaces.

More problematic are Finsler linear connections and corresponding parallelisms. It is hard to imagine what can be an analogue of the Levi-Civita connection. One can try to approximate singular Finsler norms by smooth ones and use some limiting procedure. As a consequence, the following question arises.

Question 6.2 How do the limiting versions of differential quantities depend on the choice of approximation and limiting procedures?

Of course, the version of the extension of Finsler geometry to the singular case described here is not unique. But the question is

Question 6.3 What other manners can be used to extend smooth Finsler geometry to the non-smooth or to the non-strongly convex situation?

Recall that this paper was inspired by the theorem proved in [2]: the tangent cone for a non-positively curved G-space has a structure of normed space with strictly convex norm. The theorem is proved for the reversible case. The following series of problems arises.

Problem 6.4 Let X be a non-positively curved Busemann G-space with non-reversible metric. This means that all the axioms of G-space hold except that the metric symmetry and the axiom of curvature non-positivity holds for the set of directed segments.

(a) Is it true that the topologies τ_+ and τ_- coincide?
(b) Are the spaces of directions defined independently on the asymmetry?
(c) Is the pseudometric d_* from Lemma 3.18 well defined and is it a metric?
(d) Is the tangent cone $K_p X$ homeomorphic to X and is it a normed space?

The following problem results: is the space X a singular Finsler space?

Some questions arise from the consideration of quasihyperbolic spaces. The known researches deal with quasihyperbolic planes, that is, with reversible two-dimensional quasihyperbolic spaces. Going to high dimensions and breaking the symmetry can generate extra obstructions for the description of quasihyperbolic geometry. The obvious is the fact that a quasihyperbolic space is a singular Finsler space only if its norm is strictly convex. The general problem is the following.

Problem 6.5 Let $X = \mathbb{R}^n \times \mathbb{R}_+$ be the $(n+1)$-dimensional open half-space. Consider the metrics d on X generated by the linear elements of type

$$ds = \frac{F(d\mathbf{x}, dy)}{y}$$

where the function $F(\mathbf{u}, v)$ is convex and positively homogeneous: $F(\lambda \mathbf{u}, \lambda v) = \lambda F(\mathbf{u}, v)$ for all pairs $(\mathbf{u}, v) \in \mathbb{R}^n \times \mathbb{R}_+$ and all $\lambda > 0$.

(a) Find geodesics of the metric d in the general case.
(b) Is there any kind of duality between singular Finsler metrics and quasihyperbolic metrics?

Note that these questions are unanswered for the non-reversible metrics even in the two-dimensional case.

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