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RESEARCH ARTICLE

Group actions on categories and Elagin's theorem revisited

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Abstract After recalling basic definitions and constructions for a finite group G action on a k-linear category we give a concise proof of the following theorem of Elagin: if $\mathcal{C} = \langle \mathcal{A}, \mathcal{B} \rangle$ is a semiorthogonal decomposition of a triangulated category which is preserved by the action of G, and $\mathcal{C}^G$ is triangulated, then there is a semiorthogonal decomposition $\mathcal{C}^G = \langle \mathcal{A}^G, \mathcal{B}^G \rangle$. We also prove that any G-action on $\mathcal{C}$ is weakly equivalent to a strict G-action which is the analog of the Coherence theorem for monoidal categories.

Keywords Group actions on categories · Derived categories of coherent sheaves · Elagin's theorem

Mathematics Subject Classification 14L30 · 18E30

1 Introduction

1.1 The setting of finite groups acting on categories is a well-studied ground, see e.g. [2–5,9] and references therein. A useful way to define the action is to require for every $g \in G$ an autoequivalence $\rho_g : \mathcal{C} \to \mathcal{C}$ together with a choice of isomorphisms $\rho_g \rho_h \simeq \rho_{gh}$ satisfying a cocycle condition, see 2.1. One would then study the category of equivariant objects $\mathcal{C}^G$, see 2.4.

1.2 The main goal of this paper is to give a direct proof of Elagin's theorem [3,4] stating that if $\mathcal{C} = \langle \mathcal{A}, \mathcal{B} \rangle$ is a semi-orthogonal decomposition of triangulated categories and

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G is a finite group acting on $\mathcal{C}$ by triangulated autoequivalences in such a way that the category of equivariant objects $\mathcal{C}^G$ is triangulated and preserving $\mathcal{A}$ and $\mathcal{B}$, then there is a semi-orthogonal decomposition $\mathcal{C}^G = \langle \mathcal{A}^G, \mathcal{B}^G \rangle$, see Theorem 6.2.

1.3 In our proof we construct the functors $\mathcal{C}^G \to \mathcal{A}^G$ and $\mathcal{C}^G \to \mathcal{A}^G$ adjoint to the inclusion functors. The key step in the proof is to show that if $\Phi \colon \mathcal{A} \to \mathcal{C}$ is a G-equivariant functor which admits a left or right adjoint functor Ψ, then Ψ is automatically equivariant: see Proposition 3.9.

1.4 We also prove that every G-action is G-weakly equivalent to a strict G-action, that is to an action satisfying $\rho_g \rho_h = \rho_{gh}$, see Theorem 5.4. This is analogous to the Coherence Theorem for monoidal categories: every monoidal category is equivalent to a strict monoidal category, see e.g. [7, 1.2.15].

1.5 In order to formulate and prove these facts we need to develop the language of G-functors, G-natural transformations and so on. Perhaps relevant definitions and constructions are well known to experts but we include these for completeness as we could not find the reference that fits our purpose.

1.6 All categories, functors, etc are k-linear where $\mathrm{char}(k) = 0$. Groups acting on categories are finite and we denote by $1 \in G$ the neutral element of the group.

We use the symbol "$\circ$" to denote vertical composition of natural transformations of functors, the other types of compositions are denoted by concatenation.

2 *G*-categories and equivariant objects

2.1 Definition By a *G-action on* $\mathcal{C}$ we mean the following data [4, Definition 3.1]:

- For each element $g \in G$ an autoequivalence $\rho_g \colon \mathcal{C} \to \mathcal{C}$.
- For each pair $g, h \in G$ an isomorphism of functors

$$\phi_{g,h} \colon \rho_g \rho_h \cong \rho_{gh}.$$

The data must satisfy the following associativity axiom: for all $g, h, k \in G$ the diagram of functors $\mathcal{C} \to \mathcal{C}$ is commutative:

$$\begin{array}{ccc} \rho_g \rho_h \rho_k & \xrightarrow{\rho_g \phi_{h,k}} & \rho_g \rho_{hk} \\ \downarrow{\scriptstyle \phi_{g,h}\rho_k} & & \downarrow{\scriptstyle \phi_{g,hk}} \\ \rho_{gh} \rho_k & \xrightarrow{\phi_{gh,k}} & \rho_{ghk}. \end{array}$$

