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ORIGINAL ARTICLE

Normal families concerning partially shared and proximate values

Rahul Kumar[1] · **Nikhil Bharti**[1]

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Abstract

In this paper, we discuss normality of two families of meromorphic functions concerning partially shared values. Precisely, we proved: Let $\mathcal{F}$ and $\mathcal{G}$ be two families of meromorphic functions on a domain D such that all zeros of each $f \in \mathcal{F}$ have multiplicities at least $k+1$, where $k \geq 1$ is an integer and let a, b and c be three finite distinct complex numbers. Assume that $\mathcal{G}$ is normal in D and for each $f \in \mathcal{F}$, there exists $g \in \mathcal{G}$ such that a, b, c are partially shared values of $f^{(k)}$ and g. Then $\mathcal{F}$ is normal in D. We also give examples to show that various conditions in the hypothesis of this theorem cannot be weakened. Furthermore, we introduce a notion of proximate values of meromorphic functions and obtain some normality criteria involving partially proximate values which generalize an established result of Liu, Li and Pang (Acta Math Sinica English series 29 (1) (2013), 151-158.)

Keywords Normal families · Partially shared values · Partially proximate values · Meromorphic functions

1 Introduction and main results

We set the following notations throughout the paper:

- $\mathcal{H}(D)$: the class of all holomorphic functions on the domain $D \subseteq \mathbb{C}$
- $\mathcal{M}(D)$: the class of all meromorphic functions on the domain $D \subseteq \mathbb{C}$
- $\mathbb{C}_\infty$: the extended complex plane
- $\mathbb{D}$: the open unit disk in $\mathbb{C}$.

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✉ Nikhil Bharti
nikhilbharti94@gmail.com

Rahul Kumar
rktkp5@gmail.com

[1] Department of Mathematics, University of Jammu, Jammu, Jammu and Kashmir 180006, India

A subfamily $\mathcal{F} \subset \mathcal{M}(D)$ is said to be *normal* in D if each sequence $\{f_n\} \subseteq \mathcal{F}$ eventuates a locally uniformly convergent subsequence on D with respect to the spherical metric. It is not difficult to see that the limit functions of the aforementioned subsequences are either meromorphic on D or identically ∞. Normality of a subfamily of $\mathcal{H}(D)$ is defined similarly with respect to the Euclidean metric (see [10, 12]).

Pioneered by a French mathematician Paul Montel [8], the theory of normal families of meromorphic functions has played a significant role in complex analysis ever since its creation in 1912. One of the most fascinating sufficient conditions for a family $\mathcal{F}$ of meromorphic functions to be normal in a domain D is that each $f \in \mathcal{F}$ omits three fixed distinct values in $\mathbb{C}_\infty$. Schiff [10, page 74] documented this result as the Fundamental Normality Test (FNT). Subsequently, Carathéodory [5, page 202] proved that the omitted values do not need to be fixed and they may depend on the particular function in the family as long as these omitted values are uniformly separated (see [1, Theorem 8.4]). Recently, Beardon and Minda [2] exposed that Montel had published an extension to his three omitted value theorem which gave a necessary and sufficent condition for a family of meromorphic functions to be normal: *A family $\mathcal{F} \subset \mathcal{M}(D)$ is normal in D iff there exists four ϵ-separated values in $\mathbb{C}_\infty$ such that their preimages are equi-separated on compacta.* However, this result was not well documented and there was a small error in Montel's proof which was ameliorated by Beardon and Minda (see [2, Theorem 4]).

