



Explicit soliton for the Laplacian co-flow on a solvmanifold

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Abstract

We apply the general Ansatz proposed by Lauret (Rend Semin Mat Torino 74:55–93, 2016) for the Laplacian co-flow of invariant G_2 -structures on a Lie group, finding an explicit soliton on a particular almost Abelian 7-manifold. Our methods and the example itself are different from those presented by Bagaglini and Fino (Ann Mat Pura Appl 197(6):1855–1873, 2018).

1 Introduction

Geometric flows in G_2 -geometry were first outlined by the seminal works of Bryant [5] and Hitchin [10], and have since been studied by several authors, e.g. [3,4,9,14,15]. These so-called G_2 -flows arise as a tool in the search for ultimately torsion-free G_2 -structures, by varying a non-degenerate 3-form on an oriented and spin 7-manifold M towards some $\varphi \in \Omega^3 := \Omega^3(M)$ such that the torsion $\nabla^{g_\varphi} \varphi$ vanishes, where g_φ is the natural Riemannian metric defined from φ by

$$6g_\varphi(X, Y) \cdot d\text{Vol} := (X \lrcorner \varphi) \wedge (Y \lrcorner \varphi) \wedge \varphi.$$

Such pairs (M^7, φ) solving the nonlinear PDE problem $\nabla^{g_\varphi} \varphi \equiv 0$ are called G_2 -manifolds and are very difficult to construct, especially when M is required to be compact. To this date, all known solutions stem from elaborate constructions in geometric analysis [6,11,12].

Some weaker formulations of that problem can be obtained from the classical fact, first established by Fernández and Gray [8], that the torsion-free condition is equivalent to φ being both *closed* and *coclosed*, in the sense that $d\varphi = 0$ and $d *_{\varphi} \varphi = 0$, respectively, and thus one may study each of these conditions separately. For instance, Grigorian [9] and Karigiannis and Tsui [14] considered the *Laplacian co-flow* of G_2 -structures $\{\varphi_t\}$ defined by

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$$\frac{\partial \psi_t}{\partial t} = -\Delta_{\psi_t} \psi_t, \tag{1}$$

where $\psi_t := *_t \varphi_t$ is the Hodge dual and $\Delta_{\psi_t} \psi_t := (dd^{*t} + d^{*t}d)\psi_t$ is the Hodge Laplacian of the metric g_{φ_t} on 4-forms. It is a natural process to consider among coclosed G_2 -structures, as it manifestly preserves that property, i.e., it flows ψ_t in its de Rham cohomology class. Moreover, it is the gradient flow of Hitchin’s volume functional [10].

When $M^7 = G$ is a Lie group, we propose to study this flow from the perspective introduced by Lauret [15] in the general context of geometric flows on homogeneous spaces. As a proof of principle, we apply a natural Ansatz to construct an example of invariant self-similar solution, or *soliton*, of the Laplacian co-flow. Solitons are G_2 -structures which, under the flow, simply scale monotonically and move by diffeomorphisms. In particular, they provide potential models for singularities of the flow, as well as means for desingularising certain singular G_2 -structures, both of which are key aspects of any geometric flow. We follow in spirit the approach of Karigiannis et al. [14] to obtain solitons to the Laplacian coflow from a general Ansatz for a coclosed cohomogeneity one G_2 -structure on manifolds of the form $M^7 = N^6 \times L^1$, where $L^1 = \mathbb{R}$ or S^1 and N^6 is compact and either nearly Kähler or a Calabi-Yau 3-fold. In that case, as in ours, the symmetries of the space are exploited to reduce the soliton condition to a manageable ODE.

2 Torsion forms of a G_2 -structure

Let us briefly review some elementary representation theory underlying G_2 -geometry, following the setup from [5, 13]. The natural action of $G_2 \subset \text{SO}(7)$ decomposes $\Omega^l(M)$ into G_2 -invariant irreducible subbundles:

$$\begin{aligned} \Omega^1 &= \Omega_7^1, & \Omega^2 &= \Omega_7^2 \oplus \Omega_{14}^2, & \Omega^3 &= \Omega_1^3 \oplus \Omega_7^3 \oplus \Omega_{27}^3, \\ \Omega^6 &= \Omega_7^6, & \Omega^5 &= \Omega_7^5 \oplus \Omega_{14}^5, & \Omega^4 &= \Omega_1^4 \oplus \Omega_7^4 \oplus \Omega_{27}^4, \end{aligned} \tag{2}$$

where each Ω_l^k has rank l . Studying the symmetries of torsion one finds that $\nabla\varphi \in \Omega^1 \otimes \Omega_7^3$, so that tensor lies in a bundle of rank 49 [13, Lemma 2.24]. Notice also that $\Omega_7^3 \cong \Omega^1$, so, contracting the dual 4-form $\psi = *_\varphi \varphi$ by a frame of TM , then using the Riemannian metric, one has

$$\Omega^2 \oplus S^2(T^*M) = \Omega^1 \otimes \Omega_7^3 \cong \text{End}(TM) = \mathfrak{so}(TM) \oplus \text{sym}(TM).$$

