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ORIGINAL ARTICLE

On Lat-Igusa-Todorov algebras

Marcos Barrios[1] · **Gustavo Mata**[1]

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Abstract
Lat-Igusa-Todorov algebras are a natural generalization of Igusa-Todorov algebras. They are defined using the generalized Igusa-Todorov functions given in Bravo et al. (J Algebra, 580:63–83, 2021) and also verify the finitistic dimension conjecture. In this article we give new ways to construct examples of Lat-Igusa-Todorov algebras. On the other hand we show an example of a family of algebras that are not Lat-Igusa-Todorov.

Keywords Igusa-Todorov function · Igusa-Todorov algebra · Finitistic dimension conjecture

Mathematics Subject Classification 16G10

1 Introduction

In an attempt to prove the finitistic dimension conjecture, Igusa and Todorov defined in [9] two functions from the objects of mod A (the category of right finitely generated modules over an Artin algebra A) to the natural numbers, which generalizes the notion of projective dimension. Using these functions, they showed that the finitistic dimension of Artin algebras with representation dimension at most three is finite. Nowadays, these functions are known as the Igusa-Todorov functions, ϕ and ψ.

Igusa-Todorov algebras were introduced by Wei in [13] based in the work of Igusa and Todorov (see [9]), and Xi (see [14] and [15]). In the cited article, Wei proved that Igusa-Todorov algebras verify the finitistic dimension conjecture. Wei also proved that the class of 2-Igusa-Todorov algebras is closed under taking

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✉ Gustavo Mata
gmata@fing.edu.uy

Marcos Barrios
marcosb@fing.edu.uy

1 Universidad de la República, Montevideo, Uruguay

endomorphism algebras of projective modules. Since every Artin algebra can be realized as an endomorphism algebra of a projective module over a quasi-hereditary algebra (see [7]), then in case all quasi-hereditary algebra is 2-Igusa-Todorov the finitistic dimension conjecture is true.

Later, Conde showed, based in an article of Rouquier, that the exterior algebras $\Lambda(\Bbbk^m)$ are not Igusa-Todorov algebras for $\Bbbk$ an uncontable field and $m \geq 3$ (see [6] and [12]).

In [2] Bravo, Lanzilotta, Mendoza and Vivero define the Generalized Igusa-Todorov functions and the Lat-Igusa-Todorov algebras, and prove that Lat-Igusa-Todorov algebras also verify the finitistic dimension conjecture. They also show that selfinjective algebras are Lat-Igusa-Todorov algebras, in particular the example given by Conde is a Lat-Igusa-Todorov algebra.

This article is organized as follows:

In Sect. 2, we recall the concepts given in [2] of 0-Igusa-Todorov subcategories, Lat-Igusa-Todorov algebras and its properties.

In Sects. 3 and 4, we give sufficiency conditions for an algebra being a Lat-Igusa-Todorov algebra. We prove that if an algebra A verifies that every module in $\Omega^n(\operatorname{mod} A)$ is an extension of modules of two $\mathscr{D}$-syzygy finite subcategories, then A is n-Lat-Igusa-Todorov (Corollary 2), where $\mathscr{D}$ is a 0-Igusa-Todorov subcategory. In particular, Sect. s5 is dedicated to 0-Lat-Igusa-Todorov and 1-Lat-Igusa-Todorov algebras.

In Sect. 5, we introduce the algebras with only trivial 0-Igusa-Todorov subcategories, i.e. every 0-Igusa-Todorov subcategory is a subcategory of the category of projective modules. Note that: If A has only trivial 0-Igusa-Todorov subcategories, then A is an Igusa-Todorov algebra if and only if A is Lat-Igusa-Todorov. We find some algebras that have only trivial 0-Igusa-Todorov subcategories and we also give a tool to build new family of examples (Theorem 4).

Finally, Sect. 6 is devoted to show that some algebras are not Lat-Igusa-Todorov (Example 3). The examples have only trivial 0-Igusa-Todorov subcategories and they are built from the exterior algebras of Conde example.

2 Preliminaries

Throughout this article A is an Artin algebra and $\operatorname{mod} A$ is the category of finitely generated right A-modules, $\operatorname{ind} A$ is the subcategory of $\operatorname{mod} A$ formed by all indecomposable modules, $\mathscr{P}_A \subset \operatorname{mod} A$ is the class of projective A-modules. $\mathscr{S}(A)$ is the set of isoclasses of simple A-modules and $A_0 = \oplus_{S \in \mathscr{S}(A)} S$. For $M \in \operatorname{mod} A$ we denote by $M^k = \oplus_{i=1}^k M$, by $P(M)$ its projective cover and by $\Omega(M)$ its syzygy. For a subcategory $\mathscr{C} \subset \operatorname{mod} A$, we denote by $\operatorname{findim}(\mathscr{C})$, $\operatorname{gldim}(\mathscr{C})$ its finitistic dimension and its global dimension respectively and by $\operatorname{add} \mathscr{C}$ the full subcategory of $\operatorname{mod} A$ formed by all the sums of direct summands of every $M \in \mathscr{C}$.

Given A and B algebras, if $\alpha : A \to B$ is a morphism of algebras, we know that there is an additive functor $F_\alpha : \operatorname{mod} B \to \operatorname{mod} A$ such that F_α is an embedding of $\operatorname{mod} B$ into $\operatorname{mod} A$ if α is an epimorphism.

If $Q = (Q_0, Q_1, \mathrm{s}, \mathrm{t})$ is a finite connected quiver, $\mathfrak{M}_Q$ denotes its adjacency matrix and $\Bbbk Q$ its associated path algebra. We compose paths in Q from left to right. Given ρ a path in $\Bbbk Q$, $\mathrm{l}(\rho)$, $\mathrm{s}(\rho)$ and $\mathrm{t}(\rho)$ denote the length, start and target of ρ respectively. We say that a quiver Q is strongly connected if for every $v_1, v_2 \in Q_0$ there is a $\rho \in Q_1$ such that $\mathrm{s}(\rho) = v_1$ and $\mathrm{t}(\rho) = v_2$. We denote by J the ideal of $\Bbbk Q$ generated by all the arrows.

2.1 Truncated path algebras

We say that A is a **truncated path algebra** if $A = \frac{\Bbbk Q}{J^k}$ for any $k \geq 2$. For a truncated path algebra A, we denote by $M_v^l(A)$ the ideal ρA, where $l(\rho) = l$, $t(\rho) = v$ and $M^l(A) = \oplus_{v \in Q_0} M_v^l(A)$.

Note that if $A = \frac{\Bbbk Q}{J^k}$ is a truncated path algebra, then

$$\Omega(M_v^l(A)) = \bigoplus_{\rho : \begin{cases} \mathrm{s}(\rho) = v \\ \mathrm{l}(\rho) = k - l \end{cases}} M^{k-l}_{\mathrm{t}(\rho)}(A),$$

$$\Omega^2(M_v^l(A)) = \bigoplus_{\rho : \begin{cases} \mathrm{s}(\rho) = v \\ \mathrm{l}(\rho) = k \end{cases}} M^{l}_{\mathrm{t}(\rho)}(A).$$

For a proof of the next theorem see Theorem 5.11 of [1], and for definitions of skeleton and σ-critical see [8].

Theorem 1 [1] *Let A be a truncated path algebra. If M is any nonzero left A-module with skeleton σ, then*

$$\Omega(M) \cong \bigoplus_{\rho \text{ is } \sigma\text{-critical}} \rho A.$$

Note that if Q is a strongly connected quiver, then every non projective $\frac{\Bbbk Q}{J^k}$-module has infinte projective dimension.

