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Scalar curvature estimates of constant mean curvature hypersurfaces in locally symmetric spaces

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Abstract
We derive sharp estimates for the infimum and supremum of the scalar curvature of a hypersurface immersed with constant mean curvature in a locally symmetric space obeying standard curvature constraints. Our approach is based on the well known Omori–Yau maximum principle and on the weak version of it.

Keywords Locally symmetric spaces · Constant mean curvature hypersurfaces · Scalar curvature estimates

Mathematics Subject Classification Primary 53C42 · Secondary 53B30 · 53C50

1 Introduction

The problem of characterizing hypersurfaces immersed with constant mean curvature in a Riemannian space form $\overline{M}_c^{n+1}$ of constant sectional curvature c constitutes an important and fruitful topic in the theory of isometric immersions, which has being widely approached by many authors. For instance, in a classical paper, Klotz and

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Osserman [7] characterized totally umbilical spheres and circular cylinders as the only complete surfaces immersed into the Euclidean 3-space $\mathbb{R}^3$ with constant mean curvature $H \neq 0$ and whose Gaussian curvature does not change sign. Later on, Hoffman [6] and Tribuzy [12] gave an extension of this result to the case of surfaces with constant mean curvature in the Euclidean 3-sphere $\mathbb{S}^3$ and in the hyperbolic space $\mathbb{H}^3$, respectively.

More recently, Alías and García-Martínez [2] used the weak Omori–Yau maximum principle due to Pigola et al. [10,11] to study the behavior of the scalar curvature R of a complete hypersurface immersed with constant mean curvature in $\overline{M}_c^{n+1}$, with $n \geq 3$, deriving a sharp estimate for the infimum of R. Afterwards, these same authors [3] used the Omori–Yau maximum principle [9,13] to obtain a sharp estimate for the supremum of R of a constant mean curvature hypersurface with two distinct principal curvature immersed in $\overline{M}_c^{n+1}$, with $n \geq 3$.

An interesting generalization of constant curvature spaces are the so-called locally symmetric spaces. We recall that a Riemannian manifold is said *locally symmetric* when all the covariant derivative components of its curvature tensor vanish identically. So, it is a natural question to revisit in this ambient space the known results of constant curvature spaces. In this direction, here we deal with hypersurfaces with constant mean curvature immersed into a locally symmetric Riemannian manifold obeying standard curvature constraints. Our purpose is just to extend the techniques developed in [2,3] in order to obtain sharp estimates for the infimum and supremum of the scalar curvature of such a hypersurface, treating even the case $n = 2$.

This manuscript is organized in the following way: in Sect. 2 we introduce some basic facts and notations that will appear along the paper and we recall a suitable Simons type formula which will be essential to prove our main results. Next, in Sect. 3 we establish our curvatures constraints related to a hypersurface immersed in a locally symmetric space and we quote some auxiliary results. Afterwards, in Sect. 4 we apply the weak Omori–Yau maximum principle to obtain a sharp estimate for the infimum of the scalar curvature of a stochastically complete hypersurface with constant mean curvature, proving that it must be either totally umbilical or isometric to an isoparametric hypersurface having two distinct principal curvatures, one of them being simple (see Theorems 1 and 2). Finally, in Sect. 5 we use the (strong) Omori–Yau maximum principle to obtain a sharp estimate to the supremum of the scalar curvature (see Theorems 3 and 4, for $n \geq 3$, and Theorems 5 and 6, for $n = 2$).

2 Preliminaries

In this work, we will deal with n-dimensional, orientable and connected hypersurface $\psi : \Sigma^n \to \overline{M}^{n+1}$ immersed into a $(n+1)$-dimensional Riemannian manifold $\overline{M}^{n+1}$. We choose a local field of orthonormal frame $\{e_1, \ldots, e_{n+1}\}$ in $\overline{M}^{n+1}$ with dual coframe $\{\omega_1, \ldots, \omega_{n+1}\}$ such that, at each point of Σ^n, $e_1, \ldots, e_n$ are tangent to Σ^n and e_{n+1} is normal to Σ^n. We will use the following convention for the indices:

$$1 \leq A, B, C, \ldots \leq n+1 \quad \text{and} \quad 1 \leq i, j, k, \ldots \leq n.$$

In this setting, $\overline{R}_{ABCD}$, $\overline{R}_{AC}$ and $\overline{R}$ denote respectively the Riemannian curvature tensor, the Ricci tensor and the scalar curvature of the Riemannian manifold $\overline{M}^{n+1}$. So, we have

$$\overline{R}_{AC}=\sum_B \overline{R}_{ABCB} \quad \text{and} \quad \overline{R}=\sum_A \overline{R}_{AA}.$$

Now, restricting all the tensor to Σ^n, we see that $\omega_{n+1}=0$ on Σ^n. Hence, $0=d\omega_{n+1}=-\sum_i \omega_{n+1i}\wedge\omega_i$ and as it well known we get

$$\omega_{n+1i}=\sum_j h_{ij}\omega_j, \quad h_{ij}=h_{ji}.$$

This gives the second fundamental form of Σ^n, $B=\sum_{i,j} h_{ij}\omega_i\omega_j e_{n+1}$ and its squared length $S=|B|^2=\sum_{i,j} h_{ij}^2$. Furthermore, the mean curvature H of Σ^n is defined by $H=\frac{1}{n}\sum_i h_{ii}$.

