



Explicit construction of irreducible modules for $U_q(\mathfrak{gl}_n)$

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Abstract

We construct new families of $U_q(\mathfrak{gl}_n)$ -modules by continuation from finite dimensional representations. Each such module is associated with a combinatorial object—admissible set of relations. More precisely, we prove that any admissible set of relations leads to a family of irreducible $U_q(\mathfrak{gl}_n)$ -modules. Finite dimensional and generic modules are particular cases of this construction.

Keywords Quantum group · Gelfand–Tsetlin module · Gelfand–Tsetlin basis · Tableaux realization

Mathematics Subject Classification Primary 17B67

1 Introduction

Gelfand and Graev [13] proposed a method of constructing of \mathfrak{gl}_n -modules which extend finite dimensional modules and admit a basis of tableaux with the standard action of the generators of the Lie algebra [14]. This construction is based on a choice of certain relations satisfied by the entries of the Gelfand–Tsetlin tableaux. Lemire and Patera [21] conjectured sufficient conditions under which the Gelfand–Graev’s construction defines in fact a module, proving it for $n = 3$ and $n = 4$. In [11] the authors proved this conjecture and extended the construction for a larger class of irreducible

Dedicated to the 70th birthday of Ivan Shestakov.

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\mathfrak{gl}_n -modules. The purpose of this letter is to show how to deform this construction and obtain new large families of irreducible modules for the quantum group $U_q(\mathfrak{gl}_n)$. Infinite dimensional generic and finite dimensional modules are particular cases of this construction. New irreducible modules are presented explicitly with a basis consisting of certain tableaux and with explicit action of the generators of $U_q(\mathfrak{gl}_n)$ generalizing the construction of finite dimensional representations [28]. Having such an explicit construction will be useful for possible applications.

Constructed modules belong to the category of Gelfand–Tsetlin modules with a diagonalizable action of the Gelfand–Tsetlin subalgebra. For \mathfrak{gl}_n the theory of Gelfand–Tsetlin modules has origin in the classical paper of Gelfand and Tsetlin [14]. It is related to many concepts arising in Mathematics and Physics, see for example [1, 2, 4, 9, 15–17, 19, 20]. The general theory of Gelfand–Tsetlin modules for \mathfrak{gl}_n was developed in [3, 5–7, 10, 22, 23, 25, 26], and references therein. For $U_q(\mathfrak{gl}_n)$ certain families of Gelfand–Tsetlin modules were constructed in [12, 24], while the general theory was developed in [8].

Current letter provides new information about Gelfand–Tsetlin modules for $U_q(\mathfrak{gl}_n)$. The paper is organized as follows. Section 2 contains some preliminary information. In Sect. 3 we introduce our main technical tools—admissible sets of relations and realizable sets of relations. To any realizable set of relations we associate a family of $U_q(\mathfrak{gl}_n)$ -modules. We prove the main result of this letter stating that any admissible set of relations is a realizable set of relations (Theorem 3.7). A certain effective method (RR-method) of constructing the admissible relation is described in Theorem 3.11. Finally, in Sect. 4 we study the action of the generators of the Gelfand–Tsetlin subalgebra on modules associated with admissible sets of relations. The Gelfand–Tsetlin subalgebra Γ_q is diagonalizable on all constructed modules (Theorem 4.1), moreover, it separates the basis tableaux (Proposition 4.4). Using the action of Γ_q we obtain a criterion of irreducibility of constructed admissible modules: irreducible modules correspond to maximal admissible sets of relations (Theorem 4.5).

We now fix some notation and conventions. Throughout the paper we fix an integer $n \geq 2$ and $q \in \mathbb{C}$ which is not root of unity. The ground field will be \mathbb{C} . By U_q we denote the quantum enveloping algebra of \mathfrak{gl}_n . We fix the standard Cartan subalgebra \mathfrak{h} , the standard triangular decomposition and the corresponding basis of simple roots $\alpha_1, \dots, \alpha_{n-1}$. The weights of U_q will be written as n -tuples $(\lambda_1, \dots, \lambda_n)$. For a commutative ring R , by $\text{Spec} R$ we denote the set of maximal ideals of R . For $i > 0$ by S_i we denote the i th symmetric group. Let $1(q)$ be the set of all complex x such that $q^x = 1$. Finally, for any complex number x , we set

$$(x)_q = \frac{q^x - 1}{q - 1}, \quad [x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}.$$

2 Preliminaries

We define U_q as a unital associative algebra generated by $e_i, f_i (1 \leq i \leq n)$ and $q^h (h \in \mathfrak{h})$ with the following relations:

$$q^0 = 1, q^h q^{h'} = q^{h+h'} \quad (h, h' \in \mathfrak{h}), \tag{1}$$

$$q^h e_i q^{-h} = q^{\langle h, \alpha_i \rangle} e_i, \tag{2}$$

$$q^h f_i q^{-h} = q^{-\langle h, \alpha_i \rangle} f_i, \tag{3}$$

$$e_i f_j - f_j e_i = \delta_{ij} \frac{q^{\alpha_i} - q^{-\alpha_i}}{q - q^{-1}}, \tag{4}$$

$$e_i^2 e_j - (q + q^{-1}) e_i e_j e_i + e_j e_i^2 = 0 \quad (|i - j| = 1), \tag{5}$$

$$f_i^2 f_j - (q + q^{-1}) f_i f_j f_i + f_j f_i^2 = 0 \quad (|i - j| = 1), \tag{6}$$

$$e_i e_j = e_j e_i, f_i f_j = f_j f_i \quad (|i - j| > 1). \tag{7}$$

The quantum special linear algebra $U_q(sl_n)$ is the subalgebra of U_q generated by $e_i, f_i, q^{\pm\alpha_i} (i = 1, 2, \dots, n - 1)$.