2.2 Igusa-Todorov functions and Igusa-Todorov algebras

We now recall the definition of the generalized Igusa-Todorov ϕ function from [2] and some of its basic properties. Let us start by recalling the following version of Fitting's Lemma.

Lemma 1 *Let R be a noetherian ring. Consider a left R-module M and $f \in \mathrm{End}_R(M)$. Then, for any finitely generated R-submodule X of M, there is a non-negative integer*

$$\eta_f(X) = \min\{k \text{ a non-negative integer } : f|_{f^m(X)} : f^m(X) \to f^{m+1}(X), \text{ is injective } \forall m \geq k\}.$$

Furthermore, for any R-submodule Y of X, we have that $\eta_f(Y) \leq \eta_f(X)$.

Definition 1 [9] Let $K_0(A)$ be the abelian group generated by all symbols $[M]$, with $M \in \operatorname{mod} A$, modulo the relations

1. $[M] - [M'] - [M'']$ if $M \cong M' \oplus M''$,
2. $[P]$ for each projective module P.

- For a subcategory $\mathscr{C} \subset \operatorname{mod} A$, we denote by $\langle \mathscr{C} \rangle \subset K_0(A)$ the free abelian group generated by the classes of direct summands of modules of $\mathscr{C}$.
- In particular, for an A-module M, $\langle M \rangle = \langle \operatorname{add} M \rangle$.

If $\mathscr{D} \subset \operatorname{mod} A$ is a subcategory such that $\mathscr{D} = \operatorname{add}(\mathscr{D})$ and $\Omega(\mathscr{D}) \subset \mathscr{D}$, then

- The quotient group $K_{\mathscr{D}}(A) = \frac{K_0(A)}{\langle \mathscr{D} \rangle}$ is a free abelian group.
- For a subcategory $\mathscr{C} \subset \operatorname{mod} A$, we denote by $[\mathscr{C}]_{\mathscr{D}}$ the quotient $\frac{\langle \mathscr{C} \rangle + \langle \mathscr{D} \rangle}{\langle \mathscr{D} \rangle}$.
- In particular, for an A-module M, $\overline{\langle M \rangle} = (\langle M \rangle + \langle \mathscr{D} \rangle)/\langle \mathscr{D} \rangle$.

Lemma 2 [2] *Let G be a free abelian group, D be a subgroup of G,* $L \in End_{\mathbb{Z}}(G)$ *be such that* $L(D) \subset D$ *and let k be a positive integer for which* $L : L^k(D) \to D$ *is a monomorphism. Then, for each finitely generated subgroup* $X \subset G$, *we have that*

$$\eta_L(X) \leq \eta_{\overline{L}}(\overline{X}) + k,$$

where $\overline{L} : G/D \to G/D$, $g + D \to L(g) + D$, *and* $\overline{X} = (X + D)/D$.

We define the **Generalized Igusa-Todorov functions** as follows

Definition 2 [2] Let A be an Artin algebra and $\mathscr{D} \subset modA$ be a subcategory such that $\Omega(\mathscr{D}) \subset \mathscr{D}$ and $\operatorname{add}(\mathscr{D}) = \mathscr{D}$. Let $\bar{\Omega}_{\mathscr{D}} : K_{\mathscr{D}}(A) \to K_{\mathscr{D}}(A)$ be the group endomorphism defined by $\bar{\Omega}_{\mathscr{D}}([M] + \langle \mathscr{D} \rangle) = [\Omega(M)] + \langle \mathscr{D} \rangle$. For any $M \in mod(A)$, we set

$$\phi_{[\mathscr{D}]}(M) = \eta_{\bar{\Omega}_{\mathscr{D}}}(\overline{\langle M \rangle}) \text{ and } \psi_{[\mathscr{D}]}(M) = \phi_{[\mathscr{D}]}(M) + \operatorname{findim}(\operatorname{add}(\Omega^{\phi_{[\mathscr{D}]}(M)}(M)))$$

where $\overline{\langle M \rangle} = (\langle M \rangle + \langle \mathscr{D} \rangle)/\langle \mathscr{D} \rangle$.

For $\mathscr{D} = \{0\}$ we denote by $\bar{\Omega}$ the group homomorphism $\bar{\Omega}_{\mathscr{D}}$. We also define the subgroup $K_n(A) \subset K_0(A)$ as $K_n(A) = \bar{\Omega}^1(K_{n-1}(A)) = \ldots = \bar{\Omega}^n(K_0(A))$.

Remark 1 Note that if $\mathscr{D} = \{0\}$, then $\phi_{[\mathscr{D}]} = \phi$ and $\psi_{[\mathscr{D}]} = \psi$, the Igusa-Todorov functions defined in [9].

Now we can define the **Generalized Igusa-Todorov dimensions**.

Definition 3 [2] Let A be an Artin algebra A and $\mathscr{D} \subset \operatorname{mod} A$ be a subcategory such that $\Omega(\mathscr{D}) \subset \mathscr{D}$ and $\operatorname{add}(\mathscr{D}) = \mathscr{D}$. For a subcategory $\mathscr{C} \subset \operatorname{mod} A$, we define the $\phi_{[D]}$ **-dimension** and the $\psi_{[D]}$ **-dimension** of $\mathscr{C}$, respectively, as follows:

- $\phi \dim_{[D]}(\mathscr{C}) = \sup\{\phi_{[D]}(M) : M \in \mathscr{C}\}$,
- $\psi \dim_{[D]}(\mathscr{C}) = \sup\{\psi_{[D]}(M) : M \in \mathscr{C}\}$.

We also define the $\phi_{[\mathscr{D}]}$-dimension and $\psi_{[\mathscr{D}]}$-dimension of A, respectively, as follows:

- $\phi \dim_{[D]}(A) = \phi \dim_{[D]}(\operatorname{mod} A)$,
- $\psi \dim_{[D]}(A) = \psi \dim_{[D]}(\operatorname{mod} A)$.

The following remark summarize some propierties of the Generalized Igusa-Todorov functions.

Remark 2 (Propositions 3.9, 3.10, and 3.12 of [2]) Let A be an Artin algebra and $\mathscr{D} \subset \operatorname{mod} A$ be a subcategory such that $\Omega(\mathscr{D}) \subset \mathscr{D}$ and $\operatorname{add}(\mathscr{D}) = \mathscr{D}$. Then, we have the following statements, for $X, Y, M \in \operatorname{mod} A$.

1. If $M \in \mathscr{D} \cup \mathscr{P}(A)$, then $\phi_{[\mathscr{D}]}(M) = 0$ and $\phi_{[\mathscr{D}]}(X \oplus M) = \phi_{[\mathscr{D}]}(X)$.
2. $\phi_{[\mathscr{D}]}(X) \leq \phi_{[\mathscr{D}]}(X \oplus Y)$ and $\psi_{[\mathscr{D}]}(X) \leq \psi_{[\mathscr{D}]}(X \oplus Y)$.
3. $\phi_{[\mathscr{D}]} \dim(\operatorname{add}(X)) = \phi_{[\mathscr{D}]}(X)$ and $\psi_{\mathscr{D}} \dim(\operatorname{add}(X)) = \psi_{[\mathscr{D}]}(X)$.
4. $\phi_{\mathscr{D}}(M) \leq \phi_{\mathscr{D}}(\Omega(M)) + 1$ and $\psi_{\mathscr{D}}(M) \leq \psi_{\mathscr{D}}(\Omega(M)) + 1$.
5. If Z is a direct summand of $\Omega^n(X)$, $0 \leq t \leq \phi_{[\mathscr{D}]}(X)$ and $\operatorname{pd}(Z) < \infty$, then $\operatorname{pd}(Z) + t \leq \psi_{[\mathscr{D}]}(X)$.
6. Suppose that $\phi\dim(\mathscr{D}) = 0$.

