European Journal of Mathematics (2017) 3:43–52
DOI 10.1007/s40879-016-0126-0

RESEARCH ARTICLE

On hypergeometric identities related to zeta values

Raffaele Marcovecchio[1]

Received: 22 April 2016 / Revised: 7 December 2016 / Accepted: 8 December 2016 /
Published online: 23 January 2017

Abstract Two linear forms, $\sigma_n\zeta(5)+\tau_n\zeta(3)+\varphi_n$ and $\sigma_n\zeta(2)+\tau_n/2$, with suitable rational coefficients $\sigma_n,\tau_n,\varphi_n$, are presented. As a byproduct, we obtain an identity between simple and double binomial sums, where the simple sum is the value of a terminating well-poised Saalschützian ${}_4F_3$ series. This complements a recent note of the author on two linear forms: $\alpha_n\widetilde{\zeta}(4)+\beta_n\widetilde{\zeta}(2)+\gamma_n$, based on an identity of Paule–Schneider, and $\alpha_n\zeta(2)+\beta_n$, coming from the Apéry–Beukers construction.

Keywords Zeta values · Binomial and binomial-harmonic identities

Mathematics Subject Classification 11J13 · 05A19 · 11B65 · 33C20

1 Main binomial and binomial-harmonic identities

Very well-poised hypergeometric series provide a clue in the study of diophantine properties of the values of the Riemann zeta function $\zeta(s)$ at positive integers, see e.g. [4,7,8], to quote only a few papers dealing with this important topic. Further references can be found in the bibliography of [4].

In the recent paper [5], we observed that the linear forms

$$\sum_{k=1}^{\infty}\left(k+\frac{n}{2}\right)\frac{n!^5}{(k)_{n+1}^5}(-1)^k=\alpha_n\widetilde{\zeta}(4)+\beta_n\widetilde{\zeta}(2)+\gamma_n,\qquad \alpha_n,\beta_n,\gamma_n\in\mathbb{Q},$$

✉ Raffaele Marcovecchio
raffaele.marcovecchio@unich.it

1 Dipartimento di Ingegneria e Geologia, Università degli Studi G. D'Annunzio di Chieti-Pescara, Viale Pindaro 42, 65127 Pescara, Italy

and

$$n!\sum_{k=1}^{\infty}\frac{(k-n)_n}{(k)_{n+1}^2}=A_n\zeta(2)+B_n,\qquad A_n,B_n\in\mathbb{Q},$$

have two common coefficients, namely

$$A_n=\alpha_n\qquad\text{and}\qquad B_n=\beta_n.$$

Here and in the sequel, $(x)_m=x(x+1)\cdots(x+m-1)$ if $m>0$ is an integer, and $(x)_0=1$. We denote by $\zeta(s)$ and $\widetilde{\zeta}(s)$ the following:

$$\zeta(s)=\sum_{k=1}^{\infty}\frac{1}{k^s}\qquad\text{and}\qquad\widetilde{\zeta}(s)=\sum_{k=1}^{\infty}\frac{(-1)^k}{k^s}=(2^{1-s}-1)\zeta(s).$$

The equality $A_n=\alpha_n$ was noted earlier by Paule and Schneider [6], and is a special case of [4, Proposition 1].

In the present paper we adapt the methods of [5] to the linear forms

$$\sum_{k=1}^{\infty}\left(k+\frac{n}{2}\right)\frac{n!^6}{(k)_{n+1}^6}=\sigma_n\zeta(5)+\tau_n\zeta(3)+\varphi_n,\qquad\sigma_n,\tau_n,\varphi_n\in\mathbb{Q},$$

and

$$\sum_{k=1}^{\infty}\frac{(k-n)_n(k+n+1)_n}{(k)_{n+1}^2}=S_n\zeta(2)+T_n,\qquad S_n,T_n\in\mathbb{Q}.$$

