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SPECIAL SECTION: AN HOMAGE TO MANFREDO P. DO CARMO

Certain submanifolds of complex space forms

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Abstract

After recalling the geometric meaning of the commutativity of the second fundamental tensor of a real hypersurface of a complex space form and its induced almost contact structure, we present some classification theorems for CR submanifolds of maximal CR dimension and submanifolds of real codimension two of complex space forms $\overline{M}$, under the algebraic condition on the second fundamental form of the submanifold and the endomorphism induced from the natural almost complex structure of $\overline{M}$ on the tangent bundle of the submanifold.

Keywords Complex space forms · CR submanifolds of maximal CR dimension · Submanifold of real codimension two · Second fundamental form

1 Introduction

Real hypersurfaces of Kähler manifolds (especially of complex space forms) have been an active field of research for years and many authors have described a lot of interesting geometric properties of these hypersurfaces (see for example [3, 17, 18, 21, 22] and especially [19] for the fundamental definitions and results and for further references). However, for arbitrary codimension, there are only a few recent results. Motivated by the work on the classification of hypersurfaces that satisfy the commutative condition (1.1), we recall the study of the generalization of the hypersurfaces,

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namely CR submanifolds of maximal CR dimension and submanifolds of real codimension two, in complex space forms, which satisfy the corresponding condition (1.2).

One of the aims of submanifold geometry is to classify submanifolds according to given geometric data, for example, studying the relations and the interplay between intrinsic invariants, which only depend on the submanifold as a manifold itself, and extrinsic invariants, which depend on the immersion. The structure of a submanifold in Riemannian geometry is encoded in the second fundamental form. We have characterized several important classes of submanifolds considering the condition (1.2) on the submanifold structure (represented by the second fundamental form) and structure naturally induced from the almost complex structure of the ambient space. This idea goes back at least as far as [22], where the hypersurfaces of a complex projective space were studied. Namely, a real hypersurface M of a complex space form $\overline{M}$ has two geometric structures: an almost contact structure F induced from the almost complex structure J of $\overline{M}$, and a submanifold structure represented by the second fundamental tensor A of M in $\overline{M}$. In this sense, in [18, 22], the authors studied and classified real hypersurfaces M of a complex projective space and of a complex hyperbolic space (Theorems 3.1 and 3.2), respectively, which satisfy the commutativity condition

$$FA = AF. \tag{1.1}$$

Above all, they gave the geometric meaning of the commutativity of the second fundamental tensor of the real hypersurface of a complex space form and its induced almost contact structure.

A real hypersurface is a typical example of a CR submanifold of maximal CR dimension and the generalization of some results which are valid for real hypersurfaces to CR submanifolds of maximal CR dimension may be expected, although for arbitrary codimension, less detailed results are known. For example, we refer to [5, 7, 8, 24] and the book [10], where we collected the elementary facts about complex manifolds and their submanifolds and introduced the reader to the study of CR submanifolds of complex manifolds, especially complex projective space.

On this occasion we present our results on submanifolds M of complex space forms $(\overline{M}, J, \overline{g})$ which satisfy the condition

$$h(FX, Y) + h(X, FY) = 0, \text{ for all } X, Y \in T(M), \tag{1.2}$$

where F is the skew-symmetric endomorphism acting on $T(M)$ and induced by the almost complex structure J of $\overline{M}$ and h is the second fundamental form of M. This condition corresponds to (1.1) and is equivalent to (4.14)–(4.16) for CR submanifolds of maximal CR dimension and to (5.10) for submanifolds of real codimension two.

In Sect. 3 we recall the notation and the above-mentioned study and results from the theory of hypersurfaces of complex space forms we need here and we present more details on the investigation of the condition (1.1) on real hypersurfaces of complex space forms. Moreover, we explain several examples of real hypersurfaces in a complex projective space and in a complex hyperbolic space, which will be characterized in Sects. 4 and 5 and which are commonly called ”model spaces”. Namely, it is well-known that a complex projective space does not admit either totally geodesic or totally umbilical real hypersurface and, as one can regard a complex projective space $\mathbb{C}P^n$ as a projection from S^{2n+1} with fibre S^1, H.B. Lawson in [15] exploited this idea to study a hypersurface in a complex projective space by lifting it to an S^1-invariant hypersurface of the sphere. A distinguished example of a real hypersurface of $\mathbb{C}P^n$ is $M^C_{p,q}$ which is defined as a projection $\pi(S^{2p+1} \times S^{2q+1})$ of the product of two odd-dimensional spheres in a unit sphere S^{2n+1}, where $p + q = n - 1$ and π is the Hopf map (see also [10], p. 104). The analogous construction has been done for a complex hyperbolic space and as well explained in Sect. 3.

In Sects. 4 and 5, we consider a condition (1.2) corresponding to (1.1), by studying CR submanifolds M^n of maximal CR dimension (equal to $\frac{n-1}{2}$) and submanifolds of real codimension two, respectively, both in complex space forms. In Sect. 5 we pay a special attention to some CR submanifolds of CR dimension $\frac{n-2}{2}$, as a special class of submanifolds of real codimension two. The aim of this paper is to bring together two areas of our previous work: the study of CR submanifolds of maximal CR dimension and of submanifolds of real codimension two in complex space forms, under the condition (1.2) which relates the intrinsic and extrinsic information and this work is intended as an attempt to motivate further research on his topic, for example the study of the following

Problem Investigate submanifolds M^m of real codimension two of a complex hyperbolic space, under the condition (1.2), so that there exist real hypersurfaces M^*_{m+1} or $M^H_{p,q}(r)$ such that $M \subset M^*_{m+1}$ or $M \subset M^H_{p,q}(r)$.