    (a) If $\operatorname{pd}(X) < \infty$, then $\phi_{[\mathscr{D}]}(X) = \phi(X) = \operatorname{pd}(X)$.
    (b) $\psi(X) \leq \psi_{[\mathscr{D}]}(X)$.
    (c) If $M \in \mathscr{D} \cup \mathscr{P}(A)$, then $\psi_{[\mathscr{D}]}(X \oplus M) = \psi_{[\mathscr{D}]}(X)$.
    (d) $\psi_{[\mathscr{D}]} \dim(\mathscr{D}) = 0$.

The following result shows the relation between the ϕ-dimension and the $\phi_{[\mathscr{D}]}$ -dimension.

Theorem 2 [2] *Let A be an Artin algebra and $\mathscr{D} \subset \operatorname{mod} A$ such that $\mathscr{D} = \operatorname{add}(\mathscr{D})$ and $\Omega(\mathscr{D}) \subset \mathscr{D}$. Then, for every $X \in \operatorname{mod} A$*

$$\phi(X) \leq \phi_{[\mathscr{D}]}(X) + \phi\dim(\mathscr{D}).$$

2.3 Gorenstein and stable modules

We denote by ${}^{\perp}A$ the full subcategory of $\operatorname{mod} A$ whose objects are those $M \in \operatorname{mod} A$ such that $\operatorname{Ext}_A^i(M, A) = 0$ for $i \geq 1$.

We denote by $(\,\cdot\,)^*$ the functor $\hom_A(\,\cdot\,, A) : \operatorname{mod} A \to \operatorname{mod} A^{op}$.

A finitely generated A-module G is **Gorenstein projective** if there exists an exact sequence of A-modules:

$$\cdots \longrightarrow P_{-2} \xrightarrow{p_{-2}} P_{-1} \xrightarrow{p_{-1}} P_0 \xrightarrow{p_0} P_1 \xrightarrow{p_1} P_2 \xrightarrow{p_2} \cdots$$

such that $G \cong \ker(p_0)$, P_i is projective for all $i \in \mathbb{Z}$ and the following is an exact sequence:

$$\cdots \longrightarrow P_2{}^* \xrightarrow{p_1{}^*} P_1{}^* \xrightarrow{p_0{}^*} P_0{}^* \xrightarrow{p_{-1}{}^*} P_{-1}{}^* \xrightarrow{p_{-2}{}^*} P_{-2}{}^* \xrightarrow{p_{-3}{}^*} \cdots.$$

We denote by $\mathscr{G}\mathscr{P}(A)$ the subcategory of Gorenstein projective modules. The next properties are well known (see [16]):

Remark 3 Let A be an Artin algebra. The following statements hold.

1. Every finite direct sum of modules of $\mathscr{G}\mathscr{P}(A)$ (${}^{\perp}A$) is in $\mathscr{G}\mathscr{P}(A)$ (${}^{\perp}A$)
2. Every direct summand of modules of $\mathscr{G}\mathscr{P}(A)$ (${}^{\perp}A$) is in $\mathscr{G}\mathscr{P}(A)$ (${}^{\perp}A$).
3. Every projective module is in $\mathscr{G}\mathscr{P}(A)$ (${}^{\perp}A$).
4. Every module in $\mathscr{G}\mathscr{P}(A)$ (${}^{\perp}A$) is either a projective module or its projective dimension is infinite.

Let A be an algebra. We say that A is a **Gorenstein algebra** if $\operatorname{id}(A_A) < \infty$ and $\operatorname{pd}(D({}_AA)) < \infty$. The following results will be usefull.

Proposition 1 *Let A be an Artin algebra.*

1. *If A if a Gorenstein algebra, then there is a non negative integer k such that* $\Omega^k(\operatorname{mod} A) = \mathscr{G}\mathscr{P}(A)$.
2. *If* $\operatorname{id} A_A < \infty$, *then there is a non negative integer k such that* $\Omega^k(\operatorname{mod} A) = {}^{\perp}A$.

Proposition 2 [10] *Let A be an Artin algebra, then*

$$\phi\dim(\mathscr{G}\mathscr{P}(A)) = \phi\dim({}^{\perp}A) = 0.$$

2.4 Lat-Igusa-Todorov algebras

Lat-Igusa-Todorov algebras were introduced in [2] as a generalization of Igusa-Todorov algebras (see Definition 2.2 of [13]). They also verify the finitistic dimension conjecture as can be seen in Theorem 3.

Definition 4 Let A be an Artin algebra. If $\mathscr{D} \subset \operatorname{mod} A$ is a subcategory such that

1. $\mathscr{D} = \operatorname{add}(\mathscr{D})$,
2. $\Omega(\mathscr{D}) \subset \mathscr{D}$ and
3. $\phi\dim(\mathscr{D}) = 0$,

we call it a 0-**Igusa-Todorov subcategory**.
Remark 4 Let A be an Artin algebra.

1. If $\phi\dim(A) = 0$, then $\mathscr{D} = \operatorname{mod} A$ is a 0-Igusa-Todorov subcategory.
2. If $\phi\dim(A) = 1$, then $\mathscr{D} = \Omega(\operatorname{mod} A)$ is a 0-Igusa-Todorov subcategory.
3. $\mathscr{G}\mathscr{P}(A)$ and ${}^{\perp}A$ are 0-Igusa-Todorov subcategories.

Definition 5 [2] Let A be an Artin algebra. A subcategory $\mathscr{C} \subset \operatorname{mod} A$ is called **(n, V,** $\mathscr{D}$**)-Lat-Igusa-Todorov** (for short **n- LIT**) if the following conditions are verified

– There is some 0-Igusa-Todorov subcategory $\mathscr{D} \subset \operatorname{mod} A$,
– there is some $V \in \operatorname{mod} A$ satisfying that each $M \in \mathscr{C}$ admits an exact sequence:

$$0 \longrightarrow V_1 \oplus D_1 \longrightarrow V_0 \oplus D_0 \longrightarrow \Omega^n(M) \longrightarrow 0$$

such that $V_0, V_1 \in \operatorname{add}(V)$ and $D_0, D_1 \in \mathscr{D}$.
We say that V is a **(n, V,** $\mathscr{D}$**)- Lat-Igusa-Todorov module** (for short a **n-LIT module**) for $\mathscr{C}$.
Definition 6 [2] We say that A is a **(n, V,** $\mathscr{D}$**)-Lat-Igusa-Todorov algebra** (for short a **n-LIT algebra**) if $\operatorname{mod} A$ is $(n, V, \mathscr{D})$-LIT. We say that A is a LIT algebra if A is n-LIT for some non-negative integer n.

Remark 5 [13] If $\mathscr{D} = \{0\}$ in Definition 6, we say that A is a **n-Igusa-Todorov algebra**.

Remark 6 Let A be an algebra and $\mathscr{D}$ a 0-Igusa-Todorov subcategory. If V is a n-LIT module, then $\Omega(V)$ is an $(n+1)$-LIT module.

Example 1 The following are examples of LIT algebras.

