



# An encounter of classical differential geometry with dynamical systems in the realm of structural stability of principal curvature configurations

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Accepted: 27 March 2021 / Published online: 18 May 2021  
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## Abstract

This evocative essay focuses on some landmarks that led the author to the study of principal curvature configurations on surfaces in  $\mathbb{R}^3$ , their structural stability and generic properties. The starting point was an encounter with the book of D. Struik and the reading of the references to the works of Euler, Monge and Darboux found there. The concatenation of these references with the work of Peixoto, 1962, on differential equations on surfaces, was a crucial second step. The circumstances of the convergence toward the theorems of Gutiérrez and Sotomayor, 1982–1983, are recounted here. These theorems are pointed out as the first encounter between the line of thought disclosed from the works of Monge, 1796, Dupin, 1815, and Darboux, 1896, with that transpiring and evolving from the achievements of Poincaré, 1881, Andronov–Pontrjagin, 1937, and Peixoto, 1962. Some mathematical developments as well as open problems sprouting from the 1982–1983 works are mentioned on Sect. 10.

**Keywords** Umbilic point · Principal curvature cycle · Principal curvature lines

**Mathematics Subject Classification** 53C12 · 34D30 · 53A05 · 37C75

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To the memory of Maurício M. Peixoto (1921–2019), Carlos E. Harle (1937–2020), Carlos T. Gutiérrez (1944–2008) and Daniel B. Henry (1945–2002).

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Communicated by Marco Antonio Teixeira.

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The author is a fellow of CNPq, Grant: PQ-SR-307690/2016-4.

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## 1 Monge’s Ellipsoid

This story begins on a hot October night in 1970 in Rio de Janeiro. Suffering from insomnia, I decided to snoop around the books that my wife had carefully accommodated in our shelf.

My candid and relaxed attitude contrasted with a strange tension that emanated from the ensemble of books, flooding the living room. Intrigued, I was impelled to find out the cause.

Squeezed in a corner, wrapped in an elegant green cover, nervously pulsed the Aguilar, Spanish translation of the book “*Lectures on Classical Differential Geometry*” of D. Struik.

My devotion to Geometry and to the language of Cervantes, drove me to, naively, open it. And I did so just on the page from which, as if it had been lurking, the picture of the triaxial ellipsoid, posted in Fig. 1, popped up.

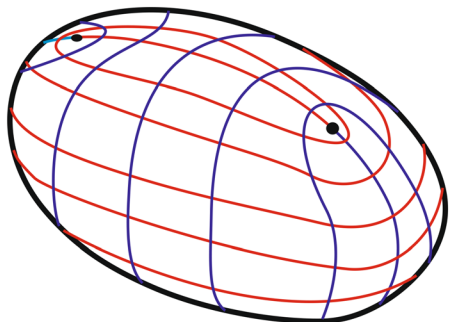
More than a century and a half had elapsed since the French mathematician Gaspard Monge conceived it, calculating its principal curvature lines, and locating its four umbilic points. Meanwhile, printed on books, restricted to an asphyxiating two-dimensional existence, the Ellipsoid had traveled thousands of kilometers, transposed mountains and crossed wide seas, so that on that singular tropical night, we could meet face to face. See also ellipsoids in Figs. 2, 3.

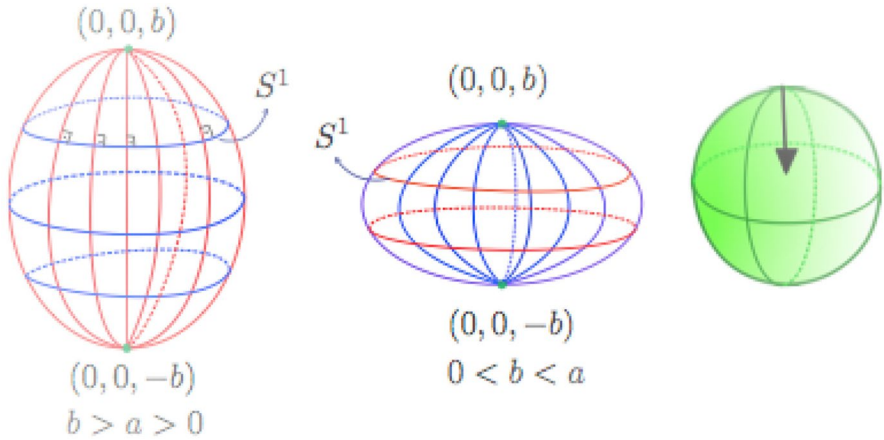
As early as in 1961 I had browsed through Struik’s book, but I did not notice the picture. At that time, I also was fond to peruse the classic “*Geometry and the Imagination*” of Hilbert and Cohn Vossen, not so much to read it but for contemplating its pictures. From this book, as Struik annotates, was taken the alluded picture.

How could I have ignored it?

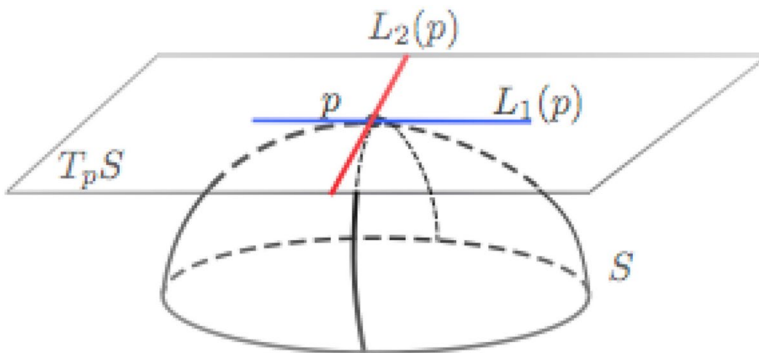
With meticulous attention I had studied two other excellent textbooks, of a comparable level to that of Struik: Willmore’s “*Introduction to Differential Geometry*” as an undergraduate in 1961, and O’Neill’s, “*Elementary Differential Geometry*”, as a young instructor in 1967. However, none of these two remarkable models of geometric exposition contains not even an outline of the fascinating picture. It should be mentioned, however, that O’Neil proposes a commented exercise in which he outlines how to locate the umbilic points of the ellipsoid, with no concern with their local principal configurations.

**Fig. 1** Monge’s Ellipsoid. Illustration of the ellipsoid with three different axes:  $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ ,  $a > b > c > 0$ , endowed with its Principal Curvature Configuration, which consists on its umbilic singular points and, outside them, its principal curvature lines. The separatrices are the curvature lines that connect the umbilic points





**Fig. 2** Ellipsoid of Revolution. Illustration of the (oblate, left) ellipsoid with two equal axes:  $x^2/a^2 + y^2/c^2 + z^2/b^2 = 1$ ,  $0 < a = c < b$ , its principal curvature lines and its pair of umbilic singular points. The (prolate, center) ellipsoid corresponds to  $0 < b < a = c$ . On the sphere, right,  $0 < c = a = b$ , all the points are umbilic



**Fig. 3** Principal directions at a point  $p$  on a surface  $S$  oriented by the inward normal field

It was like a love at first glance. The symmetry and beauty of its curves captivated me immediately. As in a ritual of mutual measuring, we stared at each other for some minutes.

I promptly read the adjacent text, as well as other relevant sections.

The reader of that singular night, however, was somewhat distant from the naive student of 1961. Having traveled along pathways of the Qualitative Theory of Differential Equations and Dynamical Systems, he had his mathematical awareness enlarged and shaped by the study of Structural Stability and Bifurcation Theory.

In an hour of active reading I made a mathematical journey that, chronologically, had spanned almost two centuries.