In order to continue the investigation of the algebraic condition (1.2) and its influence on the geometry of the submanifold, it is useful to compare how the condition (1.2) looks like for real hypersurfaces of complex manifolds (1.1), for CR submanifolds of maximal CR dimension (4.14)–(4.16) and for submanifolds of real codimension two of complex space forms (5.10). Also, we point out how the Gauss and Weingarten formulae (4.1), (4.2) (intensively used in our study) are expressed for real hypersurfaces of complex manifolds (3.1), (3.2), for CR submanifolds of maximal CR dimension (4.10)–(4.12) and for submanifolds of real codimension two of complex space forms (5.6), (5.7). Moreover, since the condition (1.2) depends on the endomorphism F induced by the almost complex structure J, we hint at its derivatives for real hypersurfaces of complex manifolds (5.5), for CR submanifolds of maximal CR dimension (4.17)–(4.19) and for submanifolds of real codimension two of complex space forms (5.8)–(5.9).

2 Preliminaries

Let M be an m-dimensional submanifold of a Riemannian manifold $(\overline{M}^{m+k}, \overline{g})$ with the immersion $\imath$ of M into $\overline{M}$, where we also denote by $\imath$ the differential of the immersion, or we omit to mention $\imath$, for brevity of notation. The Riemannian metric g of M is induced from the Riemannian metric $\bar{g}$ of $\overline{M}$ in such a way that $g(X, Y) = \overline{g}(\imath X, \imath Y)$, where $X, Y \in T(M)$. We denote by $T(M)$ and $T^{\perp}(M)$ the tangent bundle and the normal bundle of M, respectively.

Next, let us denote by $\overline{\nabla}$ and ∇ the Riemannian connection of $\overline{M}$ and M, respectively, and by D the normal connection induced from $\overline{\nabla}$ in the normal bundle of M. They are related by the following well-known **Gauss** and **Weingarten formulae**

$$\overline{\nabla}_{\iota X}\iota Y = \iota\nabla_X Y + h(X, Y), \tag{2.1}$$

$$\overline{\nabla}_{\iota X}\xi = -\iota AX + D_X\xi, \tag{2.2}$$

for any $X, Y \in T(M)$ and $\xi \in T^{\perp}M$, where h is the **second fundamental form** and A is the **shape operator**, also called **second fundamental tensor** corresponding to ξ, related by $\overline{g}(h(X, Y), \xi) = g(A_\xi X, Y)$. A submanifold M is said to be **totally geodesic** if for every geodesic $\gamma(s)$ of M, curve $\iota\gamma(s)$ is a geodesic of $\overline{M}$, or equivalently, if the second fundamental form vanishes identically. It is said that it is **totally umbilical** if there exists a normal vector field ξ such that $h(X, Y) = g(X, Y)\xi$, for any tangent vector fields X, Y. It is clear that every totally geodesic submanifold is also totally umbilical, but the converse is not true in general.

Further, let us suppose that the ambient space is a complex manifold $(\overline{M}^{\frac{m+k}{2}}, J)$ equipped with a Hermitian metric $\overline{g}$, namely, such that $\overline{g}(JX, JY) = \overline{g}(X, Y)$. It is well-known that, for any $x \in M$, the subspace $H_x(M) = JT_x(M) \cap T_x(M)$, called the holomorphic tangent space to M at x, is the maximal J-invariant subspace of the tangent space $T_x(M)$ at x. In general, the dimension of $H_x(M)$ varies with x (see [7], for example), but if the subspace $H_x(M)$ has constant dimension for any $x \in M$, the submanifold M is called the Cauchy–Riemann submanifold or briefly **CR submanifold** and the constant complex dimension of $H_x(M)$ is called the CR dimension of M ([20, 27]). It is clear that a real hypersurface is one of the typical examples of CR submanifolds whose CR dimension is $\frac{m-1}{2}$, where m is the dimension of a hypersurface. It is easily seen that if M is a CR submanifold in the sense of Bejancu's definition given in [1], M is also a CR submanifold in the sense of the above-given definition. In the case when M is a CR submanifold of CR dimension $\frac{m-1}{2}$, these definitions coincide. On the other hand, when the CR dimension is less than $\frac{m-1}{2}$, the converse is wrong. Further, we recall two more examples of CR submanifolds of maximal CR dimension of Hermitian manifolds $(\overline{M}, J, \overline{g})$:

- any real hypersurface M of a complex submanifold (i.e. J-invariant submanifold) M' of $\overline{M}$;
- any F'-invariant submanifold of a real hypersurface M' of $\overline{M}$, where F' is a skew-symmetric endomorphism induced by the almost complex structure J of $\overline{M}$ which determines the almost contact metric structure on M'.

We refer to [7, 10] for more details and examples of CR submanifolds of maximal CR dimension.

We will restrict our attention to the case when the ambient manifold $\overline{M}$ is a complex space form, i.e. a Kähler manifold whose holomorphic sectional curvature $\frac{\overline{g}(\overline{R}(X,JX)JX,X)}{\overline{g}(X,X)^2}$ is constant, for all J-invariant planes $\{X, JX\}$ in $T_x(\overline{M})$ and for all points $x \in \overline{M}$ ($\overline{R}$ is the Riemannian curvature tensor of $\overline{M}$). It is well-known that two complete, simply connected complex space forms of the same constant holomorphic sectional curvature are isometric and biholomorphic. It follows from the classification of symmetric spaces that a Kähler manifold of constant holomorphic sectional curvature c is locally isometric to one of the following spaces: **complex Euclidean space** ($c = 0$), **complex projective space** ($c > 0$) and **complex hyperbolic space** ($c < 0$).