1. If $\phi\dim(A) \leq 1$, then A is a LIT algebra (see Remark 4).

2. If A is a Gorenstein algebra, then A is a LIT algebra where $\mathscr{D} = \mathscr{GP}(A)$ (see Proposition 1).
3. If $\operatorname{id} A_A < \infty$, then A is a LIT algebra where $\mathscr{D} = {}^{\perp}A$ (see Proposition 1).

The following result show that LIT algebras verifies the finitistic dimension conjecture. For a proof see [2].

Theorem 3 [2] *Let A be a $(n, V, \mathscr{D})$-LIT algebra. Then*

$$\operatorname{findim}(A) \leq \psi_{[\mathscr{D}]}(V) + n + 1 < \infty.$$

3 LIT algebras and $\mathscr{D}$-syzygy finite subcategories

In this section we show that some algebras are LIT algebras under certain properties.

Remark 7 Let A be an Artin algebra, $\mathscr{D}$ a 0-Igusa-Todorov subcategory and $\mathscr{C} \subset \operatorname{mod} A$ a subcategory. If $[\Omega^k(\mathscr{C})]_{\mathscr{D}}$ is finitely generated, then $[\Omega^{k+1}(\mathscr{C})]_{\mathscr{D}}$ is finitely generated.

Definition 7 Let A an Artin algebra and $\mathscr{D}$ a 0-Igusa-Todorov subcategory. We say that a subcategory $\mathscr{C} \subset \operatorname{mod} A$ is $\mathscr{D}$ **-syzygy finite** if $[\Omega^k(\mathscr{C})]_{\mathscr{D}}$ is finitely generated for some non-negative integer k.

The following result generalizes Proposition 2.5 of [13].

Proposition 3 *Let A be an Artin algebra and $\mathscr{D}$ be a 0-Igusa-Todorov subcategory. If $\operatorname{mod} A$ is $\mathscr{D}$-syzygy finite, then A is a LIT algebra.*

Proof Suppose that $[\Omega^n(\operatorname{mod} A)]_{\mathscr{D}}$ is finitely generated. Then there exist $\{N_1, \ldots, N_l\} = \mathscr{N} \subset \operatorname{ind} A$ such that $\forall M \in \Omega^n(\operatorname{mod} A)$, every indecomposable summand of M belongs to $\mathscr{N}$ or $\mathscr{D}$. We deduce that $N = \oplus_{i=1}^{l} N_i$ is a n-LIT module. □

Proposition 4 *Let A be an Artin algebra and $\mathscr{D} \subset \operatorname{mod} A$ a 0-Igusa-Todorov subcategory. If $\mathscr{C}_1$, $\mathscr{C}_2$, $\mathscr{E}$ are three subcategories of A-modules such that, for any $E \in \mathscr{E}$, there is an exact sequence $0 \to C_1 \to C_2 \to E \to 0$ with $C_i \in \mathscr{C}_i$ for $i = 1, 2$, the next statements follows.*

1. *If $\mathscr{C}_1$ and $\mathscr{C}_2$ are $\mathscr{D}$-syzygy finite, then $\mathscr{E}$ is n-LIT for some non-negative integer n.*
2. *If $\mathscr{C}_1$ is $\mathscr{D}$-syzygy finite and $\operatorname{gldim}(\mathscr{C}_2) < \infty$, then $\mathscr{E}$ is $\mathscr{D}$-syzygy finite.*
3. *If $\mathscr{C}_1$ is n-LIT and $\operatorname{gldim}(\mathscr{C}_2) < \infty$, then $\mathscr{E}$ is $(n + 1)$-LIT.*

Proof For $E \in \mathscr{E}$ there is a short exact sequence $0 \to C_1 \to C_2 \to E \to 0$ with $C_i \in \mathscr{C}_i$ for $i = 1, 2$. Thus, for any $n \in \mathbb{N}$ we obtain a short exact sequence $0 \to \Omega^n(C_1) \to \Omega^n(C_2) \oplus P \to \Omega^n(E) \to 0$ for some projective P.

1. Since $[\Omega^n(\mathscr{C}_1)]_{\mathscr{D}}$ and $[\Omega^n(\mathscr{C}_2)]_{\mathscr{D}}$ are finitely generated for $n \in \mathbb{N}$, there are modules $U = \oplus_{i=1}^t U_i$ and $V = \oplus_{j=i}^s V_j$ such that if $M_1 \in \Omega^n(\mathscr{C}_1)$ and $M_2 \in \Omega^n(\mathscr{C}_2)$, then $M_1 = \oplus_{i=1}^t U_i^{\alpha_i} \oplus D_1$ and $M_2 = \oplus_{j=1}^s V_j^{\beta_j} \oplus D_2$, where $D_i \in \mathscr{D}$ for $i = 1, 2$ and $\alpha_i, \beta_j \in \mathbb{N}$. Hence for every $E \in \mathscr{E}$ there is a short exact sequence
$$0 \to U_1' \oplus D_1' \to V_1' \oplus D_2' \oplus P \to \Omega^n(E) \to 0$$
with $U_1' \in \operatorname{add}(U)$, $V_1' \in \operatorname{add}(V)$, $D_i \in \mathscr{D}$ for $i = 1, 2$ and P a projective module. We conclude that $\mathscr{E}$ is n-LIT with LIT module $U \oplus V \oplus A$.
2. Take $n \in \mathbb{N}$ such that $[\Omega^n(\mathscr{C}_1)]_{\mathscr{D}}$ is finitely generated and $\operatorname{gldim}(\mathscr{C}_2) \le n$. Then $\Omega^n(C_2)$ is projective for every $C_2 \in \mathscr{C}_2$. It follows that $\Omega^n(C_1) = \Omega^{n+1}(E) \oplus P$ for some projective P. We deduce that $[\Omega^{n+1}(\mathscr{E})]_{\mathscr{D}}$ is finitely generated.
3. Take n to be an integer such that $\mathscr{C}_1$ is n-LIT and $\operatorname{gldim}(\mathscr{C}_2) \le n$. Similarly to the proof of item (2), we obtain that $\Omega^n(C_1) = \Omega^{n+1}(E) \oplus P$ for some projective P. Note that there is an exact sequence $0 \to V_1 \oplus D_1 \to V_0 \oplus D_0 \to \Omega^n(C) \to 0$ with $V_i \in \operatorname{add}(V)$ and $D_i \in \mathscr{D}$ for $i = 0, 1$, where V is a n-LIT module. Since P is projective, we can also obtain an exact sequence $0 \to V_1' \oplus D_1' \to V_0' \oplus D_0' \to \Omega^{n+1}(E) \to 0$ with $V_i' \in \operatorname{add}(V)$ and $D_i \in \mathscr{D}$ for $i = 0, 1$. It follows that $\mathscr{E}$ is $(n+1)$-LIT with V a $(n+1)$-LIT module.

□

Remark 8 Note that in part 1 of Proposition 4, $\min\{m : [\Omega^m(\mathscr{C}_1)] \text{ and } [\Omega^m(\mathscr{C}_2)] \text{ are finitely generated}\}$ is a possible choice of n.

Corollary 1 *Let A be an Artin algebra and $\mathscr{D} \subset \operatorname{mod} A$ a 0-Igusa-Todorov subcategory. Consider $\mathscr{C}, \mathscr{F}, \mathscr{E}$ three subcategories of A-modules, such that $\operatorname{gldim}(\mathscr{F}) < \infty$ and for any $E \in \mathscr{E}$, there is an exact sequence*
$$0 \to C_1 \to F_0 \to \dots \to F_k \to E \to 0$$
with $C_1 \in \mathscr{C}$ and each $F_i \in \mathscr{F}$. If $\mathscr{C}$ is $\mathscr{D}$-syzygy-finite (n-LIT), then $\mathscr{E}$ is $\mathscr{D}$-syzygy finite ($(n+k+1)$-LIT).