Consider the following chain

$$U_q(\mathfrak{gl}_1) \subset U_q(\mathfrak{gl}_2) \subset \dots \subset U_q(\mathfrak{gl}_n).$$

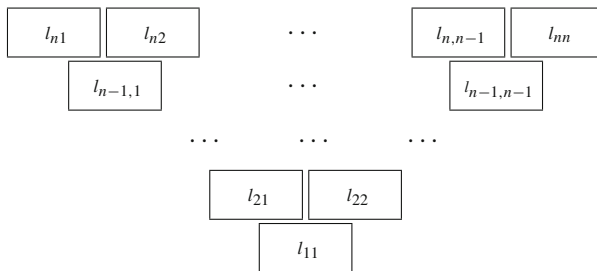
Let Z_m denote the center of $U_q(\mathfrak{gl}_m)$. The subalgebra of $U_q(\mathfrak{gl}_n)$ generated by $\{Z_m \mid m = 1, \dots, n\}$ will be called the *Gelfand–Tsetlin subalgebra* of U_q and will be denoted by Γ_q .

Definition 2.1 A finitely generated U_q -module M is called a *Gelfand–Tsetlin module* (with respect to Γ_q) if

$$M = \bigoplus_{m \in \text{Specm } \Gamma_q} M(m), \tag{8}$$

where $M(m) = \{v \in M \mid m^k v = 0 \text{ for some } k \geq 0\}$.

For a vector $L = (l_{ij})$ in $\mathbb{C}^{\frac{n(n+1)}{2}}$, by $T(L)$ we will denote the following array with entries $\{l_{ij} : 1 \leq j \leq i \leq n\}$



such an array will be called a *Gelfand–Tsetlin tableau* of height n . For any $1 \leq j \leq i \leq n - 1$, the vector $\delta^{ij} \in \mathbb{Z}^{\frac{n(n+1)}{2}}$ is defined by $(\delta^{ij})_{ij} = 1$ and all other $(\delta^{ij})_{k\ell}$ are zero. Finally, a Gelfand–Tsetlin tableau of height n is called *standard* if $l_{ki} - l_{k-1,i} \in \mathbb{Z}_{\geq 0}$ and $l_{k-1,i} - l_{k,i+1} \in \mathbb{Z}_{> 0}$ for all $1 \leq i \leq k \leq n$.

Recall the quantum version of the classical result of Gelfand and Tsetlin which provides an explicit basis in the finite dimensional case.

Theorem 2.2 ([28, Theorem 2.11] and [12, Proposition 4.3]) *Let $L(\lambda)$ be the finite dimensional irreducible module over U_q of highest weight $\lambda = (\lambda_1, \dots, \lambda_n)$, where $\lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0}$. Then there exists a basis of $L(\lambda)$ consisting of all standard tableaux $T(L)$ with fixed top row $l_{nj} = \lambda_j - j$. Moreover, the action of the generators of U_q on $L(\lambda)$ is given by the Gelfand–Tsetlin formulae:*

$$\begin{aligned}
 q^{\epsilon_k}(T(L)) &= q^{a_k}T(L), \quad a_k = \sum_{i=1}^k l_{k,i} - \sum_{i=1}^{k-1} l_{k-1,i} + k, \quad k = 1, \dots, n, \\
 e_k(T(L)) &= - \sum_{j=1}^k \frac{\prod_i [l_{k+1,i} - l_{k,j}]_q}{\prod_{i \neq j} [l_{k,i} - l_{k,j}]_q} T(L + \delta^{kj}), \\
 f_k(T(L)) &= \sum_{j=1}^k \frac{\prod_i [l_{k-1,i} - l_{k,j}]_q}{\prod_{i \neq j} [l_{k,i} - l_{k,j}]_q} T(L - \delta^{kj}).
 \end{aligned}
 \tag{9}$$

Moreover, the generators c_{mk} of Γ_q acts on $T(L)$ as multiplication by

$$\gamma_{mk}(L) = (k)_{q^{-2}}!(m - k)_{q^{-2}}!q^{k(k+1) - \frac{m(m+1)}{2}} \sum_{\tau} q^{\sum_{i=1}^k l_{m\tau(i)} - \sum_{i=k+1}^m l_{m\tau(i)}} \tag{10}$$

where $\tau \in S_m$ is such that $\tau(1) < \dots < \tau(k)$, $\tau(k + 1) < \dots < \tau(m)$.