3 Real hypersurfaces of complex space forms

Let M be a hypersurface of a Riemannian manifold $(\overline{M}, \overline{g})$ and let $\iota : M \to \overline{M}$ denote the isometric immersion. Then the Gauss formula (2.1) and Weingarten formula (2.2) reduce respectively to

$$\overline{\nabla}_{\iota X}\iota Y = \iota\nabla_X Y + h(X, Y) = \iota\nabla_X Y + g(AX, Y)\xi, \tag{3.1}$$

$$\overline{\nabla}_{\iota X}\xi = -\iota AX, \tag{3.2}$$

where ξ is a local choice of a unit normal, $X, Y \in T(M)$ and A is the shape operator in the direction of ξ. For $\overline{M}^{m+1}$ being a Kähler manifold with a Kähler structure $(J, \overline{g})$, we define a structure vector field $U \in T(M)$ by $\iota U = -J\xi$ and a skew-symmetric (1, 1)-tensor field F (from the tangential projection of J) by:

$$J\iota X = \iota FX + g(X, U)\xi. \tag{3.3}$$

Applying J to relation (3.3), we derive

$$\begin{aligned} F^2X = & -X + g(X, U)U, \quad FU = 0, \\ g(FX, FY) = & g(X, Y) - g(X, U)g(Y, U), \end{aligned} \tag{3.4}$$

which asserts that F determines an almost contact metric structure (see [2] for more details about almost contact metric structure). Taking the covariant derivative of relation (3.3) and using the Gauss and Weingarten formulae (3.1), (3.2), we conclude

$$(\nabla_X F)Y = g(U, Y)AX - g(AX, Y)U, \quad \nabla_X U = FAX. \tag{3.5}$$

We refer to [19] for necessary background material to access the study of real hypersurfaces in complex space forms, as well as a detailed construction of the important examples of hypersurfaces in complex projective and complex hyperbolic space. We also refer to [10] for the basic material about the geometry of submanifolds of complex manifolds and for further study of these examples.

Especially, if $\overline{M}$ is a complex space form of constant holomorphic sectional curvature $4c$, using the Codazzi equation

$$(\nabla_X A)Y - (\nabla_Y A)X = c(g(X, U)FY - g(Y, U)FX + 2g(X, FY)U) \tag{3.6}$$

for $A = \lambda I$, with $Y = U$, we obtain $c = 0$, since $\{X, FX, U\}$ are linearly independent. Namely, using the Codazzi equation (3.6), we conclude that complex projective and complex hyperbolic space do not admit either totally umbilical or totally geodesic hypersurfaces. We note that the nonexistence of totally umbilical hypersurfaces was first proved in [26].

We recall here that H. B. Lawson showed in [15] that, although there do not exist totally geodesic hypersurfaces in a complex projective space, there are however certain distinguished minimal hypersurfaces, denoted by $M^C_{p,q}$, of a complex projective space $\mathbb{C}P^n$, for $p, q \geq 0,\ p + q = n - 1$, which naturally generalize the equatorial hypersurfaces of spheres. These spaces admit strong intrinsic characterization which recognizes them in the class of minimal hypersurfaces. For example, H. B. Lawson showed that there exist positive constants c_n and c'_n such that if M is any compact minimal hypersurface of $\mathbb{C}P^n$ over which either the length of the second fundamental form $\|h\|$ satisfies $||h|| < c_n$ or equivalently, the scalar curvature τ satisfies $\tau \geq c'_n$, then the equality holds identically and $M \cong M^C_{p,q}$ for some p, q. Moreover, any (not necessarily compact) minimal hypersurface whose scalar curvature is identically equal to c'_n must be an open subset of $M^C_{p,q}$. The investigation of minimal submanifolds of a sphere done by S. S. Chern, M. do Carmo and S. Kobayashi in [4] has been very helpful for this study, having in mind H. B. Lawson's idea to construct a circle bundle over a real hypersurface, which is compatible with the Hopf fibration:

$$\begin{array}{ccc} S^{2p+1} \times S^{2q+1} & \xrightarrow{\imath'} & \mathbf{S}^{2n+1} \\ \pi\downarrow & & \downarrow\pi \\ M^C_{p,q} & \xrightarrow[\imath]{} & \mathbb{C}P^n \end{array}$$