It seems reasonable to expect that one can solve [9, Problem 1] by similar methods. We assume that the reader is familiar with the background contained in [4, Section 2]. In particular, we have

$$\begin{aligned}\sigma_n&=-\sum_{j=0}^{n}\frac{\mathrm{d}}{\mathrm{d}j}\left(\frac{n}{2}-j\right)\binom{n}{j}^6\\&=\sum_{j=0}^{n}\binom{n}{j}^6\left(1-6\left(\frac{n}{2}-j\right)(H_{n-j}-H_j)\right),\qquad(1)\\\tau_n&=-\frac{1}{3!}\sum_{j=0}^{n}\frac{\mathrm{d}^3}{\mathrm{d}j^3}\left(\frac{n}{2}-j\right)\binom{n}{j}^6\\&=-\sum_{j=0}^{n}\binom{n}{j}^6\left(36\left(\frac{n}{2}-j\right)(H_{n-j}-H_j)^3\right.\end{aligned}$$

$$+ 18\left(\frac{n}{2} - j\right)(H_{n-j} - H_j)\big(H_{n-j}^{(2)} + H_j^{(2)}\big)$$
$$+ 2\left(\frac{n}{2} - j\right)\big(H_{n-j}^{(3)} - H_j^{(3)}\big)$$
$$- 18(H_{n-j} - H_j)^2 - 3\big(H_{n-j}^{(2)} + H_j^{(2)}\big)\bigg), \tag{2}$$

$$S_n = (-1)^n \sum_{j=0}^{n} \binom{n+j}{n}\binom{n}{j}^2\binom{2n-j}{n}, \tag{3}$$

$$T_n = (-1)^n \sum_{j=0}^{n} \binom{n+j}{n}\binom{n}{j}^2\binom{2n-j}{n} \cdot \big(H_j^{(2)} + H_j\big(3(H_j - H_{n-j}) + H_{2n-j} - H_{n+j}\big)\big). \tag{4}$$

Throughout this paper,

$$H_j^{(k)} = 1 + \cdots + \frac{1}{j^k}, \quad H_0^{(k)} = 0, \qquad k = 1, 2, \ldots,$$

and the notation for the derivative $\mathrm{d}/\mathrm{d}j$ is taken from [4, (7.2)].

The main result of the present paper is the following:

Theorem 1.1 *We have*

$$\sigma_n = S_n, \tag{5}$$
$$\tau_n = 2T_n. \tag{6}$$

Despite the analogy between the equalities $\tau_n = 2T_n$ in (6) and $\beta_n = B_n$ in the paper [5] quoted above, we currently miss a unified proof. However, both (5) and (6), and similar observations in [5], are implicitely connected to the period structure of some multiple integrals (see [3, Section 9.5]). In particular, the linear forms Λ_n (respectively, Θ_n) in Sect. 2, and even more general linear forms, are equal to suitable 3-fold (respectively, 5-fold) multiple integrals over $[0, 1]^3$ (respectively, over $[0, 1]^5$), and similar remarks hold for the linear forms in 1 and $\zeta(2)$ and in 1, $\zeta(2)$ and $\zeta(4)$ in [5]. All the integrals alluded to above are period integrals on moduli spaces (see [3, Section 1.3]).

In Sect. 2 we provide more details on the above linear forms and coefficients, and in Sects. 3–4 we prove Theorem 1.1.

By combining (5) with another special case of [4, Proposition 1], we have

Theorem 1.2

$$\sum_{j=0}^{n} \binom{n+j}{n}\binom{n}{j}^2\binom{2n-j}{n} = \sum_{0\leqslant j\leqslant k\leqslant n} \binom{n}{j}^2\binom{n}{k}^2\binom{n+k-j}{n}. \tag{7}$$

We give an independent proof of (7) in the last section of the present paper.