2.2 It follows from the definition that there is an isomorphism of functors

$$\phi_1 \colon \rho_1 \simeq \mathrm{id}$$

obtained by post-composing $\phi_{1,1}\colon \rho_1\rho_1 \to \rho_1$ with ρ_1^{-1}. That is we have

$$\phi_{1,1} = \rho_1\phi_1.$$

Furthermore one can show that ϕ_1 satisfies [5, 2.1.1 (e)]:

$$\phi_{g,1} = \rho_g\phi_1\colon \rho_g\rho_1 \to \rho_g, \qquad \phi_{1,g} = \phi_1\rho_g\colon \rho_1\rho_g \to \rho_g$$

so that the definition 2.1 coincides with that of [5, 2.1].

On the other hand if one asks for ϕ_1 to be the identity transformation, one gets a slightly stronger definition of a G-descent datum of [8, Definition 1.1].

2.3 Using the language of monoidal functors [7, Definition 1.2.10], one can give a very concise definition of a group acting on a category. For that consider G as a monoidal category: G is discrete as a category and its monoidal structure is defined by

$$g\otimes h = gh, \qquad \mathrm{id}_g\otimes \mathrm{id}_h = \mathrm{id}_{gh}.$$

Now a G-action on $\mathcal{C}$ amounts to the same thing as an action of monoidal category G on $\mathcal{C}$ [7, Example 1.2.12], i.e. a weak monoidal functor

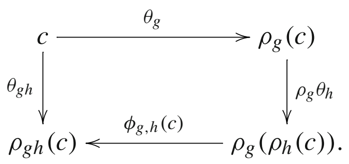
$$\rho\colon G \to [\mathcal{C}, \mathcal{C}]$$

where on the right is the category of functors $\mathcal{C} \to \mathcal{C}$ with monoidal structure given by composing functors.

2.4 Definition One defines the *category of G equivariant objects* $\mathcal{C}^G$ [4,5] as follows: objects of $\mathcal{C}^G$ are linearized objects, i.e. objects $c \in \mathcal{C}$ equipped with isomorphisms

$$\theta_g\colon c \to \rho_g(c), \qquad g \in G,$$

satisfying the condition that the following diagrams are commutative:



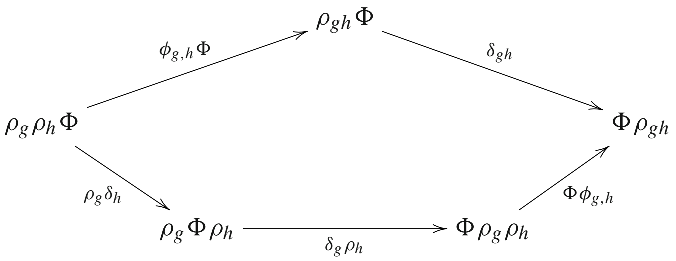
Morphisms of equivariant objects consist of those morphisms of the underlying objects in $\mathcal{C}$ which commute with all θ_g, $g \in G$.

3 *G*-functors and *G*-natural transformations

3.1 Definition Given two categories $\mathcal{C}$, $\mathcal{D}$ with G-actions and a functor $\Phi : \mathcal{C} \to \mathcal{D}$, Φ is called a *right lax G-functor* if there are given natural transformations

$$\delta_g : \rho_g \Phi \to \Phi \rho_g$$

such that the two natural transformations $\rho_g \rho_h \Phi \to \Phi \rho_{gh}$ coincide:



This commutative diagram is called the *pentagon axiom*. Similarly Φ is called a *left lax G-functor* if there are given natural transformations

$$\delta_g : \Phi \rho_g \to \rho_g \Phi$$

satisfying the dual pentagon axiom. A right (or left) lax G-functor Φ is called a *weak G-functor* if all δ_g are isomorphisms.