More precisely, the family of generalized Clifford hypersurfaces

$$M'_{p,q} = \{z = (z_1, z_2) \in \mathbb{C}^{n+1} | G_1(z_1, z_1) = r^2 - b^2, G_2(z_2, z_2) = b^2\},$$

in $S^{2n+1}(r)$, where G_1 and G_2 are the restrictions of G to $\mathbb{C}^{p+1}$ and $\mathbb{C}^{q+1}$ respectively, for $G(z, w) = \sum_{k=0}^{n} z_k \bar{w}_k$, $z = (z_0, z_1, \dots, z_n)$, $w = (w_0, w_1, \dots, w_n)$, $z, w \in \mathbb{C}^{n+1}$, $\mathbb{C}^{n+1} = \mathbb{C}^{p+1} \times \mathbb{C}^{q+1}$, $p, q \geq 0$, $p + q = n - 1 > 0$ (choosing b so that $0 < b < r$), is the Cartesian product of spheres

$$M'_{p,q} = S^{2p+1}((r^2 - b^2)^{1/2}) \times S^{2q+1}(b),$$

where the radii of the spheres have been chosen so that $M'_{p,q}$ lies in $S^{2n+1}(r)$. Therefore, we get the fibrations

$$S^1 \longrightarrow M'_{p,q} \longrightarrow M^C_{p,q},$$

compatible with the standard (canonical) projection of $S^{2n+1}(r)$ to complex projective space $\mathbb{C}P^n$

$$\pi : S^{2n+1} \to \mathbb{C}P^n.$$

The surfaces $M^C_{p,q}$ are called "generalized equators" in [15]. For the special case $p = 0$, $M^C_{0,n-1}$ is diffeomorphic to a $(2n-1)$-dimensional sphere and it is a geodesic hypersphere (see [3, 10] for more details and proofs).

The second author of this paper proved in [22] (Theorem 4.3) that the condition (1.1) for a real hypersurface M of a complex projective space occurs if and only if the shape operator of $\pi^{-1}(M)$ is parallel. From this result and Ryan's paper [25], he obtained

Theorem 3.1 (Theorem 4.4, [22]) *$M^C_{p,q}$ are the only complete real hypersurfaces of a complex projective space for which the condition* (1.1) *is satisfied.*

The construction of standard examples (the so-called "model spaces") M^*_n and $M^H_{p,q}(r)$ of real hypersurfaces in the complex hyperbolic space $\mathbb{C}H^n$ is analogous to the above explained in $\mathbb{C}P^n$, with some important differences. We refer to [9] for a more detailed explanation. Using the fibration

$$\pi : H_1^{2n+1}(r) \to \mathbb{C}H^n, \tag{3.7}$$

for anti-De Sitter space $H_1^{2n+1}(r)$ of radius r in $\mathbb{C}^{n+1}$, the so-called "horospheres" M^*_n are defined as $M^*_n = \pi'(M'_n)$ where π' is the submersion which is compatible with the fibration π in (3.7) and

$$M'_n = \{z \in \mathbb{C}^{n+1} \mid < z, z >= -r^2, |z_0 - z_1| = t\},$$

for $< z, w >=$ Re $g(z, w)$, $\quad g(z, w) = -z_0\bar{w}_0 + \sum_{k=1}^n z_k\bar{w}_k$, $\quad z = (z_0, z_1, \dots, z_n)$, $w = (w_0, w_1, \dots, w_n)$, $z, w \in \mathbb{C}^{n+1}$. The Lorentzian hypersurface of H_1^{2n+1}, denoted by $M'_{2p+1,2q+1}(r)$ for given integers p, q, with $p + q = n - 1$ and $r \in \mathbb{R}$ is defined by

$$M'_{2p+1,2q+1}(r) = \{(z_1, z_2) \in \mathbb{C}^{n+1} | g_1(z_1, z_1) = -(b^2 + r^2), g_2(z_2, z_2) = b^2\},$$

where g_1 and g_2 are the restrictions of $g(z, w)$ to $\mathbb{C}^{p+1}$ and $\mathbb{C}^{q+1}$, respectively. Then $M'_{2p+1,2q+1}$ is the Cartesian product of an anti-De Sitter space and a sphere whose radii have been chosen so that it lies in $H_1^{2n+1}(r)$ and each factor is embedded in H_1^{2n+1} in a totally umbilical way, i.e.

$$M'_{2p+1,2q+1}(r) = H_1^{2p+1}((b^2 + r^2)^{1/2}) \times S^{2q+1}(b).$$

Since $M'_{2p+1,2q+1}(r)$ is S^1-invariant, $M^H_{p,q}(r) = \pi'(M'_{2p+1,2q+1}(r))$ is a real hypersurface of $\mathbb{C}H^n$ which is complete and satisfies the condition (1.1).

Moreover, Montiel and Romero studied in [18] the same geometric characterization as in the case of a complex projective space and proved

Theorem 3.2 (Theorem 5.1, [18]) *Let M^{2n-1} be a complete real hypersurface of the complex hyperbolic space $\mathbb{C}H^n$ for which the condition* (1.1) *is satisfied. Then, we have the following possibilities*:

(a) *M^{2n-1} is congruent to $M^H_{2p+1,2q+1}(r)$, $p + q = n - 1$;*

(b) M^{2n-1} *is congruent to* M_n^*.

4 CR submanifolds of maximal CR dimension of complex space forms

In this section we recall the complete classification of CR submanifolds of maximal CR dimension of complex space forms which satisfy the condition (1.2) and for further details we refer to [9, 10]. We emphasize here that a real hypersurface is one of the examples of CR submanifolds of maximal CR dimension of complex space forms and that it is natural to replace the condition (1.1) with the condition (1.2), which is equivalent to (4.14)–(4.16).