3 Admissible relations

Set $\mathfrak{A} := \{(i, j) \mid 1 \leq j \leq i \leq n\}$. In this section we will consider certain binary relations on \mathfrak{A} . Set

$$\mathcal{R}^+ := \{((i, j); (i - 1, t)) \mid 1 \leq j \leq i, 2 \leq i \leq n, 1 \leq t \leq i - 1\} \tag{11}$$

$$\mathcal{R}^- := \{((i, j); (i + 1, s)) \mid 1 \leq j \leq i \leq n - 1, 1 \leq s \leq i + 1\} \tag{12}$$

$$\mathcal{R}^0 := \{((n, i); (n, j)) \mid 1 \leq i \neq j \leq n\} \tag{13}$$

and let $\mathcal{R} := \mathcal{R}^- \cup \mathcal{R}^0 \cup \mathcal{R}^+ \subset \mathfrak{A} \times \mathfrak{A}$. From now any $\mathcal{C} \subseteq \mathcal{R}$ will be called a *set of relations*.

Associated with any $\mathcal{C} \subseteq \mathcal{R}$ we can construct a directed graph $G(\mathcal{C})$ with set of vertices \mathfrak{A} and an arrow going from (i, j) to (r, s) if and only if $((i, j); (r, s)) \in \mathcal{C}$. For convenience we will picture the vertex set as disposed in a triangular arrangement with n rows and k -th row given by $\{(k, 1), \dots, (k, k)\}$.

Definition 3.1 Let \mathcal{C} be any set of relations.

- (i) We denote $\mathfrak{A}(\mathcal{C}) \subseteq \mathfrak{A}$ the set of all vertices in $G(\mathcal{C})$ which are starting or ending vertices of an arrow.
- (ii) \mathcal{C} is called indecomposable if $G(\mathcal{C})$ is a connected graph.
- (iii) \mathcal{C} is called a loop if $G(\mathcal{C})$ is an oriented cycle.

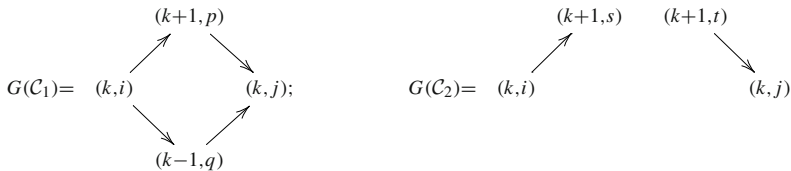
(iv) Given $(i, j), (r, s) \in \mathfrak{A}$ we will write $(i, j) \succeq_{\mathcal{C}} (r, s)$ if there exists a path in $G(\mathcal{C})$ starting in (i, j) and finishing in (r, s) .

Note that any subset of \mathcal{R} is a union of disconnected indecomposable sets and such decomposition is unique. Any $\mathcal{C} \subset \mathcal{R}$ can be written in the form $\mathcal{C} = \mathcal{C}^- \cup \mathcal{C}^0 \cup \mathcal{C}^+$, where $\mathcal{C}^- := \mathcal{R}^- \cap \mathcal{C}$, $\mathcal{C}^0 := \mathcal{R}^0 \cap \mathcal{C}$ and $\mathcal{C}^+ := \mathcal{R}^+ \cap \mathcal{C}$.

We will define now our main concept which is a slight modification of the definition of an admissible set in [11].

Definition 3.2 Let \mathcal{C} be an indecomposable set. We say that \mathcal{C} is **admissible** if it satisfies the following conditions:

- (i) \mathcal{C} does not contain loops.
- (ii) \mathcal{C} is noncritical.
- (iii) For any $1 \leq k \leq n$, $(k, i) \succeq_{\mathcal{C}} (k, j)$ if and only if $(k, i), (k, j)$ are in the same indecomposable subset of \mathcal{C} and $i < j$.
- (iv) \mathcal{C} is reduced.
- (v) There is not cross in \mathcal{C} .
- (vi) For every adjoining pair (k, i) and (k, j) , $1 \leq k \leq n - 1$, there exist p, q such that $\mathcal{C}_1 \subseteq \mathcal{C}$ or, there exist $s < t$ such that $\mathcal{C}_2 \subseteq \mathcal{C}$, where the graphs associated to \mathcal{C}_1 and \mathcal{C}_2 are as follows



An arbitrary set \mathcal{C} is admissible if every indecomposable subset of \mathcal{C} is admissible.

3.1 Tableaux realization of admissible sets of relations

In this section we will describe \mathbb{C} -vector spaces associated with sets of relations \mathcal{C} with Gelfand–Tsetlin tableaux as a bases. We will prove that we have a structure of a U_q -module on such space with the action of the generators of U_q given by the Gelfand–Tsetlin formulas (9).

Definition 3.3 Let \mathcal{C} be any set of relations and $T(L)$ any Gelfand–Tsetlin tableau.

- (i) We will say that $T(L)$ satisfies \mathcal{C} if:
 - $l_{ij} - l_{rs} \in \mathbb{Z}_{\geq 0} + \frac{1(q)}{2}$ for any $((i, j); (r, s)) \in \mathcal{C}^+ \cup \mathcal{C}^0$.
 - $l_{ij} - l_{rs} \in \mathbb{Z}_{> 0} + \frac{1(q)}{2}$ for any $((i, j); (r, s)) \in \mathcal{C}^-$.
- (ii) We say that $T(L)$ is a \mathcal{C} -realization if $T(L)$ satisfies \mathcal{C} and for any $1 \leq k \leq n - 1$ we have, $l_{ki} - l_{kj} \in \mathbb{Z} + \frac{1(q)}{2}$ if and only if (k, i) and (k, j) in the same connected component of $G(\mathcal{C})$.