2 Linear forms in 1, $\zeta(2)$ and in 1, $\zeta(3)$ and $\zeta(5)$

The following series is a linear form in 1 and $\zeta(2)$ with rational coefficients:

$$\Lambda_n = \sum_{k=1}^{\infty} \frac{(k-n)_n (k+n+1)_n}{(k)_{n+1}^2} = S_n \zeta(2) + T_n, \qquad S_n, T_n \in \mathbb{Q}.$$

It is worth noticing that Λ_n is a Saalschützian ${}_4F_3$ well-poised hypergeometric series

$$\Lambda_n = \frac{n!^3 (3n+1)!}{(2n+1)!^3} \, {}_4F_3 \left[\begin{matrix} n+1, & n+1, & n+1, & 3n+2 \\ & 2n+2, & 2n+2, & 2n+2 \end{matrix} \right].$$

Throughout the present paper we use identities between values of the function ${}_{q+1}F_q$ with the argument $z = 1$, which is customary omitted.

We have

$$\begin{aligned} S_n &= \sum_{j=0}^{n} \frac{(k-n)_n (k+n+1)_n (k+j)^2}{(k)_{n+1}^2} \bigg|_{k=-j} \\ &= (-1)^n \sum_{j=0}^{n} \binom{n+j}{n} \binom{n}{j}^2 \binom{2n-j}{n}, \end{aligned}$$

whence S_n is the right-hand side of (3) and, similarly, T_n is given as the right-hand side of (4).

The next series is a linear form in 1, $\zeta(3)$ and $\zeta(5)$ with rational coefficients:

$$\Theta_n = \sum_{j=0}^{\infty} \left(k + \frac{n}{2} \right) \frac{n!^6}{(k)_{n+1}^6} = \sigma_n \zeta(5) + \tau_n \zeta(3) + \varphi_n,$$

where

$$\begin{aligned} \sigma_n &= \sum_{j=0}^{n} \frac{\mathrm{d}}{\mathrm{d}k} \left(k + \frac{n}{2} \right) \frac{n!^6 (k+j)^6}{(k)_{n+1}^6} \bigg|_{k=-j} \\ &= \sum_{j=0}^{n} \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \left(\frac{n}{2} + \varepsilon - j \right) \frac{n!^6}{(\varepsilon - j)_j^6 (1+\varepsilon)_{n-j}^6} \bigg|_{\varepsilon=0}. \end{aligned}$$

By exchanging j with $n-j$, i.e. by inverting the order of summation, we have

$$\sigma_n = \sum_{j=0}^{n} \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \left(-\frac{n}{2} + j + \varepsilon \right) \frac{n!^6}{(1+\varepsilon)_j^6 (1-\varepsilon)_{n-j}^6} \bigg|_{\varepsilon=0}.$$

Therefore σ_n equals the left-hand side of (1), and similarly τ_n is given by one of the two equivalent sums in (2).

3 Application of Whipple's transformation

We apply the following transformation formula, due to Whipple (see [2, 4.3 (4)]):

$$\begin{aligned} &{}_7F_6\left[\begin{matrix} a,\ a/2+1, & b, & c, & d, & e, & -m \\ a/2, & 1+a-b, & 1+a-c, & 1+a-d, & 1+a-e, & 1+a+m \end{matrix}\right] \\ &\quad = \frac{(1+a)_m(1+a-d-e)_m}{(1+a-d)_m(1+a-e)_m}\,{}_4F_3\left[\begin{matrix} 1+a-b-c, & d, & e, & -m \\ & 1+a-b, & 1+a-c, & d+e-a-m \end{matrix}\right]. \end{aligned} \tag{8}$$

The coefficient σ_n can be written as

$$\sigma_n = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left(-\frac{n}{2}+\varepsilon\right){}_7F_6\left[\begin{matrix} -n+2\varepsilon, & -n/2+\varepsilon+1, & -n+\varepsilon, & -n+\varepsilon, & -n+\varepsilon, & -n+\varepsilon, & -n \\ & -n/2, & 1+\varepsilon, & 1+\varepsilon, & 1+\varepsilon, & 1+\varepsilon, & 1+2\varepsilon \end{matrix}\right]\Bigg|_{\varepsilon=0}.$$