Here $S^2(T^*M)$ denotes the symmetric bilinear forms and $\text{sym}(TM)$ the symmetric endomorphisms of TM . Both of the above splittings are G_2 -invariant, so, comparing the G_2 -irreducible decomposition $\mathfrak{so}(7) = \mathfrak{g}_2 \oplus [\mathbb{R}^7]$ and (2), we get the following identification between G_2 -irreducible summands

$$[\mathbb{R}^7] \cong \Omega_7^2 \quad \text{and} \quad \mathfrak{g}_2 \cong \Omega_{14}^2. \tag{3}$$

For $S^2(T^*M) \cong \text{sym}(TM)$, Bryant defines maps $i : S^2(T^*M) \rightarrow \Omega^3$ and $j : \Omega^3 \rightarrow S^2(T^*M)$ by

$$i(h) = \frac{1}{2} h_{il} g^{lm} \varphi_{mjk} dx^{ijk} \quad \text{and} \quad j(\eta)(u, v) = *((u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \eta), \quad (4)$$

where we adopt the familiar implicit summation convention for repeated indices and the inverse of the metric. The map i is injective [13, Corollary 2.16] and, by the G_2 -decomposition $S^2(T^*M) = \mathbb{R}g_\varphi \oplus S_0^2(T^*M)$, it identifies

$$\mathbb{R}g_\varphi \cong \Omega_1^3 \quad \text{and} \quad S_0^2(T^*M) \cong \Omega_{27}^3.$$

Accordingly, we have a decomposition for the torsion components $d\varphi \in \Omega^4$ and $d\psi \in \Omega^5$ given by

$$d\varphi = \tau_0\psi + 3\tau_1 \wedge \varphi + *\tau_3 \quad \text{and} \quad d\psi = 4\tau_1 \wedge \psi + \tau_2 \wedge \varphi,$$

where $\tau_0 \in \Omega^0$, $\tau_1 \in \Omega^1$, $\tau_2 \in \Omega_{14}^2$ and $\tau_3 \in \Omega_{27}^3$ are called the *torsion forms*. Indeed, the torsion is completely encoded in the *full torsion tensor* T , defined in coordinates by

$$\nabla_l \varphi_{abc} =: T_{lm} g^{mn} \psi_{nabc},$$

which is expressed in terms of the irreducible G_2 -decomposition of $\text{End}(TM)$ by [13, Theorem 2.27]

$$T = \frac{\tau_0}{4} g_\varphi - \tau_{27} - (\tau_1)^\sharp \lrcorner \varphi - \frac{1}{2} \tau_2,$$

where $\tau_3 := i(\tau_{27})$ and $^\sharp : \Omega^1 \rightarrow \mathcal{X}(M)$ the musical isomorphism induced by the G_2 -metric. If moreover the G_2 -structure is co-closed, the torsion tensor $T = \frac{\tau_0}{4} g_\varphi - \tau_{27}$ is totally symmetric, and the Hodge Laplacian of ψ is given by [2]

$$\Delta_\psi \psi = dd^* \psi = d\tau_0 \wedge \varphi + \tau_0^2 \psi + \tau_0 * \tau_3 + d\tau_3.$$

If moreover τ_3 vanishes, then ψ is a Laplacian eigenform and the G_2 -structure is called *nearly parallel*.

3 Invariant G_2 -structures on Lie groups

Let us briefly survey Lauret’s approach to geometric flows on homogeneous spaces [15]. Consider the action of a Lie group G on a manifold M . A (r, s) -tensor γ on M is G -invariant if $g^* \gamma = \gamma$, for each $g \in G$, where

$$g^* \gamma(X_1, \dots, X_r, \alpha_1, \dots, \alpha_s) := \gamma(g_* X_1, \dots, g_* X_r, (g^{-1})^* \alpha_1, \dots, (g^{-1})^* \alpha_s),$$

for $X_1, \dots, X_r \in \Gamma(TM)$ and $\alpha_1, \dots, \alpha_s \in \Gamma(T^*M)$. In particular, when $M = G/H$ is a reductive homogeneous space, i.e.

