European Journal of Mathematics (2016) 2:874–885
DOI 10.1007/s40879-016-0103-7

RESEARCH ARTICLE

On the lower bound of the discrepancy of Halton's sequence II

Mordechay B. Levin[1]

Received: 30 July 2015 / Revised: 1 March 2016 / Accepted: 3 March 2016 /
Published online: 31 March 2016

Abstract Let $(H_s(n))_{n\geqslant 1}$ be an s-dimensional generalized Halton's sequence. Let D_N^* be the discrepancy of the sequence $(H_s(n))_{n=1}^N$. It is known that $ND_N^* = \mathrm{O}(\ln^s N)$ as $N \to \infty$. In this paper, we prove that this estimate is exact. Namely, there exists a constant $C(H_s) > 0$ such thats

$$\max_{1\leqslant M\leqslant N} MD_M^* \geqslant C(H_s)\log_2^s N \qquad \text{for} \quad N = 2, 3, \ldots$$

Keywords Halton's sequence · Ergodic adding machine

Mathematics Subject Classification 11K38

1 Introduction

Let $(\beta_n)_{n\geqslant 1}$ be a sequence in the unit cube $[0, 1)^s$, $B_{\mathbf{y}} = [0, y_1) \times \cdots \times [0, y_s)$,

$$\Delta\big(B_{\mathbf{y}}, (\beta_n)_{n=1}^N\big) = \sum_{n=1}^{N} \big(\mathbf{1}_{B_{\mathbf{y}}}(\beta_n) - y_1 \cdots y_s\big), \tag{1}$$

where $\mathbf{1}_{B_{\mathbf{y}}}(\mathbf{x}) = 1$ if $\mathbf{x} \in B_{\mathbf{y}}$, and $\mathbf{1}_{B_{\mathbf{y}}}(\mathbf{x}) = 0$ if $\mathbf{x} \notin B_{\mathbf{y}}$.

✉ Mordechay B. Levin
mlevin@math.biu.ac.il

[1] Department of Mathematics, Bar-Ilan University, Ramat-Gan 52900, Israel

We define the *star discrepancy* of an N-point set $(\beta_n)_{n=1}^N$ as

$$D^*\big((\beta_n)_{n=1}^N\big) = \sup_{0<y_1,\ldots,y_s\leqslant 1} \left| \frac{\Delta\big(B_{\mathbf{y}}, (\beta_n)_{n=1}^N\big)}{N} \right|. \tag{2}$$

In 1954, K. Roth proved that

$$\limsup_{N\to\infty} N(\ln N)^{-s/2} D^*\big((\beta_n)_{n=1}^N\big) > 0.$$

According to the well-known conjecture (see, e.g., [1, p. 283]), this estimate can be improved to

$$\limsup_{N\to\infty} N(\ln N)^{-s} D^*\big((\beta_n)_{n=1}^N\big) > 0. \tag{3}$$

In 1972, W. Schmidt proved this conjecture for $s = 1$. For $s = 2$, Faure and Chaix [4] proved (3) for a class of (t, s)-sequences. See [2] for the most important results on this conjecture.

There exists another conjecture on the lower bound for the discrepancy function: there exists a constant $\dot{c}_3 > 0$ such that

$$ND^*\big((\beta_{k,N})_{k=0}^{N-1}\big) > \dot{c}_3(\ln N)^{s/2}$$

for all N-point sets $(\beta_{k,N})_{k=0}^{N-1}$ (see [2, p. 147]).

Definition An s-dimensional sequence $((\beta_n)_{n\geqslant 1})$ is of *low discrepancy* (abbreviated l.d.s.) if $D^*\big((\beta_n)_{n=1}^N\big) = \mathrm{O}\big(N^{-1}(\ln N)^s\big)$ for $N \to \infty$.

Let $p \geqslant 2$ be an integer,

$$n = \sum_{j\geqslant 1} e_{p,j}(n)p^{j-1}, \quad e_{p,j}(n) \in \{0, 1, \ldots, p-1\}, \quad \phi_p(n) = \sum_{j\geqslant 1} e_{p,j}(n)p^{-j}.$$

van der Corput proved that $(\phi_p(n))_{n\geqslant 0}$ is a 1-dimensional l.d.s. (see [12]). Let

$$\widehat{H}_s(n) = \big(\phi_{\hat{p}_1}(n), \ldots, \phi_{\hat{p}_s}(n)\big), \qquad n = 0, 1, 2, \ldots,$$

where $\widehat{p}_1, \ldots, \widehat{p}_s \geqslant 2$ are pairwise coprime integers. Halton proved that $(\widehat{H}_s(n))_{n\geqslant 0}$ is an s-dimensional l.d.s. (see [6]). For other examples of l.d.s. see, e.g., [1,5,11]. In [9], we proved that Halton's sequence satisfies (3). In this paper we generalize this result.

Let $Q = (q_1, q_2, \ldots)$ and $Q_j = q_1 q_2 \cdots q_j$, where $q_j \geqslant 2$, $j = 1, 2, \ldots$, is a sequence of integers. Consider Cantor's expansion of $x \in [0, 1)$:

$$x = \sum_{j=1}^{\infty} \frac{x_j}{Q_j}, \qquad x_j \in \{0, 1, \ldots, q_j - 1\}, \quad x_j \neq q_j - 1 \text{ for infinitely many } j.$$

The Q-adic representation of x is then unique. We define the *odometer transform* as

$$T_Q(x) = \frac{x_k+1}{Q_k} + \sum_{j \geqslant k+1} \frac{x_j}{Q_j}, \qquad T_Q^n(x) = T_Q\big(T_Q^{n-1}(x)\big), \tag{4}$$

$n = 2, 3, \dots, T_Q^0(x) = x$, where $k = \min\{j : x_j \neq q_i - 1\}$.

For $Q = (q, q, \dots)$, we obtain von Neumann–Kakutani's q-adic adding machine (see, e.g., [5]). As is known, the sequence $(T_Q^n(x))_{n\geq 1}$ coincides for $x = 0$ with the van der Corput sequence (see, e.g., [5, Section 2.5]).