Another intriguing idea is to look at normality from a perspective of shared values. A value $a \in \mathbb{C}_\infty$ is said to be a *shared value* of functions $f, g \in \mathcal{M}(D)$ if $f(z) = a \iff g(z) = a$ on D (without regard to multiplicities). If $f(z) = a \Rightarrow g(z) = a$, then we say that a is a partially shared value of f and g. In this direction, Schwick [11] was the first to evince correspondence between normality and shared values. Precisely, he proved that

Theorem 1 [11, *Theorem* 2] *Let $\mathcal{F} \subset \mathcal{M}(D)$ and a_1, a_2, a_3 be three distinct finite complex numbers. If for every $f \in \mathcal{F}, f$ and f' share the values a_1, a_2, a_3, then $\mathcal{F}$ is normal in D.*

Furthermore, a value $b \in \mathbb{C}_\infty$ is said to be a *proximate value* of f and g on D if there exists an $\epsilon > 0$ such that $\sigma(f(z), b) < \epsilon \iff \sigma(g(z), b) < \epsilon$ on D. Similarly, we call b a *partially proximate value* of f and g if there exists an $\epsilon > 0$ such that $\sigma(f(z), b) < \epsilon \Rightarrow \sigma(g(z), b) < \epsilon$, where σ denotes the spherical metric. It is easy to see that every shared (partially shared) value is a proximate (partially proximate) value.

In [7], Liu, Li and Pang posed the following interesting problem:

Problem 2 Given two families of meromorphic functions which share some values. If one is normal, is the other normal?

An affirmative answer to Problem 2 is obtained as below:

Theorem 3 [7, *Theorem* 1.1] *Let $\mathcal{F}, \mathcal{G} \subset \mathcal{M}(D)$ and a_1, a_2, a_3, a_4 be four distinct complex numbers. If $\mathcal{G}$ is normal, and for every $f \in \mathcal{F}$, there exists $g \in \mathcal{G}$ such that f and g share the values a_1, a_2, a_3, a_4, then $\mathcal{F}$ is normal in D.*

It is natural to ask whether Problem 2 has an affirmative answer if instead of shared values, we only have partially shared or proximate or even partially proximate values. In this paper, we dealt with such problems and obtained the following:

Theorem 4 *Let $\mathcal{F}, \mathcal{G} \subset \mathcal{H}(D)$ be such that all zeros of each $f \in \mathcal{F}$ have multiplicities at least $k+1$, where $k \geq 1$ is an integer. Let a and b be two nonzero finite distinct complex numbers. Assume that $\mathcal{G}$ is normal in D and for each $f \in \mathcal{F}$, there exists $g \in \mathcal{G}$ such that a and b are partially shared values of $f^{(k)}$ and g, that is,*

(i) $f^{(k)}(z) = a \Rightarrow g(z) = a$; *and*
(ii) $f^{(k)}(z) = b \Rightarrow g(z) = b$.

Then $\mathcal{F}$ is normal in D.

The following examples corroborate that various conditions in the hypothesis of Theorem 4 are not redundant.

Example 1 Let $k = 1$. Take $\mathcal{F} = \{f_n\}$ and $\mathcal{G} = \{g_n\}$, where

$$f_n(z) = n(z-1)^2 \text{ and } g_n(z) = \left(\frac{2n}{2n+1}\right)z$$

on the domain $D = \left\{z : |z| > \frac{1}{2}\right\}$. Clearly, the family $\mathcal{G}$ is normal in D and each $f_n \in \mathcal{F}$ has a single zero of multiplicity 2. Furthermore, $f_n'(z) = 2n(z-1)$. By simple calculation, it is easy to see that

$$f_n'(z) = 1 \Rightarrow g_n(z) = 1.$$

But $\mathcal{F}$ is not normal in D.

This example shows that none of the conditions (i) and (ii) in Theorem 4 can be dropped.

Example 2 Let $k = 1$, $\mathcal{F} = \{f_n : f_n(z) = e^{nz}\}$ and $\mathcal{G} = \{g_n\}$, where $g_n \equiv b$, $b \neq 0$, $b \in \mathbb{C}$ on $\mathbb{D}$. Since $f_n'(z) = ne^{nz}$ is a transcendental entire function which omits 0, it assumes every nonzero finite complex value infinitely often. Therefore, it follows vacuously that $f_n'(z) = 0 \Rightarrow g_n(z) = 0$ and $f_n'(z) = b \Rightarrow g_n(z) = b$. But $\mathcal{G}$ is normal and $\mathcal{F}$ is not normal in $\mathbb{D}$.