There, the classical results concerning the principal curvatures on a surface were proved. I reviewed Euler's formula that expresses the normal curvature in terms of the principal curvatures and directions.

The *principal curvature lines* are the curves, along a surface, whose tangent directions, *denominated principal directions*, make it to bend extremely in  $\mathbb{R}^3$ . The scalar measures of these extreme curvatures, are called the *principal curvatures*. One of them, the minimum, is denoted by  $k_1$ ; the other, the maximum, is designated by  $k_2$ . Their values are given by the normal curvature (the second fundamental form of the surface), evaluated at the principal directions. So,  $k_1 < k_2$ , except at the *umbilic points*, where  $k_1 = k_2$ .

Euler's formula states that the normal curvature  $k_n(\theta)$  in the direction that makes angle  $\theta$  with the minimum principal direction, is given by

$$k_n(\theta) = k_1 \cos^2 \theta + k_2 \sin^2 \theta.$$

This formula is equivalent to the diagonalization, by means of a rotation, of the Second Fundamental Form of Surface Theory.

In *Recherches sur la courbure des surfaces*, Mémoires de l'Académie de Sciences de Berlin, 16, (1767), Euler accomplished the first study of these mathematical objects, with which he inaugurated the use of Differential Calculus in the investigation of surface geometry. It is derived from this work that, in general, the principal directions are orthogonal.

At umbilic points, the principal curvatures coincide. Only outside them, the principal directions are defined and determine a pair of mutually orthogonal tangent line fields,  $L_1$  and  $L_2$ , called the principal line fields;  $L_1$ , corresponds to the minimal and  $L_2$ , to the maximal principal curvatures. The surface is assumed to be oriented. An exchange in its orientation permutes  $L_1$  and  $L_2$ .

With Struik I reviewed Dupin's Theorem that allows to determine the lines of principal curvature of surfaces that belong to a triply orthogonal family. To this end it is enough to intersect the surface under study with those of the two families orthogonal to it. In the case of Monge's Ellipsoid, one must take the hyperboloids of one and two sheets, which, together with the ellipsoids, form the so-called homofocal system of quadrics. These quadrics are also the coordinate surfaces of the so-called Ellipsoidal coordinate system in  $\mathbb{R}^3$  (Fig. 4).

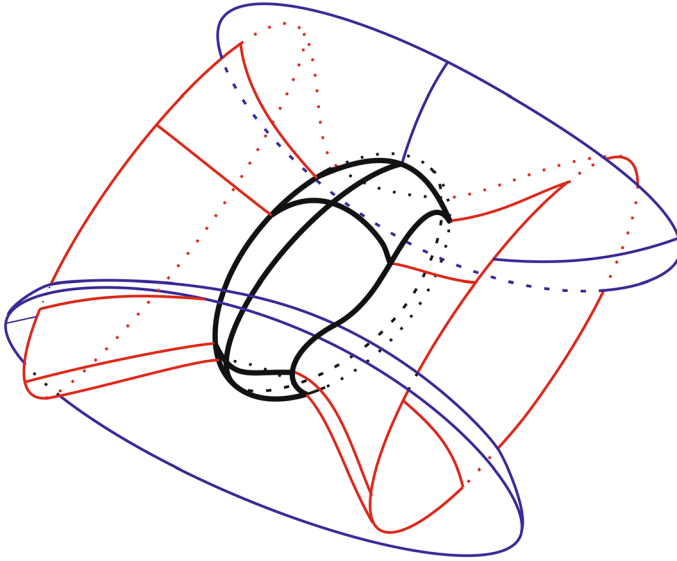
### 1.1 Report of a first encounter between classical principal curvature geometry and the structural stability of differential equations

I felt enraptured and rewarded. The browsing and reading session had allowed me to become aware of a remarkable qualitative jump in the evolution of the mathematical ideas, in three steps.

**Remark 1** *A triple qualitative jump.*

Firstly, Euler established the definitions and basic concepts.

The second step was taken by Monge, calculating a key example.



**Fig. 4** Dupin's theorem. Illustration of principal configurations for homofocal quadrics: ellipsoids and hyperboloids, with one and two sheets

Thirdly, Dupin demonstrated a general theorem that unifies the particular cases, all integrable, known at that time.

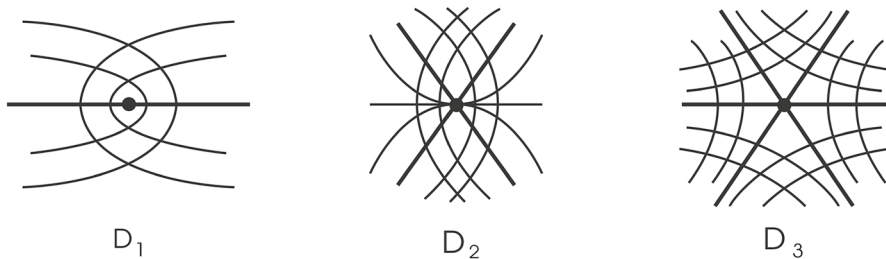
From Dupin's Theorem, a handy set of color pencils and numerous diagrams allowed me to reach the following conclusions:

**Conclusion 1** Consider the nine dimensional space  $\mathcal{Q}$  of quadratic compact surfaces, endowed with the topology defined by the, normalized, coefficients. The surfaces whose principal configurations are Structurally Stable (that is, they are not altered topologically by small perturbations of their coefficients), are precisely the ellipsoids of Monge, whose three axes are distinct. This class, denoted by  $\mathcal{E}_3$ , is an open and dense subset in  $\mathcal{Q}$ .

**Conclusion 2** Inside  $\mathcal{Q}_1 = \mathcal{Q} \setminus \mathcal{E}_3$ , those whose principal configurations are First Order Structurally Stable (that is, they are not altered by small disturbances within  $\mathcal{Q}_1$ ), are the ellipsoids of non-spherical revolution; these form a sub-variety  $\mathcal{E}_2$ , of codimension 2 in  $\mathcal{Q}$ .

**Conclusion 3** Inside the complement of  $\mathcal{E}_3 \cup \mathcal{E}_2$  in  $\mathcal{Q}$ , the spheres form a submanifold of dimension 3, that is of codimension 6.

Analogous results could also be formulated for the non compact quadrics. While I was quitting the long working session, the following inquiry struck me:



**Fig. 5** Darbouxian umbilic points. The subscripts stand for the number of umbilic separatrices for each of the principal foliations

**Problem 1** *How would it be, in general, the principally structurally stable configurations of smooth oriented compact non quadratic surfaces?*

## 2 The fundamental problem of principal curvature geometry

Two references from Struik had attracted my attention. One of them was a note in Volume III of the famous classical treatise of Gaston Darboux “*Leçons de la Théorie des Surfaces*,” Gauthier–Villars, 1888–1896, the other was an article in *Acta Mathematica*, 1904, by Alvar Gullstrand. Having slept over Problem 1, next morning, I promptly perused them at the library of IMPA.<sup>1</sup>

Both were concerned with the possible configurations of the lines of principal curvature in the vicinity of an umbilic point.

The first one had a description of the three cases of generic umbilic points, characterized by algebraic conditions, expressed as inequalities, involving the third derivatives of a function representing the surface in Monge coordinates centered at the umbilic point. See illustrations in Fig. 5. Everything suggested that principal configurations of these types could be also structurally stable, locally at the umbilic point.

As was usual, in works of around 1886, they were real analytic objects; surfaces in our case.

The details of the proofs at the first readings seemed very hard to assimilate.