Let $q_0 \geqslant 4$, $p_{i,j} \geqslant 2$, $s \geqslant i \geqslant 1$, $j \geqslant 1$, be integers, g.c.d.$(p_{i,k}, p_{j,l}) = 1$ for $i \neq j$, $\mathcal{P}_i = (p_{i,1}, p_{i,2}, \dots)$, $\boldsymbol{\mathcal{P}} = (\mathcal{P}_1, \dots, \mathcal{P}_s)$, $T_{\boldsymbol{\mathcal{P}}}(\mathbf{x}) = (T_{\mathcal{P}_1}(x_1), \dots, T_{\mathcal{P}_s}(x_s))$,

$$\widetilde{P}_{i,0} = 1, \quad \widetilde{P}_{i,j} = \prod_{1 \leqslant k \leqslant j} p_{i,k}, \quad \widetilde{P}_{i,j} \leqslant q_0^{j/2}, \qquad i \in [1, s], \quad j \geqslant 1, \tag{5}$$

$$n = \sum_{j \geqslant 1} e_{p_{i,j},j}(n)\, \widetilde{P}_{i,j-1}, \quad e_{p_{i,j},j}(n) \in \{0, 1, \dots, p_{i,j} - 1\}, \quad n = 0, 1, \dots,$$

$$\varphi_{\mathcal{P}_i}(n) = \sum_{j \geqslant 1} e_{p_{i,j},j}(n)\, \widetilde{P}_{i,j}^{-1}, \qquad H_{\boldsymbol{\mathcal{P}}}(n) = (\varphi_{\mathcal{P}_1}(n), \dots, \varphi_{\mathcal{P}_s}(n)). \tag{6}$$

We note that $H_{\boldsymbol{\mathcal{P}}}(n) = T_{\boldsymbol{\mathcal{P}}}^n(\mathbf{0})$ for $n = 0, 1, \dots$

Let $\Sigma_i = (\sigma_{i,j})_{j \geqslant 1}$ be a sequence of corresponding permutations $\sigma_{i,j}$ of $\{0, 1, \dots, p_{i,j} - 1\}$ for $j \geqslant 1$, $\boldsymbol{\Sigma} = (\Sigma_1, \dots, \Sigma_s)$, $\mathbf{x} = (x_1, \dots, x_s)$,

$$\widetilde{\boldsymbol{\Sigma}}(\mathbf{x}) = \big(\widetilde{\Sigma}_1(x_1), \dots, \widetilde{\Sigma}_s(x_s)\big), \qquad \widetilde{\Sigma}_i(x_i) = \sum_{j \geqslant 1} \frac{\sigma_{i,j}(x_{i,j})}{\widetilde{P}_{i,j}}, \quad x_i = \sum_{j \geqslant 1} \frac{x_{i,j}}{\widetilde{P}_{i,j}}.$$

We consider the following generalization of Halton's sequence (see [3,5,7]):

$$H_{\boldsymbol{\mathcal{P}}}^{\boldsymbol{\Sigma}}(n, \mathbf{x}) = \widetilde{\boldsymbol{\Sigma}}(T_{\boldsymbol{\mathcal{P}}}^n(\mathbf{x})), \qquad n = 0, 1, 2, \dots$$

We note that $(H_{\boldsymbol{\mathcal{P}}}^{\boldsymbol{\Sigma}}(n, \mathbf{x}))_{n \geqslant 0}$ coincides for $\mathbf{x} = \mathbf{0}$ and $s = 1$ with the Faure sequence S_Q^{Σ} [3]. Similarly to [11, pp. 29–31], we get that $(H_{\boldsymbol{\mathcal{P}}}^{\boldsymbol{\Sigma}}(n, \mathbf{x}))_{n \geqslant 0}$ is of low discrepancy.

2 Theorem and its proof

In this section we will prove

Theorem *Let* $s \geqslant 2$, $C_1 = s q_0^{s+1} \log_2 q_0$, $C = 8 q_0^s C_1^s$ *and* $\log_2 N \geqslant 2 q_0^s C_1$. *Then*

$$\inf_{\mathbf{x} \in [0,1)^s} \max_{1 \leqslant M \leqslant N} M D^*\big((H_{\boldsymbol{\mathcal{P}}}^{\boldsymbol{\Sigma}}(n, \mathbf{x}))_{n=1}^M\big) \geqslant C^{-1} \log_2^s N. \tag{7}$$

This result supports conjecture (3) (see also [8,10]).

The proof of Theorem is similar to the proof of [9, Theorem]. The main part of the proof in [9] and in this paper is the construction of the bounded vector $(y_1, \dots, y_s)$ and the application of the Chinese Remainder Theorem. In the paper [9], we take $y_i = \sum_{j=1}^m p_i^{-\tau_{i,j}}$, $i = 1, \dots, s$, where

$$\tau_{i,j} = \tau_i j, \quad j = 1, 2, \dots, \quad p_i^{\tau_i} \equiv 1 \left(\bmod \frac{p_1 \cdots p_s}{p_i} \right), \quad \tau_i \in [1, p_1 \cdots p_s]. \quad (8)$$

In this paper we take $y_i = \sum_{j=1}^m \widetilde{P}_{i,\tau_{i,j}}^{-1}$, with some special sequences $(\tau_{i,j})_{1 \leqslant i \leqslant s, j \geqslant 1}$. In order to obtain the 'periodic' properties similar to (8), we need a more complicated construction of $(\tau_{i,j})_{s \geqslant i \geqslant 1, j \geqslant 1}$:

- $p_{i,\tau_{i,j}} = p_{i,\tau_{i,1}}$, $j = 1, 2, \dots,$
- $\sigma_{i,\tau_{i,j}}^{-1}(0) - \sigma_{i,\tau_{i,j}}^{-1}(1) \equiv \sigma_{i,\tau_{i,1}}^{-1}(0) - \sigma_{i,\tau_{i,1}}^{-1}(1) \pmod{p_{i,\tau_{i,1}}}$, $j = 1, 2, \dots,$
- $\widetilde{P}_{i,\tau_{i,j}} \equiv \widetilde{P}_{i,\tau_{i,1}} \pmod{p_1 \cdots p_s / p_i}$, $j = 1, 2, \dots,$

in such a way that the sets $\{\tau_{i,1}, \tau_{i,2}, \dots\} \cap [1, \mathfrak{m}]$ would receive the greatest length, where $\mathfrak{m} = [2s^{-1} \log_{q_0} N]$, $s \geqslant i \geqslant 1$. We need all these conditions to prove statement (26).