Thus the condition that a and b are nonzero distinct complex numbers is necessary.

Example 3 Let $k = 2$. Consider $\mathcal{F} = \{f_n : f_n(z) = nz^2\}$ and $\mathcal{G} = \{g_n\}$, where $g_n(z) = z^n$, on the domain $\mathbb{D}$. Then $f_n^{(k)}(z) = 2n$. Obviously, $f_n^{(k)}(z) = a_i \Rightarrow g_n(z) = a_i$, where a_i $(i = 1, 2)$ is any odd integer. However, $\mathcal{G}$ is normal in $\mathbb{D}$ and $\mathcal{F}$ is not normal in $\mathbb{D}$.

This example shows that multiplicities of zeros of each $f \in \mathcal{F}$ cannot be less than $k + 1$. Moreover, a simple modification of Example 2 reveals that Theorem 4 does not hold for $k = 0$. It is readily seen from the following:

Example 4 Let $\mathcal{F} = \{f_n\}$, where $f_n(z) = e^{nz} + 1$ and let $\mathcal{G} = \{g_n\}$, where $g_n(z) \equiv b, b \neq 1, b \in \mathbb{C}$ on the domain $\mathbb{D}$. By the same token as in Example 2, we have $f_n(z) = 1 \Rightarrow g_n(z) = 1$ and $f_n(z) = b \Rightarrow g_n(z) = b$. But $\mathcal{G}$ is normal and $\mathcal{F}$ is not normal in $\mathbb{D}$.

The meromorphic analogue of Theorem 4 obtained as below:

Theorem 5 *Let $\mathcal{F}, \mathcal{G} \subset \mathcal{M}(D)$ be such that all zeros of each $f \in \mathcal{F}$ have multiplicities at least $k + 1$, where $k \geq 1$ is an integer. Let a, b and c be three finite distinct complex numbers. Assume that $\mathcal{G}$ is normal in D and for each $f \in \mathcal{F}$, there exists $g \in \mathcal{G}$ such that*

(i) $f^{(k)}(z) = a \Rightarrow g(z) = a$;
(ii) $f^{(k)}(z) = b \Rightarrow g(z) = b$; *and*
(iii) $f^{(k)}(z) = c \Rightarrow g(z) = c$.

Then $\mathcal{F}$ is normal in D.

Example 5 The idea behind this example is inspired by [6, Example 1.4]. Let n, k be positive integers and define a_n by

$$\frac{k!\, a_n^{k+1}}{n} = 1.$$

Set

$$f_n(z) = \frac{(a_n z + 1)^{k+1}}{nz},\ n \in \mathbb{N}$$

and let $\mathcal{F} = \{f_n\}$ on $\mathbb{D}$. Clearly, each function in $\mathcal{F}$ has a single zero of multiplicity $k + 1$. Also,

$$f_n^{(k)}(z) = 1 + \frac{(-1)^k\, k!}{nz^{k+1}}.$$

Evidently, $f_n^{(k)}(z) \neq 1,\ \forall f_n \in \mathcal{F}$.

Take $\mathcal{G} = \{g_n\}$ such that $g_n(z) = b, b \neq 1, b \in \mathbb{C}$ for all $z \in \mathbb{D}$ and $n \in \mathbb{N}$. Then $\mathcal{G}$ is normal in $\mathbb{D}$. It follows that $f_n^{(k)}(z) = 1 \Rightarrow g_n(z) = 1$.

Also, in particular for $k = 2$, $f_n^{(2)}(z) = b \Rightarrow g_n(z) = b$. But $\mathcal{F}$ is not normal in $\mathbb{D}$.

This example shows that the requirement of three partially shared values of $f^{(k)}$ and g in Theorem 5 is necessary.