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m} \quad \text{such that} \quad \text{Ad}(h)\mathfrak{m} \subset \mathfrak{m}, \quad \forall h \in H,$$

any G -invariant tensor γ is completely determined by its value γ_{x_0} at the point $x_0 = [1_G] \in G/H$, where γ_{x_0} is an $\text{Ad}(H)$ -invariant tensor at $\mathfrak{m} \cong T_{x_0}M$, i.e. $(\text{Ad}(h))^*\gamma_{x_0} = \gamma_{x_0}$ for each $h \in H$. Given $x = [gx_0] \in G/H$, clearly $\gamma_x = (g^{-1})^*\gamma_{x_0}$. Consider now a geometric flow on M of the general form

$$\frac{\partial}{\partial t} \gamma_t = q(\gamma_t). \tag{5}$$

Then, if $M = G/H$, requiring G -invariance of γ_t , for all t , reduces the flow to an ODE for a one-parameter family γ_t of $\text{Ad}(H)$ -invariant tensors on the vector space \mathfrak{m} :

$$\frac{d}{dt} \gamma_t = q(\gamma_t).$$

Now, we fix $\dim G = 7$ and $H = \{1\}$ the trivial subgroup. For any G_2 -structure $\varphi_0 \in \Lambda^3(\mathfrak{g})^*$ on $\mathfrak{g} = \text{Lie}(G)$, non-degeneracy means that, for each non-zero vector $v \in \mathfrak{g}$, the 2-form $\iota(v)\varphi_0$ is symplectic on the vector space $\mathfrak{g}/\langle v \rangle$. We also know that the $\text{Gl}(\mathfrak{g})$ -orbit of the dual 4-form $\psi_0 = *\varphi_0$ is open in $\Lambda^4(\mathfrak{g})^*$ under the natural action

$$h \cdot \psi_0 := (h^{-1})^*\psi_0 = \psi_0(h^{-1}\cdot, h^{-1}\cdot, h^{-1}\cdot, h^{-1}\cdot), \quad h \in \text{Gl}(\mathfrak{g}).$$

Denoting by $\theta : \mathfrak{gl}(\mathfrak{g}) \rightarrow \text{End}(\Lambda^4(\mathfrak{g})^*)$ the infinitesimal representation $\theta(A)\psi_0 := \frac{d}{dt}(e^{tA} \cdot \psi_0)|_{t=0}$, we have

$$\theta(\mathfrak{gl}(\mathfrak{g}))\psi_0 = \Lambda^4(\mathfrak{g})^*, \tag{6}$$

and the Lie algebra of the stabilizer

$$G_{\psi_0} := \{h \in \text{Gl}(\mathfrak{g}) ; h \cdot \psi_0 = \psi_0\}$$

is characterised by

$$\mathfrak{g}_{\psi_0} := \text{Lie}(G_{\psi_0}) = \{A \in \mathfrak{gl}(\mathfrak{g}) ; \theta(A)\psi_0 = 0\}.$$

Indeed, the orbit $\text{Gl}(\mathfrak{g}) \cdot \psi_0$ is parametrised by the homogeneous space $\text{Gl}(\mathfrak{g})/G_{\psi_0}$. Using the reductive decomposition $\mathfrak{gl}(\mathfrak{g}) = \mathfrak{g}_{\psi_0} \oplus \mathfrak{q}_{\psi_0}$ from Eq. (6), we have

$$\theta(\mathfrak{q}_{\psi_0})\psi_0 = \Lambda^4(\mathfrak{g})^*. \tag{7}$$

In particular, for the Laplacian $\Delta_0\psi_0$, there exists a unique $Q_0 \in \mathfrak{q}_{\psi_0}$ such that $\theta(Q_0)\psi_0 = \Delta_0\psi_0$. Now, for any other $\psi = h \cdot \psi_0 \in \text{Gl}(\mathfrak{h}) \cdot \psi_0$,

$$G_\psi = G_{h \cdot \psi_0} = h^{-1}G_{\psi_0}h \quad \text{and} \quad \mathfrak{g}_\psi = \mathfrak{g}_{h \cdot \psi_0} = \text{Ad}(h^{-1})\mathfrak{g}_{\psi_0}.$$

where $\text{Ad} : \text{Gl}(\mathfrak{g}) \rightarrow \text{Gl}(\mathfrak{gl}(\mathfrak{g}))$. Moreover, we have the following relations.