A fact well known is that the Riemannian curvature tensor of the hypersurface Σ^n can be described in terms of the second fundamental form and of the curvature tensor of the ambient space $\overline{M}^{n+1}$ by the Gauss equation given by

$$R_{ijkl}=\overline{R}_{ijkl}+(h_{ik}h_{jl}-h_{il}h_{jk}),$$

where R_{ijkl} are the components of the Riemannian curvature tensor of Σ^n. In particular, it follows from Gauss equation that the Ricci curvature and the scalar curvature of Σ^n are given, respectively, by

$$R_{ij}=\sum_k \overline{R}_{ikjk}+nHh_{ij}-\sum_k h_{ik}h_{kj} \tag{2.1}$$

and

$$R=\sum_i R_{ii}. \tag{2.2}$$

From (2.1) and (2.2) we obtain

$$R=\sum_{i,j}\overline{R}_{ijij}+n^2H^2-S. \tag{2.3}$$

We also remember that the Codazzi equation of the hypersurface Σ^n is given by

$$h_{ijk}-h_{ikj}=-\overline{R}_{n+1ijk}, \tag{2.4}$$

where h_{ijk} denote the first covariant derivatives of h_{ij} and satisfies

$$\sum h_{ijk}\omega_k=dh_{ij}+\sum h_{kj}\omega_{ki}+\sum h_{ik}\omega_{kj}. \tag{2.5}$$

Observe that

$$|\nabla B|^2 = \sum_{i,j,k} h_{ijk}^2, \tag{2.6}$$

Taking a local orthonormal frame $\{e_1, \ldots, e_n\}$ on Σ^n such that $h_{ij} = \lambda_i \delta_{ij}$, from equation (2.8) of [4] (see also equation (2.10) of [5]) we have the following Simons type formula:

$$\begin{aligned}\frac{1}{2}\Delta S = &|\nabla B|^2 + \sum_i \lambda_i (nH)_{ii} + nH \sum_i \lambda_i^3 - S^2 \\ &- \sum_{i,j,k} h_{ij}(\overline{R}_{(n+1)ijk;k} + \overline{R}_{(n+1)kik;j}) + \sum_i \overline{R}_{(n+1)i(n+1)i}\,(nH\lambda_i - S) \\ &+ \sum_{i,j} (\lambda_i - \lambda_j)^2 \overline{R}_{ijij}.\end{aligned} \tag{2.7}$$

3 Locally symmetric spaces and auxiliary results

Proceeding within the context of the previous section, we will assume that there exist constants c_1 and c_2 such that the sectional curvature $\overline{K}$ of the ambient space $\overline{M}^{n+1}$ satisfies the following two constraints

$$\overline{K}(\eta, v) = \frac{c_1}{n}, \tag{3.1}$$

for vectors $\eta \in T^{\perp}\Sigma$ and $v \in T\Sigma$, and either

$$\overline{K}(u, v) \geq c_2, \tag{3.2}$$

or

$$\overline{K}(u, v) \leq c_2, \tag{3.3}$$

for vectors $u, v \in T\Sigma$.

From now on, we consider $\overline{M}^{n+1}$ a locally symmetric Riemannian manifold. Recall that a Riemannian manifold is said *locally symmetric* when all the covariant derivative components $\overline{R}_{ABCD;E}$ of its curvature tensor vanish identically.

Remark 1 Obviously, when the ambient manifold $\overline{M}^{n+1}$ has constant sectional curvature $\overline{c}$, then it is locally symmetric and the curvature conditions (3.1), (3.2) and (3.3) are satisfied for every hypersurface Σ^n immersed in $\overline{M}^{n+1}$, with $c_1/n = c_2 = \overline{c}$. Therefore, in some sense our assumptions are a natural generalization of the case where the ambient space has constant sectional curvature. Moreover, when the ambient manifold is a Riemannian product of two Riemannian manifolds of constant sectional curvature, say $\overline{M} = M_1(\kappa_1) \times M_2(\kappa_2)$, then $\overline{M}$ is locally symmetric and, if $\kappa_1 = 0$ and $\kappa_2 \geq 0$ (resp. $k_2 \leq 0$), then every hypersurface of type $\Sigma = \Sigma_1 \times M_2(\kappa_2)$, where Σ_1 is an orientable and connected hypersurface immersed in $M_1(\kappa_1)$, satisfies the curvature constraints (3.1) and (3.2) (resp. (3.3)) with $c_1 = c_2 = 0$ (for more details, see Remark 3.1 of [4]). Moreover, it is not difficult to see that the equality $\overline{R}_{n+1ijk} = 0$ holds on Σ. Then by Codazzi equation we get that the second fundamental form B of hypersurface Σ must be a Codazzi tensor, that is, the covariant differential ∇B is symmetric in all indices. In particular, this justifies the study of geometry of hypersurfaces such that its second fundamental form is a Codazzi tensor.