The following lemma is a useful criterion for a weak G-functor.

3.2 Lemma *Let Φ be a right (or left) lax G-functor. The following conditions are equivalent*:

(i) *The natural transformation $\delta_1 : \rho_1 \Phi \to \Phi \rho_1$ is an isomorphism.*

(ii) Φ *satisfies the identity element axiom*:

$$\Phi\phi_1 \circ \delta_1 = \phi_1 \Phi : \rho_1 \Phi \to \Phi.$$

(iii) Φ *is a weak G-functor.*

Proof Implications (iii) ⇒ (i), (ii) ⇒ (i) are obvious. Let us prove that (i) ⇒ (iii). Consider the case of the right lax G-functor. Applying the pentagon axiom to the pair (g^{-1}, g) gives

$$\delta_{g^{-1}} \rho_g \circ \rho_{g^{-1}} \delta_g = \Phi \phi^{-1}_{g^{-1},g} \circ \delta_1 \circ \phi_{g^{-1},g} \Phi.$$

Since the natural transformation on the right-hand side is an isomorphism (note that δ_1 is an isomorphism by the identity element axiom) and ρ_g, $\rho_{g^{-1}}$ are equivalences, it follows that $\delta_{g^{-1}}$ is left invertible and δ_g is right invertible. Thus we see that all δ_g are isomorphisms.

Now we prove (i) ⇒ (ii). Consider the natural transformation

$$\varepsilon = \Phi\phi_1 \circ \delta_1 \circ \phi_1^{-1}\Phi \colon \Phi \to \Phi.$$

We are given that ε is an isomorphism and we need to prove that ε is in fact an identity.

We use Lemma 3.3 applied to the trivial group $H = \{1\}$ and the composition

$$(\mathcal{C}, \mathrm{id}) \xrightarrow{(\mathrm{id},\, \phi_1)} (\mathcal{C}, \rho_1) \xrightarrow{(\Phi,\, \delta_1)} (\mathcal{D}, \rho_1) \xrightarrow{(\mathrm{id},\, \phi_1^{-1})} (\mathcal{D}, \mathrm{id})$$

which gives a lax G-functor $(\mathcal{C}, \mathrm{id}) \xrightarrow{(\Phi,\varepsilon)} (\mathcal{D}, \mathrm{id})$. The pentagon axiom for this functor yields

$$\varepsilon^2 = \varepsilon$$

and we deduce that $\varepsilon = \mathrm{id}$. □

3.3 Lemma *If* $(\Phi, \delta^{\Phi}) \colon \mathcal{C} \to \mathcal{D}$, $(\Psi, \delta^{\Psi}) \colon \mathcal{D} \to \mathcal{E}$ *are right/left/weak G-functors, then their composition* $(\Psi\Phi, \Phi\delta^{\Psi} \circ \delta^{\Phi}\Psi)$ *is a right/left/weak G-functor.*

For the proof one needs to check that the composition satisfies the pentagon and/or the identity element axioms; this is a straightforward check.

3.4 Lemma *A weak G-functor $\Phi \colon \mathcal{C} \to \mathcal{D}$ induces a functor on the categories of equivariant objects $\Phi^G \colon \mathcal{C}^G \to \mathcal{D}^G$ such that the following diagram is commutative*:

$$\begin{array}{ccc} \mathcal{C}^G & \xrightarrow{\Phi^G} & \mathcal{D}^G \\ \downarrow & & \downarrow \\ \mathcal{C} & \xrightarrow{\Phi} & \mathcal{D}. \end{array}$$

Proof For $(c, \theta) \in \mathcal{C}^G$ we define linearization on $\Phi(c)$ as a composition of isomorphisms

$$\Phi(c) \to \Phi\rho_g(c) \to \rho_g\Phi(c)$$

of $\Phi\theta_g$ with δ_g. It is now a standard check that $\Phi(c)$ becomes an equivariant object and that Φ^G is a functor. □