Proof Denote $\mathscr{E}_0 = \mathscr{C}$, and by induction, $\mathscr{E}_{i+1} = \{M : \exists\, 0 \to C \to F \to M \to 0 \text{ with } C \in \mathscr{E}_i \text{ and } F \in \mathscr{F}\}$. Then by hypothesis and Proposition 4, inductively we obtain that each E_i is $\mathscr{D}$-syzygy finite ($(n+i)$-LIT). Note that $\mathscr{E} \subset \mathscr{E}_{k+1}$, so $\mathscr{E}$ is also $\mathscr{D}$-syzygy finite ($(n+k+1)$-LIT). □

Proposition 5 *Let A an Artin algebra, $\mathscr{D} \subset \operatorname{mod} A$ a 0-Igusa-Todorov subcategory, and two $\mathscr{D}$-syzygy finite subcategories $\mathscr{C}_1$ and $\mathscr{C}_2$. Consider $\mathscr{E} \subset \operatorname{mod} A$ a subcategory such that $\forall M \in \mathscr{E}$ there exists a short exact sequence $0 \to C_1 \to M \to C_2 \to 0$ with $C_i \in \mathscr{C}_i$ for $i = 1, 2$, then $\mathscr{E}$ is n-LIT for some $n \in \mathbb{Z}^+$.*

Proof Suppose that for $n \in \mathbb{N}$ $[\Omega^n(\mathscr{C}_1)]_{\mathscr{D}}$ and $[\Omega^n(\mathscr{C}_2)]_{\mathscr{D}}$ are finitely generated. For any $M \in \mathscr{C}$ there are $C_i \in \mathscr{C}_i$ such that $0 \to C_1 \to M \to C_2 \to 0$ is a short exact sequence. Consider the following pullback diagram obtained from that short exact sequence.

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \Omega(C_2) & \xrightarrow{1} & \Omega(C_2) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_1 & \longrightarrow & C_1 \oplus P(C_2) & \longrightarrow & P(C_2) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C_1 & \longrightarrow & M & \longrightarrow & C_2 & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

It is easy to check that $\Omega^n(\mathscr{C})$ is n-LIT, just apply part 1 of Proposition 4 to the middle column in the above diagram. □

The following result follows directly from the previous proposition.

Corollary 2 *Let A an Artin algebra, $\mathscr{D}$ a 0-Igusa-Todorov subcategory for* $\operatorname{mod} A$. *If there are two $\mathscr{D}$-syzygy finite subcategories $\mathscr{C}_1$ and $\mathscr{C}_2$ such that for every $M \in \operatorname{mod} A$ there is a short exact sequence*

$$0 \to C_1 \to \Omega^n(M) \to C_2 \to 0$$

with $C_i \in \mathscr{C}_i$, then A is a n-LIT algebra.

4 Small LIT algebras

Throughout this section, we identify 0-LIT and 1-LIT algebras under conditions in the category of modules, in quotients, and its categories of modules.

The first result is a generalization of Proposition 3.2 from [13]. This result allows us to identify 0-LIT algebras.

Proposition 6 *Let A be an Artin algebra and $\mathscr{D} \subset \operatorname{mod} A$ a 0-Igusa-Todorov subcategory. Consider two ideals I, J with $JI = 0$. Then A is a 0-LIT algebra provided that the following two statements are valid.*

1. $\operatorname{ind} \frac{A}{I} \setminus \mathscr{D} \subset \operatorname{mod} A$ *and* $\operatorname{ind} \frac{A}{J} \setminus \mathscr{D} \subset \operatorname{mod} A$ *are finite sets.*

2. $\operatorname{ind}\frac{A}{I}\setminus\mathscr{D}\subset \operatorname{mod}A$ *is finite*, $\frac{A}{J}$ *is projective in* $\operatorname{mod}A$ *and* $[\Omega(\operatorname{mod}\frac{A}{J})]_{\mathscr{D}}$ *is finitely generated*.

Proof For any $N\in \operatorname{mod}A$, we have a short exact sequence $0\to NJ\to N\to \frac{N}{NJ}\to 0$. Note that $(NJ)I=0$ and $(\frac{N}{NJ})J=0$, so NJ is also in $\operatorname{mod}\frac{A}{I}$ and $\frac{N}{NJ}$ is also in $\operatorname{mod}\frac{A}{J}$.

Consider the following pullback diagram obtained from the above short exact sequence.

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \Omega(\frac{N}{NJ}) & \xrightarrow{1} & \Omega(\frac{N}{NJ}) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & NJ & \longrightarrow & NJ\oplus P(\frac{N}{NJ}) & \longrightarrow & P(\frac{N}{NJ}) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle 1} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & NJ & \longrightarrow & N & \longrightarrow & \frac{N}{NJ} & \longrightarrow & 0\\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Both items follow by Remark 8 applied to the middle row in the diagram. □

The following two results are generalizations of Theorem 3.4 and Corollary 3.5 of [13] respectively.

Proposition 7 *Let A be an Artin algebra, $\mathscr{D}\subset \operatorname{mod}A$ a 0-Igusa-Todorov subcategory and I an ideal with* $\operatorname{rad}(A)I=0$. *If* $\operatorname{mod}\frac{A}{I}\subset \operatorname{mod}A$ *is 0-LIT, then A is a 1-LIT algebra*.

Proof By hypothesis, for any $M\in \operatorname{mod}A$, we have that $\Omega(M)I\subset \operatorname{rad}(P(M))I=0$. Then $\Omega(M)$ is also an $\frac{A}{I}$-module. Since $\operatorname{mod}\frac{A}{I}\subset \operatorname{mod}A$ is 0-LIT with a LIT-module V, then we obtain an exact sequence of A-modules $0\to V_1\oplus D_1\to V_0\oplus D_0\to \Omega(M)\to 0$ with $V_0,V_1\in \operatorname{add}(V)$ and $D_0,D_1\in\mathscr{D}$. Hence, we conclude that A is a 1-LIT algebra with a LIT module V. □

Corollary 3 *Let A be an Artin algebra and $\mathscr{D}\subset \operatorname{mod}A$ a 0-Igusa-Todorov subcategory. If* $\operatorname{rad}^{2n+1}(A)=0$ *and* $\operatorname{ind}\frac{A}{\operatorname{rad}^n(A)}\setminus\mathscr{D}\subset \operatorname{mod}A$ *is finite, then A is 1-LIT*.

Proof We have the following embeddings of module categories

$$\operatorname{mod}\frac{A}{\operatorname{rad}^n(A)}\subset \operatorname{mod}\frac{A}{\operatorname{rad}^{2n}(A)}\subset \operatorname{mod}A$$

Consider $I = J = \frac{\operatorname{rad}^n A}{\operatorname{rad}^{2n}(A)}$ ideal of $\frac{A}{\operatorname{rad}^{2n}(A)}$. Observe that $IJ = 0$. If $M \in \operatorname{mod} \frac{A}{\operatorname{rad}^{2n}(A)}$, then $JM \in \operatorname{mod} \frac{A}{\operatorname{rad}^{n}(A)}$ and $\frac{M}{JM} \in \operatorname{mod} \frac{A}{\operatorname{rad}^{n}(A)}$ and by Proposition 6 we conclude that the subcategory $\operatorname{mod} \frac{A}{\operatorname{rad}^{2n}(A)} \subset \operatorname{mod} A$ is 0-LIT. Finally, by Proposition 7 A is 1-LIT. □

5 Algebras with only trivial 0-Igusa-Todorov subcategories

In this section we build algebras with only trivial 0-Igusa-Todorov subcategories. We will use these results in Sect. 6 to construct examples of non LIT algebras.