Example 6 Let $\mathcal{F} = \{f_n\}$ and $\mathcal{G} = \{g_n\}$, where

$$f_n(z) = \tan nz \text{ and } g_n(z) = b,\ b \in \mathbb{C},\ b \neq i, -i$$

on the domain $\mathbb{D}$. Clearly

(i) $f_n(z) = i \Rightarrow g_n(z) = i$;
(ii) $f_n(z) = -i \Rightarrow g_n(z) = -i$; and
(iii) $f_n(z) = b \Rightarrow g_n(z) = b$.

But $\mathcal{F}$ is not normal in $\mathbb{D}$ and $\mathcal{G}$ is normal in $\mathbb{D}$.

Example 6 shows that Theorem 5 does not hold if $k = 0$. In view of Example 3, it follows immediately that the condition 'all zeros of each $f \in \mathcal{F}$ have multiplicities at least $k+1,\ k \geq 1$' in Theorem 5 cannot be weakened.

What follows is an generalization of Theorem 3:

Theorem 6 *Let $\mathcal{F}, \mathcal{G} \subset \mathcal{M}(D)$ and $\epsilon > 0$. Assume that $\mathcal{G}$ is normal on D and for each $f \in \mathcal{F}$, there exist $g \in \mathcal{G}$ and four distinct values a_1, a_2, a_3, a_4 in $\mathbb{C}$ with $\sigma(a_i, a_j) > \epsilon$, $(i \neq j,\ i,j \in \{1,2,3,4\})$ such that*

$$\sigma(f(z), a_i) < \frac{\epsilon}{3} \Rightarrow \sigma(g(z), a_i) < \frac{\epsilon}{3},\ i = 1,2,3,4.$$

Then $\mathcal{F}$ is normal in D.

Note that in Theorem 6, a_1, a_1, a_3, a_4 are partially proximate values of each $f \in \mathcal{F}$ and $g \in \mathcal{G}$. If these four values happen to be shared values, then Theorem 6 reduces to Theorem 3. In fact, Theorem 6 is a special case of the following more general result which shows that the four partially proximate values in Thoerem 6 may change with change of $f \in \mathcal{F}$.

Theorem 7 *Let $\mathcal{F}, \mathcal{G} \subset \mathcal{M}(D)$ and $\epsilon > 0$. Assume that $\mathcal{G}$ is normal in D and for each $f \in \mathcal{F}$, there exist $g \in \mathcal{G}$ and four distinct values $a_{1f}, a_{2f}, a_{3f}, a_{4f}$ with $\sigma(a_{if}, a_{jf}) > \epsilon$, $(i \neq j,\ i,j \in \{1,2,3,4\})$ such that*

$$\sigma(f(z), a_{if}) < \frac{\epsilon}{3} \Rightarrow \sigma(g(z), a_{if}) < \frac{\epsilon}{3},\ i = 1,2,3,4.$$

Then $\mathcal{F}$ is normal in D.

The following theorem asserts that the requirement of four values in Theorem 7 can be reduced to three under suitable conditions.

Theorem 8 *Let $\mathcal{F}, \mathcal{G} \subset \mathcal{M}(D)$ and $\epsilon > 0$. Suppose that a is a totally ramified value of f for each $f \in \mathcal{F}$. Further, assume that $\mathcal{G}$ is normal in D and for each*

$f \in \mathcal{F}$, there exist $g \in \mathcal{G}$ and three distinct values a_{1f}, a_{2f}, a_{3f} with $\sigma(a_{if}, a_{jf}) > \epsilon$, $(i \neq j,\ i,j \in \{1,2,3\})$ and $\sigma(a_{if}, a) > \epsilon$ such that

$$\sigma(f(z), a_{if}) < \frac{\epsilon}{3} \Rightarrow \sigma(g(z), a_{if}) < \frac{\epsilon}{3},\ i = 1,2,3.$$

Then $\mathcal{F}$ is normal in D.