Consider a surface in Monge form:

$$z = (k/2)(x^2 + y^2) + (a/6)x^3 + (b/2)xy^2 + (c/6)y^3 + O[(x^2 + y^2)^2],$$

in which the coefficient of the term  $x^2y$  has been eliminated by means of a rotation.

The conditions that define the Darbouxian umbilic points are written as follows:

### Definition 1

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T) Transversality Condition:  $b(b - a) \neq 0$ ;

D) Discriminant Conditions:

$$D_1: \frac{a}{b} > \left(\frac{c}{2b}\right)^2 + 2;$$

$$D_2: \left(\frac{c}{2b}\right)^2 + 2 > \frac{a}{b} > 1; a \neq 2b;$$

$$D_3: \frac{a}{b} < 1.$$

These expressions first appeared in print in 1983 in the first step carried out by Gutiérrez and Sotomayor, cited at the end of Sect. 8, in an endeavour to solve Problem 1.

Being illiterate in German, from the pictures and formulas, I concluded that Gullstrand's work, after reviewing Darboux's contribution, proposed a description of the initial non-generic cases.

That seemed a sort of the starting for an investigation of the bifurcations of the umbilic points. The potential connections with bifurcation problems of differential equations captured my interest.

After the contact with Darboux and Gullstrand references, my geometric imagery enriched remarkably.

Back home, for a long while, I contemplated and tried to organize coherently the precious antique pieces I had already collected.

I quitted to rest earlier than usual.

I woke up at dawn and, certain of being contributing to the fourth stage in the multiple mathematical qualitative jump witnessed the previous night, annotated in Remark 1, without hesitation, I wrote:

**Problem 2** The fundamental problem on the study of the Principal Curvature Configurations of the differentiable, compact and oriented surfaces immersed into  $\mathbb{R}^3$ , consists in establishing the following two theorems:

**Theorem 4** *The necessary and sufficient conditions for a surface to have its Principal Configuration Structurally Stable, with respect to  $C^3$  small deformations of its immersion into  $\mathbb{R}^3$ , are the following:*

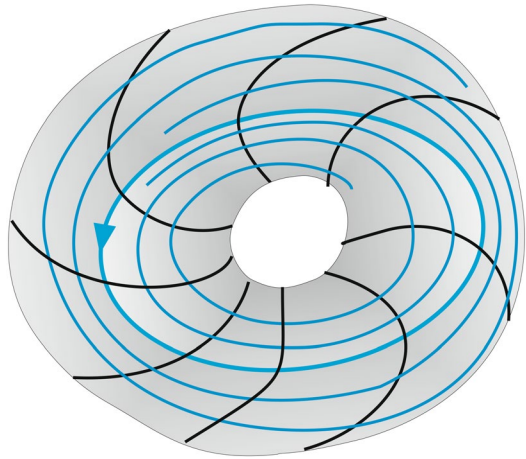
4a) *The umbilic points must all be Darbouxian.*

4b) *The periodic curvature lines (principal cycles) must be all hyperbolic. That is, their Poincaré Transformation, or First Return Map, must have a derivative different of 1 (Fig. 6).*

4c) *It must not admit connections, or self-connections, of umbilical separatrices.*

4d) *The limit sets of any non-periodic principal curve must be an umbilic point or a principal cycle.*

**Fig. 6** A periodic principal curvature line, i.e. a principal cycle, which is hyperbolic. Illustration of lines of curvature neighboring a principal curve, with first return map with derivative less than 1. The arrows are placed to indicate a local conventional orientation. The lines of curvature, generally, are not globally orientable. See, for example, a neighborhood of the umbilic points in Fig. 5



**Theorem 5** *Within the space of immersions of compact oriented surfaces, endowed with the  $C^3$  topology, the surfaces that satisfy the conditions (4a) to (4d) form a set that is 5a: open and 5b: dense.*

A rough scheme of demonstration for Theorems 4 and 5, revealed that part (5a) of Theorem 5, as well as the sufficiency of conditions (4a) to (4d) for Theorem 4, seemed feasible. However, serious difficulties in establishing global aspects of the necessity of the conditions and, above all, the density, part (5b) of Theorem 5, became apparent.

At that time no example was known of a compact surface satisfying the conditions (4a) to (4d).

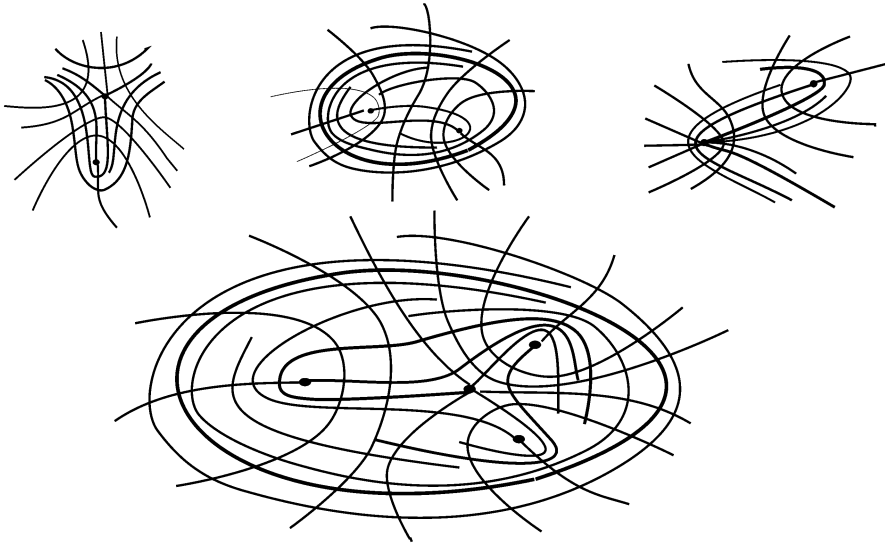
The influence that Peixoto's work on the genericity of the *Structurally Stable vector fields on two-dimensional manifolds*, Topology, 1962, was crucial in the formulation of the Fundamental Problem 2. This matter has been evoked in the author's essay:

- *On a list of ordinary differential equations problems, São Paulo Journal of Mathematical Sciences*, <https://doi.org/10.1007/s40863-018-0110-3>. Zbl1417.01036 Zbl1028.34001. MR3947401.
- The article <https://www.maths.ox.ac.uk/about-us/departamental-art/theory/differential-geometry>, in an Oxford University webpage, contains a very objective outline of Differential Geometry, with pertinent links.

### 3 Principal curvature configurations as a research area

A library search for examples led nowhere. Not a single clue for an isolated principal cycle, much less hyperbolic, was encountered (Fig. 7).





**Fig. 7** Illustrations of principally structurally stable patterns

The same for surfaces with recurrent principal curvature lines, that is with those lines that violate condition (4d) in Theorem 4, as it is the case of integral curves of non singular vector fields with *irrational rotation number* on the torus.

The Qualitative Theory of Differential Equations and Dynamical Systems, founded by Poincaré, almost 100 years before, had not penetrated this aspect of Principal Configurations in Classical Differential Geometry, contrasting with the progress that dynamic ideas had in the study of geodesics.

A first round of calculations led to a formula for the derivative of the Poincaré First Return Map, in terms of the Mean Curvature  $H = (k_1 + k_2)/2$ , the Gaussian Curvature:  $K = k_1 k_2$  and the Christoffel Symbols. The elimination of these symbols was achieved later using Codazzi integrability conditions.