Definition 8 Let A be an Artin algebra. We say that A has **only trivial 0-Igusa-Todorov subcategories** if for all 0-Igusa-Todorov subcategory $\mathscr{D}$, $\mathscr{D} \subset \mathscr{P}_A$.

Definition 9 Let A be an Artin algebra. For $M \in \operatorname{mod} A$ we define

$$\gamma(M) = \phi\mathrm{dim}(\operatorname{add}\{N\colon N \text{ is a direct summand of } \Omega^n(M) \text{ for some non-negative integer } n\}).$$

Proposition 8 *Let A be an Artin algebra. The following statements are equivalent*

1. *A has only trivial* 0*-Igusa-Todorov subcategories.*
2. $\min\{\gamma(M)\colon \text{such that } M \in \operatorname{mod} A \setminus \mathscr{P}_A\} \geq 1$.
3. $\min\{\gamma(M)\colon \text{such that } M \in \operatorname{ind} A \setminus \mathscr{P}_A\} \geq 1$.

Proof We prove the equivalences.

(1 ⇒ 2) Consider $M \in \operatorname{mod} A \setminus \mathscr{P}_A$. It is clear that the following class

$$\mathscr{C}_M = \{N\colon N | \Omega^n(M) \text{ for some non-nogative integer } n\}$$

verifies the first two axioms for a 0-Igusa-Todorov subcategory. Since A has only trivial 0-Igusa-Todorov subcategories, $\phi\mathrm{dim}(\mathscr{C}_M) = \gamma(M) \geq 1$.

(2 ⇒ 3) It is a particular case.

(3 ⇒ 1) Let $\mathscr{D}$ be a non trivial subgategory such that is closed by syzygies and direct summands. Then there is a non projective indecomposable module $M \in \mathscr{D}$. By hypothesis $\gamma(M) \geq 1$ so there is $N \in \mathscr{D}$ such that $\phi(N) \geq 1$. We deduce that $\mathscr{D}$ is not a 0-Igusa-Todorov subcategory. □

Proposition 9 *The following algebras have only trivial* 0*-Igusa-Todorov subcategories*

1. *If* $A = \frac{\Bbbk Q}{J^2}$ *is a non selfinjective radical square zero algebra such that* Q *is strongly connected and the adjacence matrix* $\mathfrak{M}_Q$ *of* Q *is not invertible.*
2. *If* $A = \frac{\Bbbk Q}{J^k}$ *is a truncated path algebra such that* Q *is strongly connected algebra with at least one loop and the adjacence matrix* $\mathfrak{M}_Q$ *of* Q *is not invertible.*

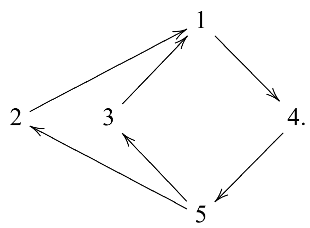
Proof

1. By Proposition 4.14 and Theorem 4.32 of [11], $\phi(A_0) \geq 1$. If $M \in \operatorname{ind} A \setminus \mathscr{P}_A$, then $\Omega(M) \subset \operatorname{add}(A_0)$. Since Q is strongly connected quiver, A_0 has no projective summands. On the other hand, since Q is strongly connected, then $A_0 \in \operatorname{add}(\oplus_{k=1}^{n} \Omega^k(M))$, and it follows the thesis.
2. By Remark 11 of [4], $\phi(M^l(A) \oplus M^{k-l}(A)) \geq 1$ for every $1 \leq l \leq k-2$. If M is not a projective module, then $\Omega(M) = M^l_v(A) \oplus N$ for some $1 \leq l \leq k-2$, $v \in Q_0$. On the other hand, since Q is strongly connected and has a loop, then $M^l(A) \oplus M^{k-l}(A) \in \operatorname{add}(\oplus_{k=1}^{n} \Omega^k(M))$, and it follows the thesis.

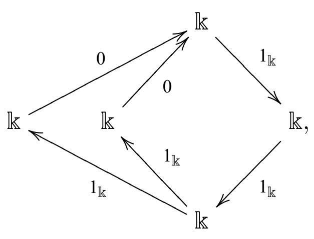
□

The following example shows that it is necessary to have at least one loop in the case of truncated path algebras of the above proposition.

Example 2 Consider the algebra $A = \frac{\Bbbk Q}{J^8}$, with Q the following quiver



Let M be the A-module given by the representation below



then $\Omega(M) = M \oplus M$, and $\gamma(M) = \phi(M) = 0$. We conclude that A does not have only trivial 0-Igusa-Todorov subcategories.

Definition 10 Let $A = \frac{\Bbbk Q}{I}$ a finite dimensional algebra. If $\bar{Q}$ is a full subquiver of Q and $B = \frac{\Bbbk \bar{Q}}{I \cap \Bbbk \bar{Q}}$, then we denote by $\pi_B : \operatorname{mod} A \to \operatorname{mod} B$ the restriction functor.

Theorem 4 *Let $A = \frac{\Bbbk Q}{I}$ a finite dimensional algebra such that there are two disjoint full subquivers Γ and $\bar{\Gamma}$ of Q which verifies*:

- $\bar{\Gamma}$ has no sinks.
- $Q_0 = \Gamma_0 \cup \bar{\Gamma}_0$.
- For all $v \in \Gamma_0$ there is an arrow $\alpha_v \in Q_1$ such that $\mathrm{s}(\alpha_v) = v$ and $\mathrm{t}(\alpha_v) = w \in \bar{\Gamma}_0$.
- There are no arrows $\alpha \in Q_1$ with $\mathrm{s}(\alpha) \in \bar{\Gamma}_0$ and $\mathrm{t}(\alpha) \in \Gamma_0$.
- For all $\alpha \in Q_1$ such that $\mathrm{s}(\alpha) \in \Gamma_0$ and $\mathrm{t}(\alpha) \in \bar{\Gamma}_0$ then $\alpha\beta = 0 = \delta\alpha$ for all $\beta, \delta \in Q_1$.

If $C = \frac{\Bbbk \bar{\Gamma}}{I \cap \Bbbk \bar{\Gamma}}$ has only trivial 0-Igusa-Todorov subcategories, then A has only trivial 0-Igusa-Todorov subcategories.

Proof Let B and C be the algebras $C = \frac{\Bbbk \bar{\Gamma}}{I \cap \Bbbk \bar{\Gamma}}$ and $B = \frac{\Bbbk \Gamma}{I \cap \Gamma}$ respectively. It is easy to see that $\Omega(\bmod A) \subset \bmod B \oplus \bmod C \oplus \{\oplus P_v : v \in \Gamma_0\}$. Notice that $\bmod C$ has no simple projective modules. Consider $\mathscr{D}$ a 0-Igusa-Todorov subcategory for A.

Claim: $\mathscr{D} \cap \bmod C$ is a 0-Igusa-Todorov subcategory for C.

Since $\mathscr{P}_C \subset \mathscr{P}_A$, then $\Omega_C(M) = \Omega_A(M)$ for all $M \in \bmod C$. Hence $\Omega_C(M) \in \mathscr{D} \cap \bmod C$ and $\phi_C(M) = \phi_A(M) = 0$ for all $M \in \mathscr{D} \cap \bmod C$. On the other hand consider $M \in \bmod C$, if N is a direct summand of M in $\bmod A$, it is clear that $N \in \bmod C$.