- (iii) Suppose that $T(L)$ satisfies \mathcal{C} . By $\mathcal{B}_{\mathcal{C}}(T(L))$ we denote the set of all tableaux of the form $T(L + z)$, $z \in \mathbb{Z}_0^{\frac{n(n+1)}{2}}$ satisfying \mathcal{C} . By $V_{\mathcal{C}}(T(L))$ we denote the complex vector space spanned by $\mathcal{B}_{\mathcal{C}}(T(L))$.
- (iv) We call \mathcal{C} **realizable** if for any $T(L)$ satisfying \mathcal{C} , the space $V_{\mathcal{C}}(T(L))$ is a Gelfand–Tsetlin module with the action of the generators of U_q given by the Gelfand–Tsetlin formulas (9).

Example 3.4 Set $\mathcal{S} = \mathcal{S}^+ \cup \mathcal{S}^-$ where

$$\begin{aligned} \mathcal{S}^+ &:= \{(i + 1, j); (i, j) \mid 1 \leq j \leq i \leq n - 1\} \\ \mathcal{S}^- &:= \{(i, j); (i + 1, j + 1) \mid 1 \leq j \leq i \leq n - 1\}. \end{aligned}$$

It follows from the definition that \emptyset and \mathcal{S} are admissible and realizable sets of relations. Moreover, the set of all tableaux satisfying \mathcal{S} coincides with the set of all standard tableaux and the set of all tableaux satisfying \emptyset coincide with the set of all generic tableaux.

Our goal is to show that any admissible set of relations \mathcal{C} is realizable and leads to a family of U_q -modules. In fact, for any $T(L)$ satisfying \mathcal{C} we will prove that $V_{\mathcal{C}}(T(L))$ is a Gelfand–Tsetlin module with the action of the generators of U_q given by the Gelfand–Tsetlin formulas (9). For this we will need some technical lemmas.

Set

$$e_{ki}(w) = \begin{cases} 0, & \text{if } T(w) \notin \mathcal{B}_{\mathcal{C}}(T(L)) \\ -\frac{\prod_{j=1}^{k+1} [w_{ki} - w_{k+1, j}]_q}{\prod_{j \neq i}^k [w_{ki} - w_{kj}]_q}, & \text{if } T(w) \in \mathcal{B}_{\mathcal{C}}(T(L)) \end{cases} \tag{14}$$

$$f_{ki}(w) = \begin{cases} 0, & \text{if } T(w) \notin \mathcal{B}_{\mathcal{C}}(T(L)) \\ \frac{\prod_{j=1}^{k-1} [w_{ki} - w_{k-1, j}]_q}{\prod_{j \neq i}^k [w_{ki} - w_{kj}]_q}, & \text{if } T(w) \in \mathcal{B}_{\mathcal{C}}(T(L)) \end{cases} \tag{15}$$

$$h_k(w) = \begin{cases} 0, & \text{if } T(w) \notin \mathcal{B}_{\mathcal{C}}(T(L)) \\ q^{2 \sum_{i=1}^k w_{ki} - \sum_{i=1}^{k-1} w_{k-1, i} - \sum_{i=1}^{k+1} w_{k+1, i} - 1}, & \text{if } T(w) \in \mathcal{B}_{\mathcal{C}}(T(L)) \end{cases} \tag{16}$$

$$\Phi(L, z_1, \dots, z_m) = \begin{cases} 1, & \text{if } T(L + z_1 + \dots + z_t) \in \mathcal{B}_{\mathcal{C}}(T(L)) \text{ for any } 1 \leq t \leq m \\ 0, & \text{otherwise.} \end{cases} \tag{17}$$

We will denote by $T(v)$ the tableau with variable entries v_{ij} .

Lemma 3.5 *Let \mathcal{C} be admissible, $T(L)$ any tableau satisfying \mathcal{C} .*

- (i) *If $T(L + \delta^{kj}) \notin \mathcal{B}_{\mathcal{C}}(T(L))$ and $l_{k, i} - l_{k, j} \notin 1 + \frac{1(q)}{2}$ for any i , then*

$$\lim_{v \rightarrow l} e_{kj}(v) f_{kj}(v + \delta^{kj}) = 0.$$

(ii) If $T(L - \delta^{kj}) \notin \mathcal{B}_C(T(L))$ and $l_{k,j} - l_{k,i} \notin 1 + \frac{1(q)}{2}$ for any i , then

$$\lim_{v \rightarrow l} f_{kj}(v)e_{kj}(v - \delta^{kj}) = 0.$$

(iii) If $l_{k,i} - l_{k,j} \notin 1 + \frac{1(q)}{2}$, then $T(L + \delta^{k,j}), T(L - \delta^{k,i}) \notin \mathcal{B}_C(T(L))$, and

$$\lim_{v \rightarrow l} e_{kj}(v)f_{kj}(v + \delta^{k,j}) - f_{ki}(v)e_{ki}(v - \delta^{k,i}) = 0.$$

