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ORIGINAL ARTICLE

Remark on "When are all the zeros of a polynomial real and distinct?"

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Abstract

The purpose of this note is to point out that Chamberland's theorem is implicitly contained in the elementary lore of the theory of orthogonal polynomials.

Keywords Geronimus–Wendroff's theorem · Polynomials · Real zeros

Mathematics Subject Classification 30C15

In [1], Chamberland proved the following theorem:

Let P be a polynomial of degree $n \geq 2$ with real coefficients. Then the zeros of P are real and distinct if and only if

$$\left(\left(P^{(n-j-1)}(x)\right)'\right)^2 - P^{(n-j-1)}(x)\left(P^{(n-j-1)}(x)\right)'' > 0 \tag{1}$$

for all $x \in \mathbb{R}$ and $j \in \{1, \dots, n-1\}$.

The purpose of this note is to point out that this result is implicitly contained in the elementary lore of the theory of orthogonal polynomials on the real line (OPRL).

($\Rightarrow$) By Geronimus–Wendroff's theorem [2, Exercise 5.5, p. 30],[1] there are probability measures on $\mathbb{R}$, μ_j, with finite moments so that $P_{j+1} = P^{(n-j-1)}$ and $P_j = P'_{j+1}$ are among the OPRL for μ_j. Since P_{j+1} and P_j have leading coefficients of the same sign, then (see [2, (4.13), p. 24])

[1] Geronimus–Wendroff's theorem is just a footnote in [J. Geronimus, On the trigonometric problem, Ann. of Math. 47 (1946) 742–761] whose proof is immediate from Favard's theorem.

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$$P'_{j+1}(x)P_j(x) - P'_j(x)P_{j+1}(x) > 0 \tag{2}$$

for all $x \in \mathbb{R}$ and $j \in \{1, \ldots, n-1\}$.

($\Leftarrow$) Set $P_j = P^{(n-j)}$ for all $j \in \{1, \ldots, n\}$. Without loss of generality we can assume P monic. Suppose that the zeros of P_j are real and distinct. From (2) we see that P_{j+1} has at least one zero between two zeros of P_j, and so P_{j+1} has at least $j-1$ real and distinct zeros. Suppose that the other two zeros of P_{j+1} are not real, and therefore they appear as a complex conjugate pair. Let a be the largest zero of P_j. Clearly, $P_{j+1}(x) > 0$ for all $x > a$. However, by (2), we have $P_{j+1}(a) < 0$, which leads to a contradiction. From this we conclude that if P_j has real and distinct zeros, then the same holds for P_{j+1}. Since $P_1 = P^{(n-1)}$ has a single real zero, we infer successively that $P_2, \ldots, P_n = P$ have real and distinct zeros.

We emphasize for the reader's convenience that any polynomial with real and distinct zeros is an element of a sequence of OPRL, and so any sentence starting with *"the zeros of a polynomial are real and distinct if and only if"* is virtually talking about an element of a certain sequence of OPRL.

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