The derivative,  $T'$ , of the first return map,  $T$ , for a periodic curvature line  $\gamma$ , after simplification achieved the following expression:

$$\log(T') = \pm \frac{1}{2} \int_{\gamma} dH / (H^2 - K)^{1/2}.$$

I accumulated a large collection of bibliographic references on principal curvature lines and umbilic points, none of them elucidating Problem 2. Among them were several papers relative to the Carathéodory Conjecture:

**Conjecture 6** *Every  $C^2$  convex and compact surface has at least two umbilic points.*

Years later I found out that this conjecture was considered demonstrated for the case of analytic surfaces. The conjecture is acknowledged to be open for  $C^\infty$  surfaces.

For the generic case it is trivial. In fact, having the Darbouxian umbilic points indices  $\pm 1/2$ , there must be at least four of them to be able to add up to give the Euler-Poincaré Characteristic of the surface (in this case equal to 2).

Intrigued about the circumstances that could have led to the formulation of this astonishing conjecture, with the aid of German colleagues, I researched for its sources. However, no written record of this formulation by the author was found. It is presumed that this was done orally in seminars.<sup>2</sup>

I learned that Gullstrand, the author for me initially unknown, had been awarded the 1911 Nobel Prize in Medicine, for his contribution to Ophthalmology. The Acta Mathematica paper that I had browsed through was the mathematical part of the work that made him worthy of the award.

Discussions about aspects of my project with colleagues, more experienced than me, in the fields of Dynamical Systems and Differential Geometry, led to no mathematical reward. Unfortunately, Peixoto was traveling, outside Rio de Janeiro, during that crucial semester.

I was lucky to meet the distinguished young American Geometer Herbert Blaine Lawson, visiting IMPA. He was very receptive and made the following thoughtful and stimulating comment:

*“If what you propose works, it will be opening a new research area.”*

## 4 A promenade with Monge, Darboux and Gullstrand

### 4.1 São Paulo

In March 1971, I visited the University of São Paulo. I was asked by colleagues to deliver a seminar lecture presenting the subject of my research.

This prompted me to organize the pertinent material I had gathered, starting with the classical background: Euler, Monge, Dupin, then continuing with the geometric transformations that preserve the principal configurations: rigid motions, inversions and small parallel displacements along the normals to the surface, etc. Concluding with a discussion of the statements of Theorems 4 and 5, the Darbouxian Umbilics and, to include a personal contribution, the formula for the derivative of the Poincaré return map that I had found.

Although the seminar lecture on Principal Configurations did not happen because of unexpected administrative obstacles, I could have rewarding discussions of local aspects of the project with Waldyr Oliva and Edgard Harle<sup>3</sup> (1937–2020).

In particular, Edgard Harle, Geometer and fluent in German, helped me to understand the article of J. Fischer, Deutsche Math., 1935, written in the Gothic alphabet,

<sup>2</sup> The biography of C. Carathéodory by the historian Maria Georgiadou, Springer, 2004, suggests that she had elucidated the origin of this conjecture. However, the reference given leads to a work on umbilic points unrelated to the conjecture-problem 6.

<sup>3</sup> <https://www.ime.usp.br/instituto/nossos-mestres/carlos-harle>.

giving an example of a principal configuration with spiraling behavior around an umbilic point.

The surfaces that verified the conditions (4b) and (4d) would be of this sort, around principal cycles.

Note that in the case of quadrics, surfaces of revolution and all the other examples established in Classical Differential Geometry, by the fact of deriving from a calculation, there is always a first integral.

The visit to São Paulo and the discussions, along the preparation for a seminar presentation, helped me to reinforce the conviction that the subject was not devoid of interest.

## 4.2 Salvador

The International Colloquium on Dynamical Systems of Salvador arrived, in July 1971. There I met the French mathematician René Thom (1923–2002) with whom I discussed the expectations I had around The Fundamental Problem 2.

It was an interesting dialog.

For him, the umbilics represented catastrophes, within the focal set: the envelope of the family of normal lines to the surface, that is the caustic of the surface.

When I asked something about the umbilical separatrices associated with the points of Darboux, he answered mentioning properties about the “ridges” associated with the hyperbolic and elliptic, (focal) umbilics of the generic surfaces.

When I mentioned the nets of lines of principal curvature on surfaces, he answered me with the focal set located in the ambient space.

## 4.3 Trieste and Paris

In July 1972, just before traveling to Trieste, to the International Center for Theoretical Physics (ICTP), and to Paris, for a visit to the Institut de Hautes Études Scientifiques (IHES), Carlos Gutiérrez, a doctoral candidate at IMPA, asked me for a suggestion of a thesis problem. Without thinking twice, I took out of my briefcase a copy of Darboux’s article (which always accompanied me) and I proposed:

**Problem 3** Give a modern proof of this theorem, so that mortals can understand it.

Prove that conditions (4a) to (4d) in the Fundamental Problem, Theorems 4 and 5, are generic.

Pay special attention to the last condition (4d).

On that trip I carried with me a good part of the bibliographic material relevant to Problem 3, in case I had to correspond with Gutiérrez.

To every Swedish mathematician I met in Trieste I commented about Gullstrand paper. I inquired for the medical journal in which he published his contribution to Ophthalmology. More than one colleague promised to mail me a copy, which never arrived to my hands.

By the end of November, I received a letter from Gutiérrez. He had decided to consider a more general form of Line Fields in two-dimensional manifolds. In that context, he already had a coherent theory involving a weak form of  $C^1$  density for a class of Line Fields defined by conditions resembling, topologically, those formulated in (4a) to (4d), Theorem 4.

His topologically general theory, however, did not bring anything enlightening on the genericity of the principal line fields and their singularities (the umbilic points), as stated in the Fundamental Problem 2.

I encouraged him to continue on the path he had set out, which I also found interesting.

When we met again in January 1973, we quickly discussed the difficulties encountered in attacking the problem incorporating the modification he suggested. We went on considering the more abstract approach, though quite distant from Classical Differential Geometry.

In this direction he converged to writing a Doctoral Thesis, IMPA 1974. In this endeavor, Gutiérrez became an exceptional expert in recurrent integral curves of line fields. A fact that will be crucial for the unfolding, in Sect. 8, of the story recounted in this essay.

For several years we did not speak again about umbilic points and principal curvature configurations.

## 5 Beaune and the Hotel Monge in Dijon

During the second semester of 1975 I visited Dijon for the first time. After perambulating several hotels, I ended lodged at the small Monge Hotel,<sup>4</sup> located on the street with the same illustrious name, in the heart of the charming neighborhood called “*Dijon Historique*”. I stayed there for a week.

My host, Robert Roussarie from the University of Dijon, took me to visit the famous vineyards of the Burgundy “*Côte d’Or*”.

We extended the visit to the historic village of Beaune. There, after a tour by impressive medieval buildings, I came upon, face to face, with an imposing statue of Gaspard Monge, [https://commons.wikimedia.org/wiki/File:Beaune\\_-\\_Monument\\_de\\_Gaspard\\_Monge.jpg](https://commons.wikimedia.org/wiki/File:Beaune_-_Monument_de_Gaspard_Monge.jpg), who was originally from that town.

The picture of the ellipsoid encountered in 1970 in Rio de Janeiro, as evoked in Sect. 1, reappeared in my memory. The cultural tour had excited my imagination. I was intrigued with the coincidence of my encounter with the statue and with the name of my hotel. The memory of the latent related problems 1, 2 and 3, came back.

The coincidence of historical, cultural and mathematical circumstances intrigued me along the rest of my visit to Dijon, while being lodged at the Hotel Monge. I profited

<sup>4</sup> Presently the restaurant “Le Marrakech” operates in the building that hosted the hotel at 20, rue Monge. See <https://www.yelp.com/biz/le-marrakech-dijon-2> and <https://www.yelp.com/biz/hotel-monge-dijon>.

from the occasion to browse Monge's book *Une application d'analyse à la géométrie, 1795*, at the University of Dijon Library, in the section of rare publications. There he studies the general properties of principal curvatures and direction fields. After perusing some of its sections, I got convinced that as a representative of the European Mathematics of the eighteenth century, Monge would have had a hard time to understand my gibberish and problematics, proposed in 1, 2 and 3, typical of an anonymous tropical redoubt of the decade of the sixties, in the 20th century.