Denoting by $\overline{R}_{AB}$ the components of the Ricci tensor of a locally symmetric Riemannian manifold $\overline{M}^{n+1}$ satisfying condition (3.1), the scalar curvature $\overline{R}$ of $\overline{M}^{n+1}$ is given by

$$\overline{R} = \sum_{A=1}^{n+1} \overline{R}_{AA} = \sum_{i,j=1}^{n} \overline{R}_{ijij} + 2\sum_{i=1}^{n} \overline{R}_{(n+1)i(n+1)i} = \sum_{i,j=1}^{n} \overline{R}_{ijij} + 2c_1. \tag{3.4}$$

Moreover, it is well known that the scalar curvature of a locally symmetric Riemannian manifold is constant. Thus, $\sum_{i,j} \overline{R}_{ijij}$ is a constant naturally attached to a locally symmetric Riemannian manifold satisfying condition (3.1). So, for sake of simplicity, in the course of this work we will denote the constant $\frac{1}{n(n-1)} \sum_{i,j} \overline{R}_{ijij}$ by $\overline{\mathcal{R}}$ and $c := 2c_2 - \dfrac{c_1}{n}$.

Given $\Phi_{ij} = h_{ij} - H\delta_{ij}$, we will also consider the following symmetric tensor

$$\Phi = \sum_{i,j} \Phi_{ij}\omega_i \otimes \omega_j.$$

Let $|\Phi|^2 = \sum_{i,j} \Phi_{ij}^2$ be the square of the length of Φ. It is not difficult to check that Φ is traceless and $|\Phi|^2 = S - nH^2 \geq 0$, with equality if and only if Σ^n is totally umbilical. For that reason, Φ is called the total umbilicity tensor of Σ^n. Moreover, from (2.3) we get

$$|\Phi|^2 = n(n-1)\left(H^2 + \overline{\mathcal{R}}\right) - R. \tag{3.5}$$

In order to establish our characterization results, we recall two classic algebraic lemmas. The first one is the well known Okumura Lemma due to Okumura in [8], and completed with the equality case proved by Alencar and do Carmo in [1].

Lemma 1 *Let $\kappa_1, \ldots, \kappa_n$, $n \geq 3$, be real numbers such that $\sum_i \kappa_i = 0$ and $\sum_i \kappa_i^2 = \beta^2$, where $\beta \geq 0$. Then*

$$-\frac{(n-2)}{\sqrt{n(n-1)}}\beta^3 \leq \sum_i \kappa_i^3 \leq \frac{(n-2)}{\sqrt{n(n-1)}}\beta^3,$$

and equality holds if, and only if, either at least $n-1$ of the numbers κ_i are equal.

The next auxiliary result will give a suitable formula for squared length of the covariant differential of a symmetric tensor with two distinct eigenvalues (for more details, see Lemma 10 in [3])

Lemma 2 *Let Σ^n be an n-dimensional Riemannian manifold, $n \geq 3$, and let $T : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ be a Codazzi tensor on Σ^n with two distinct eigenvalues, one of them being simple, such that* $\mathrm{tr}(T) = 0$. *Then,*

$$|\nabla T|^2 = \frac{n+2}{n} |\nabla |T||^2. \tag{3.6}$$

For the proof of our results, we will also make use of the well known Omori–Yau maximum principle. Let us recall that, following the terminology introduced by Pigola et al. in [11], the Omori–Yau maximum principle is said to hold on a (not necessarily complete) n-dimensional Riemannian manifold Σ^n if, for any smooth function $u \in C^2(\Sigma)$ with $u^* = \sup u < +\infty$ there exists a sequence of points $(p_k) \subset \Sigma^n$ satisfying

$$u(p_k) > u^* - \frac{1}{k}, \quad |\nabla u(p_k)| < \frac{1}{k} \quad \text{and} \quad \Delta u(p_k) < \frac{1}{k}.$$

Equivalently, for any smooth function $u \in C^2(\Sigma)$ with $u_* = \inf u > -\infty$ there exists a sequence of points $(p_k) \subset \Sigma^n$ satisfying

$$u(p_k) < u_* + \frac{1}{k}, \quad |\nabla u(p_k)| < \frac{1}{k} \quad \text{and} \quad \Delta u(p_k) > -\frac{1}{k}.$$

In this sense, the classical result given by Omori and Yau in [9,13] states that Omori–Yau maximum principle holds on every complete Riemannian manifold with Ricci curvature bounded from below.