Lemma 3.1 *Let $\psi = h \cdot \psi_0$ for $h \in \text{Gl}(\mathfrak{g})$, denote $*$ the Hodge star and Δ the Laplacian operator of ψ , then*

$$* = (h^{-1})^* *_0 h^* \quad \text{and} \quad h^* \circ \Delta = \Delta_0 \circ h^*,$$

where $*_0$ and Δ_0 are the Hodge star and the Laplacian operator of ψ_0 , respectively.

Proof The inner products on \mathfrak{g} and \mathfrak{g}^* induced by a G_2 -structure $\varphi = h \cdot \varphi_0$ are $g = (h^{-1})^* g_0$ and $g = h^* g_0$, respectively, where g_0 is the inner product induced by φ_0 . So, for $\alpha \in \Lambda^k(\mathfrak{g})^*$ we have

$$\begin{aligned} \alpha \wedge *\alpha &= g(\alpha, \alpha) \text{ vol} \\ &= (h^* g_0)(\alpha, \alpha) (h^{-1})^* \text{ vol}_0 \\ &= (h^{-1})^* (g_0(h^* \alpha, h^* \alpha) \text{ vol}_0) \\ &= \alpha \wedge (h^{-1})^* *_0 h^* \alpha, \end{aligned}$$

which gives the first claimed relation. In particular,

$$*\psi = (h^{-1})^* *_0 h^* \psi = (h^{-1})^* *_0 \psi_0 = h \cdot \varphi_0 = \varphi.$$

Applying again the first relation to the operator $d^* = (-1)^{7k} * d*$, we have $d^* = (h^{-1})^* \circ d^{*0} \circ h^*$, which yields the claim because d commutes with the pullback h^* . \square

As consequence of the above Lemma, we can relate $Q_\psi \in \mathfrak{q}_\psi$ to $Q_0 \in \mathfrak{q}_{\psi_0}$:

$$\begin{aligned} \theta(Q_\psi)\psi &= \Delta_\psi \psi = \Delta_\psi ((h^{-1})^* \psi_0) = (h^{-1})^* (\Delta_0 \psi_0) \\ &= (h^{-1})^* \theta(Q_0)\psi_0 = (h^{-1})^* \theta(Q_0)h^* \psi \\ &= (h^{-1})^* \frac{d}{dt} (e^{tQ_0} \cdot (h^{-1} \cdot \psi))|_{t=0} = \frac{d}{dt} ((he^{tQ_0}h^{-1}) \cdot \psi)|_{t=0} \\ &= \frac{d}{dt} ((e^{t \text{Ad}(h)Q_0}) \cdot \psi)|_{t=0} = \theta(\text{Ad}(h)Q_0)\psi, \end{aligned}$$

since $\mathfrak{g}_\psi \cap \mathfrak{q}_\psi = 0$. Therefore,

$$Q_\psi = \text{Ad}(h)Q_0. \tag{8}$$

We will address the flow (5) in the particular case $(M, \gamma_t) = (G, \psi_t)$ and $q = -\Delta_\psi$, i.e. under the Laplacian co-flow (1). In particular, a G -invariant solution of the Laplacian co-flow is given by a 1-parameter family in \mathfrak{g} solving

$$\frac{d}{dt} \psi_t = -\Delta_t \psi_t. \tag{9}$$

Writing $\psi_t =: h_t^{-1} \cdot \psi_0$ for $h_t \in \text{Gl}(\mathfrak{g})$, we have

$$\begin{aligned} \frac{d}{dt} \psi_t &= \psi_0(h'_t \cdot, h_t \cdot, h_t \cdot, h_t \cdot) + \psi_0(h_t \cdot, h'_t \cdot, h_t \cdot, h_t \cdot) \\ &\quad + \psi_0(h_t \cdot, h_t \cdot, h'_t \cdot, h_t \cdot) + \psi_0(h_t \cdot, h_t \cdot, h_t \cdot, h'_t \cdot) \\ &= \psi_t(h_t^{-1} h'_t \cdot, \cdot, \cdot, \cdot) + \psi_t(\cdot, h_t^{-1} h'_t \cdot, \cdot, \cdot) + \psi_t(\cdot, \cdot, h_t^{-1} h'_t \cdot, \cdot) + \psi_t(\cdot, \cdot, \cdot, h_t^{-1} h'_t \cdot) \\ &= -\theta(h_t^{-1} h'_t) \psi_t, \end{aligned}$$

thus the evolution of h_t under the flow (9) is given by

$$\frac{d}{dt} h_t = h_t Q_t. \tag{10}$$

4 Lie bracket flow

The *Lie bracket flow* is a dynamical system defined on the variety of Lie algebras, corresponding to an invariant geometric flow under a natural change of variables. It is introduced in [15] as a tool for the study of regularity and long-time behaviour of solutions.