3.5 Definition A natural transformation between two weak G-functors $\mu \colon \Phi_1 \to \Phi_2 \colon \mathcal{C} \to \mathcal{D}$ is called a *G-natural transformation* if for every $g \in G$ the following diagram commutes:

$$\begin{array}{ccc} \rho_g\Phi_1 & \xrightarrow{\rho_g\mu} & \rho_g\Phi_2 \\ {\scriptstyle \delta_{1,g}}\downarrow & & \downarrow{\scriptstyle \delta_{2,g}} \\ \Phi_1\rho_g & \xrightarrow{\mu\rho_g} & \Phi_2\rho_g. \end{array}$$

3.6 Lemma *A G-natural transformation μ between two weak G-functors $\Phi_1, \Phi_2 \colon \mathcal{C} \to \mathcal{D}$ induces a natural transformation $\mu^G \colon \Phi_1^G \to \Phi_2^G$.*

Proof To prove that μ descends to a natural transformation $\mu^G \colon \Phi_1^G \to \Phi_2^G$ we check that for every $(c, \theta) \in \mathcal{C}^G$ the morphism $\mu \colon \Phi_1(c) \to \Phi_2(c)$ commutes with linearizations:

$$\begin{array}{ccc}
\rho_g \Phi_1(c) & \xrightarrow{\rho_g \mu(c)} & \rho_g \Phi_2(c) \\
{\scriptstyle \delta_1} \downarrow \simeq & & {\scriptstyle \delta_2} \downarrow \simeq \\
\Phi_1 \rho_g(c) & \xrightarrow{\mu \rho_g(c)} & \Phi_2 \rho_g(c) \\
{\scriptstyle \Phi_1 \theta_g} \uparrow & & \uparrow {\scriptstyle \Phi_2 \theta_g} \\
\Phi_1(c) & \xrightarrow{\mu(c)} & \Phi_2(c).
\end{array}$$

The transformation μ^G is natural since the original transformation μ is natural and the forgetful functor $\mathcal{C}^G \to \mathcal{C}$ is faithful. □

3.7 Definition Two weak G-functors $\Phi \colon \mathcal{C} \to \mathcal{D}$, $\Psi \colon \mathcal{D} \to \mathcal{C}$ are called *G-adjoint* if they are adjoint and the unit $\varepsilon \colon \mathrm{id} \to \Phi\Psi$ and counit $\eta \colon \Psi\Phi \to \mathrm{id}$ of the adjunction are G-natural transformations.

3.8 Lemma *A G-adjoint pair of functors Φ, Ψ induces an adjoint pair Φ^G, Ψ^G between the categories of equivariant objects.*

Proof From 3.6 it follows that we have natural transformations $\varepsilon^G \colon \mathrm{id} \to \Phi^G \Psi^G$, $\eta^G \colon \Phi^G \Psi^G \to \mathrm{id}$. The condition for Ψ and Φ to be adjoint is that two compositions

$$\Phi\eta \circ \varepsilon\Phi \colon \Phi \to \Phi\Psi\Phi \to \Phi$$

and

$$\eta\Phi \circ \Psi\varepsilon \colon \Psi \to \Psi\Phi\Psi \to \Psi$$

are identities. Since the forgetful functor $\mathcal{C}^G \to \mathcal{C}$ is faithful, the same holds for Φ^G, Ψ^G. □

3.9 Proposition *A left or right adjoint Ψ to a weak G-functor Φ can be made into a weak G-functor in such a way that* Ψ and Φ *become G-adjoint.*

Proof Let Ψ be the left adjoint to $\Phi \colon \mathcal{C} \to \mathcal{D}$. We construct the structure of a left lax G-functor on Ψ using the structure of a right lax G-functor on Φ.