On the other hand, as observed also by Pigola et al. in [11], the validity of Omori–Yau maximum principle on Σ^n does not depend on curvatures bounds as much as one would expect. For instance, the Omori–Yau maximum principle holds on every Riemannian manifolds which is properly immersed into a Riemannian space form with controlled mean curvature (see [11], Example 1.14). In particular, it holds for every constant mean curvature hypersurface properly immersed into a Riemannian space form.

More generally, and following again the terminology introduced in [11], the weak Omori–Yau maximum principle is said to hold on a (not necessarily complete) n-dimensional Riemannian manifold Σ^n if, for any smooth function $u \in C^2(\Sigma)$ with $u^* < +\infty$ there exists a sequence of points $(p_k) \subset \Sigma^n$ with the properties

$$u(p_k) > u^* - \frac{1}{k} \quad \text{and} \quad \Delta u(p_k) < \frac{1}{k}.$$

As proved by Pigola et al. [10,11], the fact that the weak Omori–Yau maximum principle holds on Σ^n is equivalent to the stochastic completeness of the manifold, that is,

Lemma 3 *A Riemannian manifold Σ^n is stochastically complete if and only if for every $u \in C^2(\Sigma)$ satisfying* $\sup_\Sigma u < +\infty$, *there exists a sequence of points $(p_k) \subset \Sigma^n$*

such that

$$\lim u(p_k) = \sup_{\Sigma} u \quad \text{and} \quad \limsup \Delta u(p_k) \leq 0.$$

In particular, the weak Omori–Yau maximum principle holds on every parabolic Riemannian manifold.

4 Estimates for the infimum of scalar curvature

This section is devoted to establish our results concerning to estimates for the infimum of the scalar curvature of a stochastically complete hypersurface with constant mean curvature immersed into a locally symmetric Riemannian manifold. So, we state our first result.

Theorem 1 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$, $n \geq 2$, be a stochastically complete hypersurface immersed into a locally symmetric Riemannian manifold $\overline{M}^{n+1}$ satisfying curvature conditions* (3.1) *and* (3.2). *Suppose that Σ^n has constant mean curvature H with $H^2 + c > 0$, where $c = 2c_2 - c_1/n$. Then,*

(i) *either* $\inf_\Sigma R = n(n-1)(H^2 + \overline{\mathcal{R}})$ *and Σ^n is a totally umbilical hypersurface,*
(ii) *or*

(a) $\inf_\Sigma R \leq 2(\overline{\mathcal{R}} - c)$, *if $n = 2$,*
(b) *and if $n \geq 3$,*

$$\inf_{\Sigma} R \leq \frac{n(n-2)}{2(n-1)} \left(2(n-1)c_0 + nH^2 + |H|\sqrt{n^2H^2 + 4(n-1)c} \right),$$

where the constant c_0 is given by

$$c_0 = \frac{\overline{R} - c_1 - 2nc_2}{n(n-2)}.$$

Moreover, if the equality holds and this infimum is attained at some point of Σ^n, then Σ^n is an isoparametric hypersurface with (in the case $c > 0$, assume that $H \neq 0$) two distinct principal curvatures one of which is simple.

Remark 2 We observe that when $\overline{M}^{n+1}$ is, for instance, a space form with constant curvature $\overline{c}$, then the constants $\overline{\mathcal{R}}$ and c_0 in Theorem 1 agree with $\overline{c}$.

We note that Theorem 1 generalizes Theorems 2 and 3 of [2] for the context of hypersurfaces immersed with constant mean curvature in a locally symmetric spaces. On the other hand, it follows from (3.5) that $\inf_\Sigma R = n(n-1)(H^2 + \overline{\mathcal{R}}) - \sup_\Sigma |\Phi|^2$. Hence, Theorem 1 can be rewritten equivalently in terms of the total umbilicity tensor as follows.