For each $h \in \text{Gl}(\mathfrak{g})$, consider the following Lie bracket in \mathfrak{g} :

$$\mu = [\cdot, \cdot]_h := h \cdot [\cdot, \cdot] = h[h^{-1} \cdot, h^{-1} \cdot]. \tag{11}$$

Indeed, $(\mathfrak{g}, [\cdot, \cdot]) \xrightarrow{h} (\mathfrak{g}, \mu)$ defines a Lie algebra isomorphism, and consequently an equivalence between invariant structures

$$\eta : (G, \psi_\mu) \rightarrow (G_\mu, \psi),$$

where G_μ is the 1-connected Lie group with Lie algebra (\mathfrak{g}, μ) , η is an automorphism such that $d\eta_1 = h$ and $\psi_\mu = \eta^* \psi$. In particular, by Lemma 3.1, $\Delta_\mu \psi_\mu = \eta^* \Delta_\psi \psi$, or, equivalently, $Q_\mu = h Q_\psi h^{-1}$, by Eq. (8).

Lemma 4.1 [15, §4.1] *Let $\{h_t\} \subset \text{Gl}(\mathfrak{g})$ be a solution of (10), then the bracket $\mu_t := [\cdot, \cdot]_{h_t}$ evolves under the flow*

$$\frac{d}{dt} \mu_t = -\delta_{\mu_t}(Q_{\mu_t}), \tag{12}$$

in which $\delta_\mu : \text{End}(\mathfrak{g}) \rightarrow \Lambda^2(\mathfrak{g})^* \otimes \mathfrak{g}$ is the infinitesimal representation of the $\text{Gl}(\mathfrak{g})$ -action (11), defined by

$$\delta_\mu(A) := -A\mu(\cdot, \cdot) + \mu(A \cdot, \cdot) + \mu(\cdot, A \cdot).$$

Proof Setting $Q_{\mu_t} := h_t Q_t h_t^{-1}$, we compute:

$$\begin{aligned} \frac{d}{dt} \mu_t &= h'_t[h_t^{-1} \cdot, h_t^{-1} \cdot] + h_t[(h_t^{-1})' \cdot, h_t^{-1} \cdot] + h_t[h_t^{-1} \cdot, (h_t^{-1})' \cdot] \\ &= h'_t h_t^{-1} \mu_t(\cdot, \cdot) - \mu_t(h'_t h_t^{-1} \cdot, \cdot) - \mu_t(\cdot, h'_t h_t^{-1} \cdot) \\ &= -\delta_{\mu_t}(h'_t h_t^{-1}) = -\delta_{\mu_t}(h_t Q_t h_t^{-1}) = -\delta_{\mu_t}(Q_{\mu_t}), \end{aligned}$$

since $(h_t^{-1})' = -h_t^{-1} h'_t h_t^{-1}$. □

Remark Notice that, if $\{h_t\} \subset \text{Gl}(\mathfrak{g})$ solves

$$\frac{d}{dt} h_t = Q_{\mu_t} h_t,$$

then μ_t solves the bracket flow (12).

5 Self similar solutions

We say that a 4-form ψ flows self-similarly along the flow (1) if the solution ψ_t starting at ψ has the form $\psi_t = b_t f_t^* \psi$, for some one-parameter families $\{f_t\} \subset \text{Diff}(G)$ and time-dependent non-vanishing functions $\{b_t\}$. This is equivalent to the relation

$$-\Delta \psi = \lambda \psi + \mathcal{L}_X \psi,$$

for some constant $\lambda \in \mathbb{R}$ and a complete vector field X . Suppose that the infinitesimal operator defined by $\Delta \psi = \theta(Q_\psi) \psi$ had the particular form

$$Q_\psi = cI + D \quad \text{for } c \in \mathbb{R} \quad \text{and} \quad D \in \text{Der}(\mathfrak{g}). \tag{13}$$

Then we have

$$\begin{aligned} \theta(Q_\psi) \psi &= -4c\psi + \theta(D)\psi = -4c\psi - \frac{d}{dt} \left((e^{tD})^* \psi \right) \Big|_{t=0} \\ &= -4c\psi - \mathcal{L}_{X_D} \psi, \end{aligned}$$

where X_D is a vector field on \mathfrak{g} defined by the 1-parameter group of automorphisms $e^{tD} \in \text{Aut}(\mathfrak{g})$.