2 Preparations for the proof of main results

In this section, we describe some preliminary results that are crucial to prove main results of this paper. First recall that if f be a meromorphic function in $\mathbb{C}$ and $a \in \mathbb{C}_\infty$, then a is said to be totally ramified value of f if $f - a$ has no simple zeros. Nevanlinna (see [3, page 84]) proved the following widely known result concerning multiplicities of a-points of a meromorphic function. This result plays a pivotal role in the proof of Theorem 8:

Theorem 9 (Nevanlinna's Theorem) *Let f be a non-constant meromorphic function, $a_1, a_2, \ldots, a_q \in \mathbb{C}_\infty$ and $m_1, m_2, \ldots, m_q \in \mathbb{N}$. Suppose that all a_j-points of f have multiplicity at least m_j, for $j = 1, 2, \ldots, q$. Then*

$$\sum_{j=1}^{q} \left(1 - \frac{1}{m_j}\right) \leq 2.$$

If f omits the value a_j, then $m_j = \infty$.

Lemma 10 [9, *Lemma* 2] *Let $\mathcal{F}$ be a family of meromorphic functions on the unit disk $\mathbb{D}$, all of whose zeros have multiplicities at least k, and suppose that there exists $M \geq 1$ such that for each $f \in \mathcal{F}$, $|f^{(k)}(z)| \leq M$ whenever $f(z) = 0$. If $\mathcal{F}$ is not normal on $\mathbb{D}$, then for each $0 \leq \alpha \leq k$, there exist*

(i) *a number $0 < r < 1$;*
(ii) *points $z_n : |z_n| < r$;*
(iii) *functions $f_n \in \mathcal{F}$; and*
(iv) *positive numbers $\rho_n \longrightarrow 0^+$,*

such that $g_n(\zeta) = \rho_n^{-\alpha} f_n(z_n + \rho_n \zeta) \stackrel{\sigma}{\longrightarrow} g(\zeta)$ locally uniformly on $\mathbb{C}$, where g is a non-constant meromorphic function on $\mathbb{C}$ such that for every $\zeta \in \mathbb{C}$, $g^{\#}(\zeta) \leq g^{\#}(0) = kM + 1$.

Here $g^{\#}(\zeta) = |g'(\zeta)|/[1 + |g(\zeta)|^2]$ is the spherical derivative of g and $\stackrel{\sigma}{\longrightarrow}$ indicates that the convergence is with respect to the spherical metric.

Lemma 10 is commonly known as Zalcman-Pang Lemma.

Lemma 11 [4, *Theorem* 3] *Let f be a transcendental meromorphic function of finite order on* $\mathbb{C}$, *all of whose zeros have multiplicity at least* $k+1$, *where k is a positive integer, then* $f^{(k)}$ *assumes every nonzero complex number b infinitely many times on* $\mathbb{C}$.

Lemma 12 [6, *Lemma* 2.2] *Let k be a positive integer and f be a meromorphic function on* $\mathbb{C}$ *such that* $f^{(k)}$ *omits two values in* $\mathbb{C}$. *Then* $f^{(k)}$ *is constant.*

3 Proof of main theorems

Proof of Theorem 4 Suppose that $\mathcal{F}$ is not normal on D. Then by Lemma 10, there exist sequences $\{z_n\} \subset D$, $z_n \longrightarrow z_0 \in D$, $\{f_n\} \subset \mathcal{F}$ and positive numbers $\rho_n \longrightarrow 0^+$ such that

$$h_n(\zeta) = \frac{f_n(z_n + \rho_n\zeta)}{\rho_n^k} \longrightarrow h(\zeta),$$

where h is a non-constant entire function, all of whose zeros have multiplicity at least $k+1$.

By hypothesis, there exists $g_n \in \mathcal{G}$ such that

$$f_n^{(k)}(z) = a \Rightarrow g_n(z) = a \text{ and } f_n^{(k)}(z) = b \Rightarrow g_n(z) = b.$$

Also, since $\mathcal{G}$ is normal on D, we can obtain a subsequence of $\{g_n\}$, again denoted by $\{g_n\}$, such that $g_n \longrightarrow g$ locally uniformly on D. Now we consider the following two cases:

Case I: If h is a polynomial, then $h^{(k)}$ is also a non-constant polynomial owing to the fact that each zero of h has multiplicity at least $k+1$. Thus $h^{(k)}$ assume the values a and b on $\mathbb{C}$. Suppose $h^{(k)}(\zeta_0) = a$ and $h^{(k)}(\zeta_0^*) = b$. Then by Hurwitz's theorem, there exist sequences $\zeta_n \longrightarrow \zeta_0$ and $\zeta_n^* \longrightarrow \zeta_0^*$ such that $h_n^{(k)}(\zeta_n) = a$ and $h_n^{(k)}(\zeta_n^*) = b$. This implies that $f_n^{(k)}(z_n + \rho_n\zeta_n) = a$ and $f_n^{(k)}(z_n + \rho_n\zeta_n^*) = b$. Hence $g_n(z_n + \rho_n\zeta_n) = a$ and $g_n(z_n + \rho_n\zeta_n^*) = b$. Taking the limit as $n \longrightarrow \infty$, we find that $g(z_0) = a$ and $g(z_0) = b$, a contradiction.

Case II: If h is a transcendental entire function, then in view of Lemma 11, $h^{(k)}$ assumes a and b on $\mathbb{C}$. By a similar argument as in Case I, we get a contradiction.

Proof of Theorem 5 Suppose that $\mathcal{F}$ is not normal on D. Then by Lemma 10, there exist sequences $\{z_n\} \subset D$, $z_n \longrightarrow z_0 \in D$, $\{f_n\} \subset \mathcal{F}$ and positive numbers $\rho_n \longrightarrow 0^+$ such that

$$h_n(\zeta) = \frac{f_n(z_n + \rho_n\zeta)}{\rho_n^k} \xrightarrow{\sigma} h(\zeta)$$

locally uniformly, where h is a non-constant meromorphic function, all of whose zeros have multiplicity at least $k+1$.

By hypothesis, there exists $g_n \in \mathcal{G}$ such that

(i) $f_n^{(k)}(z) = a \Rightarrow g_n(z) = a$;
(ii) $f_n^{(k)}(z) = b \Rightarrow g_n(z) = b$; and
(iii) $f_n^{(k)}(z) = c \Rightarrow g_n(z) = c$.

Also, since $\mathcal{G}$ is normal, we may assume that $g_n \xrightarrow{\sigma} g$ normally. Furthermore, since zeros of h have multiplicity at least $k+1$, we conclude that $h^{(k)}$ is also a non-constant meromorphic function. In view of Lemma 12, it follows that $h^{(k)}$ can omit at most one value in $\mathbb{C}$ and hence it must assume at least two values from a, b, c. Without loss of generality, suppose that $h^{(k)}(\zeta_0) = b$ and $h^{(k)}(\zeta_0^*) = c$. Then by Hurwitz's theorem, there exist sequences $\zeta_n \longrightarrow \zeta_0$ and $\zeta_n^* \longrightarrow \zeta_0^*$ such that $h_n^{(k)}(\zeta_n) = b$ and $h_n^{(k)}(\zeta_n^*) = c$. This implies that $f_n^{(k)}(z_n + \rho_n\zeta_n) = b$ and $f_n^{(k)}(z_n + \rho_n\zeta_n^*) = c$. Hence $g_n(z_n + \rho_n\zeta_n) = b$ and $g_n(z_n + \rho_n\zeta_n^*) = c$. Taking the limit as $n \longrightarrow \infty$, we find that $g(z_0) = b$ and $g(z_0) = c$, a contradiction.

Proof of Theorem 6 Follows immediately from Theorem 7.