Theorem 2 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$, $n \geq 2$, be a stochastically complete hypersurface immersed into a locally symmetric Riemannian manifold $\overline{M}^{n+1}$ satisfying curvature* *conditions* (3.1) *and* (3.2). *Suppose that* Σ^n *has constant mean curvature* H *with* $H^2 + c > 0$, *where* $c = 2c_2 - c_1/n$. *Then,*

(i) *either* $\sup_\Sigma |\Phi| = 0$ *and* Σ^n *is a totally umbilical hypersurface,*
(ii) *or*

$$\sup_\Sigma |\Phi| \geq \alpha_{|H|,c} = \frac{\sqrt{n}}{2\sqrt{n-1}} \left(\sqrt{n^2H^2 + 4(n-1)c} - (n-2)|H| \right) > 0.$$

Moreover, if the equality holds and this supremum is attained at some point of Σ^n, *then* Σ^n *is an isoparametric hypersurface with (in the case* $c > 0$, *assume that* $H \neq 0$*) two distinct principal curvatures one of which is simple.*

Proof Firstly, taking a local orthonormal frame field $\{e_1, \ldots, e_n\}$ in Σ^n such that

$$h_{ij} = \lambda_i \delta_{ij} \quad \text{and} \quad \Phi_{ij} = \kappa_i \delta_{ij},$$

we can check that

$$\sum_i \kappa_i = 0, \quad \sum_i \kappa_i^2 = |\Phi|^2 \quad \text{and} \quad \sum_i \lambda_i^3 = \sum_i \kappa_i^3 + 3H|\Phi|^2 + nH^3.$$

Now, since $\overline{M}^{n+1}$ is locally symmetric and Σ^n has constant mean curvature, it follows from (2.7) that

$$\begin{aligned}\frac{1}{2}\Delta|\Phi|^2 &= \frac{1}{2}\Delta S \\ &= |\nabla B|^2 + nH\sum_i \lambda_i^3 - S^2 + \sum_i \overline{R}_{(n+1)i(n+1)i}\,(nH\lambda_i - S) \\ &\quad + \sum_{i,j}(\lambda_i - \lambda_j)^2 \overline{R}_{ijij}. \end{aligned} \tag{4.1}$$

From curvature conditions (3.1) and (3.2), we get

$$\sum_i \overline{R}_{(n+1)i(n+1)i}(nH\lambda_i - S) = c_1(nH^2 - S) = -c_1|\Phi|^2 \tag{4.2}$$

and

$$\begin{aligned}\sum_{i,j} \overline{R}_{ijij}(\lambda_i - \lambda_j)^2 &\geq c_2 \sum_{i,j}(\lambda_i - \lambda_j)^2 \\ &= 2nc_2(S - nH^2) = 2nc_2|\Phi|^2. \end{aligned} \tag{4.3}$$

Moreover, when $n \geq 3$, we can apply Lemma 1 to the real numbers $\kappa_1, \ldots, \kappa_n$ and to obtain

$$
\begin{aligned}
nH\sum_i \lambda_i^3 - S^2 &= n^2H^4 + 3nH^2|\Phi|^2 + nH\sum_i \kappa_i^3 - \left(|\Phi|^2 + nH^2\right)^2 \\
&\geq n^2H^4 + 3nH^2|\Phi|^2 - n|H|\left|\sum_i \kappa_i^3\right| - |\Phi|^4 - 2nH^2|\Phi|^2 - n^2H^4 \\
&\geq -|\Phi|^4 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\Phi|^3 + nH^2|\Phi|^2. \qquad (4.4)
\end{aligned}
$$

In the case that $n = 2$, a straightforward computation gives

$$
\begin{aligned}
2H\sum_i \lambda_i^3 - S^2 &= 2H(3H|\Phi|^2 + 2H^3) - \left(|\Phi|^2 + 2H^2\right)^2 \\
&= -|\Phi|^4 + 2H^2|\Phi|^2. \qquad (4.5)
\end{aligned}
$$

In any case, since $c = 2c_2 - c_1/n$, inserting (4.2), (4.3), (4.4) and (4.5) into (4.1) we obtain that

$$
\begin{aligned}
\frac{1}{2}\Delta|\Phi|^2 &\geq |\nabla B|^2 - |\Phi|^4 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\Phi|^3 + n(H^2+c)|\Phi|^2 \\
&\geq -|\Phi|^2 P_{|H|,c}(|\Phi|), \qquad (4.6)
\end{aligned}
$$

where

$$
P_{|H|,c}(x) = x^2 + \frac{n(n-2)}{\sqrt{n(n-1)}}|H|x - n(H^2+c).
$$

Observe that, since $H^2 + c > 0$, the polynomial $P_{|H|,c}(x)$ has an unique positive root given by

$$
\alpha_{|H|,c} = \frac{\sqrt{n}}{2\sqrt{n-1}}\left(\sqrt{n^2H^2 + 4(n-1)c} - (n-2)|H|\right).
$$

If $\sup_\Sigma |\Phi| = +\infty$, then (ii) holds trivially and there is nothing to prove. If $\sup_\Sigma |\Phi| < +\infty$, then we can apply Lemma 3 to the function $|\Phi|^2$ to assures that there exists a sequence $(p_k) \subset \Sigma^n$ such that

$$
\lim |\Phi|(p_k) = \sup_\Sigma |\Phi| \quad \text{and} \quad \limsup \Delta|\Phi|^2(p_k) \leq 0,
$$

which jointly with (4.6) implies

$$
\left(\sup_\Sigma |\Phi|\right)^2 P_{|H|,c}\left(\sup_\Sigma |\Phi|\right) \geq 0.
$$