Let $\varepsilon \colon \mathrm{id} \to \Phi\Psi$ and $\eta \colon \Psi\Phi \to \mathrm{id}$ be the unit and the counit of the adjunction. Given a right lax G-structure $\delta_g \colon \rho_g \Phi \to \Phi \rho_g$ on Φ, we define the left lax G-structure $\delta'_g \colon \Psi \rho_g \to \rho_g \Psi$ on Ψ as a mate of δ_g with respect to the adjunction [6, Proposition 2.1], [7, pp. 185–186], i.e.

$$\delta'_g = \eta \rho_g \Psi \circ \Psi \delta_g \Psi \circ \Psi \rho_g \varepsilon \colon \Psi \rho_g \to \Psi \rho_g \Phi \Psi \to \Psi \Phi \rho_g \Psi \to \rho_g \Psi.$$

The pentagon axiom can be expressed as an equality of certain compositions in the double category of [6, p. 86], hence is preserved under taking mates by [6, Proposition 2.2]. Checking the identity axiom for δ_1' is straightforward.

Now by 3.2, Ψ becomes a weak G-functor. The proof for right adjoints is analogous.

We now need to prove that the unit and counit transformations ε, η are G-natural. We do the proof for the unit ε. We need to check that the following diagram commutes:

$$\begin{array}{ccc} \rho_g \mathrm{id} & \xrightarrow{\varepsilon} & \rho_g \Phi\Psi \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \delta_{\Phi\Psi,g}} \\ \mathrm{id}\rho_g & \xrightarrow{\varepsilon} & \Phi\Psi\rho_g . \end{array}$$

Here $\delta_{\Phi\Psi}$ is defined using 3.3. Unraveling the definitions we are left with checking the diagram (where we use simplified notation for the natural transformations to denote the obvious compositions)

$$\begin{array}{ccccccc} \rho_g & \xrightarrow{\varepsilon} & \rho_g\Phi\Psi & \underset{\simeq}{\xrightarrow{\delta_g}} & \Phi\rho_g\Psi & \xrightarrow{=} & \Phi\rho_g\Psi \\ {\scriptstyle \varepsilon}\downarrow & & {\scriptstyle \varepsilon}\downarrow & & {\scriptstyle \varepsilon}\downarrow & \nearrow{\scriptstyle \eta} & \\ \Phi\Psi\rho_g & \xrightarrow{\varepsilon} & \Phi\Psi\rho_g\Phi\Psi & \underset{\simeq}{\xrightarrow{\delta_g}} & \Phi\Psi\Phi\rho_g\Psi & & \end{array}$$

which is easily seen to commute. □

3.10 Corollary *Let $\Phi : \mathcal{C} \to \mathcal{D}$ be a weak G-functor. Then the following conditions are equivalent*:

(a) *Φ is an equivalence of categories*
(b) *There exists a weak G-functor $\Psi : \mathcal{D} \to \mathcal{C}$ and G-natural isomorphisms $\Psi \circ \Phi \simeq \mathrm{id}_{\mathcal{C}}$, $\Phi \circ \Psi \simeq \mathrm{id}_{\mathcal{D}}$.*

In this case we will call Φ a weak G-equivalence.

Proof We only need to prove (a) ⇒ (b) as the opposite implication is trivial. Let $\Psi : \mathcal{D} \to \mathcal{C}$ be the quasi-inverse functor to Φ. In particular Ψ and Φ are adjoint (both ways) so that by 3.9 Ψ has a structure of a weak G-functor with compositions G-isomorphic to identity functors. □

4 Example: *G*-actions on the category of vector spaces

4.1 In this section we review a well-known example of how equivalence classes of G-actions on the category of k-vector spaces correspond bijectively to cohomology classes $H^2(G, k^*)$.

4.2 Let $\mathcal{C} = \mathrm{Vect}_k$ be the category of k-vector spaces, and let ρ be the G-action on Vect_k. As every autoequivalence of $\mathcal{C}$ is isomorphic to the identity functor, let us assume $\rho_g = \mathrm{id}$ for every $g \in G$. In this setup the data of the G-action ρ defined in 2.1 is equivalent to specifying a cocycle $\phi \in Z^2(G, k^*)$.