The link [https://books.google.com.br/books?id=aSEOAAAAQAAJ&redir\\_esc=y](https://books.google.com.br/books?id=aSEOAAAAQAAJ&redir_esc=y), leads presently to the above cited book of Monge.

## 6 Plastic transparency and extrapolation exercises

Toward 1976, I had the help of mathematicians fluent in German, that translated for me substantial portions of Gullstrand's paper and helped me to interpret his results. It was clear that the author was strongly interested in the focal normal set, specially in how its ridge singular curves approached the umbilic points.

However, he did not address the analytic foundations to justify the principal configuration pictures of Darboux and, much less, the other more degenerate ones that he also considered in his work. However, issues such as the uniqueness of the umbilical separatrices, for example, were not touched at all.

I packed the bibliographic material pertinent to the Fundamental Problem 2, Theorems 4 and 5, and also my handwritten notes, in a plastic bag, which I carried up and down with me. For long periods, when my briefcase was too heavy, I used to leave the bag resting at home. Months later I would take it along with me again.

The transparency of the plastic prompted me to locate and have the Fundamental Problem present. The transportation ceremony ritual gave me a strange sensation of possession and of being working on it, secretly.

With uncertain periodicity, however, I engaged in longer intuitive exercises, without directly attacking its fundamental mathematical difficulties.

I particularly enjoyed extrapolating conditions 4a to 4d, weakening and adapting them to higher order ones, predicting the generic bifurcations of principal curvature configurations on surfaces evolving subject to one parameter spatial deformations. This amounted to an additional *stage in the sequence of qualitative jumps* discussed in Sects. 1 and 2.

A bolder, higher order, extrapolation to which the previous predictions pointed to were the principal configurations of 3-dimensional manifolds immersed into  $\mathbb{R}^4$ .

In Sect. 10 will be given references to works achieved later in the lines of research outlined above.

## 7 Collaboration in research teams

At the beginning of 1980 we received a determination from the CNPq Central Administration: Researchers should be organized in teams, as in soccer, and present joint projects. This was a novelty. We were accustomed to receiving requests to fill out new cadastre forms every time there was a change in the administrative cadres.

This naive administrative measure ended up being beneficial for me, for Carlos Gutiérrez and also for the Fundamental Problem on Principal Curvature Configurations.

I engaged Gutiérrez to join forces with me to work in the Issue.

The progress was surprisingly fast, taking into account that each of us worked simultaneously on individual projects. The two of us, and the Problem, had gained maturity.

We updated the formula for the derivative,  $T'$ , of the first return transformation,  $T$ , of a periodic curvature line  $\gamma$ .

In terms of the mean  $H$  and Gaussian  $K$  curvatures, the following beautiful expression, mentioned also in Sect. 3, holds:

$$\log(T') = \pm \frac{1}{2} \int_{\gamma} dH / (H^2 - K)^{1/2}.$$

In fact, chronologically, it was obtained after its equivalent version

$$\log(T') = \pm \int_{\gamma} dk_2 / (k_2 - k_1),$$

in terms of the principal curvatures  $k_1 < k_2$ , since  $H = (k_1 + k_2)/2$  and  $K = k_1 k_2$ ,

from which we derived a perturbation method to hyperbolize periodic curvature lines.

Using a method of resolution (“blowing up”) of singularities, very close to the traditional one, used to reduce the study of complicated singular points to simpler ones, we fully justified, for the case of surfaces of class  $C^4$ , the Darbouxian principal configuration pictures in Fig. 5.

This was a novelty in relation to the analytic case considered by the French Geometer.

On the recurrent principal lines of curvature, Carlos Gutiérrez, formulated the following diagnosis:

*It will be possible to get an approximation in class  $C^2$  that eliminates them. The increasing from class  $C^2$  to  $C^3$ , however, will be very difficult.*

We produced an example of a Toroidal Surface in  $\mathbb{R}^3$  without umbilic points and with dense recurrent curvature lines. Its immersion, however, was quite distant from the standard Torus of revolution.

The pieces of the puzzle seemed to fit. We had a sustainable version!

The time had come to make a broad communication of the results. The 1981 International Dynamical Systems Symposium inaugurating the new installations of IMPA, constituted an auspicious occasion.

## 8 The first international presentation on principal curvature configurations

On the eve of the lecture, I stayed until later than usual discussing with Gutiérrez what the presentation would be like. Some delicate points emerged.

It became explicit that we did not have examples of principal recurrences other than that in the aforementioned, free of umbilics, Torus. In particular, we did not know immersions of spheres, necessarily carrying umbilic points, exhibiting principal recurrences. Furthermore, at that time the Torus example seemed too technical to be quickly explained geometrically during the lecture.

We were in the uncomfortable position of being able to eliminate all the recurrent curvature lines, replacing them after an approximation  $C^2$ -small, by periodic lines of curvature or, in some cases, by connections of umbilical separatrices. Specifically we knew only the example of the Torus.

What if someone, interested in concrete examples, raised the question? Worst still, what if there were no more examples of recurrences than those on the Torus?

In that case, our results would be considerably weakened.

The search for an example of principal recurrences on a surface of genus zero lasted a long while.

It was then that Monge's Ellipsoidal reappeared; this time it was more flexible than ever. It allowed more daring deformations and contortions, including non-analytic ones. So, bending more here, rotating there, we stumbled over an example.

The lecture was ready! It had been scheduled to open the next morning session.

I started projecting Monge's Ellipsoid (Fig. 1) and reviewed how to explain its principal configuration, using the Theorem of Dupin (Fig. 4). I commented on the absence of examples of Global Principal Configurations, distinct from that one (Fig. 8).

After the preparatory setting concerning the space of Immersions of oriented two-manifolds into  $\mathbb{R}^3$ , I proposed the problem of recognizing globally the immersion with Structurally Stable principal configurations and establishing their genericity. Then came the statement of our result, insisting on the limitation in class  $C^2$  for the density approach. I ended up summoning the listeners to raise this class to  $C^3$ , thus solving the only problem that remained open in relation to the initial program, as stated in Theorems 1 and 2.

**Remark 2** This problem remains open until now.

Reaching the end of the lecture, none of the experts in recurrences on Dynamical Systems in the audience asked for specific examples. This was disappointing.

Fortunately, Dan Henry<sup>5</sup> (1945–2002), an expert in Differential Equations in Infinite Dimensions, formulated the propitious question:

<sup>5</sup> <https://www.ime.usp.br/map/dhenry/danhenry/texto01.htm>.



**Fig. 8** No photo of the lecture at IMPA, 1981, was found. Here is one from a lecture on Principal Configurations delivered in Montevideo, 2012. Photo by A. Chenciner. The ellipsoid picture was borrowed from E. Ghys site

*“I do not know why you make the hypothesis (4.d). I do not know any surface that does not satisfy it.”*

I drew an ellipsoid of revolution (Fig. 2, left). I deformed it slightly so that, in both polar caps, it was like one of Monge, with their three axes distinct, while around the equator it continued being one of revolution.

Then, I rotated around the major axis only the upper hemisphere of the surface. Because of its equatorial rotational symmetry, this was a family of  $C^\infty$  surfaces  $E_\theta$ , depending on the parameter  $\theta$ , designating the angle of rotation.