Proof Since $T(L + \delta^{kj}) \notin \mathcal{B}_C(T(L))$, we have $\{(k + 1, s); (k, j)\} \subseteq \mathcal{C}$ or $\{(k - 1, t); (k, j)\} \subseteq \mathcal{C}$. Suppose $\{(k + 1, s); (k, j)\} \subseteq \mathcal{C}$ and $T(L + \delta^{kj}) \notin \mathcal{B}_C(T(L))$. Then $l_{k+1,s} - l_{k,j} \notin \frac{1(q)}{2}$ and by direct computation one has $\lim_{v \rightarrow l} e_{kj}(v)f_{kj}(v + \delta^{kj}) = 0$. Suppose $T(L + \delta^{kj})$ does not satisfies the relation $l_{k-1,t} - l_{k,j} \in \mathbb{Z}_{>0} + \frac{1(q)}{2}$. Then we have $l_{k-1,t} - l_{k,j} \in 1 + \frac{1(q)}{2}$ and

$$\lim_{v \rightarrow l} e_{kj}(v)f_{kj}(v + \delta^{kj}) = 0.$$

The proof of (ii) is similar to (i).

It is clear that $T(L - \delta^{k,j}), T(L + \delta^{k,j+1}) \notin \mathcal{B}_C(T(L))$ if $l_{k,j} - l_{k,j+1} \in 1 + \frac{1(q)}{2}$. It is easy to see that $\#\{i' \mid r_{k+1,i'} = r_{kj}\} + \#\{j' \mid r_{k-1,j'} = r_{ki}\} \geq 2$. By direct computation one has

$$\lim_{v \rightarrow l} e_{kj}(v)f_{kj}(v + \delta^{k,j+1}) - f_{kj}(v)e_{kj}(v - \delta^{k,j}) = 0.$$

□

Lemma 3.6 [11, Lemma 4.32] *Let \mathcal{C} be admissible, $T(L)$ any \mathcal{C} -realization and $z^{(1)}, z^{(2)} \in \mathbb{Z}^{\frac{n(n-1)}{2}}$. Denote $I_1 = \{(i, j) \mid z_{ij}^{(1)} \neq 0\}$, $I_2 = \{(i, j) \mid z_{ij}^{(2)} \neq 0\}$. If $I_1 \cap I_2 = \emptyset$ and for any $(i_1, j_1) \in I_1, (i_2, j_2) \in I_2$ there is no relation between (i_1, j_1) and (i_2, j_2) , then $T(R + z^{(1)} + z^{(2)}) \in \mathcal{B}_C(T(L))$ if and only if $T(R + z^{(1)}) \in \mathcal{B}_C(T(L))$ and $T(R + z^{(2)}) \in \mathcal{B}_C(T(L))$.*

3.2 U_q -modules defined by admissible relations

Theorem 3.7 *If \mathcal{C} is a admissible set of relations and $T(L)$ is tableau satisfying \mathcal{C} , then \mathcal{C} is realizable, i.e. the vector space $V_C(T(L))$ has a structure of a U_q -module, endowed with the action of U_q given by the Gelfand–Tsetlin formulas (9).*

Proof It is sufficient to consider the case when \mathcal{C} is a union of two disconnected indecomposable admissible sets. Suppose $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$.

Let $T(L)$ be any \mathcal{C} -realization. In order to prove that $V_C(T(L))$ is a U_q -module one needs to verify all the defining relations (2–7) for any $T(R) \in \mathcal{B}_C(T(L))$.

First we show that $(e_i^2 e_j - (q + q^{-1})e_i e_j e_i + e_j e_i^2)T(R) = 0$ ($|i - j| = 1$).

$$\begin{aligned}
 & (e_i^2 e_j - (q + q^{-1})e_i e_j e_i + e_j e_i^2)T(R) \\
 &= \sum_{r,s,t} \Phi(R, \delta^{jr}, \delta^{is})e_{jr}(R)e_{is}(R + \delta^{jr})e_{it}(R + \delta^{jr} + \delta^{is})T(R + \delta^{jr} + \delta^{is} + \delta^{it}) \\
 &+ \sum_{r,s,t} \Phi(R, \delta^{is}, \delta^{it})e_{is}(R)e_{it}(R + \delta^{is})e_{jr}(R + \delta^{is} + \delta^{it})T(R + \delta^{jr} + \delta^{is} + \delta^{it}) \\
 &- [q]_2 \sum_{r,s,t} \Phi(R, \delta^{is}, \delta^{jr})e_{is}(R)e_{jr}(R + \delta^{is})e_{it}(R + \delta^{is} + \delta^{jr})T(R + \delta^{jr} + \delta^{is} + \delta^{it}).
 \end{aligned}
 \tag{18}$$

Now we consider the coefficients by nonzero tableau $T(R + \delta^{jr} + \delta^{is} + \delta^{it})$.

(i) Let $s = t$.