In that case, (G, ψ) is a soliton for the Laplacian co-flow with

$$-\Delta_\psi \psi = 4c\psi + \mathcal{L}_{X_D} \psi,$$

where X_D also denotes the invariant vector field on G defined by the 1-parameter subgroup β_t in $\text{Aut}(G)$ such that $d(\beta_t)_1 = e^{tD} \in \text{Aut}(\mathfrak{g})$.

A G_2 -structure whose underlying 4-form ψ satisfies (13) is called an algebraic soliton, and we say that it is expanding, steady, or shrinking if λ is positive, zero, or negative, respectively.

Lemma 5.1 Given $\psi_2 = c\psi_1$ with $c \in \mathbb{R}^*$, the Laplacian operator satisfies the scaling property

$$\Delta_2\psi_2 = c^{1/4}\Delta_1\psi_1 \tag{14}$$

Proof Notice that $c\psi_1 = (c^{1/4}I)^*\psi_1$ and apply Lemma 3.1. □

Lemma 5.2 If ψ is an algebraic soliton with $Q_\psi = cI + D$, then $\psi_t = b_t h_t^* \psi$ is a self-similar solution for the Laplacian co-flow (9), with

$$b_t = (3ct + 1)^{4/3} \text{ and } h_t = e^{s_t D}, \text{ for } s_t = -\frac{1}{3c} \log(3ct + 1). \tag{15}$$

Moreover,

$$Q_t = b_t^{-3/4} Q_\psi.$$

Proof Applying Lemmas 3.1 and 5.1, we have

$$\begin{aligned} \Delta_t \psi_t &= b_t^{1/4} h_t^* \Delta \psi = b_t^{1/4} h_t^* \theta(Q_\psi) \psi \\ &= b_t^{1/4} h_t^* \left(-4c\psi + \theta(D)\psi \right) \\ &= -4cb_t^{1/4} h_t^* \psi + \theta(b_t^{1/4} h_t^{-1} D h_t) h_t^* \psi. \end{aligned}$$

On the other hand,

$$\begin{aligned} \frac{d}{dt} \psi_t &= b_t' h_t^* \psi + b_t (h_t^* \psi)' \\ &= b_t' h_t^* \psi + b_t \theta(h_t^{-1} h_t') h_t^* \psi. \end{aligned}$$

Replacing the above expressions in (9) and comparing terms we obtain the ODE system

$$\begin{cases} b_t' = 4cb_t^{1/4}, & b(0) = 1 \\ b_t h_t' = -b_t^{1/4} D h_t, & h(0) = I \end{cases}$$

the solutions of which are as claimed.

Finally, we have

$$\begin{aligned} \theta(Q_t)\psi_t &= \Delta_t \psi_t = b_t^{1/4} h_t^* \Delta \psi = b_t^{1/4} h_t^* \theta(Q_\psi) \psi \\ &= b_t^{1/4} \theta(h_t^{-1} Q_\psi h_t) h_t^* \psi = \theta(b_t^{-3/4} h_t^{-1} Q_\psi h_t) \psi_t, \end{aligned}$$

so $Q_t = b_t^{-3/4} h_t^{-1} Q_\psi h_t$, which yields the second claim, since $Q_\psi h_t = h_t Q_\psi$. □

In terms of the bracket flow, we have $Q_{\mu_t} = h_t Q_t h_t^{-1} = b_t^{-3/4} Q_\psi$. Then, replacing in (12) the Ansatz

$$\mu_t = \left(\frac{1}{c(t)} I \right) \cdot [\cdot, \cdot] = c(t) [\cdot, \cdot] \text{ for } c(t) \neq 0 \text{ and } c(0) = 1, \tag{16}$$

we obtain $c'_t = cb_t^{-3/4}c_t$, which has solution $c_t = e^{c \cdot s_t}$, with s_t as above.

Indeed, there is an equivalence between the time-dependent Lie bracket given in (16) and the corresponding soliton given in Lemma 5.2:

Theorem 5.3 [15, Theorem 6] *Let (G, φ) be a 1-connected Lie group with an invariant G_2 -structure. The following conditions are equivalent:*

(i) *The bracket flow solution starting at $[\cdot, \cdot]$ is given by*

$$\mu_t = \left(\frac{1}{c(t)} I \right) \cdot [\cdot, \cdot] \text{ for } c(t) > 0, c(0) = 1.$$

(ii) *The operator $Q_t \in \mathfrak{q}_\psi \subset \text{End}(\mathfrak{g})$, such that $\Delta_\psi \psi = \theta(Q_\psi)\psi$, satisfies*

$$Q_\psi = cI + D, \text{ for } c \in \mathbb{R} \text{ and } D \in \text{Der}(\mathfrak{g}).$$