It follows from here that either $\sup_\Sigma |\Phi| = 0$, which means that $|\Phi|$ vanishes identically and the hypersurface is totally umbilical, or $\sup_\Sigma |\Phi| > 0$ and then $P_{|H|,c}(\sup_\Sigma |\Phi|) \geq 0$. In the latter case, it must be $\sup_\Sigma |\Phi| \geq \alpha_{|H|,c}$, which gives the inequality in (ii). Moreover, assume that equality $\sup_\Sigma |\Phi| = \alpha_{|H|,c}$ holds. In this case, $P_{|H|,c}(|\Phi|) \leq 0$ on Σ^n, which jointly with (4.6) implies that function $|\Phi|^2$ is subharmonic on Σ^n. Therefore, if this supremum is attained at some point of Σ^n, it follows from stronger maximum principle that $|\Phi| = \alpha_{|H|,c}$ is constant. Thus, (4.6) becomes trivially an equality,

$$\frac{1}{2}\Delta|\Phi|^2 = 0 = -|\Phi|^2 P_{|H|,c}(|\Phi|).$$

From here we obtain that $|\nabla B|^2 = 0$ and, consequently, from (2.6) we conclude that Σ^n is isoparametric hypersurface. Finally, using once more the equality (4.6) we also obtain the equality in Lemma 1, which implies that the hypersurface has exactly two distinct principal curvatures one of which is simple. This finishes the proof from theorem. □

In the particular case where Σ^n is complete, we obtain from Theorem 1 the following consequence

Corollary 1 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$, $n \geq 2$, be a complete hypersurface immersed into a locally symmetric Riemannian manifold $\overline{M}^{n+1}$ satisfying curvature conditions* (3.1) *and* (3.2). *Suppose that Σ^n has constant mean curvature H with $H^2 + c > 0$, where $c = 2c_2 - c_1/n$. Then,*

(i) *either* $\inf_\Sigma R = n(n-1)(H^2 + \overline{\mathcal{R}})$ *and Σ^n is a totally umbilical hypersurface,*
(ii) *or*
  (a) $\inf_\Sigma R \leq 2(\overline{\mathcal{R}} - c)$, *if $n = 2$,*
  (b) *and if $n \geq 3$,*

$$\inf_\Sigma R \leq \frac{n(n-2)}{2(n-1)}\left(2(n-1)c_0 + nH^2 + |H|\sqrt{n^2H^2 + 4(n-1)c}\right),$$

where the constant c_0 is given by

$$c_0 = \frac{\overline{R} - c_1 - 2nc_2}{n(n-2)}.$$

Moreover, if the equality holds and this infimum is attained at some point of Σ^n, then Σ^n is an isoparametric hypersurface with (in the case $c > 0$, assume that $H \neq 0$) two distinct principal curvatures one of which is simple.

As mentioned before, we can rewritten Corollary 1 equivalently in terms of the total umbilicity tensor as follows.

Corollary 2 *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$, $n \geq 2$, be a complete hypersurface immersed into a locally symmetric Riemannian manifold $\overline{M}^{n+1}$ satisfying curvature conditions* (3.1) *and* (3.2). *Suppose that Σ^n has constant mean curvature H with $H^2 + c > 0$, where $c = 2c_2 - c_1/n$. Then,*

(i) *either* $\sup_\Sigma |\Phi| = 0$ *and* Σ^n *is a totally umbilical hypersurface,*
(ii) *or*

$$\sup_\Sigma |\Phi| \geq \alpha_{|H|,c} = \frac{\sqrt{n}}{2\sqrt{n-1}}\left(\sqrt{n^2H^2 + 4(n-1)c} - (n-2)|H|\right) > 0.$$

Moreover, if the equality holds and this supremum is attained at some point of Σ^n*, then* Σ^n *is an isoparametric hypersurface with (in the case* $c > 0$*, assume that* $H \neq 0$*) two distinct principal curvatures one of which is simple.*

Proof We note that when $\sup_\Sigma |\Phi| = +\infty$ the result is clearly true. So, we can suppose that $\sup_\Sigma |\Phi| < +\infty$. In this case, since Σ^n has constant mean curvature, we have that $\sup S < +\infty$. Hence, from the Gauss equation and our hypothesis on sectional curvature of $\overline{M}^{n+1}$, we get

$$R_{ii} \geq (n-1)c_2 - nH \sup_\Sigma \sqrt{S} - \sup_\Sigma S > -\infty,$$

that is, the Ricci curvature of Σ^n is bounded from below. In particular, Σ^n is stochastically complete and the result follows from Theorem 2. □

Other consequence from Theorem 1 (or Theorem 2) is the following result for complete parabolic hypersurfaces in locally symmetric spaces.