As a consequence of the claim, it is clear that for $M \in \mathscr{D} \setminus \mathscr{P}_A$, if $N \in \bmod C$ is a direct summand of $\Omega(M)$, then $N \in \mathscr{P}_C$.

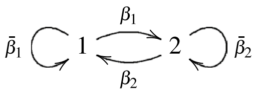
Suppose $M \in \mathscr{D} \setminus \mathscr{P}_A$, then $\Omega(M)$ is not projective. Hence $\Omega(M)$ has a non projective direct summand in $\bmod B$. Since there is a simple C-module S such that S is a direct summand of $\Omega^2(M)$, then $\Omega^2(M)$ has a non projective direct summand in $\bmod C$. Finally if we apply the claim to $\Omega(M)$ is a projective module, and this is absurd. □

Remark 9 The algebras from Theorem 4 are a particular case of the algebras from Theorem 5.2 of [3].

6 Examples of non LIT algebras

In this section, we give an example of a family of finite dimensional algebras that are not LIT.

Example 3 Let $B = \frac{\Bbbk Q}{I_B}$ be a finite dimensional $\Bbbk$-algebra and $C = \frac{\Bbbk Q'}{J^2}$, where Q' is the following quiver



Consider $A = \frac{\Bbbk\Gamma}{I_A}$, with

- $\Gamma_0 = Q_0 \cup Q_0'$,
- $\Gamma_1 = Q_1 \cup Q_1' \cup \{\alpha_i : i \to 1\ \forall i \in Q_0\}$ and
- $I_A = \langle I_B, J_C^2, \{\lambda\alpha_i, \alpha_i\lambda\ \forall \lambda \text{ such that } 1(\lambda) \geq 1\}\rangle$.

Note that

- If $M \in \operatorname{mod} A$, then $\operatorname{pd} M = \begin{cases} 0, & \text{or} \\ \infty. \end{cases}$
- $K_1(A) \subset \langle [M] : M \in \operatorname{mod} B \subset \operatorname{mod} A\rangle \times \langle [S_1]\rangle \times \langle [S_2]\rangle$.
- If $M \in \operatorname{mod} B$, then $\Omega_A(M) = \Omega_B(M) \oplus S_1^{\dim_{\Bbbk}(\operatorname{Top}(M))}$.
- If $M \in \operatorname{mod} A$ and $\operatorname{pd}(M) = \infty$, then S_1 and S_2 are direct summands of $\Omega_A^3(M)$. As a consequence A is a *LIT* algebra if and only if A is an Igusa-Todorov algebra (Use Theorem 4 and Proposition 9).

Remark 10 Let A be an algebra as in Example 3 where B is a selfinjective algebra. If $0 \to V_B \oplus S \to P \to W_B \oplus \bar{S} \to 0$ is a short exact sequence in $\operatorname{mod} A$ with $V_B, W_B \in \operatorname{mod} B \setminus \mathscr{P}_B$, $P \in \mathscr{P}_A$ and $S, \bar{S} \in \operatorname{add}(S_1 \oplus S_2)$, then there is a short exact sequence $0 \to V_B \to \bar{P} \to W_B \to 0$ in $\operatorname{mod} A$ with $\bar{P} \in \mathscr{P}_B$.

Remark 11 Let A be an algebra as in Example 3 where B is a selfinjective algebra. If A is an 1-Igusa-Todorov algebra, then B is also an 1-Igusa-Todorov algebra.

Lemma 3 *Let A be an algebra as in Example 3 where B is a selfinjective algebra, then*

$$K_1(A) = \langle [M] : M \in \operatorname{mod} B \setminus \mathscr{P}_B \subset \operatorname{mod} A\rangle \times \langle [S_1]\rangle \times \langle [S_2]\rangle.$$

Proof It easy to see that $S_1, S_2 \in K_1(A)$, and If $P \in \mathscr{P}_B$ then $P \notin K_1(A)$. On the other hand consider $V_B \in \operatorname{mod} B \setminus \mathscr{P}_B$. Since B is a selfinjective algebra, there is a short exact sequence in $\operatorname{mod} B$ as follows

$$0 \to V_B \to P \to W_B \to 0,$$

where $P \in \mathscr{P}_B$. From the previous short exact sequence, we can construct the following short exact sequence in $\operatorname{mod} A$.

$$0 \to V_B \oplus S_1^{\dim_{\Bbbk}(\operatorname{Top}(W_B))} \to \bar{P} \to W_B \to 0,$$

where $\bar{P} \in \mathscr{P}_A$. We deduce that $V_B \in K_1(A)$. $\square$

As a consequence of the proof of Lemma 3 we have the next result.

Corollary 4 *Let A be an algebra as in Example 3 where B is a selfinjective algebra. Then the next statements follows*

1. *If* $V \in \Omega_A(\operatorname{mod} A)$, *there is a semisimple* $S \in \operatorname{mod} A$ *and a short exact sequence* $0 \to V \oplus S \to P \to W \to 0$, *with* $P \in \mathscr{P}_A$ *and* $W \in \Omega_A(\operatorname{mod} A)$.
2. $\bar{\Omega}|_{\langle [M] : M \in \operatorname{mod} B \setminus \mathscr{P}_B \rangle}$ *is injective*.

Proof

1. The A-module V can be decomposed into $V = V_B \oplus S_1^{m_1} \oplus S_2^{m_2}$ with $V_B \in \operatorname{mod} B$.

   Let W_B be a preimage of V_B, and $\bar{W}_B$ a preimage of W_B as in Lemma 3. It is easy to see that $\Omega(S_1) = \Omega(S_2) = S_1 \oplus S_2$, then

$$\Omega(W_B \oplus S_1^{\operatorname{Top}(\bar{W}_B)+m_1+m_2}) = V_B \oplus S_1^{\operatorname{Top}(W_B)+\operatorname{Top}(\bar{W}_B)+m_1+m_2} \oplus S_2^{\operatorname{Top}(\bar{W}_B)+m_1+m_2}$$

2. Is a direct consequence of Lemma 3

□

Proposition 10 *Let A as in Example 3 where B is a selfinjective algebra. If A is m-Igusa-Todorov, then A is 1-Igusa-Todorov.*