4.3 Consider two G-actions on Vect_k given by cocycles $\phi, \phi' \in Z^2(G, k^*)$. For the G-actions to be equivalent there needs to exist a weak G-functor

$$\Phi\colon (\mathrm{Vect}_k, \phi) \to (\mathrm{Vect}_k, \phi')$$

which is an equivalence of categories. Then the pentagon axiom 3.1 requires existence of an element $\delta = (\delta_g)_{g \in G} \in Z^1(G, k^*)$ such that $\phi'_{g,h} = \delta_g \delta_h \delta_{gh}^{-1} \phi_{g,h}$ for all g, h. Thus G-categories (Vect_k, ϕ) and (Vect_k, ϕ') are equivalent if and only if $[\phi] = [\phi'] \in H^2(G, k^*)$.

4.4 The category of equivariant objects $(\mathrm{Vect}_k, \phi)^G$ is the category of ϕ-twisted G-representations with objects given by vector spaces V together with isomorphism $\theta_g\colon V \to V$ satisfying $\theta_{gh} = \phi(g,h)\theta_g\theta_h$ and G-equivariant morphisms. In particular, if ϕ is the trivial cocycle, so that G-action on Vect_k is trivial, Vect_k^G is the category of G-representations.

5 Strictifying *G*-actions

5.1 Let $\Omega(G)$ denote the category with one object for every element $g \in G$ with $\mathrm{Hom}(g, g) = k$ and $\mathrm{Hom}(g, h) = 0$ for $g \neq h$.

5.2 Let $\mathcal{C}$ be a category with a G-action. Consider the category of weak G-functors and G-natural transformations from $\Omega(G)$ to $\mathcal{C}$

$$\mathcal{C}' = \mathrm{Hom}_G(\Omega(G), \mathcal{C}).$$

We endow $\mathcal{C}'$ with the strict G-action induced by the G-action on $\Omega(G)$.

5.3 Explicitly the objects of $\mathcal{C}'$ consist of families $(c_g \in \mathcal{C})_{g \in G}$ together with isomorphisms $\delta_{h,g}\colon \rho_h c_g \simeq c_{hg}$ satisfying the cocycle condition that two ways of getting an isomorphism $\rho_k \rho_h c_g \simeq c_{khg}$ coincide. The morphisms from $(c_g)_{g \in G}$ to $(d_g)_{g \in G}$ are morphisms $f_g\colon c_g \to d_g$ satisfying the condition that the two natural ways of forming a morphism $\rho_h c_g \to d_{hg}$ coincide.

5.4 Theorem *The functor $\Phi\colon \mathcal{C}' \to \mathcal{C}$ sending $(c_g)_{g \in G}$ to c_1 is a weak G-equivalence. Hence, every G-action is weakly equivalent to a strict G-action.*

Proof We need to check that Φ has a structure of a weak G-functor and that Φ is fully faithful and essentially surjective.

The structure of a weak G-functor on Φ is in fact simply given by the structure maps $\delta_{h,g}$. That is we have functorial isomorphisms

$$\rho_g \Phi(c) = \rho_g(c_1) \xrightarrow{\delta_{g,1}} c_g = \Phi \rho_g(c)$$

and the pentagon axiom follows from the cocycle condition on δ.

To check that Φ is essentially surjective, one checks that for any $c \in \mathcal{C}$ the family $(\rho_g(c))$ has a structure of an object from $\mathcal{C}$. Furthermore, one can see that any object $(c_g)_{g\in G}$ is isomorphic to $(\rho_g(c_1))_{g\in G}$.