It is evident that the curvature lines of  $E_\theta$  define in their second return to the equatorial circle a rotation of angle  $2\theta$ . So, for angles incommensurable with respect to  $2\pi$ , the lines of curvature become all dense in  $E_\theta$ . See illustration in Fig. 9.

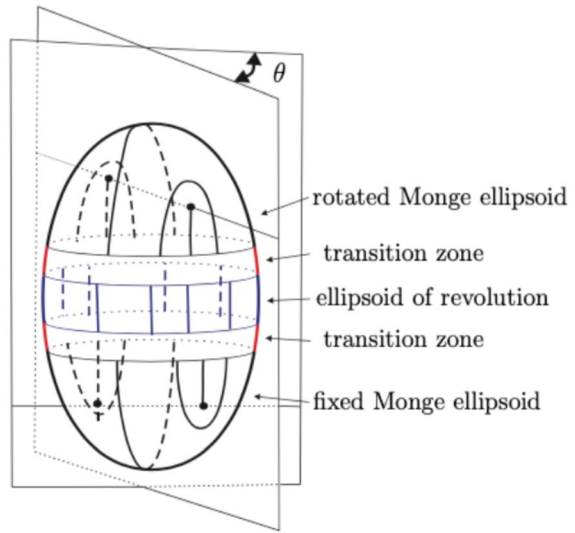
The written presentation of the work was divided in two parts.

The first one, establishing that conditions (4a) to (4d) define a  $C^3$  open set consisting in Structural Stability immersed surfaces, was published in *Astérisque*, Vol. 98-99, 1982.

The second part, which demonstrates that any compact and oriented surface can be arbitrarily  $C^2$  approximated by one that verifies the four above mentioned conditions, appeared in *Springer Lecture Notes in Mathematics*, Vol. 1007, 1983.



**Fig. 9** Ellipsoidal Surface  $E_\theta$  with oscillatory recurrent principal curvature lines



**Remark 3** The papers above are pointed out as the documentation recording the first mathematical encounter between the line of thought disclosed from Principal Curvature Geometry, represented by the works of Monge, 1796, Dupin, 1815, and Darboux, 1896, with that of Structural Stability of Dynamical Systems, transpiring from the achievements of Poincaré, 1881, Andronov–Pontrjagin, 1937, and Peixoto, 1962.

## 9 A Herald of the qualitative theory of differential equations

In the first half of 1990, on the occasion of the “*Année Spéciale de Systèmes Dynamiques*”, sponsored by the CNRS (National Research Council) of France, I delivered a short course at the University of Dijon. The subject was *Umbilic Points and Principal Curvature Lines*.

During that visit I had access to the original work of Monge, entitled *Sur les lignes de courbure de la surface de l’Ellipsoïde*, published in *Journal de l’Ecole Polytechnique., II cah, 1796*.

The contemplation of Monge’s original pictures, supplemented with some additional readings, led me to make explicit an intuitive observation about the history of mathematical ideas, which, in an embryonic form, was already present in my initial motivation, after the first contact with the Ellipsoid and the subsequent formulation of the Fundamental Problem 1.

There is no bibliographic record that Euler, responsible for the conceptualization of the principal curvature line fields, would have integrated them, visualizing the Principal Configuration.

Monge was the first mathematician to recognize the importance of this structure, providing the first non-trivial example, for global integration of the differential equations of the lines of curvature (that is, of the line fields  $L_1$  and  $L_2$ ) in the case of the Ellipsoid.

Let's inquire: *What would have led him to establish a result of this nature, which in our days fits perfectly within the Qualitative Theory of Ordinary Differential Equations, considering that this happened almost one hundred years before Poincaré founded it and defined its goals?* Monge's motivation derived from a complex interaction of aesthetic and practical considerations, and also the explicit intention to apply the results of his mathematical research.

At Monge's time was more tenuous the artificial separation between Mathematics and Applications. The lines of curvature were discovered studying the problem of transportation of debris in the construction of embankments and fortifications, called the "*deblais et remblais*" problem. His ellipsoidal picture was proposed in the architectural project for the construction of the vault, over an elliptical terrain, of the Legislative Assembly Building of the French Revolution Government: the lines of curvature would be the guiding curves for the placement of the stone bricks, the umbilics would serve as supporting points for hanging the sources of illumination, under one of which would be located the rostrum for the speakers. The architectural project was never put into effect.

**Remark 4** If Poincaré, for the scope and depth of its contribution, is recognized as the founder, Monge has the merit of being regarded as a precursor—a herald—of the Qualitative Theory of Differential Equations and of Foliations with Singularities Theory.

However, it should be emphasized that the novelty of the Qualitative Theory of Poincaré lies in the methods developed for the study of the phase portrait of general non-integrable differential equations, that he established for polynomial equations, in the generic case. Keeping in mind that this was a laboratory model for the far reaching and more complex problems of Celestial Mechanics that he was investigating.

The connection between the works of Monge and Poincaré, outlined above, was overlooked in the accomplished historical work of René Tatón "*L'Oeuvre Scientifique de Monge*," Presses Univ. de France, 1951.

## 10 Books, later developments and related references

### 10.1 A short course in 1991

The project of writing an expository book based on the lectures delivered at the 1990 short course in Dijon, was formulated on that occasion. See the beginning of Sect. 9.

In 1991, I engaged Gutiérrez to collaborate with me in the project above, which was proposed as the lecture notes for a course in the 18th Brazilian Mathematics Colloquium.

Besides the results on Structural Stability and Approximation established in the original papers, the lecture notes included more details concerning the theoretical foundations and the motivation for the study of Principal Curvature Configurations. The examples of principal recurrences were improved and reformulated in more

conceptual terms, especially the one devoted to the immersed Toroidal Surface.<sup>6</sup> The bibliographic references were also updated. See:

- C. GUTIERREZ and J. SOTOMAYOR, Lines of Curvature and Umbilical Points on Surfaces, 18th Brazilian Math. Colloquium, Rio de Janeiro, IMPA, (1991). Reprinted, with some upgrading, as Structurally Stable Configurations of Lines of Curvature and Umbilic Points on Surfaces, Lima, Monografias del IMCA, (1998). MR2007065.

## 10.2 A book in 2009

A book of broader scope than that of 1991, with more topics on the rich interaction of Classical Differential Geometry and Differential Equations, was published in 2009.

- R. GARCIA and J. SOTOMAYOR, *Differential Equations of Classical Geometry, a Qualitative Theory*, Publicações Matemáticas, 27º Colóquio Brasileiro de Matemática, IMPA, (2009). Zbl 1180.53002. MR2532372.

## 10.3 Further developments on surfaces in $\mathbb{R}^3$

Without claiming to be complete, some additional works that are pertinent to the present essay are listed below in chronological order.

- C. GUTIERREZ and J. SOTOMAYOR, *Principal lines on surfaces immersed with constant mean curvature*. Trans. Amer. Math. Soc. **293**, 1986, no. 2, 751–766. MR816323. Zbl 0598.53007.
- J. W. BRUCE and D. L. FIDAL, *On binary differential equations and umbilic points*, Proc. Royal Soc. Edinburgh **111A**, 1989, 147–168. Zbl 0685.34004.
- C. GUTIERREZ and J. SOTOMAYOR, *Periodic lines of curvature bifurcating from Darbouxian umbilical connections*. Bifurcations of planar vector fields (Luminy, 1989), 196–229, Lecture Notes in Math., 1455, Springer, Berlin, 1990. MR1094381.
- B. SMYTH and F. XAVIER, *A sharp geometric estimate for the index of an umbilic on a smooth surface*. Bull. London Math. Soc. **24**, (1992), no. 2, 176–180. MR1148679.
- B. SMYTH and F. XAVIER, *Eigenvalue estimates and the index of Hessian fields*. Bull. London Math. Soc. **33**, (2001), no. 1, 109–112. MR1798583.
- R. GARCIA and J. SOTOMAYOR, *Lines of curvature near singular points of implicit surfaces*. Bull. Sci. Math. **117** (1993), no. 3, 313–331. MR1228948.