6 Example of a co-flow soliton

We now apply the previous theoretical framework to construct an explicit co-flow soliton from a natural Ansatz. Let $\mathfrak{g} = \mathbb{R} \times_\rho \mathbb{R}^6$ be the Lie algebra defined by $\rho(t) = \exp(tA) \in \text{Aut}(\mathfrak{g})$, with

$$A = \left(\begin{array}{ccc|ccc} & & & & & 1 \\ & & & & 0 & \\ & & & 1 & & \\ \hline & & 0 & & & \\ & & 1 & & & \end{array} \right).$$

The canonical $SU(3)$ -structure on \mathbb{R}^6 with respect to the orthonormal basis $\{e_1, e_6, e_2, e_5, e_3, e_4\}$ is

$$\omega = e^{16} + e^{25} + e^{34}, \quad \rho_+ = e^{123} + e^{145} + e^{356} - e^{246}$$

and the standard complex structure of \mathbb{R}^6 is

$$J = \begin{pmatrix} 0 & -I_3 \\ I_3 & 0 \end{pmatrix}.$$

We also have the natural 3-form

$$\rho_- := J \cdot \rho_+ = -e^{135} + e^{124} + e^{236} + e^{456}.$$

The structure equations of \mathfrak{g}^* with respect to the dual basis of $\{e_1, e_6, e_2, e_5, e_3, e_4, e_7\}$ are

$$de^1 = e^{67}, \quad de^6 = e^{17}, \quad de^3 = e^{47}, \quad de^4 = e^{37}, \quad de^j = 0 \text{ for } j = 2, 5.$$

From the above, we have

$$d\omega = 0, \quad d\rho_+ = 2(e^{1357} + e^{4567}), \quad \text{and} \quad d\rho_- = 2(e^{2467} + e^{1237}).$$

There is a natural co-closed G_2 -structure on \mathfrak{g} , given by

$$\varphi := \omega \wedge e^7 - \rho_- = e^{167} + e^{257} + e^{347} + e^{135} - e^{124} - e^{236} - e^{456}, \quad (17)$$

with dual 4-form

$$\psi = *\varphi = \frac{\omega^2}{2} + \rho_+ \wedge e^7 = e^{1256} + e^{1346} + e^{2345} + e^{1237} + e^{1457} + e^{3567} - e^{2467}. \quad (18)$$

Clearly $\tau_1 = 0$ and $\tau_2 = 0$, and

$$d\varphi = -d\rho_- = -2(e^{2467} + e^{1237}) = *\tau_3,$$

since $d\varphi \wedge \varphi = 0$, i.e. $\tau_0 = 0$. Therefore, using (4),

$$\tau_3 = 2(e^{135} + e^{456}) \quad \text{or, alternatively,} \quad \tau_{27} = (e^1)^2 + (e^3)^2 - ((e^4)^2 + (e^6)^2).$$

The Laplacian of ψ is

$$\begin{aligned} \Delta\psi &= d * d * \psi + * d * d\psi = d * d\varphi \\ &= d\tau_3 = 4(e^{1457} + e^{3567}). \end{aligned}$$

Consider the derivation $D = \text{diag}(a, b, c, c, d, a, 0) \in \text{Der}(\mathfrak{g})$, and take the vector field on \mathfrak{g}

$$X_D(x) = \frac{d}{dt}(\exp(tD)(x)), \quad \text{for } x \in \mathfrak{g}.$$

Then we have

$$\begin{aligned} \mathcal{L}_{X_D}\psi &= \frac{d}{dt}(\exp(-tD)*\psi)|_{t=0} = -\theta(D)\psi \\ &= (2a + b + d)e^{1256} + (2a + 2c)e^{1346} + (b + 2c + d)e^{2345} + (a + b + c)e^{1237} \\ &\quad + (a + c + d)e^{1457} + (a + c + d)e^{3567} - (a + b + c)e^{2467}. \end{aligned}$$

From the soliton equation $-\Delta\psi = \mathcal{L}_{X_D}\psi + \lambda\psi$, we obtain a system of linear equations

$$\begin{cases} 2a + b + d + \lambda = 0 \\ 2a + 2c + \lambda = 0 \\ a + b + c + \lambda = 0 \\ a + c + d + \lambda = -4 \end{cases},$$

which has solution $D = \text{diag}(2, 4, 2, 2, 0, 2, 0)$ and $\lambda = -8$. In particular, for the matrix $Q_\psi = D + \frac{\lambda}{4}I_7$, we have $\Delta\psi = \theta(Q_\psi)\psi$. By Lemma 5.2, the functions

$$c(t) = (1 - 6t)^{4/3} \quad \text{and} \quad s(t) = \frac{1}{6} \log(1 - 6t) \quad \text{for} \quad \frac{1}{6} > t,$$

yield the family of 4-forms $\{\psi_t = c(t)(f(t)^{-1})^*\psi\}$, where

$$\begin{aligned} f(t)^{-1} &= \exp(-s(t)D) \\ &= \text{diag}((1-6t)^{-1/3}, (1-6t)^{-2/3}, (1-6t)^{-1/3}, (1-6t)^{-1/3}, 1, (1-6t)^{-1/3}, 1). \end{aligned}$$