By applying (8), we obtain

$$\sigma_n = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left(-\frac{n}{2}+\varepsilon\right)\frac{(1-n+2\varepsilon)_n(n+1)_n}{(1+\varepsilon)_n^2}\,{}_4F_3\left[\begin{matrix} n+1, & -n+\varepsilon, & -n+\varepsilon, & -n \\ & 1+\varepsilon, & 1+\varepsilon, & -2n \end{matrix}\right]\Bigg|_{\varepsilon=0}.$$

Since

$$\left(-\frac{n}{2}+\varepsilon\right)(1-n+2\varepsilon)_n = \varepsilon(-n+2\varepsilon)_n, \tag{9}$$

we have

$$\begin{aligned} \sigma_n &= \frac{(-n)_n(n+1)_n}{(1)_n^2}\,{}_4F_3\left[\begin{matrix} n+1, & -n, & -n, & -n \\ & 1, & 1, & -2n \end{matrix}\right] \\ &= (-1)^n\sum_{j=0}^{n}\binom{n+j}{n}\binom{n}{j}^2\binom{2n-j}{n}. \end{aligned}$$

Therefore (5) is proved.

Let $a, b, c, d, \alpha, \beta, \gamma, \delta$ be eight complex parameters to be chosen later. We consider the following functions of ε:

$$\begin{aligned} f_{n,j}(\varepsilon) = {}&\left(-\frac{n}{2}+j+\varepsilon\right)\binom{n}{j}\frac{n!}{(1+2\varepsilon)_j(1-2\varepsilon)_{n-j}}\,\frac{n!}{(1+a\varepsilon)_j(1-\alpha\varepsilon)_{n-j}} \\ &\cdot\frac{n!}{(1+b\varepsilon)_j(1-\beta\varepsilon)_{n-j}}\,\frac{n!}{(1+c\varepsilon)_j(1-\gamma\varepsilon)_{n-j}}\,\frac{n!}{(1+d\varepsilon)_j(1-\delta\varepsilon)_{n-j}}. \end{aligned}$$

We have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\big(f_{n,j}(\varepsilon)\big)_{\varepsilon=0} \\ &\quad = \binom{n}{j}^6 + \left(-\frac{n}{2}+j\right)\binom{n}{j}^6 \\ &\qquad\qquad \cdot\big((2+\alpha+\beta+\gamma+\delta)H_{n-j} - (2+a+b+c+d)H_j\big), \end{aligned}$$

and similar expressions for the second and third order derivatives of $f_{n,j}(\varepsilon)$ at $\varepsilon = 0$. With the choice $a = \beta = 1+i, b = \alpha = 1-i, c = d = \gamma = \delta = 1$, where $i = \sqrt{-1}$, we have

$$3!\,\tau_n = \sum_{j=0}^{n} \frac{\mathrm{d}^3}{\mathrm{d}\varepsilon^3} \big(f_{n,j}(\varepsilon)\big)_{\varepsilon=0}.$$

Therefore,

$$\tau_n = \frac{1}{3!}\frac{\mathrm{d}^3}{\mathrm{d}\varepsilon^3} \frac{(-1)^n(-n/2+\varepsilon)n!^5}{(-n+2\varepsilon)_n(-n+(1+i)\varepsilon)_n(-n+(1-i)\varepsilon)_n(-n+\varepsilon)_n^2} \cdot {}_7F_6\left[\begin{matrix} -n+2\varepsilon, & -n/2+\varepsilon+1, & -n+(1+i)\varepsilon, & -n+(1-i)\varepsilon, & -n+\varepsilon, & -n+\varepsilon, & -n \\ & -n/2+\varepsilon, & 1+(1-i)\varepsilon, & 1+(1+i)\varepsilon, & 1+\varepsilon, & 1+\varepsilon, & 1+2\varepsilon \end{matrix}\right]\Bigg|_{\varepsilon=0}.$$