<sup>6</sup> This example of 1991 has only one of the principal foliations with dense curves. In R. GARCIA and J. SOTOMAYOR, *Tori embedded in  $\mathbb{R}^3$  with dense principal lines*. Bull. Sci. Math., **133:4** (2009), 348–354, was given an example in which both principal foliations have its lines dense.

- R. GARCIA and J. SOTOMAYOR, *Lines of curvature near hyperbolic principal cycles*. Dynamical systems (Santiago, 1990), 255–262, Pitman Res. Notes Math. Ser., 285, Longman Sci. Tech., Harlow, 1993. MR1213951.
- R. GARCIA and J. SOTOMAYOR, *Lines of Curvature on Algebraic Surfaces*, Bull. Sciences Math. **120**, (1996), 367–395. MR1411546.
- J. SOTOMAYOR, *Lines of curvature and an integral form of Mainardi-Codazzi equations*. An. Acad. Brasil. Ciênc. **68** (1996), no. 2, 133–137. MR1751266.
- T. MAEKAWA, F. E. WOLTER and N. M. PATRIKALAKIS, *Umbilics and lines of curvature for shape interrogation*, Comput. Aid. Geometr. Des. **13** (1996), 133–161. Zbl 0875.68858.
- R. GARCIA and J. SOTOMAYOR, *Structural stability of parabolic points and periodic asymptotic lines*, Matemática Contemporânea, **12**, (1997), 83–102. MR163442.
- C. GUTIERREZ and J. SOTOMAYOR, *Lines of Curvature, Umbilical Points and Carathéodory Conjecture*, Resenhas IME-USP, **03**, (1998), 291–322. MR1633013.
- R. GARCIA, C. GUTIERREZ and J. SOTOMAYOR, *Lines of principal curvature around umbilics and Whitney umbrellas*. Tohoku Math. J. (2) **52:2** (2000), 163–172. MR1756092.
- R. GARCIA and C. GUTIERREZ, *Ovaloids of  $\mathbb{R}^3$  and their umbilics: a differential equation approach*. J. Differential Equations **168**, (2000), no. 1, 200–211. MR1801351.
- R. GARCIA and J. SOTOMAYOR, *Structurally stable configurations of lines of mean curvature and umbilic points on surfaces immersed in  $\mathbb{R}^3$* , Publ. Matemáticas. **45:2**, (2001), 431–466. MR1876916. Zbl 0875.68858.
- R. GARCIA and J. SOTOMAYOR, *Umbilic and tangential singularities on configurations of principal curvature lines*. An. Acad. Brasil. Ciênc. **74**, (2002), no. 1, 1–17. MR1882514.
- B. SMYTH and F. XAVIER, *A sharp geometric estimate for the index of an umbilic on a smooth surface*. Bull. London Math. Soc. **24**, (1992), no. 2, 176–180. MR1148679.
- B. SMYTH and F. XAVIER, *Eigenvalue estimates and the index of Hessian fields*. Bull. London Math. Soc. **33**, (2001), no. 1, 109–112. MR1798583.
- B. SMYTH, *The nature of elliptic sectors in the principal foliations of surface theory*. EQUADIFF 2003, 957–959, World Sci. Publ., Hackensack, NJ, 2005.
- R. GARCIA and J. SOTOMAYOR, *Lines of Geometric Mean Curvature on surfaces immersed in  $\mathbb{R}^3$* , Annales de la Faculté des Sciences de Toulouse, **11**, (2002), 377–401. MR2015760.
- R. GARCIA and J. SOTOMAYOR, *Lines of Harmonic Mean Curvature on surfaces immersed in  $\mathbb{R}^3$* , Bull. Bras. Math. Soc., **34**, (2003), 303–331. MR1992644.
- R. GARCIA and J. SOTOMAYOR, *Lines of Mean Curvature on surfaces immersed in  $\mathbb{R}^3$* , Qualit. Theory of Dyn. Syst. **5**, (2004), 137 - 183. MR2129722.
- R. GARCIA, C. GUTIERREZ and J. SOTOMAYOR, *Bifurcations of Umbilic Points and Related Principal Cycles*. Journ. Dyn. and Diff. Eq. **16, 2**, (2004), 321–346. MR2129722.

- R. GARCIA and J. SOTOMAYOR, *On the patterns of principal curvature lines around a curve of umbilic points*. An. Acad. Brasil. Ciênc. 77 (2005), no. 1, 13–24. MR2114929
- R. GARCIA and J. SOTOMAYOR, *Lines of principal curvature near singular end points of surfaces in  $\mathbb{R}^3$* . Singularity theory and its applications, 437–462, Adv. Stud. Pure Math., **43**, Math. Soc. Japan, Tokyo, (2006). MR2325150.
- R. GARCIA and J. SOTOMAYOR, *Tori embedded in  $\mathbb{R}^3$  with dense principal lines*. Bull. Sci. Math., **133:4** (2009), 348 - 354. MR2503006.
- R. GARCIA, R. LANGEVIN and P WALCZAK, *Foliations making a constant angle with principal directions on ellipsoids*. Annales Polonici Mathematici, **113** (2015), 165–173. MR3312099.
- R. GARCIA and J. SOTOMAYOR, *Historical Comments on Monge’s Ellipsoid and the Configurations of Lines of Curvature on Surfaces*, Antiquitates Mathematicae, **10(1)** (2016), 348–354. Zbl 1426.53008. MR3613151.
- V. V. IVANOV, *An analytic conjecture of Carathéodory*. Siberian Math. J. **43**, no. 2, (2002), 251–322. MR1902826.

#### 10.4 Principal configurations of hypersurfaces in $\mathbb{R}^4$

An important achievement that must be mentioned is the extension of the generic properties of the classical lines of curvature on surfaces to the case of three-dimensional manifolds immersed in  $\mathbb{R}^4$ , first obtained by Ronaldo García in his Thesis (IMPA, 1989).

In this line of research the principal singularities at which two, of the three principal curvatures coincide, are generically smooth curves, along which the transversal structure of the principal curvatures resemble that of one parameter families of umbilic points of surfaces into  $\mathbb{R}^3$ .

The initial steps on this promising, though intricate, line of research can be found in the works below.

- R. GARCIA *Principal Curvature Lines near Partially Umbilic Points in hypersurfaces immersed in  $\mathbb{R}^4$* , *Comp. and Appl. Math.*, 20, 121 - 148, 2001.

An integral expression for the hyperbolicity condition of periodic principal lines, extending to the hypersurfaces in  $\mathbb{R}^4$  case, the one provided in Sect. 7, for surfaces, is not known. The study of the periodic linear  $2 \times 2$  system of differential equations which defines the derivative of the first return map, led R. Garcia to establish, in his IMPA Thesis, the hyperbolicity of periodic principal lines also for hypersurfaces in  $\mathbb{R}^4$ , holds generically.

The following more recent developments should be added:

- D. LOPES, J. SOTOMAYOR and R. GARCIA, *Umbilic and Partially umbilic singularities of Ellipsoids of  $\mathbb{R}^4$* . Bulletin of the Brazilian Mathematical Society, **45**, (2014), 453–483.