Let us consider CR submanifolds M^m of complex space forms $\overline{M}^{m+k}$ whose CR dimension is maximal, that is, $\dim_{\mathbb{R}} H_x(M) = \dim_{\mathbb{R}}(JT_x(M) \cap T_x(M)) = m-1$. Then it follows that M is odd-dimensional and that there exists a unit vector field ξ normal to M such that $JT_x(M) \subset T_x(M) \oplus \text{ span } \{\xi_x\}$, for any $x \in M$. Hence, for any $X \in T(M)$, choosing a local orthonormal basis $\xi, \xi_1, \dots, \xi_{k-1}$ of vectors normal to M, we have the following decomposition into tangential and normal components:

$$J\imath X = \imath FX + u(X)\xi\,, \tag{4.1}$$

$$J\xi = -\imath U + P\xi, \tag{4.2}$$

$$J\xi_a = -\imath U_a + P\xi_a \qquad (a = 1, \dots, k-1). \tag{4.3}$$

Here F and P are skew-symmetric endomorphisms acting on $T(M)$ and $T^{\perp}(M)$, respectively, $U, U_a, a = 1, \dots, k-1$ are tangent vector fields and u is one form on M. Furthermore, using (4.1), (4.2) and (4.3), the Hermitian property of J implies

$$g(U, X) = u(X)\,, \quad J\xi = -\imath U\,, \tag{4.4}$$

$$F^2 X = -X + u(X)U, \tag{4.5}$$

$$u(FX) = 0\,, \quad FU = 0\,, \quad u(U) = 1\,, \tag{4.6}$$

$$g(FX, FY) = g(X, Y) - u(X)u(Y), \tag{4.7}$$

$$J\xi_a = P\xi_a \quad (a = 1, \dots, k-1)\,, \tag{4.8}$$

which means that (F, u, U, g) is an almost contact metric structure on M (see [2]).

We proved in [6] that if a CR submanifold of maximal CR dimension of a Kähler manifold satisfies the condition

$$h(FX, Y) - h(X, FY) = g(FX, Y)\eta, \tag{4.9}$$

for all $X,\ Y \in T(M)$, where $\eta = \rho\xi + \sum_{a=1}^{q}(\rho^a \xi_a + \rho^{a^*}\xi_{a^*}) \in T^{\perp}(M)$ and $\rho \neq 0$, then M is a contact manifold. We denote the orthonormal basis of $T^{\perp}(M)$ by

$\xi, \xi_1, \dots, \xi_q, \xi_{1^*}, \dots, \xi_{q^*}$, where $\xi_{a^*} = J\xi_a$ and $q = \frac{k-1}{2}$, as $\{\eta \in T^\perp(M), \eta \perp \xi\}$ is J-invariant.

Denoting by $\overline{\nabla}$ and ∇ the Riemannian connection of $\overline{M}$ and M, respectively, and by D the normal connection induced from $\overline{\nabla}$ in the normal bundle of M and using the above constructed basis $\xi, \xi_1, \dots, \xi_q, \xi_{1^*}, \dots, \xi_{q^*}$, the Weingarten formulae can be written as follows

$$\begin{aligned}\overline{\nabla}_{\iota X}\xi &= -\iota AX + D_X\xi \\ &= -\iota AX + \sum_{a=1}^{q}\{s_a(X)\xi_a + s_{a^*}(X)\xi_{a^*}\},\end{aligned} \tag{4.10}$$

$$\begin{aligned}\overline{\nabla}_{\iota X}\xi_a &= -\iota A_aX + D_X\xi_a = -\iota A_aX - s_a(X)\xi \\ &+ \sum_{b=1}^{q}\{s_{ab}(X)\xi_b + s_{ab^*}(X)\xi_{b^*}\},\end{aligned} \tag{4.11}$$

$$\begin{aligned}\overline{\nabla}_{\iota X}\xi_{a^*} &= -\iota A_{a^*}X + D_X\xi_{a^*} \\ &= -\iota A_{a^*}X - s_{a^*}(X)\xi + \sum_{b=1}^{q}\{s_{a^*b}(X)\xi_b + s_{a^*b^*}(X)\xi_{b^*}\},\end{aligned} \tag{4.12}$$

where s's are the coefficients of the normal connection D and A, A_a, A_{a^*} are the shape operators for the normals ξ, ξ_a, ξ_{a^*}, respectively, related to the second fundamental form by

$$h(X,Y) = g(AX,Y)\xi + \sum_{a=1}^{q}\{g(A_aX,Y)\xi_a + g(A_{a^*}X,Y)\xi_{a^*}\}. \tag{4.13}$$

Therefore, since F is skew-symmetric, the relation (1.2) is equivalent to

$$AF = FA, \tag{4.14}$$

$$A_aF = FA_a, \tag{4.15}$$

$$A_{a^*}F = FA_{a^*}. \tag{4.16}$$

Also, differentiating relation (4.1), we get

$$(\nabla_X F)Y = u(Y)AX - g(AX,Y)U, \tag{4.17}$$

$$(\nabla_X u)(Y) = g(FAX,Y), \tag{4.18}$$

$$\nabla_X U = FAX. \tag{4.19}$$

If $\overline{M}$ is a Kähler manifold, the condition (4.14) implies that U is an eigenvector of the shape operator A. Furthermore, if $\overline{M}$ is a complex space form and the condition (1.2) is satisfied, using the Codazzi equations, we proved in [9] that one of the following holds

- the distinguished normal vector field ξ is parallel with respect to the normal connection,
- the ambient manifold $\overline{M}$ is a complex Euclidean space and M is locally isometric to a Euclidean space.