Application of (8) yields

$$\tau_n = \frac{1}{3!}\frac{\mathrm{d}^3}{\mathrm{d}\varepsilon^3} \frac{(-1)^n(-n/2+\varepsilon)n!^5}{(-n+2\varepsilon)_n(-n+(1+i)\varepsilon)_n(-n+(1-i)\varepsilon)_n(-n+\varepsilon)_n^2} \cdot \frac{(1-n+2\varepsilon)_n(1+n)_n}{(1+\varepsilon)_n^2}\, {}_4F_3\left[\begin{matrix} 1+n, & -n+\varepsilon, & -n+\varepsilon, & -n \\ & 1+(1-i)\varepsilon, & 1+(1+i)\varepsilon, & -2n \end{matrix}\right]\Bigg|_{\varepsilon=0}.$$

Using (9) again,

$$\tau_n = \frac{1}{2!}\frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2} \frac{(-1)^n n!^6}{(1+\varepsilon)_n^2(-n+(1+i)\varepsilon)_n(-n+(1-i)\varepsilon)_n(-n+\varepsilon)_n^2} \cdot \sum_{j=0}^{n} \binom{n+j}{n}\binom{2n-j}{n} \frac{(-n+\varepsilon)_j^2}{(1+(1-i)\varepsilon)_j(1+(1+i)\varepsilon)_j}\Bigg|_{\varepsilon=0}.$$

Taking

$$g_{n,j}(\varepsilon) = \frac{n!^6}{(1+\varepsilon)_n^2(-n+(1+i)\varepsilon)_n(-n+(1-i)\varepsilon)_n(-n+\varepsilon)_n^2} \cdot \frac{(-n+\varepsilon)_j^2}{(1+(1+i)\varepsilon)_j(1+(1-i)\varepsilon)_j},$$

and computing its first and second derivatives at $z = 0$, after a few simplifications we obtain

$$(-1)^n\tau_n = \sum_{j=0}^{n} \binom{n+j}{n}\binom{n}{j}^2\binom{2n-j}{n}\big(H_n^{(2)} + H_{n-j}^{(2)} + 2(H_{n-j} - H_j)^2\big).$$

4 End of the proof of Theorem 1.1

In this section we denote by a, b, α, β four real parameters to be chosen later. Let $h_n(\varepsilon, \omega)$ be defined by

$$h_n(\varepsilon,\omega) = \frac{(1+\alpha\varepsilon)_n(-2n+\beta\varepsilon)_n}{n!^2} \cdot {}_4F_3\left[\begin{matrix} n+1+a\varepsilon, & -n+b\varepsilon, & -n+(\alpha+\beta+1)\varepsilon+\omega, & -n \\ & 1+(a+b+1)\varepsilon+\omega, & 1+\alpha\varepsilon, & -2n+\beta\varepsilon \end{matrix}\right].$$

By applying (see [2, 7.2 (1)])

$${}_4F_3\left[\begin{matrix} x, & y, & z, & -n \\ & u, & v, & w \end{matrix}\right] = \frac{(v-z)_n(w-z)_n}{(v)_n(w)_n}\, {}_4F_3\left[\begin{matrix} u-y, & u-x, & z, & -n \\ & u, & 1-w+z-n, & 1-v+z-n \end{matrix}\right],$$

valid for $u+v+w = x+y+z-n+1$, with

$$x = n+1+a\varepsilon, \qquad y = -n+b\varepsilon, \qquad z = -n+(\alpha+\beta+1)\varepsilon+\omega$$

and

$$u = 1+(a+b+1)\varepsilon+\omega, \qquad v = 1+\alpha\varepsilon, \qquad w = -2n+\beta\varepsilon,$$

we have

$$h_n(\varepsilon,\omega) = \frac{(1+(\alpha+1)\varepsilon+\omega)_n(-2n+(\beta+1)\varepsilon+\omega)_n}{n!^2} \cdot {}_4F_3\left[\begin{matrix} n+1+(a+1)\varepsilon+\omega, & -n+(b+1)\varepsilon+\omega, & -n+(\alpha+\beta+1)\varepsilon+\omega, & -n \\ & 1+(a+b+1)\varepsilon+\omega, & 1+(\alpha+1)\varepsilon+\omega, & -2n+(\beta+1)\varepsilon+\omega \end{matrix}\right].$$