- D. LOPES, J. SOTOMAYOR and R. GARCIA, *Partially umbilic singularities of hypersurfaces of  $\mathbb{R}^4$* . Bulletin de Sciences Mathématiques, **139**, (2015), 421–472.
- R. GARCIA and J. SOTOMAYOR, Lines of curvature on quadric hypersurfaces of  $\mathbb{R}^4$ , Lobachevskii Journal of Mathematics **37**, (2016), 288–306.

An initial study of the generic bifurcations of the one-parameter principal configurations on hypersurfaces in  $\mathbb{R}^4$ , pertinent to partially umbilic singularities, was carried out in

- D. LOPES, Pontos Parcialmente Umbílicos em famílias a um parâmetro de superfícies em  $\mathbb{R}^4$ . <https://teses.usp.br/teses/disponiveis/45/45132/tde-28012014-144141/pt-br.php> DOI 10.11606/T.45.2012.tde-28012014-144141

### 10.5 Axial configurations on surfaces in $\mathbb{R}^4$

The name *axial* refers to the association with the axes of the *ellipse of normal curvature* defined at the regular points of surfaces mapped into  $\mathbb{R}^4$ . The *axial umbilic singular points* occur at the points where the *ellipse of normal curvature* degenerates to a circle. Outside these points, the curves along which the normal curvature is at a vertex of the *ellipse of normal curvature* define a pair of fields of *normal crossings*, one corresponding to each ellipse axes: the *the principal* and the *mean*, corresponding, respectively, to the large and small axes of the *ellipse of curvature*.

The the geometric foundations of these normal crossings fields and their integral curves as well as their generic patterns of approach to the *axial umbilic singular points* and periodic integral curves was carried out in:

- R. GARCIA and J. SOTOMAYOR, Lines of axial curvature on surfaces immersed in  $\mathbb{R}^4$ , Differential Geom. Appl. **12**, (2000), 253 - 269. MR1764332. Zbl 0992.53010.

Other pertinent related extensions and promising directions of research can be found in:

- L. F. MELLO, *Mean directionally curved lines on surfaces immersed in  $\mathbb{R}^4$* , Publ. Mat. **47**, (2003), 415–440.
- R. GARCIA, L. MELLO and J. SOTOMAYOR, *Principal mean curvature foliations on surfaces immersed in  $\mathbb{R}^4$* , EQUADIFF (2003), 939–950, World Sci. Publ., Hackensack, NJ, 2005. Zbl 1116.57024.
- R. GARCIA and J. SOTOMAYOR, *Lines of axial curvature at critical points on surfaces mapped into  $\mathbb{R}^4$* , Sao Paulo J Math Sci **6**, (2012), no. 2, 277–300.
- R. GARCIA, J. SOTOMAYOR and F. SPINDOLA, *Axiumbilic Singular Points on Surfaces Immersed in  $\mathbb{R}^4$  and their Generic Bifurcations*, Journal of Singularities **10**, (2014), 124–146.

- F. SPINDOLA, *Bifurcação de pontos axiumbílicos e ciclos axiais de superfícies imersas em  $\mathbb{R}^4$* , <https://teses.usp.br/teses/disponiveis/45/45132/tde-25012016-153923/pt-br.php> DOI 10.11606/T.45.2016.tde-25012016-153923

## 10.6 Two open problems related to works cited in this essay

### 10.6.1 Increase in the density class from $C^2$ to $C^3$ , as inquired in Theorem 2

### 10.6.2 Recurrence on cubic deformations of Monge's ellipsoid: From exercise 3.6.3, p. 83, of the 2009, Garcia and Sotomayor book

For  $\rho$  small, consider the cubic surface  $S_\rho = f_\rho^{-1}(0)$

$$f_\rho(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + z^2 + \rho xyz - 1; \quad a > 0, \quad b > 0, \quad (a-1)(b-1)(a-b) \neq 0.$$

From simulations of the possible global behaviors of principal foliations of  $S_\rho$ , formulate a conjecture about the possibility of dense principal lines on algebraic surfaces of spherical type.

The surface  $S_\rho$ , for  $\rho \neq 0$  is the simplest algebraic one that exhibits the twisting effect depicted in the smooth example in Fig. 9. This effect, determined by the cubic term  $\rho xyz$ , was first studied in Garcia and Sotomayor, Bull. Sci. Math. 1993, for the case of the cubic surface

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - z^2 + \rho xyz = 0; \quad a > 0, \quad b > 0, \quad (a-1)(b-1)(a-b) \neq 0,$$

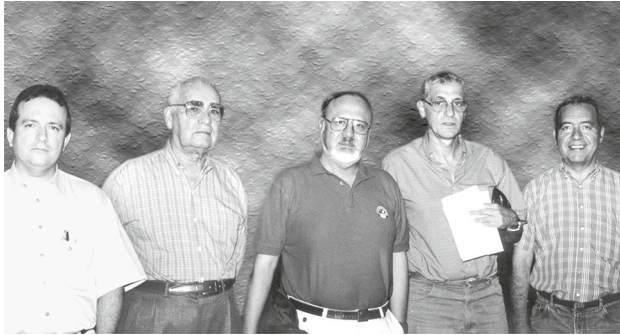
locally conoidal at its critical point  $(0, 0, 0)$ . It also appears in the study of the general case of principal configurations on algebraic surfaces. See Garcia and Sotomayor, Bull. Sci. Math. 1996.

No mathematical proof of the existence of recurrences on  $S_\rho$ , for  $\rho \neq 0$  has been published yet.

## 10.7 Three pertinent presentations

Three lectures listed below, overlapping with the subject of this essay, delivered by the author, have been posted in:

- <https://youtu.be/JX2pHiCvaxw>, Workshop Internacional de Sistemas Dinâmicos, 2011, Celebration of the 90th anniversary of Mauricio Peixoto.
- [https://www.youtube.com/watch?v=oVg-jyB\\_2Dw](https://www.youtube.com/watch?v=oVg-jyB_2Dw), in New Trends in Onedimensional Dynamics, 2016, Symposium in Celebration of the 70th anniversary of Wellington de Melo.
- <https://youtu.be/IscUm7UHv50>, 2018, A lecture dedicated to the memory of Carlos Gutiérrez (Fig. 10).



**Fig. 10** At UNICAMP, 2000, three mathematical generations: right to left, C. T. Gutiérrez, M. A. Teixeira, J. Sotomayor, M. M. Peixoto and R. A. Garcia

### 10.8 On the genesis of this essay

This work is an English version, which, after considerable revision, upgrading and elaboration, evolved from

- *O Elipsóide de Monge*, Portuguese, Mat. Univ., 15, 1993, and its Spanish enriched presentation and preliminary pertinent adaptation and updating published in
- *Materials Matemàtiques*, 2007: <http://mat.uab.cat/matmat/PDFv2007/v2007n01.pdf>.

### 10.9 Closing words evoking Poincaré

*If we wish to foresee the future of Mathematics,  
the right approach is to study its history and present condition.  
For us mathematicians, is it not this procedure, to some extent, routine?  
We are accustomed to extrapolation, which is a method  
of deducing the future from the past and the present;  
and, since we are well aware of its limitations, we run no risk  
of deluding ourselves as to the scope of the results it provides.*

H. Poincaré, in *The Future of Mathematics (L'Avenir des Mathématiques)*, speech read by G. Darboux at the ICM, Rome, 1908.

**Acknowledgements** The author is grateful to M. O. Sotomayor, C. P. Moromisato, F. E. Wolter and L. F. Mello for helpful style suggestions on a previous version and to R. A. Garcia for his mathematical comments and substantial aid in the production of the pictures in this work.

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