- (a) Suppose there is no relation between (i, s) and (j, r) . Then $\Phi(R, \delta^{jr}, \delta^{is}) = \Phi(R, \delta^{is}, \delta^{is}) = \Phi(R, \delta^{is}, \delta^{jr}) = 1$ by Lemma 3.6. Then the coefficient of $T(R + \delta^{jr} + 2\delta^{is})$ is the limit of the coefficient of $T(v + \delta^{jr} + 2\delta^{is})$ when $v \rightarrow R$ (here $T(v)$ again is a tableau with variable entries). Thus the coefficient of $T(R + \delta^{jr} + 2\delta^{is})$ is zero.
- (b) Suppose there exists a relation between (i, s) and (j, r) . Without loss of generality we assume that this relation is $C' = \{((i, s); (j, r))\}$. Let $T(v')$ be the tableau with $v'_{s't'} = l_{s't'}$ if $(s', t') = (i, s)$ or (j, r) , and variable entries otherwise. Then $T(v')$ is a C' -realization and $V_{C'}(T(v'))$ is a module for arbitrary generic values of free variables in v' . Let $z^{(1)}, z^{(2)} \in \{\delta^{jr}, \delta^{is}\}$. Then $\Phi(R, z^{(1)}, z^{(2)}) = \Phi(v, z^{(1)}, z^{(2)})$ where $z^{(1)} = z^{(2)}$ only if $z^{(1)} = z^{(2)} = \delta^{is}$. Therefore the coefficient of $T(R + \delta^{jr} + 2\delta^{is})$ is the limit of the coefficient of $T(v + \delta^{jr} + 2\delta^{is})$ when $v \rightarrow R$, hence, it is zero.

(ii) Suppose $s \neq t$. Then there is no relation between (i, s) and (i, t) .

- (a) Suppose there is no relation between (j, r) and (i, s) or between (j, r) and (i, t) . Then the value of the function Φ that appears along with $T(R + \delta^{jr} + \delta^{is} + \delta^{it})$ is 1 by Lemma 3.6. Thus the coefficient of $T(R + \delta^{jr} + \delta^{is} + \delta^{it})$ is zero similarly to (a) in (i).
- (b) Suppose there is a relation between (j, r) and one of $\{(i, s), (i, t)\}$. Similarly to (b) in (i), one has that the coefficient of $T(R + \delta^{jr} + \delta^{is} + \delta^{it})$ is zero.
- (c) Suppose there exist relations between (j, r) and both $\{(i, s), (i, t)\}$. In this case $(j, r), (i, s), (i, t)$ are in the same indecomposable set. If $r_{is} - r_{it} = 1$ then $r_{jr} = r_{it}$ and there exists r' such that $\{((i, s); (i - 1, r')), ((i - 1, r'); (i, t))\} \subseteq C$ and $r_{i-1,r'} = r_{is}$. It contradicts with $T(R + \delta^{jr} + \delta^{is} + \delta^{it})$ nonzero. Therefore $r_{is} - r_{it} \in \frac{1(q)}{2} + \mathbb{Z}_{>0}$. Then $r_{is} - r_{jr} \in \frac{1(q)}{2} + \mathbb{Z}_{>0}$ or $r_{jr} - r_{it} \in \frac{1(q)}{2} + \mathbb{Z}_{>0}$. Without loss of generality we assume that $r_{jr} - r_{it} \in \frac{1(q)}{2} + \mathbb{Z}_{>0}$. Let $C' = \{(i, s) \geq (j, r)\}$ and $T(v')$ the tableau with $v'_{s't'} = l_{s't'}$ if $(s', t') = (i, s)$ or (j, r) and variable entries otherwise. Then $T(v')$ is a C' -realization and $V_{C'}(T(v'))$ is a module. Let $z^{(1)}, z^{(2)} \in \{\delta^{jr}, \delta^{is}, \delta^{it}\}$. One has that $\Phi(R, z^{(1)}, z^{(2)}) = \Phi(v, z^{(1)}, z^{(2)})$ whenever $z^{(1)} \neq z^{(2)}$. Therefore

the coefficient of $T(R + \delta^{jr} + 2\delta^{is})$ is the limit of the coefficient of $T(v + \delta^{jr} + 2\delta^{is})$ when $v \rightarrow R$, which is zero.

In the following we show that $(e_i f_j - f_j e_i)T(R) = \delta_{ij} \frac{q^{\alpha_i} - q^{-\alpha_i}}{q - q^{-1}} T(R)$. We have

$$\begin{aligned}
 (e_i f_j - f_j e_i)T(R) &= \sum_{r=1}^j \sum_{s=1}^i \Phi(R, -\delta^{jr}) f_{jr}(R) e_{is}(R + \delta^{jr}) T(R - \delta^{jr} + \delta^{is}) \\
 &\quad - \sum_{r=1}^j \sum_{s=1}^i \Phi(R, \delta^{is}) e_{is}(R) f_{jr}(R + \delta^{is}) T(R - \delta^{jr} + \delta^{is}).
 \end{aligned}
 \tag{19}$$

Now we consider the coefficients of nonzero tableaux $T(L - \delta^{jr} + \delta^{is})$. If $(i, r) \neq (j, s)$ then the coefficient of $T(L - \delta^{jr} + \delta^{is})$ is zero similarly to the above case and, hence, $[e_i, f_j]T(R) = 0$ if $i \neq j$.

Suppose $i = j = k$. The coefficient of $T(R - \delta^{ir} + \delta^{is})$ is zero if $r \neq s$.