Considering the case when the distinguished normal vector ξ is parallel with respect to the normal connection D, we concluded that the shape operators A_a, A_{a^*} with respect to the normals ξ_a, ξ_{a^*}, respectively, vanish:

$$A_a = 0 = A_{a^*}, \quad a = 1, \dots, q, \; q = \frac{k-1}{2}.$$

Therefore, under the above conditions, using the codimension reduction theorems for complex space forms ([13] for a complex Euclidean space, [23] for a complex projective space and [14] for a complex hyperbolic space), we concluded that there exists a (corresponding) totally geodesic complex space form M' of $\overline{M}$ such that M is a real hypersurface of M'. Denoting by A' and ϕ the shape operator (second fundamental tensor) and the almost contact metric structure of the hypersurface M', naturally induced from the almost complex structure of $\overline{M}$, we showed that $A' = A$ and $\phi = F$. Relation (4.14) then implies

$$\phi A' = A' \phi, \tag{4.20}$$

which is exactly the relation (1.1) and which enabled us to apply the results from the hypersurface theory, Theorems 3.1 and 3.2, and prove

Theorem 4.1 [9] *Let M be a complete m-dimensional CR submanifold of maximal CR dimension of a complex space form $\overline{M}^{\frac{m+k}{2}}$. If the condition (1.2) is satisfied, where F is the induced almost contact structure (defined by (4.1)) and h is the second fundamental form of M, then, depending on the ambient space, one of the following three statements holds*:

- *M is a complete m-dimensional CR submanifold of CR dimension $\frac{m-1}{2}$ of a complex Euclidean space $\mathbb{C}^{\frac{m+k}{2}}$ and then M is isometric to $\mathbb{E}^m$, $\mathbb{S}^m$* or $\mathbb{S}^{2p+1} \times \mathbb{E}^{m-2p-1}$;
- *M is a complete m-dimensional CR submanifold of CR dimension $\frac{m-1}{2}$ of a complex projective space $\mathbb{C}P^{\frac{m+k}{2}}$ and then M is isometric to $M^C_{p,q}$, for some p, q satisfying $2p + 2q = m - 1$;*
- *M is a complete m-dimensional CR submanifold of CR dimension $\frac{m-1}{2}$ of a complex hyperbolic space $\mathbb{C}H^{\frac{m+k}{2}}$ and then M is isometric to M^*_m or $M^H_{p,q}(r)$, for some p, q satisfying $2p + 2q = m - 1$.*

5 Submanifolds of real codimension two of complex space forms

In this section we continue our study of submanifolds of complex space forms which satisfy the condition (1.2).

Let $\overline{M}$ be a real $(n+2)$-dimensional complex manifold, J its natural almost complex structure and $\overline{g}$ its Hermitian metric. For M being an n-dimensional submanifold of $\overline{M}$ with the isometric immersion $\imath$ of M into $\overline{M}$ and ξ_1, ξ_2 mutually orthogonal unit normals to M, we have the formulae analogous to (4.1)–(4.3):

$$J\imath X = \imath FX + u^1(X)\xi_1 + u^2(X)\xi_2, \tag{5.1}$$

$$J\xi_1 = -\imath U_1 + \lambda\xi_2, \tag{5.2}$$

$$J\xi_2 = -\imath U_2 - \lambda\xi_1, \tag{5.3}$$

where F is a skew-symmetric endomorphism acting on $T(M)$, U_a, $a = 1, 2$ are local tangent vector fields and u^a, $a = 1, 2$ are local one-forms on M. We note that u^1 and u^2 depend on the choice of normals ξ_1 and ξ_2, but the function λ^2, where $\lambda = \overline{g}(J\xi_1, \xi_2)$, does not depend on the choice of ξ_1 and ξ_2 (see [11] for the proof). For some special λ, we obtained the well-known examples of submanifolds, which are also characterized by Theorem 5.1:

Proposition 5.1 [11] *Let M be a submanifold of real codimension 2 of a complex manifold $\overline{M}$ and let λ be the function defined by* (5.2)–(5.3). *Then*:

(1) *M is a complex hypersurface if and only if $\lambda^2(x) = 1$ for any $x \in M$.*
(2) *M is a CR submanifold of CR dimension $\frac{n-2}{2}$ if $\lambda(x) = 0$ for any $x \in M$.*

Remark 1 We mention here, citing the Example 2.1 in [11], that in the item (2) of Proposition 5.1 the converse is not true, that is, for a CR submanifold of CR dimension $\frac{n-2}{2}$ the function λ does not always vanish.

Remark 2 Let us note here that in the Example 5.1 in [11] we have provided a large class of submanifolds of real codimension two of complex space forms satisfying $\lambda = 0$.

Now, applying J to (5.1), using (5.2)–(5.3) and comparing the tangential parts, we obtain

$$F^2X = -X + u^1(X)U_1 + u^2(X)U_2, \tag{5.4}$$

$$FU_1 = -\lambda U_2, \qquad FU_2 = \lambda U_1. \tag{5.5}$$

Denoting by $\overline{\nabla}$ the covariant differentiation with respect to the Hermitian metric $\overline{g}$ of $\overline{M}$, the Gauss and Weingarten formulae (2.1) and (2.2) read

$$\overline{\nabla}_X \imath Y = \imath \nabla_X Y + h(X, Y) = \imath \nabla_X Y + g(A_1 X, Y)\xi_1 + g(A_2 X, Y)\xi_2, \tag{5.6}$$

$$\overline{\nabla}_X \xi_1 = -\imath A_1 X + s(X)\xi_2, \qquad \overline{\nabla}_X \xi_2 = -\imath A_2 X - s(X)\xi_1, \tag{5.7}$$

where $h(X, Y)$ is the second fundamental form and A_a is the shape operator with respect to the normal ξ_a.