In order to construct $(\tau_{i,j})_{1 \leqslant i \leqslant s, j \geqslant 1}$, we define auxiliary sequences $\mathcal{L}_{i,j}^{(\mathfrak{m})}$, $L_i^{(\mathfrak{m})}$, $l_{i,j}$, $\mathcal{F}_{i,b}^{(\mathfrak{m})}, \dots$

2.1 Construction of the sequence $(\tau_{i,j})$

Let $\mathfrak{m} = [2s^{-1} \log_{q_0} N]$, $\mathfrak{L}_i^{(\mathfrak{m})} = \{1 \leqslant k \leqslant \mathfrak{m} : p_{i,k} \leqslant q_0\}$. By (5), we get

$$q_0^{\mathfrak{m}/2} \geqslant \prod_{j \in [1,\mathfrak{m}] \setminus \mathfrak{L}_i^{(\mathfrak{m})}} p_{i,j} > q_0^{\mathfrak{m} - \# \mathfrak{L}_i^{(\mathfrak{m})}}.$$

Hence

$$\# \mathfrak{L}_i^{(\mathfrak{m})} > \frac{\mathfrak{m}}{2}.$$

Let $a_{i,j} \equiv \sigma_{i,j}^{-1}(0) - \sigma_{i,j}^{-1}(1) \pmod{p_{i,j}}$, $a_{i,j} \in \{1, \dots, p_{i,j} - 1\}$, $\mathfrak{a} \in \{1, \dots, q_0\}$,

$$\begin{aligned} \mathcal{L}_{i,j,\mathfrak{a}}^{(\mathfrak{m})} &= \left\{ k \in \mathfrak{L}_i^{(\mathfrak{m})} : p_{i,k} = p_{i,j},\ a_{i,k} = \mathfrak{a} \right\}, \\ L_i^{(\mathfrak{m})} &= \max_{1 \leqslant j \leqslant \mathfrak{m}, 1 \leqslant \mathfrak{a} \leqslant q_0} \# \mathcal{L}_{i,j,\mathfrak{a}}^{(\mathfrak{m})}, \qquad 1 \leqslant i \leqslant s. \end{aligned} \quad (9)$$

It is easy to see that there exist $g_{i,\mathfrak{m}} \in [1, \mathfrak{m}]$ and $\mathfrak{a}_i = \mathfrak{a}_{i,m} \in [1, q_0]$ such that

$$\# \mathcal{L}_{i,g_{i,\mathfrak{m}},\mathfrak{a}_i}^{(\mathfrak{m})} = L_i^{(\mathfrak{m})}, \qquad 1 \leqslant i \leqslant s.$$

We enumerate the set $\mathcal{L}^{(\mathfrak{m})}_{i,g_{i,\mathfrak{m}},\mathfrak{a}_i}$:

$$\mathcal{L}^{(\mathfrak{m})}_{i,g_{i,\mathfrak{m}},\mathfrak{a}_i} = \{l_{i,1} < \cdots < l_{i,L_i^{(\mathfrak{m})}}\}.$$

For $i \in [1, s]$ we have

$$L_i^{(\mathfrak{m})} \geqslant \frac{\# \mathcal{L}_i^{(\mathfrak{m})}}{q_0^2} \geqslant \frac{\mathfrak{m}}{2q_0^2}, \qquad l_{i,L_i^{(\mathfrak{m})}} \leqslant \mathfrak{m}, \qquad a_{i,l_{i,j}} = \mathfrak{a}_i, \quad j \in [1, L_i^{(\mathfrak{m})}]. \tag{10}$$

Let $p_i = p_i^{(\mathfrak{m})} = p_{i,g_{i,\mathfrak{m}}} \leqslant q_0$, $p_0 = p_0^{(\mathfrak{m})} = p_1 p_2 \cdots p_s \leqslant q_0^s$, $\dot{p}_i = p_0/p_i \leqslant q_0^{s-1}$ and

$$\mathcal{F}^{(\mathfrak{m})}_{i,b} = \{1 \leqslant k \leqslant L_i^{(\mathfrak{m})} : \widetilde{P}^{-1}_{i,l_{i,k}} \equiv b \ (\mathrm{mod}\ \dot{p}_i)\}. \tag{11}$$

We define F_i, m and $b_i = b_i^{(\mathfrak{m})}$ as follows:

$$F_i = F_i^{(\mathfrak{m})} = \# \mathcal{F}^{(\mathfrak{m})}_{i,b_i} = \max_{0 \leqslant b < \dot{p}_i} \# \mathcal{F}^{(\mathfrak{m})}_{i,b}, \qquad m = \min_{1 \leqslant i \leqslant s} F_i^{(\mathfrak{m})}. \tag{12}$$

We enumerate the set $F^{(\mathfrak{m})}_{i,b_i}$:

$$\mathcal{F}^{(\mathfrak{m})}_{i,b_i} = \{f_{i,1} < \cdots < f_{i,F_i}\}, \qquad \mathfrak{m} = [2s^{-1} \log_{q_0} N].$$