By Corollary 3.5, the coefficient of $T(R)$ is

$$\begin{aligned}
 &\lim_{v \rightarrow l} \left(\sum_{r=1}^k \sum_{s=1}^k f_{kr}(v) e_{ks}(v + \delta^{kt}) - \sum_{r=1}^k \sum_{s=1}^k e_{ks}(v) f_{kr}(v + \delta^{ks}) \right) \\
 &= \lim_{v \rightarrow R} h_k(v) = h_k(R).
 \end{aligned}$$

Hence $(e_i f_j - f_j e_i)T(R) = \delta_{ij} \frac{q^{\alpha_i} - q^{-\alpha_i}}{q - q^{-1}} T(R)$.

All other relations can be verified similarly. Thus $V_{\mathcal{C}}(T(L))$ is a U_q -module. \square

Remark 3.8 The realizable sets of relations for \mathfrak{gl}_n were all obtained in [11] (Theorem 4.33). Here we only deform the definition of a tableau satisfying a set of relations, i.e. replace \mathbb{Z} by $\mathbb{Z} + \frac{1(q)}{2}$. As in the non quantum case, if we only consider irreducible modules, all realizable sets of relations are admissible. Thus the converse of Theorem 3.7 holds.

An effective method of constructing of realizable sets of relations was introduced in [11], called *relations removal method (RR-method)* for short). We will show that the same method can be applied to construct admissible sets of relations in the quantum case.

Definition 3.9 Let \mathcal{C} be any set of relations. We call $(k, i) \in \mathfrak{A}(\mathcal{C})$ maximal if there exist no $(s, t) \in \mathfrak{A}(\mathcal{C})$ such that $(s, t) \succeq_{\mathcal{C}} (k, i)$. The minimal pair can be defined similarly.

Definition 3.10 (RR-method) Let $(i, j) \in \mathfrak{A}(\mathcal{C})$ be a maximal or a minimal pair. Denote by \mathcal{C}_{ij} the set of relations obtained from \mathcal{C} by removing all relations that involve (i, j) . We say that $\tilde{\mathcal{C}} \subsetneq \mathcal{C}$ is obtained from \mathcal{C} by the RR-method if it is obtained by a sequence removing of relations of the form $\mathcal{C}' \rightarrow \mathcal{C}'_{ij}$ for different indexes.

Theorem 3.11 *Let \mathcal{C}_1 be any realizable subset of \mathcal{R} . If \mathcal{C}_2 is obtained from \mathcal{C}_1 by the RR-method then \mathcal{C}_2 is realizable.*

Proof Analogous to the proof of Theorem 4.9 in [11]. □

We immediately obtain the following statement for generic modules which was shown in [12, Theorem 5.2] (cf. [29, Theorem 2]).

Corollary 3.12 *Let $T(L)$ be a generic Gelfand–Tsetlin tableau of height n . Then $V_{\emptyset}(T(L))$ has a structure of a U_q -module with the action of the generators of U_q given by the Gelfand–Tsetlin formulas (9).*

Proof By Theorem 2.2 the set \mathcal{S} is realizable and applying the RR-method to \mathcal{S} , after finitely many steps we can remove all the relations in \mathcal{S} , then \emptyset is realizable by Theorem 3.11. □

We call $V_{\mathcal{C}}(T(L))$ *admissible Gelfand–Tsetlin module* associated with the admissible set of relations \mathcal{C} . Note that $V_{\mathcal{C}}(T(L))$ is infinite dimensional if \mathcal{C} does not imply \mathcal{S} .

4 Action of Gelfand–Tsetlin subalgebra

From now on we will assume that \mathcal{C} is an admissible subset of \mathcal{R} and consider the U_q -module $V_{\mathcal{C}}(T(L))$. We will analyze the action of the Gelfand–Tsetlin subalgebra Γ_q on modules $V_{\mathcal{C}}(T(L))$.

The action of U_q on this irreducible module is given by the Gelfand–Tsetlin formulas (9). Moreover, the action of Γ_q is given by (10). Then as in the non quantum case, we have the following theorem:

Theorem 4.1 *For any admissible \mathcal{C} the module $V_{\mathcal{C}}(T(L))$ is a Gelfand–Tsetlin module with diagonalizable action of the generators of the Gelfand–Tsetlin subalgebra given by the formula (10).*

Proof Essentially repeats the proof of Theorem 5.3 in [11]. □

Remark 4.2 Note that for any $x \in 1(q)$ we have $\gamma_{mk}(L) = \gamma_{mk}(L + x\delta^{ij})$ for any m, k and any $1 \leq j \leq i \leq n$. In particular, the tableaux $T(L)$ and $T(L + x\delta^{ij})$ define the same Gelfand–Tsetlin character.

Remark 4.3 Let \mathcal{C} be any realizable set of relations and $T(L)$ satisfying \mathcal{C} . Then for any $x \in 1(q)$ the tableau $T(L + x\delta^{ij})$ is also a \mathcal{C} -realization and the Gelfand–Tsetlin modules $V_{\mathcal{C}}(T(L + x\delta^{ij}))$ and $V_{\mathcal{C}}(T(L))$ are isomorphic (see Remark 4.2). On the other hand, when $\frac{x}{2} \notin 1(q)$, the two modules are not isomorphic. In general, every admissible set of relations defines infinitely many nonisomorphic modules.