Hence,

$$\psi_t = e^{1256} + e^{1346} + e^{2345} + e^{1237} + (1 - 6t)^{2/3}(e^{1457} + e^{3567}) - e^{2467} \quad (19)$$

defines a soliton of the Laplacian co-flow:

$$\frac{d\psi_t}{dt} = -4(1 - 6t)^{-1/3}(e^{1457} + e^{3567}) = -c(t)^{1/4}(f(t)^{-1})^*\Delta\psi = -\Delta_t\psi_t.$$

Corollary 6.1 *The relevant geometric structures associated to the 4-form given in (19) are:*

(i) *the G_2 -structure*

$$\varphi_t = c(t)^{1/4}(e^{167} + e^{257} + e^{347} + e^{135} - e^{456}) - c(t)^{-1/4}(e^{124} + e^{236});$$

(ii) *the G_2 -metric*

$$g_t = (e^1)^2 + (e^3)^2 + (e^4)^2 + (e^6)^2 + c(t)^{-1/2}(e^2)^2 + c(t)^{1/2}((e^5)^2 + (e^7)^2);$$

(iii) *the volume form*

$$\text{vol}_t = c(t)^{1/4} \text{vol}_\psi;$$

(iv) *the torsion form and the full torsion tensor*

$$\tau_3(t) = 2(e^{135} + e^{456}) \quad \text{and} \quad T(t) = c(t)^{-1/4}(- (e^1)^2 - (e^3)^2 + (e^4)^2 + (e^6)^2);$$

(v) *the Ricci tensor and the scalar curvature*

$$\text{Ric}(g_t) = -4c(t)^{-1/2}(e^7)^2 \quad \text{and} \quad R_t = -\frac{1}{2}|\tau_3(t)|^2 = -4c(t)^{-1/2};$$

(vi) *the bracket flow solution*

$$\mu_t = c(t)^{-1/4}[\cdot, \cdot].$$

Remark 6.2

- (1) From Corollary 6.1 (iv) and (v), if $t \rightarrow -\infty$ then $\text{Ric}(g_t) \rightarrow 0$, $T(t) \rightarrow 0$ and $\mu_t \rightarrow 0$. Since G_{μ_t} is solvable for each t [15, Proposition 6], (G_{μ_t}, ψ) smoothly converges to the flat G_2 -structure $(\mathbb{R}^7, \varphi_0)$.
- (2) Since $\text{Ric}(g_\psi) = \text{diag}(0, 0, 0, 0, 0, 0, -4) + 4I_7 \in \text{Der}(\mathfrak{g})$, the metric g_ψ is a shrinking Ricci soliton (cf. [1]).

7 Afterword

We would like to conclude with two questions for future work.

- (1) If the cover Lie group G admits a co-compact discrete subgroup Γ , it would be interesting to determine whether the corresponding co-closed G_2 -structure on the compact quotient G/Γ is also a Laplacian co-flow soliton.
- (2) When the full torsion tensor $T = -\tau_{27}$ is traceless symmetric, the scalar curvature of the corresponding G_2 -metric is nonpositive, and it vanishes if, and only if, the structure is torsion-free (c.f. [5, (4.28)] or [13, (4.21)]). This fact was first pointed out by Bryant for a closed G_2 -structure, in order to explain the absence of closed Einstein G_2 -structures (other than Ricci-flat ones) on compact 7-manifolds, giving rise to the concept of *extremally Ricci-pinched closed G_2 -structure* [5, Remark 13]. Later on, Fernández et al. showed that a 7-dimensional (non-flat) Einstein solvmanifold (S, g) cannot admit any left-invariant co-closed G_2 -structure φ such that $g_\varphi = g$ [7].

In that context, it would be interesting to study pinching phenomena for the Ricci curvature of solvmanifolds with a co-closed (non-flat) left-invariant G_2 -structure and traceless torsion. In our present construction, for instance, we can see from Corollary 6.1 that

$$F(t) = \frac{R_t^2}{|\text{Ric}(g_t)|^2} = 1.$$

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