Corollary 3 *Let* $\psi : \Sigma^n \to \overline{M}^{n+1}$*,* $n \geq 2$*, be a complete parabolic hypersurface immersed into a locally symmetric Riemannian manifold* $\overline{M}^{n+1}$ *satisfying curvature conditions* (3.1) *and* (3.2)*. Suppose that* Σ^n *has constant mean curvature* H *with* $H^2 + c > 0$*, where* $c = 2c_2 - c_1/n$*. Then,*

(i) *either* $\inf_\Sigma R = n(n-1)(H^2 + \overline{\mathcal{R}})$ *and* Σ^n *is a totally umbilical hypersurface,*
(ii) *or*
   (a) $\inf_\Sigma R \leq 2(\overline{\mathcal{R}} - c)$*, if* $n = 2$*,*
   (b) *and if* $n \geq 3$*,*

$$\inf_\Sigma R \leq \frac{n(n-2)}{2(n-1)}\left(2(n-1)c_0 + nH^2 + |H|\sqrt{n^2H^2 + 4(n-1)c}\right),$$

where the constant c_0 *is given by*

$$c_0 = \frac{\overline{R} - c_1 - 2nc_2}{n(n-2)}.$$

Moreover, if the equality holds, then Σ^n *is an isoparametric hypersurface with (in the case* $c > 0$*, assume that* $H \neq 0$*) two distinct principal curvatures one of which is simple.*

Equivalently, we can to proof the following result

Corollary 4 *Let* $\psi : \Sigma^n \to \overline{M}^{n+1}$*,* $n \geq 2$*, be a complete parabolic hypersurface immersed into a locally symmetric Riemannian manifold* $\overline{M}^{n+1}$ *satisfying curvature conditions* (3.1) *and* (3.2)*. Suppose that* Σ^n *has constant mean curvature* H *with* $H^2 + c > 0$*, where* $c = 2c_2 - c_1/n$*. Then,*

(i) *either* $\sup_\Sigma |\Phi| = 0$ *and* Σ^n *is a totally umbilical hypersurface,*
(ii) *or*

$$\sup_\Sigma |\Phi| \geq \alpha_{|H|,c} = \frac{\sqrt{n}}{2\sqrt{n-1}} \left(\sqrt{n^2H^2 + 4(n-1)c} - (n-2)|H| \right) > 0.$$

Moreover, if the equality holds, then Σ^n *is an isoparametric hypersurface with (in the case* $c > 0$*, assume that* $H \neq 0$*) two distinct principal curvatures one of which is simple.*

Proof Firstly, we recall that every parabolic Riemannain manifold is stochastically complete. Then, by the first part of Theorem 2 we obtain that either $\sup_\Sigma |\Phi| = 0$ and Σ^n is totally umbilical hypersurface, or $\sup_\Sigma |\Phi| \geq \alpha_{|H|,c}$. Moreover, if equality $\sup_\Sigma |\Phi| = \alpha_{|H|,c}$ holds, then as in the proof above we have $P_{|H|,c}(|\Phi|) \leq 0$ and $|\Phi|^2$ is a subharmonic function on Σ^n which is bounded from above. Since Σ^n is parabolic, it must be constant $|\Phi| = \alpha_{|H|,c}$. Therefore, at this point we can reason in a similar way to the proof of Theorem 2. □

5 Estimates for the supremum of scalar curvature

Proceeding, this section we devoted to present our estimates for the supremum of scalar curvature of constant mean curvature hypersurfaces having two distinct principal curvatures in a locally symmetric space. We point out that our results are natural generalizations of Theorems 3, 4 and 7 in [3].

In what follows, we will consider constant mean curvature hypersurfaces immersed into a locally symmetric Riemannian manifold whose its second fundamental form is a Codazzi tensor. For instance, the hypersurfaces given in Remark 1 satisfy this additional assumption and, in this sense, it is a mild hypothesis. So, we are in position to state our next result.

Theorem 3 *Let* $\psi : \Sigma^n \to \overline{M}^{n+1}$*,* $n \geq 3$*, be a hypersurface immersed into a locally symmetric Riemannian manifold* $\overline{M}^{n+1}$ *satisfying curvature conditions* (3.1) *and* (3.3)*. Suppose that* Σ^n *has constant mean curvature* H *and its second fundamental form is a Codazzi tensor with two distinct principal curvatures, one of them being simple. Assume that the Omori–Yau maximum principle holds on* Σ^n*.*