Thus to check that Φ is fully faithful, we may take two objects $(c_g)_{g\in G} = (\rho_g(c_1))$ and $(d_g)_{g\in G} = (\rho_g(d_1))$ and a morphism $f_g : c_g \to d_g$ between them. It is then easy to see that $f_g = \rho(f_1)$ and that conversely for any $f_1 : c_1 \to d_1$, the collection $\rho_g(f_1)$ defines a morphism between c and d. □

6 Elagin's theorem

6.1 If $\mathcal{C}$ is a triangulated category and G acts by triangulated autoequivalences, then $\mathcal{C}^G$ is endowed with a shift functor and a set of distinguished triangles: these are the triangles that are distinguished after applying the forgetful functor $\mathcal{C}^G \to \mathcal{C}$. Furthermore under some mild technical assumptions this gives $\mathcal{C}^G$ the structure of a triangulated category [4, Theorem 6.9], for instance existence of a dg-enhancement of $\mathcal{C}$ is a sufficient condition for $\mathcal{C}^G$ to be triangulated [4, Corollary 6.10].

6.2 Theorem *Let $\mathcal{C} = \langle \mathcal{A}, \mathcal{B} \rangle$ be a semi-orthogonal decomposition of triangulated categories. Let G act on $\mathcal{C}$ by triangulated autoequivalences which preserve $\mathcal{A}$ and $\mathcal{B}$. Assume that the equivariant category $\mathcal{C}^G$ is triangulated with respect to triangles coming from $\mathcal{C}$. Then $\mathcal{A}^G, \mathcal{B}^G \subset \mathcal{C}^G$ are triangulated and there is a semi-orthogonal decomposition*

$$\mathcal{C}^G = \langle \mathcal{A}^G, \mathcal{B}^G \rangle.$$

Proof The existence of an adjoint pair between $\mathcal{C}$ and $\mathcal{C}^G$ [4, Lemma 3.7] implies that $\mathcal{B}^G = {}^{\perp}\mathcal{A}^G$ and $\mathcal{A}^G = \mathcal{B}^{G\perp}$. In particular, $\mathcal{A}^G$ and $\mathcal{B}^G$ are triangulated subcategories of $\mathcal{C}^G$.

Now in order to establish the semi-orthogonal decomposition $\mathcal{C}^G = \langle \mathcal{A}^G, \mathcal{B}^G \rangle$ it suffices to show that the embedding $i^G : \mathcal{A}^G \to \mathcal{C}^G$ has a left adjoint [1, 1.5]. This holds true by 3.9, 3.8: the functor $i : \mathcal{A} \to \mathcal{C}$ is (strictly) G-equivariant, hence its left adjoint $p : \mathcal{C} \to \mathcal{A}$ induces an adjoint p^G to the embedding $i^G : \mathcal{A}^G \to \mathcal{C}^G$. □

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References

1. Bondal, A.I., Kapranov, M.M.: Representable functors, Serre functors, and mutations. Math. USSR Izv. **35**(3), 519–541 (1990)
2. Deligne, P.: Action du groupe des tresses sur une catégorie. Invent. Math. **128**(1), 159–175 (1997)
3. Elagin, A.D.: Descent theory for semiorthogonal decompositions. Sb. Math. **203**(5–6), 645–676 (2012)
4. Elagin, A.: On equivariant triangulated categories (2014). arXiv:1403.7027

5. Ganter, N., Kapranov, M.: Symmetric and exterior powers of categories. Transform. Groups **19**(1), 57–103 (2014)
6. Kelly, G.M., Street, R.: Review of the elements of 2-categories. In: Dold, A., Eckmann, B. (eds.) Category Seminar. Lecture Notes in Mathematics, vol. 420, pp. 75–103. Springer, Berlin (1974)
7. Leinster, T.: Higher Operads, Higher Categories, vol. 298. Cambridge University Press, Cambridge (2004)
8. Niwa, M.: Theory of *G*-categories. Mem. Fac. Ed. Shiga Univ. Nat. Sci. Ped. Sci. **40** (1990), 1–24 (1991) (in Japanese)
9. Sosna, P.: Linearisations of triangulated categories with respect to finite group actions. Math. Res. Lett. **19**(5), 1007–1020 (2012)