Bearing in mind that $\log_2 N \geqslant 2q_0^s C_1$ and $C_1 = sq_0^{s+1} \log_2 q_0$, we have

$$m \geqslant \min_{1 \leqslant i \leqslant s} \frac{L_i^{(\mathfrak{m})}}{\dot{p}_i} \geqslant \min_{1 \leqslant i \leqslant s} \frac{\mathfrak{m}}{2q_0^2 \dot{p}_i} \geqslant \frac{\mathfrak{m} q_0^{-s-1}}{2} \geqslant C_1^{-1} \log_2 N \geqslant 2q_0^s \geqslant 2p_0. \tag{13}$$

Let $\mathbf{k} = (k_1, \ldots, k_s)$, $\tau_{i,j} = l_{i,f_{i,j}}$, $\boldsymbol{\tau}_{\mathbf{k}} = (\tau_{1,k_1}, \ldots, \tau_{s,k_s})$, $P_{i,k} = \widetilde{P}_{i,\tau_{i,k}}$,

$$P_{\mathbf{k}} = \prod_{i=1}^{s} P_{i,k_i}, \qquad M_{i,\mathbf{k}} = \widetilde{M}_{i,\boldsymbol{\tau}_{\mathbf{k}}}, \quad \text{with} \ \ \widetilde{M}_{i,\mathbf{k}} \equiv \prod_{\substack{1 \leqslant j \leqslant s \\ j \neq i}} \widetilde{P}^{-1}_{j,k_j} \ (\mathrm{mod}\ \widetilde{P}_{i,k_i}). \tag{14}$$

Applying (10), we get $\tau_{i,m} = l_{i,f_{i,m}} \leqslant l_{i,F_i^{(\mathfrak{m})}} \leqslant l_{i,L_i^{(\mathfrak{m})}} \leqslant \mathfrak{m}$. Let $\mathbf{m} = (m, \ldots, m)$. From (5) and (14), we derive

$$2P_{\mathbf{m}} \leqslant 2 \prod_{i=1}^{s} \widetilde{P}_{i,\tau_{i,k}} \leqslant 2q_0^{s\mathfrak{m}/2} = q_0^{s[2s^{-1} \log_{q_0} N]/2} \leqslant N. \tag{15}$$

We will need the following properties of integers $\mathfrak{a}_i$, $1 \leqslant i \leqslant s$, (see (16), (17)): By (11), we have that $(b_i, \dot{p}_i) = 1$ and $(b_j, p_i) = 1$ for $i \neq j$, $i, j = 1, \ldots, s$. Let

$c_i \equiv \prod_{1 \leqslant j \leqslant s, j \neq i} b_j \pmod{p_i}$. According to (10), (11) and (14), we obtain

$$(c_i, p_i) = 1, \quad M_{i,\mathbf{k}} \equiv c_i \pmod{p_i}, \quad a_{i,\tau_{i,j}} = \mathfrak{a}_i, \qquad j \geqslant 1, \quad i \in [1, s]. \tag{16}$$

Let

$$\widetilde{p}_i = \text{g.c.d.}(\mathfrak{a}_i, p_i), \qquad \widehat{p}_i = \frac{p_i}{\widetilde{p}_i}, \qquad \widehat{a}_i = \frac{\mathfrak{a}_i}{\widetilde{p}_i}, \qquad d_i \equiv c_i \mathfrak{a}_i \pmod{\widehat{p}_i},$$

$d_i \in \{1, \dots, \widehat{p}_i - 1\}$. Hence

$$\frac{d_i}{\widehat{p}_i} \equiv c_i \frac{\mathfrak{a}_i}{p_i} \pmod 1, \quad (d_i, \widehat{p}_i) = 1, \quad \widehat{p}_i > 1, \qquad i = 1, \dots, s. \tag{17}$$

2.2 Using the Chinese Remainder Theorem

Let $x_i = \sum_{j \geqslant 1} x_{i,j} \widetilde{P}_{i,j}^{-1}$, with $x_{i,j} \in \{0, 1, \dots, p_{i,j} - 1\}$, $i = 1, \dots, s$. We define the truncation

$$[x_i]_r = \sum_{1 \leqslant j \leqslant r} x_{i,j} \widetilde{P}_{i,j}^{-1}, \qquad r \geqslant 1.$$

If $x = (x_1, \dots, x_s) \in [0, 1)^s$, then the truncation $[\mathbf{x}]_{\mathbf{r}}$ is defined coordinatewise, that is, $[\mathbf{x}]_{\mathbf{r}} = ([x_1]_{r_1}, \dots, [x_s]_{r_s})$, where $\mathbf{r} = (r_1, \dots, r_s)$.

By (6), we have

$$[\varphi_{\mathcal{P}_i}(n)]_{r_i} = [x_i]_{r_i} \quad \Longleftrightarrow \quad n \equiv \sum_{1 \leqslant j \leqslant r} x_{i,j} \widetilde{P}_{i,j-1} \pmod{\widetilde{P}_{i,r}}.$$

Applying (14) and the Chinese Remainder Theorem, we get

$$[H_{\mathfrak{P}}(n)]_{\mathbf{r}} = [\mathbf{x}]_{\mathbf{r}} \quad \Longleftrightarrow \quad n \equiv \check{x}_{\mathbf{r}} \pmod{\widetilde{P}_{\mathbf{r}}}, \tag{18}$$

$$\check{x}_{\mathbf{r}} \equiv \sum_{i=1}^{s} \widetilde{M}_{i,\mathbf{r}} \widetilde{P}_{\mathbf{r}} \widetilde{P}_{i,r_i}^{-1} \sum_{1 \leqslant j \leqslant r} x_{i,j} \widetilde{P}_{i,j-1} \pmod{\widetilde{P}_{\mathbf{r}}}, \qquad \check{x}_{\mathbf{r}} \in [0, \widetilde{P}_{\mathbf{r}}). \tag{19}$$