Assuming that the ambient manifold $\overline{M}$ is a Kähler manifold, applying $\overline{\nabla}$ to $J\imath Y$, using (5.1), (5.2), (5.3), (5.6), (5.7) and comparing the tangential and normal components of the obtained relations, we obtain

$$(\nabla_X F)Y = u^1(Y)A_1 X - g(A_1 X, Y)U_1 + u^2(Y)A_2 X - g(A_2 X, Y)U_2, \tag{5.8}$$

$$\nabla_X U_1 = FA_1 X - \lambda A_2 X + s(X)U_2, \quad \nabla_X U_2 = FA_2 X + \lambda A_1 X - s(X)U_1. \tag{5.9}$$

As F is a skew-symmetric endomorphism, it follows that the condition (1.2) is equivalent to

$$A_1 F = FA_1, \qquad A_2 F = FA_2, \tag{5.10}$$

that is, the linear map F commutes with both shape operators, A_1 and A_2.

It is natural to begin our investigation with the case when the submanifold M is a complex hypersurface of a Kähler manifold $\overline{M}$, i.e. when the tangent space $T_x(M)$ and the normal space $T^{\perp}(M)$ are J-invariant. Consequently, we can choose the orthonormal vectors ξ_1, ξ_2 which are normal to M in such a way that $\xi_2 = J\xi_1$ and prove that if the condition (1.2) is satisfied, then M^n is a totally geodesic submanifold of $\overline{M}^{n+2}$ (Theorem 3.1 in [11]).

Having in mind that the studies of hypersurfaces and CR submanifolds of maximal CR dimension of complex manifolds were more efficient in the case of complex space forms, we proceed considering submanifolds of real codimension two of complex space forms $\overline{M}(c)$. First we look more closely at non-Euclidean complex space forms, i.e. complex projective and complex hyperbolic spaces and recall

Theorem 5.1 [11] *Let $\overline{M}$ be a non-Euclidean complex space form. If a submanifold M of real codimension two satisfies the condition* (1.2), *then one of the following holds.*

(1) *M is a totally geodesic complex hypersurface.*
(2) *M is a CR submanifold of CR dimension $\frac{n-2}{2}$ with $\lambda = 0$.*

Before presenting our result in the case when the ambient space $\overline{M}$ is a complex projective space, we recall our achievements when $\overline{M}$ is a complex Euclidean space.

Theorem 5.2 [11] *Let M be a connected submanifold of real codimension two of a complex Euclidean space $\overline{M} = \mathbb{C}^{\frac{n+2}{2}}$. If M satisfies the condition* (1.2), *then M is one of the following*:

- *n-dimensional sphere $\mathbb{S}^n$,*
- *n-dimensional Euclidean space $\mathbb{E}^n$,*
- *product manifold of an r-dimensional sphere and an $(n-r)$-dimensional Euclidean space $\mathbb{S}^r \times \mathbb{E}^{n-r}$, where r is an even number.*
- *CR submanifold of CR dimension $\frac{n-2}{2}$ with $\lambda = 0$.*

We briefly recall that in one part of the proof of Theorem 5.2 we consider the open submanifold of M defined by $M_0 = \{x \in M | \lambda(x)(1 - \lambda^2(x)) \neq 0\}$. If the condition (1.2) is satisfied, then U_1 and U_2 (defined by (5.2) and (5.3)) are eigenvectors of both A_1 and A_2, namely

$$A_a U_1 = \alpha_a U_1, \qquad A_a U_2 = \alpha_a U_2, \qquad a = 1, 2.$$

For the case when $\alpha_1^2 + \alpha_2^2 \neq 0$, applying the codimension reduction theorem by Erbacher from [13] we prove that, under the above assumptions, there exists an $(n+1)$-dimensional totally geodesic Euclidean subspace $\mathbb{E}^{n+1}$ of $\mathbb{C}^{\frac{n+2}{2}}$ such that M_0 is a hypersurface of $\mathbb{E}^{n+1}$. Using the theory of hypersurfaces in the Euclidean space (see for example Theorem 11.4 in [10]), we conclude that M_0 is an open submanifold of an n-dimensional hypersphere, of an n-dimensional hyperplane or of the product manifold of an r-dimensional sphere and an $(n-r)$-dimensional Euclidean space. For the case when $\alpha_1^2 + \alpha_2^2 = 0$ we prove that M_0 is totally geodesic and that it is an open submanifold of an n-dimensional Euclidean space.

Since we have been interested in CR submanifolds and also having in mind the proof of Theorem 5.1, we continued our study of submanifolds M^n of a complex Euclidean space assuming that $\lambda = 0$ and that there exist some special hypersurfaces M' of $\mathbb{C}^{\frac{n+2}{2}}$ such that $M \subset M'$ and we proved:

Theorem 5.3 [11] *Let M be a submanifold of real codimension two of a complex Euclidean space $\mathbb{C}^{\frac{n+2}{2}}$ with $\lambda = 0$ which satisfies the condition* (1.2).