Here we used $(\xi)_n = (-1)^n(1-n-\xi)_n$ with $\xi = v-z$ and $\xi = w-z$. By choosing $a=-1$, $b=-2$, $\alpha=1$, $\beta=-1$ and comparing the two expressions of

$$\frac{\partial^2}{\partial\varepsilon\,\partial\omega}(h_n(\varepsilon,\omega))_{(\varepsilon,\omega)=(0,0)}$$

we find out that

$$\begin{aligned}\sum_{j=0}^{n}\binom{n+j}{n}\binom{n}{j}^2\binom{2n-j}{n} \cdot \Big(2H_j^{(2)} &- \big(H_{n-j}^{(2)}-H_n^{(2)}\big) - 2H_n^{(2)} - \big(H_{n-j}^{(2)}-H_n^{(2)}\big) \\ &+ H_{n-j}^{(2)} - H_n^{(2)} + 2H_j^{(2)} - 2H_j^{(2)} \\ &+ \big(2H_{n-j} + H_{n+j} - H_n - H_{2n-j} - 2H_j\big) \\ &\qquad \cdot \big(2H_n - (H_{n-j}-H_n) + (H_{n-j}-H_n) - 2H_j + 2H_j\big) \\ &- (H_{n-j} - H_n - H_j)\end{aligned}$$

$$\cdot \big(H_n - (H_n - H_{2n}) - (H_{n+j} - H_n) - 2(H_{n-j} - H_n) + H_{n-j} - H_n + 2H_j - H_j - (H_{2n} - H_{2n-j})\big)\Big) = 0.$$

By using

$$\sum_{j=0}^{n} \binom{n+j}{n}\binom{n}{j}^2\binom{2n-j}{n} H_n (H_{k+n-j} - H_{k+j}) = 0$$

with $k = 0$ and $k = n$, the above sum simplifies to

$$\sum_{j=0}^{n} \binom{n+j}{n}\binom{n}{j}^2\binom{2n-j}{n} \cdot \big(2H_j^{(2)} - H_n^{(2)} - H_{n-j}^{(2)} - (H_{n-j} - H_j)(H_j + H_{2n-j} - H_{n+j} - H_{n-j})\big) = 0,$$

hence

$$(-1)^n \tau_n = \sum_{j=0}^{n} \binom{n+j}{n}\binom{n}{j}^2\binom{2n-j}{n} \cdot \big(2H_j^{(2)} - (H_{n-j} - H_j)(3H_{n-j} - 3H_j - H_{2n-j} + H_{n+j})\big).$$

Since

$$\sum_{j=0}^{n} \binom{n+j}{n}\binom{n}{j}^2\binom{2n-j}{n} H_{n-j}\big(3H_{n-j} - 3H_j - H_{2n-j} + H_{n+j}\big) = -\sum_{j=0}^{n} \binom{n+j}{n}\binom{n}{j}^2\binom{2n-j}{n} H_j\big(3H_{n-j} - 3H_j - H_{2n-j} + H_{n+j}\big),$$

we have

$$(-1)^n \tau_n = 2\sum_{j=0}^{n} \binom{n+j}{n}\binom{n}{j}^2\binom{2n-j}{n} \cdot \big(H_j^{(2)} - H_j(3H_{n-j} - 3H_j + H_{n+j} - H_{2n-j})\big),$$

and (6) is proved.

5 Application of Sheppard's transformation

In this section we give a direct proof of (7). A similar argument was applied in [4, Section 7] to the double binomial sum in the middle of [4, (7.1)].