Now we will find the relation between $T_{\mathfrak{P}}^n(\mathbf{x})$ and $H_{\mathfrak{P}}(n)$ (see (20). It is easy to verify that if $r_i' \geqslant r_i$, $i = 1, \dots, s$, then $\check{x}_{\mathbf{r}'} \equiv \check{x}_{\mathbf{r}} \pmod{\widetilde{P}_{\mathbf{r}}}$. According to (4), we get

$$\text{if} \quad [\mathbf{w}]_{\mathbf{r}} = [\mathbf{x}]_{\mathbf{r}}, \quad \text{then} \quad [T_{\mathfrak{P}}^n(\mathbf{w})]_{\mathbf{r}} = [T_{\mathfrak{P}}^n(\mathbf{x})]_{\mathbf{r}}, \quad n = 0, 1, \dots$$

From (4), (6) and (18), we obtain

$$\left[T_{\mathfrak{P}}^W(\mathbf{0})\right]_{\mathbf{r}} = [H_{\mathfrak{P}}(W)]_{\mathbf{r}} = [\mathbf{x}]_{\mathbf{r}}, \qquad W = \check{x}_{\mathbf{r}}.$$

Hence

$$[T^n_{\mathcal{P}}(\mathbf{x})]_{\mathbf{r}} = \big[T^n_{\mathcal{P}}\big(T^W_{\mathcal{P}}(\mathbf{0})\big)\big]_{\mathbf{r}} = \big[T^{n+W}_{\mathcal{P}}(\mathbf{0})\big]_{\mathbf{r}} = [H_s(n+W)]_{\mathbf{r}}.$$

Let $W_{\mathbf{m}}(\mathbf{x}) = \check{x}_{\mathbf{m}} \in [0,\, P_{\mathbf{m}})$. Therefore

$$[T^n_{\mathcal{P}}(\mathbf{x})]_{\mathbf{r}} = [H_{\mathcal{P}}(n+W_{\mathbf{m}}(\mathbf{x}))]_{\mathbf{r}}, \qquad 1 \leqslant r_i \leqslant m, \quad 1 \leqslant i \leqslant s, \quad n \geqslant 0. \tag{20}$$

2.3 Construction of boundary points $y_1, \ldots, y_s$ and $u_1, \ldots, u_s$

Let $\mathbf{y} = (y_1, \ldots, y_s)$ with $y_i = \sum_{1\leqslant j\leqslant m} P^{-1}_{i,j}$, and let $\ddot{y}_{i,k_i} = \sum_{1\leqslant j\leqslant k_i} P^{-1}_{i,j}$, $k_i \geqslant 1$, $i = 1, \ldots, s$, $\mathbf{k} = (k_1, \ldots, k_s)$,

$$B_{\mathbf{y}} = [0,\, y_1) \times \cdots \times [0,\, y_s), \qquad B^{(\mathbf{k})} = \prod_{i=1}^{s} \big[\ddot{y}_{i,k_i} - P^{-1}_{i,k_i},\, \ddot{y}_{i,k_i}\big). \tag{21}$$

We deduce

$$B_{\mathbf{y}} = \bigcup_{k_1,\ldots,k_s=1}^{m} B^{(\mathbf{k})}, \qquad \mathbf{1}_{B_{\mathbf{y}}}(\mathbf{z}) - y_1 \cdots y_s = \sum_{k_1,\ldots,k_s=1}^{m} \big(\mathbf{1}_{B^{(\mathbf{k})}}(\mathbf{z}) - P^{-1}_{\mathbf{k}}\big). \tag{22}$$

Consider the following condition:

$$H^{\boldsymbol{\Sigma}}_{\mathcal{P}}(n, \mathbf{x}) \in B^{(\mathbf{k})}. \tag{23}$$

In order to express this condition in terms of the sequence $(H_{\mathcal{P}}(n))_{n\geqslant 1}$, we will construct boundary points $u_1, \ldots, u_s$. Next we will construct auxiliary sequences $\mathbf{u}^{(\mathbf{k})}$, $\check{u}^{(\mathbf{k})}$, $A_{\mathbf{k}}, \ldots$ Applying (18), we will get in (26) the solution of (23).

Let $\mathbf{u} = (u_1, \ldots, u_s)$, $u_i = \sum_{j\geqslant 1}^{\tau_{i,m}} u_{i,j}\, \widetilde{P}^{-1}_{i,j}$ with $u_{i,j} = \sigma^{-1}_{i,j}(y_{i,j})$, $u^*_{i,j} = \sigma^{-1}_{i,j}(0)$,

$$\mathbf{u}^{(\mathbf{k})} = \big(u_1^{(k_1)}, \ldots, u_s^{(k_s)}\big), \qquad u_i^{(k_i)} = \sum_{j=1}^{\tau_{i,k_i}-1} u_{i,j}\, \widetilde{P}^{-1}_{i,j} + u^*_{i,\tau_{i,k_i}}\, \widetilde{P}^{-1}_{\tau_{i,k_i}},$$

$$\check{u}^{(\mathbf{k})} \equiv \sum_{i=1}^{s} M_{i,\mathbf{k}} P_{\mathbf{k}} P^{-1}_{i,k_i} \Bigg(\sum_{j=1}^{\tau_{i,k_i}-1} u_{i,j}\, \widetilde{P}_{i,j-1} + u^*_{i,\tau_{i,k_i}}\, \widetilde{P}_{i,\tau_{i,k_i}-1} \Bigg) \pmod{P_{\mathbf{k}}},$$

$$\check{u}_{\mathbf{k}} \equiv \sum_{i=1}^{s} M_{i,\mathbf{k}} P_{\mathbf{k}} P^{-1}_{i,k_i} \sum_{j=1}^{\tau_{i,k_i}} u_{i,j}\, \widetilde{P}_{i,j-1} \pmod{P_{\mathbf{k}}}, \qquad \check{u}^{(\mathbf{k})},\, \check{u}_{\mathbf{k}} \in [0,\, P_{\mathbf{k}}). \tag{24}$$

According to (9)–(14), we have $p_{i,\tau_{i,k_i}} = p_i$, $k_i = 1, \ldots, m$, $i = 1, \ldots, s$. By (9), we get $a_{i,\tau_{i,k_i}} \equiv \sigma^{-1}_{i,\tau_{i,k_i}}(0) - \sigma^{-1}_{i,\tau_{i,k_i}}(1) \equiv u^*_{i,\tau_{i,k_i}} - u_{i,\tau_{i,k_i}} \pmod{p_i}$.