Proof If A is a m-Igusa-Todorov algebra with $m > 1$, we can assume, by Remark 6, that there exist an Igusa-Todorov module V such that $V \subset \Omega_A(\operatorname{mod} A)$. Assume that A_0 is a direct summand of V. Given the short exact sequences

$$0 \to V_1 \xrightarrow{u_m} V_0 \xrightarrow{v_m} \Omega^m(M) \to 0, \text{ and } 0 \to \Omega^m(M) \xrightarrow{i_{m-1}} P_{m-1} \xrightarrow{p_{m-1}} \Omega^{m-1}(M) \to 0,$$

we can construct the following commutative diagram with exact columns and rows

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V_1 & \xrightarrow{u_m} & V_0 & \xrightarrow{v_m} & \Omega^m(M) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \gamma_{m-1}u_m} & & \downarrow{\scriptstyle \mu_{m-1}} & & \downarrow{\scriptstyle i_{m-1}} & & \\
0 & \longrightarrow & Q_{m-1} & \xrightarrow{\iota_{m-1}} & Q_{m-1} \oplus P_{m-1} & \xrightarrow{\Pi_{m-1}} & P_{m-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle p_{m-1}} & & \\
0 & \longrightarrow & W_1 & \longrightarrow & W_0 & \longrightarrow & \Omega^{m-1}(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where the maps ι_{m-1} and Π_{m-1} are the canonical inclusion and projection respectively, and $\mu_{m-1} = \begin{pmatrix} \gamma_{m-1} \\ i_{m-1}v_m \end{pmatrix}$. Consider $S \in \mathrm{mod}\,A$ a semisimple module and $\lambda_{m-1} : S \to Q_{m-1}$ such that $\delta_{m-1} : V_0 \oplus S \to Q_{m-1} \oplus P_{m-1}$, given by $\begin{pmatrix} \gamma_{m-1} & \lambda_{m-1} \\ i_{m-1}v_m & 0 \end{pmatrix}$, is a monomorphism and $(Soc(Q_{m-1}), 0) \subset \mathrm{Im}\,(\delta_{m-1})|_{Soc(V_0)\oplus S}$. Consider $\epsilon_{m-1} : V_1 \oplus S \to Q_{m-1}$ given by $\epsilon_{m-1} = (\gamma_{m-1}u_m, \lambda_{m-1})$.

Claim: The map ϵ_{m-1} is a monomorphism.

Suppose there exist $v \in V_1$ and $s \in S$ such that $\epsilon_{m-1}(v,s) = \gamma_{m-1}u_m(v) + \lambda_{m-1}(s) = 0$. Since $u_m(v) \in V_0$ and $s \in S$, then

$$\delta_{m-1}(u_m(v), s) = \begin{pmatrix} \gamma_{m-1} & \lambda_{m-1} \\ i_{m-1}v_m & 0 \end{pmatrix}\begin{pmatrix} u_m(v) \\ s \end{pmatrix} = (\gamma_{m-1}u_m(v) + \lambda_{m-1}(s), i_{m-1}v_m u_m(v)) = (0,0)$$

Since δ_{m-1} and u_m are monomorphisms, then $v = 0$ and $s = 0$.

From the above diagram and the maps ϵ_{m-1}, λ_{m-1} and $\bar{v}_m = (v_m, 0)$, by Lemma 3×3, we obtain the following diagram.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V_1 \oplus S & \xrightarrow{u_m \oplus 1_S} & V_0 \oplus S & \xrightarrow{\bar{v}_m} & \Omega^m(M) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \epsilon_{m-1}} & & \downarrow{\scriptstyle \delta_{m-1}} & & \downarrow{\scriptstyle i_{m-1}} & & \\
0 & \longrightarrow & Q_{m-1} & \longrightarrow & Q_{m-1} \oplus P_{m-1} & \xrightarrow{\Pi_{m-1}} & P_{m-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle q_{m-1}} & & \downarrow{\scriptstyle p_{m-1}} & & \\
0 & \longrightarrow & \bar{W}_1 & \longrightarrow & \bar{W}_0 & \xrightarrow{\omega_{m-1}} & \Omega^{m-1}(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

We denote by $\bar{W}_0 = ((W_0)^i, T_\alpha)$, $Q_{m-1} = ((Q_{m-1})^i, \bar{T}_\alpha)$ and $P_{m-1} = ((P_{m-1})^i, \tilde{T}_\alpha)$ as representations.

Claim: $[\bar{W}_0] \in K_1(A)$.

Let $w \in \bar{W}_0$ such that $w \neq 0$ and $e_1 w = w$ (the case $e_2 w = w$ is easier and left to the reader). We want to prove that $w \notin \mathrm{Im}\,\sum_{\alpha:j\to 1} T_\alpha$ and $T_{\beta_1}(w) = T_{\bar{\beta}_1}(w) = 0$.

Suppose there exists $w' \in W_0$ such that $\sum_{\alpha:j\to 1} T_\alpha(w') = w$, then $\omega_{m-1}(w) = 0$. Since q_{m-1} is an epimorphism, there exist $x, x' \in Q_{m-1} \oplus P_{m-1}$ where $q_{m-1}(x) = w$, $q_{m-1}(x') = w'$ and $\sum_{\alpha:j\to 1} \bar{T}_\alpha + \tilde{T}_\alpha(x') = x$. We deduce that $x \in S_1 \subset \mathrm{Soc}\,(Q_{m-1} \oplus P_{m-1})$.

Now consider $y, y' \in P_{m-1}$ such that $\Pi_{m-1}(x) = y$ and $\Pi_{m-1}(x') = y'$, since $(Soc(Q_{m-1}), 0) \subset \mathrm{Im}\,(\delta_{m-1})|_{Soc(V_0)\oplus S}$ it is clear that $y \neq 0$. By the previous diagram there is an element $z \in S_1 \subset \mathrm{Soc}\,(\Omega^m(M))$ such that $i_{m-1}(z) = y$.

Since $\bar{v}_m$ is an epimorphism there is an element $v \in S_1 \subset \mathrm{Soc}\,(V_0)$ such that $\bar{v}_m(v) = z$. Again, by the previous diagram $\Pi_{m-1}(x - \delta_{m-1}(v)) = 0$, then $x - \delta_{m-1}(v) \in Q_{m-1}$. Since $x, \delta_{m-1}(v) \in \mathrm{Soc}\,(Q_{m-1} \oplus P_{m-1})$, it is clear that $x - \delta_{m-1}(v) \in (\operatorname{Soc}(Q_{m-1}), 0)$. Therefore there exists $v' \in \operatorname{Soc}(V_0 \oplus S)$ such that $\delta_{m-1}(v') = x - \delta_{m-1}(v)$. It is an absurd since $0 = q_{m-1}\delta_{m-1}(v') = q_{m-1}(x - \delta_{m-1}(v)) = q_{m-1}(x) = w \neq 0$.

Now, if we suppose that $T_{\beta_1}(w) \neq 0$ $(T_{\beta_2}(w) \neq 0)$. Consider $x = T_{\beta_2}(w)$ $(x = T_{\beta_1}(w))$ and the proof follows as above.

Finally, by Remark 4, there is a semisimple module $\bar{S}$ such that $\bar{W}_0 \oplus \bar{S} \in \Omega_A(\operatorname{mod} A)$.

From the below short exact sequence of the previous commutative diagram we build the following short exact sequence

$$0 \to \bar{W}_1 \oplus \bar{S} \to \bar{W}_0 \oplus \bar{S} \to \Omega^m(M) \to 0.$$

Since $\Omega^{m-1}(M)$ and $\bar{W}_0 \oplus \bar{S}$ belong to $\Omega_A(\operatorname{mod} A)$, then $\bar{W}_1 \oplus \bar{S} \in \Omega_A(\operatorname{mod} A)$. By Remark 10, there exist $W \in \Omega_A(\operatorname{mod} A)$ such that $\bar{W}_1$, $\bar{W}_1$ belong to $\operatorname{add}(W)$ for all $M \in modA$ and the thesis follows. □

We finally give an example of an Artin algebra that is not Lat-Igusa-Todorov.

Example 4 Let A as in Example 3 where B is a selfinjective algebra. If B is not an Igusa-Todorov algebra, for instance $B = \Lambda(\mathbb{k}^n)$ for $n \geq 3$ (see 4.2.10 of [6] and Corollary 4.4 of [12]), then A is not a LIT algebra. However, by Theorem 5.2 of [3] $\phi\dim(A) \leq 3$ (in fact $\phi\dim(A) = 2$), and A verifies the finitistic dimension conjecture.

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Declarations

Data Availability Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

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