- *If there exists a totally geodesic hypersurface M' of $\mathbb{C}^{\frac{n+2}{2}}$ such that $M \subset M'$, then M is one of the following*:
  - *n-dimensional hyperplane $\mathbb{E}^n$,*
  - *product manifold of an odd-dimensional sphere and a Euclidean space*: $\mathbb{S}^{2p+1} \times \mathbb{E}^{n-2p-1}$.
- *If there exists a totally umbilical hypersurface M' of $\mathbb{C}^{\frac{n+2}{2}}$, such that $M \subset M'$, then M is a product of two odd-dimensional spheres.*

Motivated by Theorem 5.1 from [11], we have proceeded (in [12]) our study of submanifolds M^n of real codimension two in complex projective space, under the condition (1.2), considering submanifolds which are not totally geodesic complex hypersurfaces. These assumptions and Theorem 5.1 imply that M^n is a CR submanifold of CR dimension $\frac{n-2}{2}$ with $\lambda = 0$. We have left the case when the ambient space is a complex hyperbolic space as a Problem, stated in the Introduction. It is to be expected that the corresponding results could be proved.

Inspired by the assumptions of Theorem 5.3, we have restricted our attention to submanifolds M^n of real codimension two in complex projective space $\mathbb{C}P^{\frac{n+2}{2}}$, such that there exists a significant real hypersurface $M^C_{p,q}$ of $\mathbb{C}P^{\frac{n+2}{2}}$ so that $M \subset M^C_{p,q}$, because a complex projective space does not admit either totally umbilic or totally geodesic real hypersurfaces. In Sect. 3, remembering that that one can regard a complex projective space $\mathbb{C}P^{\frac{n+2}{2}}$ as a projection from $\mathbb{S}^{n+3}$ with fibre $\mathbb{S}^1$, we recalled the definition of $M^C_{p,q}$ as a projection $\pi(\mathbb{S}^{2p+1} \times \mathbb{S}^{2q+1})$ of the product of two odd-dimensional spheres in a unit sphere $\mathbb{S}^{n+3}(1)$, where π is the Hopf map $\pi : \mathbb{S}^{n+3} \to \mathbb{C}P^{\frac{n+2}{2}}$. Therefore, in [12], we have first studied both the product of two odd-dimensional spheres $\mathbb{S}^{2p+1} \times \mathbb{S}^{2q+1}$ as a hypersurface of the unit sphere $\mathbb{S}^{n+3}(1)$ and its hypersurfaces. We note that in [16, 28] the authors restricted their attention to the case $p = q$. Further, we investigated certain hypersurfaces N of $\mathbb{S}^{2p+1}(|\cos\theta|) \times \mathbb{S}^{2q+1}(|\sin\theta|)$ which satisfy the condition

$$F^N A^N = A^N F^N, \tag{5.11}$$

where A_N is the shape operator with respect to $\xi_N \in T^{\perp}(N)$ and F^N is the skew-symmetric endomorphism acting on $T(N)$ defined as the tangent component of the skew-symmetric endomorphism F' on $T(\mathbb{S}^{2p+1}(|\cos\theta|) \times \mathbb{S}^{2q+1}(|\sin\theta|))$ induced by the almost contact structure $\tilde{F}$ of $\mathbb{S}^{n+3}(1)$ naturally generated from the almost complex structure J of the Euclidean space $\mathbb{C}^{\frac{n+4}{2}}$. We find it interesting to compare the condition (5.11) with (1.1). We proved that, under the above assumptions, together with (5.11), N is congruent to $\mathbb{S}^{2p+1}(|\cos\theta|) \times \mathbb{S}^{2r+1}(|\cos\varphi \sin\theta|) \times \mathbb{S}^{2s+1}(|\sin\varphi \sin\theta|)$, $q = r + s + 1$, for some constant φ. Finally, considering submanifolds M^n of real codimension two of the complex projective space $\mathbb{C}P^{\frac{n+2}{2}}$ which satisfy the condition (1.2), which are not totally geodesic complex hypersurfaces, namely, which are CR submanifolds of CR dimension $\frac{n-2}{2}$ with $\lambda = 0$, we proved the following

Theorem 5.4 [12] *Let M^n be a submanifold of real codimension two of a complex projective space, which is not its totally geodesic complex hypersurface and let M satisfy the condition (1.2). If there exists a real hypersurface $M^C_{p,q}$ such that $M \subset M^C_{p,q}$, then M is congruent to $\pi(\mathbb{S}^{2p+1} \times \mathbb{S}^{2r+1} \times \mathbb{S}^{2s+1})$, where $p + r + s = \frac{n-2}{2}$.*

$$\begin{array}{ccccc}
\pi^{-1}(M) & \xrightarrow{\imath_1'} & \mathbb{S}^{2p+1} \times \mathbb{S}^{2q+1} & \xrightarrow{\imath_2'} & \mathbb{S}^{n+3} \\
\pi\downarrow & & \downarrow\pi & & \downarrow\pi \\
M & \xrightarrow[\imath_1]{} & M^C_{p,q} & \xrightarrow[\imath_2]{} & \mathbb{C}P^{\frac{n+2}{2}}
\end{array}$$

Although the following theorem could be interpreted as a special case of Theorem 5.4 (since the geodesic sphere $M^C_{0,n-1}$ could be interpreted as a special case of $M^C_{p,q}$, which we recalled in Sect. 3), we encourage the reader to prove it using another method and to do this before solving the Problem stated in the Introduction.

Theorem 5.5 *Let M^n be a submanifold of real codimension two of a complex projective space, which is not its totally geodesic complex hypersurface and let M satisfy the condition (1.2). If there exists a geodesic sphere $M^C_{0,n-1}$ such that $M \subset M^C_{0,n-1}$, then M is congruent to $\pi(\mathbb{S}^1 \times \mathbb{S}^{2r+1} \times \mathbb{S}^{2s+1})$, where $r + s = \frac{n-2}{2}$.*

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