From (16), we obtain $a_{i,\tau_{i,k_i}} = \mathfrak{a}_i$, $k_i = 1, \ldots, m$, $i = 1, \ldots, s$. Hence

$$\breve{u}^{(\mathbf{k})} \equiv \breve{u}_{\mathbf{k}} + A_{\mathbf{k}} \ (\bmod\ P_{\mathbf{k}}), \qquad \text{where} \quad A_{\mathbf{k}} \equiv \sum_{i=1}^{s} M_{i,\mathbf{k}} P_{\mathbf{k}} p_i^{-1} \mathfrak{a}_i \ (\bmod\ P_{\mathbf{k}}) \tag{25}$$

with $A_{\mathbf{k}} \in [0, P_{\mathbf{k}})$.

Let $\mathbf{w} = (w_1, \ldots, w_s) = H_{\mathfrak{P}}^{\mathbf{\Sigma}}(n, \mathbf{x}) = \widetilde{\mathbf{\Sigma}}(T_{\mathfrak{P}}^n(\mathbf{x}))$. We see from (21) and (24) that

$$\begin{aligned}
\mathbf{w} \in B^{(\mathbf{k})} \quad &\Longleftrightarrow \quad w_{i,j} = y_{i,j}, \quad j \in [1, \tau_{i,k_i}), \qquad w_{i,\tau_{i,k_i}} = 0, \quad i \in [1, s] \\
&\Longleftrightarrow \quad \begin{array}{ll} \sigma_{i,j}(w_{i,j}) = u_{i,j}, & 1 \leqslant j \leqslant \tau_{i,k_i} - 1, \\ \sigma_{i,j}(w_{i,\tau_{i,k_i}}) = u^{*}_{i,\tau_{i,k_i}}, & i = 1, \ldots, s \end{array} \\
&\Longleftrightarrow \quad [T_{\mathfrak{P}}^n(\mathbf{x})]_{\boldsymbol{\tau}_{\mathbf{k}}} = \mathbf{u}^{(\mathbf{k})}.
\end{aligned}$$

Applying (18), (19), (20), (24) and (25), we have

$$\begin{aligned}
H_{\mathfrak{P}}^{\mathbf{\Sigma}}(n, \mathbf{x}) \in B^{(\mathbf{k})} \quad &\Longleftrightarrow \quad [T_{\mathfrak{P}}^n(\mathbf{x})]_{\boldsymbol{\tau}_{\mathbf{k}}} = \mathbf{u}^{(\mathbf{k})} \\
&\Longleftrightarrow \quad [H_{\mathfrak{P}}(n + W_{\mathbf{m}}(\mathbf{x}))]_{\boldsymbol{\tau}_{\mathbf{k}}} = \mathbf{u}^{(\mathbf{k})} \\
&\Longleftrightarrow \quad n + W_{\mathbf{m}}(\mathbf{x}) \equiv \breve{u}^{(\mathbf{k})} \ (\bmod\ P_{\mathbf{k}}) \\
&\Longleftrightarrow \quad n \equiv v_m + A_{\mathbf{k}} \ (\bmod\ P_{\mathbf{k}}),
\end{aligned}$$

where $v_m \equiv -W_{\mathbf{m}}(\mathbf{x}) + \breve{\mathbf{u}}_{\mathbf{m}} \equiv -W_{\mathbf{m}}(\mathbf{x}) + \breve{\mathbf{u}}_{\mathbf{k}} \ (\bmod\ P_{\mathbf{k}})$ and $v_m \in [0, P_{\mathbf{m}})$.

Hence

$$H_{\mathfrak{P}}^{\mathbf{\Sigma}}(n, \mathbf{x}) \in B^{(\mathbf{k})} \quad \Longleftrightarrow \quad n \equiv v_m + A_{\mathbf{k}} \ (\bmod\ P_{\mathbf{k}}), \quad v_m \in [0, P_{\mathbf{m}}), \ n \geqslant 0. \tag{26}$$

2.4 Completion of the proof of Theorem

Bearing in mind that

$$\max_{1 \leqslant M \leqslant N} MD^{*}\big((H_{\mathfrak{P}}^{\mathbf{\Sigma}}(n, \mathbf{x}))_{n=1}^{M}\big) \geqslant N^{-1} \sum_{1 \leqslant M \leqslant N} MD^{*}\big((H_{\mathfrak{P}}^{\mathbf{\Sigma}}(n, \mathbf{x}))_{n=1}^{M}\big),$$

we get that it is sufficient to find the lower bound of the main value of discrepancy function to prove Theorem.

Lemma 1 *Let*

$$\alpha_m = \frac{1}{P_{\mathbf{m}}} \sum_{M=1}^{P_{\mathbf{m}}} \Delta\big(B_{\mathbf{y}}, (H_{\mathfrak{P}}^{\mathbf{\Sigma}}(n, \mathbf{x}))_{n=v_m}^{v_m+M-1}\big). \tag{27}$$

Then

$$\alpha_m = \sum_{1 \leqslant k_1,\ldots,k_s \leqslant m} \left(\frac{1}{2} - \frac{A_{\mathbf{k}}}{P_{\mathbf{k}}} - \frac{1}{2P_{\mathbf{k}}} \right). \tag{28}$$

