São Paulo Journal of Mathematical Sciences (2018) 12:246–251
https://doi.org/10.1007/s40863-018-0095-y

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A geometric proof of the Poincaré-Birkhoff-Witt Theorem

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Published online: 1 August 2018

Abstract
We use that the n-sphere for $n \geq 2$ is simply-connected to prove the Poincaré-Birkhoff-Witt Theorem.

Keywords Poincaré-Birkhoff-Witt · Universal enveloping algebra · Symmetric group

Mathematics Subject Classification 16S30

There are several equivalent statements of the Poincaré-Birkhoff-Witt Theorem. The version we shall prove is as follows.

Theorem *Let* $\mathfrak{g}$ *be a Lie algebra and define an equivalence relation on the tensor algebra* $\bigotimes \mathfrak{g}$ *by imposing the relations that*

$$a \otimes b - b \otimes a = [a, b] \qquad (\star\star\star)$$

as a two-sided ideal in $\bigotimes \mathfrak{g}$*. Write the resulting associative algebra as* $\mathfrak{U}(\mathfrak{g})$ *and write* $ab \cdots d$ *for the equivalence class of* $a \otimes b \otimes \cdots \otimes d$*. Pick a basis for* $\mathfrak{g}$ *and declare that an element* $ab \cdots d \in \mathfrak{U}(\mathfrak{g})$ *is in 'canonical form' if and only if* $a, b, \ldots, d$ *are basis elements with* $a \leq b \leq \cdots \leq d$ *with respect to the ordering of the basis. Then elements in* $\mathfrak{U}(\mathfrak{g})$ *may be consistently and uniquely written as linear combinations of elements in canonical form.*

An algebraic proof may be found, for example, in [3]. The rest of this article is devoted to a geometric proof.

Dedicated to Joe Wolf on the occasion of his 80th birthday.

This work was also supported by the Simons Foundation grant 346300 and the Polish Government MNiSW 2015–2019 matching fund. It was written whilst the author was at the Banach Centre at IMPAN in Warsaw for the Simons Semester 'Symmetry and Geometric Structures.' This proof was presented in the author's lectures on 'Invariant Differential Operators' at the start of this Simons Semester.

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To understand what the Poincaré-Birkhoff-Witt Theorem says, let us consider the case of three elements $a, b, c \in \mathfrak{g}$, which we suppose are basis elements in this order $a \leq b \leq c$, and that we would like to rewrite the element $cba \in \mathfrak{U}(\mathfrak{g})$ (given in the 'wrong' order) as a linear combination of canonically ordered elements. Certainly, we can use the equivalence relation ($\star\star\star$) to try to reorder this element:

$$\begin{aligned} cba &= cab - c[a, b] \\ &= acb - [a, c]b - c[a, b] \\ &= abc - a[b, c] - [a, c]b - c[a, b], \end{aligned}$$

where we have firstly swopped b and a (and then followed our noses). The only problem is that one can firstly swop c and b instead:

$$\begin{aligned} cba &= bca - [b, c]a \\ &= bac - b[a, c] - [b, c]a \\ &= abc - [a, b]c - b[a, c] - [b, c]a, \end{aligned}$$

which is consistent if and only if the 'second order' remainder terms agree:

$$a[b, c] + [a, c]b + c[a, b] = [a, b]c + b[a, c] + [b, c]a.$$

Fortunately, this is exactly the Jacobi identity:

$$[a, [b, c]] + [[a, c], b] + [c, [a, b]] = 0.$$

We may arrange these calculations on a circle:

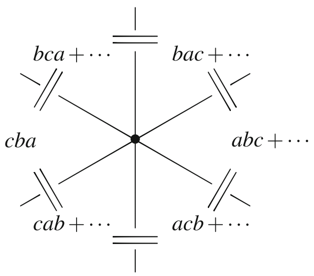


Figure 1

where $\cdots$ denotes second order terms. Otherwise said, the Jacobi identity is exactly what is needed so that an excursion through the symmetric group $\mathfrak{S}_3$ on three letters

$$abc \rightsquigarrow bac \rightsquigarrow bca \rightsquigarrow cba \rightsquigarrow cab \rightsquigarrow acb \rightsquigarrow abc$$

is consistent in $\mathfrak{U}(\mathfrak{g})$. One can think of this as saying that there is no 'holonomy' around the circle depicted in Figure 1.

If we attempt a similar proof for four basis element $a \le b \le c \le d$, then we run into trouble because there is no 'follow your nose' method for reordering elements of the symmetric group $\mathfrak{S}_4$. Instead, we may picture $\mathfrak{S}_4$ as 24 countries in the plane arranged like this:

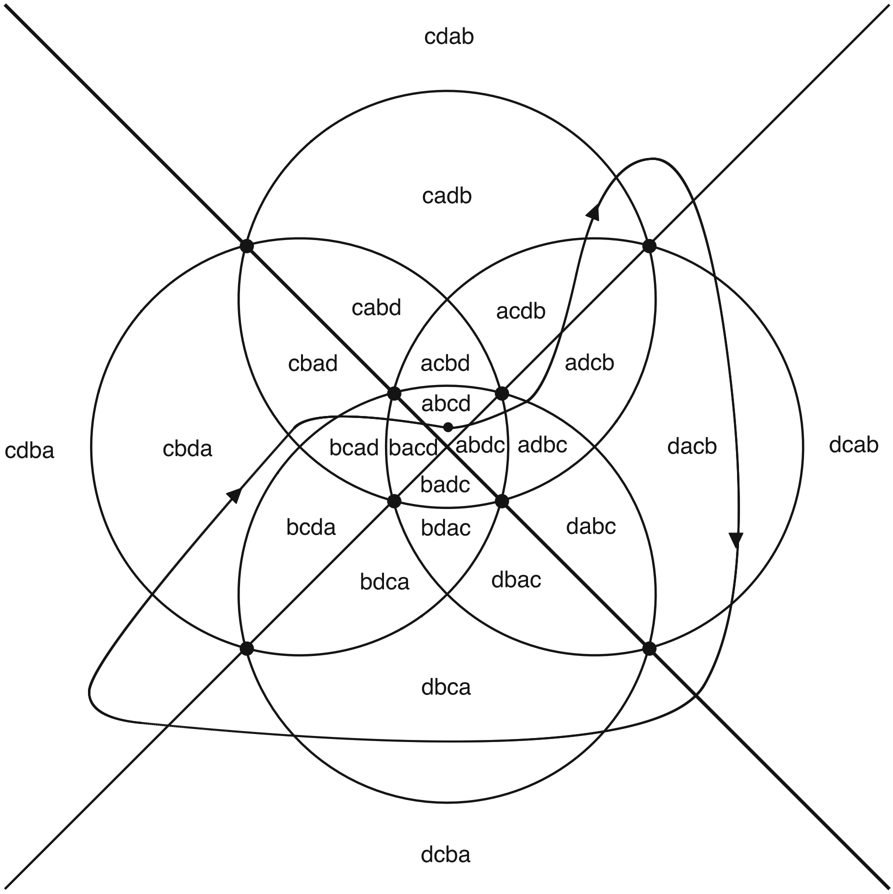


Figure 2

Also depicted is a typical excursion through $\mathfrak{S}_4$ starting and finishing at *abcd*, namely

$$\begin{array}{ccccccccccccccc}
abcd & \rightsquigarrow & abdc & \rightsquigarrow & adbc & \rightsquigarrow & adcb & \rightsquigarrow & acdb & \rightsquigarrow & cadb & \rightsquigarrow & cdab & \rightsquigarrow & dcab \\
\uparrow & & & & & & & & & & & & & & \downarrow \\
bacd & & & & & & & & & & & & & & dacb \\
\uparrow & & & & & & & & & & & & & & \downarrow \\
bcad & \leftsquigarrow & cbad & \leftsquigarrow & cbda & \leftsquigarrow & cdba & \leftsquigarrow & dcba & \leftsquigarrow & dbca & \leftsquigarrow & dcba & \leftsquigarrow & dcab
\end{array}$$

We would like to see that this excursion is consistent. There are just 8 points in the plane where 6 countries come together. For example:

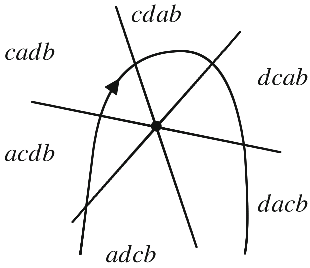
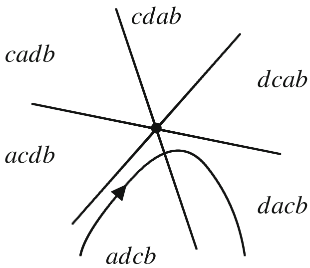


Figure 3

These are the eight points where countries of the form

$$a{*}{*}{*} \text{ or } b{*}{*}{*} \text{ or } c{*}{*}{*} \text{ or } d{*}{*}{*} \text{ or } {*}{*}{*}a \text{ or } {*}{*}{*}b \text{ or } {*}{*}{*}c \text{ or } {*}{*}{*}d$$

meet at a vertex and these are marked by • in Figure 2. The picture above is of the vertex $***b$ and one recognises the circle from Figure 1 save that the elements a, b, c have been relabelled d, c, a. We saw earlier that this circle corresponds to a consistent identity for three elements in $\mathfrak{U}(\mathfrak{g})$ and now we obtain a consistent identity for four elements in which b simply goes along for the ride. Geometrically, it means we may replace the path in Figure 3 by



to obtain an alternative but simpler excursion through $\mathfrak{S}_4$, which is consistent if and only if the original excursion is consistent. If we can similarly pull paths through the other 5 vertices where just four countries come together, then we can reduce any excursion through $\mathfrak{S}_4$ to the trivial excursion (by a series of 'simple jerks' in the terminology of [1]) and our proof is complete. A typical example is