Proof of Theorem 7 Suppose $\mathcal{F}$ is not normal at $z_0 \in \mathbb{D}$. Then by Lemma 10, there are sequences $\{z_n\} \subset D$ with $z_n \longrightarrow z_0$; $\{f_n\} \subset \mathcal{F}$ and positive numbers $\rho_n \longrightarrow 0^+$ such that

$$h_n(\zeta) = f_n(z_n + \rho_n\zeta) \xrightarrow{\sigma} h(\zeta)$$

locally uniformly, where h is a non-constant meromorphic function on $\mathbb{C}$. By given hypothesis, there exists $g_n \in \mathcal{G}$ and four values $a_{1f_n}, a_{2f_n}, a_{3f_n}, a_{4f_n}$ with $\sigma(a_{if_n}, a_{jf_n}) > \epsilon$ $(i \neq j)$ such that

$$\sigma(f_n(z), a_{if_n}) < \frac{\epsilon}{3} \Rightarrow \sigma(g_n(z), a_{if_n}) < \frac{\epsilon}{3} \ (i = 1, 2, 3, 4).$$

We may assume that $a_{if_n} \longrightarrow a_i$. Since $\sigma(a_{if_n}, a_{jf_n}) > \epsilon$, it follows that a_1, a_2, a_3 and a_4 are four distinct complex numbers. Moreover, since $\mathcal{G}$ is normal, we may assume that $g_n \longrightarrow g$ normally. We claim that h assumes at most one value from $\{a_1, a_2, a_3, a_4\}$. Suppose on the contrary that $h(\zeta_0) = a_1$ and $h(\zeta_0^*) = a_2$. Then by Hurwitz's theorem, there exist sequences $\zeta_n \longrightarrow \zeta_0$ and $\zeta_n^* \longrightarrow \zeta_0^*$ such that $f_n(z_n + \rho_n\zeta_n) - a_{1f_n} = 0$ and $f_n(z_n + \rho_n\zeta_n^*) - a_{2f_n} = 0$. Thus

$$\sigma(f_n(z_n + \rho_n\zeta_n), a_{1f_n}) < \frac{\epsilon}{3} \ \text{ and } \ \sigma(f_n(z_n + \rho_n\zeta_n^*), a_{2f_n}) < \frac{\epsilon}{3}.$$

By hypothesis, we have

$$\sigma(g_n(z_n + \rho_n\zeta_n), a_{1f_n}) < \frac{\epsilon}{3} \ \text{ and } \ \sigma(g_n(z_n + \rho_n\zeta_n^*), a_{2f_n}) < \frac{\epsilon}{3}.$$

Taking the limit as $n \longrightarrow \infty$, we obtain

$$\sigma(g(z_0), a_1) \leq \frac{\epsilon}{3} \ \text{ and } \ \sigma(g(z_0), a_2) \leq \frac{\epsilon}{3}.$$

Now $\sigma(a_1, a_2) \le \sigma(a_1, g(z_0)) + \sigma(g(z_0), a_2) \le 2\epsilon/3$ yields a contradiction. Therefore the claim is true and hence h is a constant, a contradiction.

Proof of Theorem 8 If $\mathcal{F}$ is not normal in D, then by the same token as in the proof of Theorem 7, we obtain sequences $\{z_n\} \subset D$ with $z_n \longrightarrow z_0$, $\{f_n\} \subset \mathcal{F}$, positive numbers $\rho_n \longrightarrow 0^+$ and values a_i $(i = 1, 2, 3)$ such that $a_{if} \longrightarrow a_i$ and

$$h_n(\zeta) = f_n(z_n + \rho_n \zeta) \xrightarrow{\sigma} h(\zeta)$$

locally uniformly, where h is a non-constant meromorphic function on $\mathbb{C}$. Moreover, h assumes at most one value from $\{a_1, a_2, a_3\}$. Without loss of generality, we may assume that h omits a_1 and a_2. Let m_1 and m_2 be the multiplicities of zeros of $h - a_1$ and $h - a_2$ respectively. Then $m_1 = m_2 = \infty$. Since a is a totally ramified value of f_n, it follows that a is a totally ramified value of h. If m_3 is the multiplicity of zeros of $h - a$, then clearly $m_3 \ge 2$.

By simple calculation, we find that

$$\sum_{j=1}^{3} \left(1 - \frac{1}{m_j}\right) > 2,$$

a contradiction to Nevanlinna's Theorem.

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Declarations

Conflict of interest The authors declare that they have no conflict of interests.

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