Now we can show that Γ_q separates basis tableaux in all constructed modules $V_{\mathcal{C}}(T(L))$ and hence, in their irreducible quotients.

Proposition 4.4 For any $m \in \text{Specm } \Gamma_q$ from the Gelfand–Tsetlin support of $V_{\mathcal{C}}(T(L))$, the Gelfand–Tsetlin multiplicity of m is one.

Proof The action of Γ_q is given by the formulas (10), and hence determined by the values of symmetric polynomials on the entries of the rows of the tableaux. Given two Gelfand–Tsetlin tableaux $T(R_1)$ and $T(R_2)$ in $\mathcal{B}_{\mathcal{C}}(T(L))$, we have $c_{rs}(T(R_1)) = c_{rs}(T(R_2))$ for any $1 \leq s \leq r \leq n$ if and only if $R_1 = \sigma(R_2)$ for some $\sigma \in G$. But if $\sigma \neq 1$ then $T(R_1)$ and $T(R_2)$ have different order among the entries in r th row. Since both $T(R_1)$ and $T(R_2)$ satisfy \mathcal{C} and $T(R_1) = T(R_2 + z)$ for some $z \in \mathbb{Z}^{\frac{n(n-1)}{2}}$ we come to a contradiction. \square

Therefore, we have an explicit basis of $V_{\mathcal{C}}(T(L))$, and of its irreducible quotients, parametrized by different Gelfand–Tsetlin tableaux with an explicit basis of the generators of U_q and of Γ_q .

It was proved in [12] that the irreducible module containing generic tableau $T(R)$ has a basis of tableaux

$$\mathcal{I}(T(R)) = \{T(S) \in \mathcal{B}(T(R)) : \Omega^+(T(S)) = \Omega^+(T(R))\},$$

where

$$\Omega^+(T(S)) := \left\{ (i, j, k) \mid s_{i,j} - s_{i-1,k} \in \frac{1(q)}{2} + \mathbb{Z}_{\geq 0} \right\}.$$

The following statement is a generalization of this result. Recall that \mathcal{C} is a maximal set of relations satisfied by $T(L)$ if $T(L)$ is a \mathcal{C} -realization and for any admissible set of relations \mathcal{C}' satisfied by $T(L)$, \mathcal{C} implies \mathcal{C}' .

Theorem 4.5 Admissible Gelfand–Tsetlin module $V_{\mathcal{C}}(T(L))$ is irreducible if and only if \mathcal{C} is a maximal set of relations satisfied by $T(L)$.

Proof Let $T(R)$ be any tableau in $\mathcal{B}_{\mathcal{C}}(T(L))$ and \mathcal{C} a maximal set of relations satisfied by $T(L)$. One can show easily that $U_q T(R) \subseteq V_{\mathcal{C}}(T(L))$. If $T(R')$ is another tableau in $V_{\mathcal{C}}(T(L))$ then there exist $\{(i_s, j_s)\} \subseteq \{(i, j) \mid 1 \leq j \leq i \leq n-1\}$ and t such that, for any $k \leq t$, $T(R_k) = T(R + \sum_{s=1}^k p_s \delta^{i_s, j_s}) \in \mathcal{B}_{\mathcal{C}}(T(L))$, where $p_i \in \{1, -1\}$ and $T(R_0) = T(R)$, $T(R_t) = T(R')$. It is sufficient to show that we can obtain $T(R_s)$ from $T(R_{s-1})$. If $p_i = 1$ (resp. $p_i = -1$) then acting by e_{i_s} (resp. f_{i_s}) on $T(R_{s-1})$ the coefficient of $T(R_s)$ in the image is not zero. By Theorem 4.1, there exists an element in Γ_q which annihilates all other tableaux except $T(R_s)$. We conclude that $V_{\mathcal{C}}(T(L)) \subseteq U_q T(R)$.

Conversely, assume $T(L)$ satisfies \mathcal{C} and \mathcal{C} is not maximal. Let \mathcal{C}' be the maximal set of relations satisfied by $T(L)$. Then $V_{\mathcal{C}'}(T(L))$ is a subquotient of $V_{\mathcal{C}}(T(L))$ and $V_{\mathcal{C}'}(T(L)) \neq V_{\mathcal{C}}(T(L))$. It contradicts the irreducibility of $V_{\mathcal{C}}(T(L))$. \square

We conclude with an example of the family of highest weight modules that can be realized as $V_{\mathcal{C}}(T(L))$ for some admissible set of relations \mathcal{C} .

Proposition 4.6 Set $\lambda = (\lambda_1, \dots, \lambda_n)$. The irreducible highest weight module $L(\lambda)$ is admissible Gelfand–Tsetlin module if $\lambda_i - \lambda_j \notin \mathbb{Z}$ or $\lambda_i - \lambda_j > i - j$ for any $1 \leq i < j \leq n - 1$.

Proof Let $T(L)$ be a tableau such that $l_{ij} = \lambda_j - j$, \mathcal{C} be the maximal set of relations satisfied by $T(L)$. Then \mathcal{C} is admissible and $V_{\mathcal{C}}(T(L))$ is irreducible admissible module. Moreover, $T(L)$ is a highest weight vector and $V_{\mathcal{C}}(T(L))$ is isomorphic to $L(\lambda)$. \square

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Compliance with ethical standards

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest

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