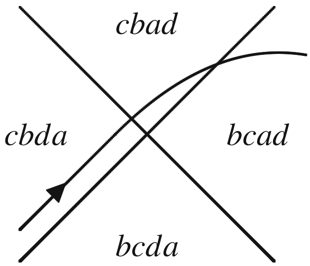


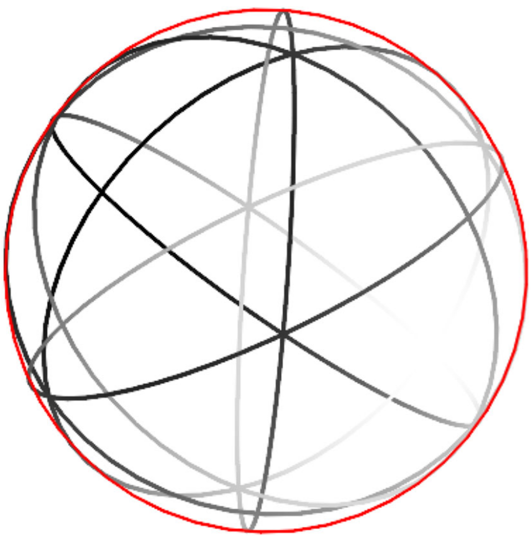
Figure 4

but vertices like this evidently have consistent holonomy

$$\begin{array}{ccc} & cbad - cb[a,d] & \\ /\!\!/ & & \backslash\!\!\backslash \\ cbda & & bcad - [b,c]ad - bc[a,d] + [b,c][a,d] \\ \backslash\!\!\backslash & & /\!\!/ \\ & bcda - [b,c]da & \end{array}$$

without using the Jacobi identity. It is because we are transposing the first two and the last two of four letters, and such transpositions commute in $\mathfrak{S}_4$.

So now, we may consistently reorder any four elements in $\mathfrak{U}(\mathfrak{g})$ and we ask about five elements and so on. We need a similar picture of the symmetric groups $\mathfrak{S}_N$ for all $N \geq 4$. To obtain such a picture, we now admit that Figure 2 was obtained from a tessellation of the 2-sphere by 24 geodesic triangles with angles $(\pi/2, \pi/3, \pi/3)$. Specifically, it was obtained by stereographic projection so that great circles on the sphere are mapped to circles or straight lines on the plane whilst angles are preserved.



Therefore, a better viewpoint on Figure 2 is as a triangulation of the 2-sphere. From this point of view there is one more ‘easy vertex,’ as in Figure 4, out at infinity. The fact that one can contract any excursion in $\mathfrak{S}_4$ to the trivial excursion is due to there being no obstructions

- at the 6 ‘easy vertices’ (commuting transpositions),
- at the 8 ‘tricky vertices’ (from the $\mathfrak{S}_3$ case),

and the fact that the 2-sphere is simply-connected. This triangulation of the 2-sphere is well-known in a different guise. It is obtained by letting the Weyl group of the A_3 root system act on $\mathbb{R}^3$, as described, for example, in [2]. The triangulation is obtained by intersecting the 24 Weyl chambers with the unit sphere in $\mathbb{R}^3$. Since the Weyl group of A_3 may be identified with $\mathfrak{S}_4$, one can pick a triangle to be called the ‘fundamental triangle’ and use the Weyl group action to identify any element of $\mathfrak{S}_4$ with the triangle obtained as the corresponding image of the fundamental triangle. This is how Figure 2 was obtained.

It is evident how to extend this to $\mathfrak{S}_N$ for all $N \geq 4$ and, for the general pattern, it suffices to make sure that $\mathfrak{S}_5$ behaves as it should. The corresponding tessellation of the unit 3-sphere is by 120 tetrahedra having $(\pi/2, \pi/2, \pi/2, \pi/3, \pi/3, \pi/3)$ as dihedral angles (each dihedral angle corresponds to a pair of vertices from the Dynkin diagram •—•—•—•, which are either adjacent (angle $\pi/3$) or not (angle $\pi/2$)). To use the simple connectivity of the 3-sphere it now suffices to be able to move a path on the 3-sphere through any edge of this tessellation.

As on the 2-sphere, there are two cases. Firstly, there are the ‘easy edges,’ where just 4 tetrahedra meet at right angles. On the Dynkin diagram, edges of this type correspond to striking out all but two 2 non-adjacent nodes

×—•—×—• •—×—×—• •—×—•—×,

in effect leaving the Weyl group of $A_1 \times A_1$ as in Figure 4. It is just the Abelian group $\mathbb{Z}_2 \times \mathbb{Z}_2$. The ‘tricky edges’ are when 6 tetrahedron meet at angle $\pi/3$. Tricky edges may be recorded on the Dynkin diagram by striking out all but two adjacent nodes

×—×—•—• $\longleftrightarrow$ permuting $ab{*}{*}{*}$ with a, b held fixed.
×—•—•—× $\longleftrightarrow$ permuting $a{*}{*}{*}b$ with a, b held fixed.
•—•—×—× $\longleftrightarrow$ permuting ${*}{*}{*}ab$ with a, b held fixed.

The tricky edges are not obstructed since the previous reasoning using the Jacobi identity applies (notice that we are left with $A_2 =$ •—• and the Weyl group of A_2 is $\mathfrak{S}_3$). Looking back, we see that Figure 1 is the root diagram for A_2.

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