Proof Let $\mathcal{H}_n = H_{\mathfrak{P}}^{\mathbf{\Sigma}}(n, \mathbf{x})$. Using (26), we have

$$\sum_{n=v_m+M_1 P_{\mathbf{k}}}^{v_m+(M_1+1)P_{\mathbf{k}}-1} \left(\mathbf{1}_{B^{(\mathbf{k})}}(\mathcal{H}_n) - P_{\mathbf{k}}^{-1} \right) = 0 \tag{29}$$

and

$$\begin{aligned}
\sum_{n=v_m+M_1 P_{\mathbf{k}}}^{v_m+M_1 P_{\mathbf{k}}+M_2-1} \left(\mathbf{1}_{B^{(\mathbf{k})}}(\mathcal{H}_n) - P_{\mathbf{k}}^{-1} \right) &= \sum_{n \in [v_m, v_m+M_2)} \left(\mathbf{1}_{B^{(\mathbf{k})}}(\mathcal{H}_n) - P_{\mathbf{k}}^{-1} \right) \\
&= \sum_{\substack{n \in [v_m, v_m+M_2) \\ n = v_m + A_{\mathbf{k}}}} 1 - M_2 P_{\mathbf{k}}^{-1} \\
&= \mathbf{1}_{[0,M_2)}(A_{\mathbf{k}}) - M_2 P_{\mathbf{k}}^{-1},
\end{aligned}$$

with $M_1 \geqslant 0$ and $M_2 \in [0, P_{\mathbf{k}})$, $M_1, M_2 \in \mathbb{Z}$. From (1) and (22), we get

$$\begin{aligned}
\Delta\left(B_{\mathbf{y}}, (\mathcal{H}_n)_{n=v_m}^{v_m+M-1}\right) &= \sum_{n=v_m}^{v_m+M-1} \left(\mathbf{1}_{B_{\mathbf{y}}}(\mathcal{H}_n) - y_1 \cdots y_s \right) = \sum_{k_1,\ldots,k_s=1}^{m} \rho(\mathbf{k}, M), \\
&\text{where} \quad \rho(\mathbf{k}, M) = \sum_{n=v_m}^{v_m+M-1} \left(\mathbf{1}_{B^{(\mathbf{k})}}(\mathcal{H}_n) - P_{\mathbf{k}}^{-1} \right).
\end{aligned} \tag{30}$$

By (27), we obtain

$$\alpha_m = \sum_{1 \leqslant k_1,\ldots,k_s \leqslant m} \alpha_{m,\mathbf{k}}, \quad \text{where} \quad \alpha_{m,\mathbf{k}} = \frac{1}{P_{\mathbf{m}}} \sum_{M=1}^{P_{\mathbf{m}}} \rho(\mathbf{k}, M). \tag{31}$$

Bearing in mind (29)–(30), we derive

$$\begin{aligned}
\alpha_{m,\mathbf{k}} = \frac{1}{P_{\mathbf{m}}} \sum_{M_1=0}^{P_{\mathbf{m}}/P_{\mathbf{k}}-1} \sum_{M_2=1}^{P_{\mathbf{k}}} \Bigg(&\sum_{n=v_m}^{v_m+M_1 P_{\mathbf{k}}-1} \left(\mathbf{1}_{B^{(\mathbf{k})}}(\mathcal{H}_n) - P_{\mathbf{k}}^{-1} \right) \\
&+ \sum_{n=v_m+M_1 P_{\mathbf{k}}}^{v_m+M_1 P_{\mathbf{k}}+M_2-1} \left(\mathbf{1}_{B^{(\mathbf{k})}}(\mathcal{H}_n) - P_{\mathbf{k}}^{-1} \right) \Bigg)
\end{aligned}$$

$$= \frac{1}{P_{\mathbf{m}}} \sum_{M_1=0}^{P_{\mathbf{m}}/P_{\mathbf{k}}-1} \sum_{M_2=1}^{P_{\mathbf{k}}} \left(\mathbf{1}_{[0,M_2)}(A_{\mathbf{k}}) - M_2 P_{\mathbf{k}}^{-1}\right)$$
$$= \frac{1}{P_{\mathbf{k}}} \sum_{M_2=1}^{P_{\mathbf{k}}} \left(\mathbf{1}_{[0,M_2)}(A_{\mathbf{k}}) - M_2 P_{\mathbf{k}}^{-1}\right)$$
$$= \frac{P_{\mathbf{k}} - A_{\mathbf{k}}}{P_{\mathbf{k}}} - \frac{P_{\mathbf{k}}(P_{\mathbf{k}}+1)}{2P_{\mathbf{k}}^2} = \frac{1}{2} - \frac{A_{\mathbf{k}}}{P_{\mathbf{k}}} - \frac{1}{2P_{\mathbf{k}}}.$$

Using (31), we have

$$\alpha_m = \sum_{1 \leqslant k_1,\dots,k_s \leqslant m} \left(\frac{1}{2} - \frac{A_{\mathbf{k}}}{P_{\mathbf{k}}} - \frac{1}{2P_{\mathbf{k}}}\right). \qquad \square$$

Lemma 2 *With notations as above,*

$$|\alpha_m| \geqslant \frac{m^s}{4p_0} \quad \textit{for} \quad m \geqslant 2p_0.$$

Proof From (16) and (25), we get

$$[0,1) \ni \frac{A_{\mathbf{k}}}{P_{\mathbf{k}}} \equiv \sum_{1 \leqslant i \leqslant s} \frac{M_{i,\mathbf{k}} P_{\mathbf{k}} p_i^{-1} \mathfrak{a}_i}{P_{\mathbf{k}}} \equiv \frac{c_1 \mathfrak{a}_1}{p_1} + \cdots + \frac{c_s \mathfrak{a}_s}{p_s} \pmod 1.$$

Applying (17) and (28), we derive

$$\alpha_m = m^s \left(\frac{1}{2} - \{\alpha\}\right) - \sum_{1 \leqslant k_1,\dots,k_s \leqslant m} \frac{1}{2P_{\mathbf{k}}}, \qquad \text{where} \quad \alpha = \frac{d_1}{\widehat{p}_1} + \cdots + \frac{d_s}{\widehat{p}_s}, \tag{32}$$