We start with the double sum in (7), and rewrite it in the form

$$\sum_{i=0}^{n} \binom{n}{j}^4 {}_3F_2\left[\begin{matrix} n+1, & -n+j, & -n+j \\ & 1+j, & 1+j \end{matrix}\right]. \tag{10}$$

Let us apply Sheppard's transformation (see [1, Corollary 3.3.4] and [2, Section 3.9]):

$${}_3F_2\left[\begin{matrix} -m, & a, & b \\ & d, & e \end{matrix}\right] = \frac{(d-a)_m\,(e-a)_m}{(d)_m\,(e)_m}\,{}_3F_2\left[\begin{matrix} -m, & a, & a+b-m-d-e+1 \\ & a-m-d+1, & a+1-m-e \end{matrix}\right].$$

We obtain

$${}_3F_2\left[\begin{matrix} -n+j, & n+1, & -n+j \\ & 1+j, & 1+j \end{matrix}\right] = \frac{(-n+j)_{n-j}^2}{(1+j)_{n-j}^2}\,{}_3F_2\left[\begin{matrix} -n+j, & n+1, & -n \\ & 1, & 1 \end{matrix}\right].$$

Hence the sum (10) is equal to

$$\sum_{j=0}^{n} \binom{n}{j}^2 {}_3F_2\left[\begin{matrix} -n+j, & n+1, & -n \\ & 1, & 1 \end{matrix}\right] = \sum_{j=0}^{n} \binom{n}{j}^2 \sum_{l=0}^{n-j} \binom{n+l}{n}\binom{n}{l}\binom{n-j}{l}.$$

Exchanging the order of summation and using

$$\binom{n}{j}\binom{n-j}{l} = \binom{n}{l}\binom{n-l}{j},$$

the last double sum becomes

$$\sum_{l=0}^{n} \binom{n+l}{n}\binom{n}{l}^2 \sum_{j=0}^{n-l} \binom{n}{j}\binom{n-l}{j} = \sum_{l=0}^{n} \binom{n+l}{n}\binom{n}{l}^2 {}_2F_1\left[\begin{matrix} -n, & -n+l \\ & 1 \end{matrix}\right].$$

The inner sum ${}_2F_1$ can be evaluated by the Chu–Vandermonde convolution formula (see e.g. [2, Section 1.3]):

$${}_2F_1\left[\begin{matrix} -n, & -n+l \\ & 1 \end{matrix}\right] = \binom{2n-l}{n}.$$

Therefore (7) is established.

Acknowledgements The author is indebted to the referee for helpful comments and suggestions, and for pointing out to him the reference [3].

References

1. Andrews, G.E., Askey, R., Roy, R.: Special Functions. Encyclopedia of Mathematics and Its Applications, vol. 71. Cambridge University Press, Cambridge (1999)
2. Bailey, W.N.: Generalized Hypergeometric Series. Cambridge University Press, Cambridge (1935)
3. Brown, F.: Irrationality proofs for zeta values, moduli spaces and dinner parties. Mosc. J. Comb. Number Theory **6**(2–3) (2016)
4. Krattenthaler, C., Rivoal, T.: Hypergéométrie et Fonction Zêta de Riemann. Memoirs of the American Mathematical Society, vol. **186**(875). American Mathematical Society, Providence (2007)
5. Marcovecchio, R.: Simultaneous approximations to $\zeta(2)$ and $\zeta(4)$ (submitted)
6. Paule, P., Schneider, C.: Computer proofs of a new family of harmonic numbers identities. Adv. Appl. Math. **31**(2), 359–378 (2003)
7. Zudilin, W.: Well-poised hypergeometric service for diophantine problems of zeta values. J. Théor. Nombres Bordeaux **15**(2), 593–626 (2003)
8. Zudilin, W.: Arithmetic of linear forms involving odd zeta values. J. Théor. Nombres Bordeaux **16**(1), 251–291 (2004)
9. Zudilin, W.: A hypergeometric problem. J. Comput. Appl. Math. **233**(3), 856–857 (2009)