$(d_i, \widehat{p}_i) = 1$, $\widehat{p}_i > 1$, $i = 1, \dots, s$, and $\{x\}$ is the fractional part of x. We have that if $\widehat{p}_0 = \widehat{p}_1 \widehat{p}_2 \cdots \widehat{p}_s \not\equiv 0 \pmod 2$ then $\alpha \not\equiv 1/2 \pmod 1$. Let $\widehat{p}_\nu \equiv 0 \pmod 2$ for some $\nu \in [1, s]$, and let $\alpha \equiv 1/2 \pmod 1$. Then

$$\frac{\widehat{p}_\nu/2 - d_\nu}{p_\nu} \equiv \sum_{\substack{1 \leqslant i \leqslant s \\ i \neq \nu}} \frac{d_i}{\widehat{p}_i} \pmod 1, \qquad a_1 \equiv a_2 \pmod{p_0},$$

with $a_1 = \widehat{p}_0(\widehat{p}_\nu/2 - d_\nu)/\widehat{p}_\nu$ and $a_2 = \sum_{i \neq \nu} \widehat{p}_0 d_i / \widehat{p}_i$. Let $j \in [1, s]$ and $j \neq \nu$. We see that $a_1 \equiv 0 \pmod{\widehat{p}_j}$ and $a_2 \not\equiv 0 \pmod{\widehat{p}_j}$. We get a contradiction. Hence $\alpha \not\equiv 1/2 \pmod 1$. We have

$$\left|\frac{1}{2} - \{\alpha\}\right| = \left|\frac{1}{2} - \left\{\frac{d_1}{\widehat{p}_1} + \cdots + \frac{d_s}{\widehat{p}_s}\right\}\right| = \frac{|a|}{2\widehat{p}_0} \qquad \text{for some integer } a.$$

Thus $|1/2 - \{\alpha\}| \geqslant 1/(2\widehat{p}_0) \geqslant 1/(2p_0)$ with $p_0 = p_1 \cdots p_s$, $(p_0, \widehat{p}_0) = \widehat{p}_0$.

Bearing in mind that $P_{\mathbf{k}} \geqslant 2^{k_1+k_2+\cdots+k_s}$, we obtain from (32) that

$$|\alpha_m| \geqslant \frac{m^s}{2p_0} - \frac{1}{2} = \frac{m^s}{2p_0}\left(1 - \frac{p_0}{m^s}\right) \geqslant \frac{m^s}{4p_0} \quad \text{for} \quad m \geqslant 2p_0. \tag{33}$$

This completes the proof. □

Going back to the proof of Theorem, by (7) and (13), we get

$$m^s(4p_0)^{-1} \geqslant (4p_0)^{-1}C_1^{-s}\log_2^s N \geqslant 2C^{-1}\log_2^s N,$$
$$m \geqslant C_1^{-1}\log_2 N \geqslant 2p_0,$$

where $C_1 = sq_0^{s+1}\log_2 q_0$, $C = 8q_0^s C_1^s$ and $q_0^s \geqslant p_0$.

Using (15) and (26), we have that $v_m + P_{\boldsymbol{\tau} m} \leqslant 2P_{\mathbf{m}} \leqslant N$. According to (33), (27) and (2), we obtain

$$\begin{aligned} 2C^{-1}\log_2^s N \leqslant m^s(4p_0)^{-1} \leqslant |\alpha_m| &\leqslant \max_{1\leqslant M\leqslant P_{\mathbf{m}}} MD^*\left(\left(H_{\mathfrak{P}}^{\boldsymbol{\Sigma}}(n,\mathbf{x})\right)_{n=v_m}^{v_m+M-1}\right) \\ &\leqslant \max_{1\leqslant L, L+M\leqslant 2P_{\mathbf{m}}} MD^*\left(\left(H_{\mathfrak{P}}^{\boldsymbol{\Sigma}}(n,\mathbf{x})\right)_{n=L}^{L+M-1}\right) \\ &\leqslant 2\max_{1\leqslant M\leqslant N} MD^*\left(\left(H_{\mathfrak{P}}^{\boldsymbol{\Sigma}}(n,\mathbf{x})\right)_{n=1}^{M}\right). \end{aligned}$$

Hence Theorem is proved.

Acknowledgments The author is very grateful to the referee for corrections and suggestions which improved this paper.

References

1. Beck, J., Chen, W.W.L.: Irregularities of Distribution. Cambridge Tracts in Mathematics, vol. 89. Cambridge University Press, Cambridge (1987)
2. Bilyk, D.: On Roth's orthogonal function method in discrepancy theory. Unif. Distrib. Theory **6**(1), 143–184 (2011)
3. Faure, H.: Discrépances de suites associées à un système de numération (en dimension un). Bull. Soc. Math. France **109**(2), 143–182 (1981)
4. Faure, H., Chaix, H.: Minoration de discrépance en dimension deux. Acta Arith. **76**(2), 149–164 (1996)
5. Faure, H., Kritzer, P., Pillichshammer, F.: From van der Corput to modern constructions of sequences for quasi-Monte Carlo rules. Indag. Math. (N.S.) **26**(5), 760–822 (2015)
6. Halton, J.H.: On the efficiency of certain quasi-random sequences of points in evaluating multi-dimensional integrals. Numer. Math. **2**, 84–90 (1960)
7. Hellekalek, P.: Regularities in the distribution of special sequences. J. Number Theory **18**(1), 41–55 (1984)
8. Levin, M.B.: On the lower bound in the lattice point remainder problem for a parallelepiped. Discrete Comput. Geom. **54**(4), 826–870 (2015)
9. Levin, M.B.: On the lower bound of the discrepancy of Halton's sequences: I. C. R. Math. Acad. Sci. Paris Sér. I Math. (to appear)
10. Levin, M.B.: On the lower bound of the discrepancy of (t, s) sequences: II (2015). arXiv:1505.04975v2

11. Niederreiter, H.: Random Number Generation and Quasi-Monte Carlo Methods. CBMS-NSF Regional Conference Series in Applied Mathematics, vol. 63. Society for Industrial and Applied Mathematics (SIAM), Philadelphia (1992)
12. van der Corput, J.G.: Verteilungsfunktionen I-II. Proceedings. Akadamie van Wetenschappen Amsterdam **38**, 813–821, 1058–1066 (1935)