(i) *If* $H^2 + c \geq 0$*, where* $c = 2c_2 - c_1/n$*, then*

$$B_{|H|,c} \leq \sup_\Sigma R \leq n(n-1)(H^2 + \overline{\mathcal{R}}),$$

where

$$B_{|H|,c} = \frac{n(n-2)}{2(n-1)}\left(2(n-1)c_0 + nH^2 - |H|\sqrt{n^2H^2 + 4(n-1)c}\right)$$

and the constant c_0 *is given by*

$$c_0 = \frac{\overline{R} - c_1 - 2nc_2}{n(n-2)}.$$

(ii) *If* $H^2 + c < 0$ *(with* $c < 0$*), where* $c = 2c_2 - c_1/n$*, then either* $\sup_\Sigma R = n(n-1)(H^2 + \overline{\mathcal{R}})$ *or* $-\frac{4(n-1)}{n^2}c \leq H^2 < -c$ *and*

$$B_{|H|,c} \leq \sup_\Sigma R \leq \hat{B}_{|H|,c} \leq n(n-1)(H^2 + \overline{\mathcal{R}}),$$

where

$$\hat{B}_{|H|,c} = \frac{n(n-2)}{2(n-1)}\left(2(n-1)c_0 + nH^2 + |H|\sqrt{n^2H^2 + 4(n-1)c}\right)$$

and the constant c_0 *is given by*

$$c_0 = \frac{\overline{R} - c_1 - 2nc_2}{n(n-2)}.$$

Moreover, if the equality $\sup_\Sigma R = B_{|H|,c}$ *holds and this supremum is attained at some point of* Σ^n*, then* Σ^n *is an isoparametric hypersurface with two distinct principal curvatures one of which is simple.*

Using once more equation (3.5), we proof the following equivalent result for the infimum of the square length of the total umbilicity tensor Φ.

Theorem 4 *Let* $\psi : \Sigma^n \to \overline{M}^{n+1}$*,* $n \geq 3$*, be a hypersurface immersed into a locally symmetric Riemannian manifold* $\overline{M}^{n+1}$ *satisfying curvature conditions* (3.1) *and* (3.3)*. Suppose that* Σ^n *has constant mean curvature* H *and its second fundamental form is a Codazzi tensor with two distinct principal curvatures, one of them being simple. Assume that the Omori–Yau maximum principle holds on* Σ^n*.*

(i) *If* $H^2 + c \geq 0$*, where* $c = 2c_2 - c_1/n$*, then*

$$0 \leq \inf_\Sigma |\Phi| \leq \beta_{|H|,c},$$

where

$$\beta_{|H|,c} = \frac{\sqrt{n}}{2\sqrt{n-1}}\left((n-2)|H| + \sqrt{n^2H^2 + 4(n-1)c}\right).$$

(ii) *If* $H^2 + c < 0$ *(with* $c < 0$*), where* $c = 2c_2 - c_1/n$*, then either* $\inf_\Sigma |\Phi| = 0$ *or* $-\frac{4(n-1)}{n^2}c \leq H^2 < -c$ *and*

$$0 < \hat{\beta}_{|H|,c} \leq \inf_\Sigma |\Phi| \leq \beta_{|H|,c},$$

where

$$\hat{\beta}_{|H|,c} = \frac{\sqrt{n}}{2\sqrt{n-1}}\left((n-2)|H| - \sqrt{n^2H^2 + 4(n-1)c}\right).$$

Moreover, if the equality $\inf_\Sigma |\Phi| = \beta_{|H|,c}$ *holds and this infimum is attained at some point of* Σ^n*, then* Σ^n *is an isoparametric hypersurface with two distinct principal curvatures one of which is simple.*

Proof In what follows, we keep the notation established on the proof of Theorem 2. So, since $\mathrm{tr}(\Phi) = 0$ and Σ^n has two distinct principal curvatures with multiplicities $n-1$ and 1, it follows that $|\Phi|$ is a positive smooth function on Σ^n and, by Lemma 1,

$$\sum_i \kappa_i^3 = \pm\frac{(n-2)}{\sqrt{n(n-1)}}|\Phi|^3.$$

Thus, Eqs. (4.2), (4.3) and (4.4) can be rewritten, respectively, as

$$\sum_i \overline{R}_{(n+1)i(n+1)i}(nH\lambda_i - S) = c_1(nH^2 - S) = -c_1|\Phi|^2,$$

$$\begin{aligned}\sum_{i,j} \overline{R}_{ijij}(\lambda_i - \lambda_j)^2 &\leq c_2\sum_{i,j}(\lambda_i - \lambda_j)^2\\ &= 2nc_2(S - nH^2) = 2nc_2|\Phi|^2.\end{aligned}$$

and

$$\begin{aligned}nH\sum_i \lambda_i^3 - S^2 &= n^2H^4 + 3nH^2|\Phi|^2 + nH\sum_i \kappa_i^3 - \left(|\Phi|^2 + nH^2\right)^2\\ &= -|\Phi|^4 \pm \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi|^3 + nH^2|\Phi|^2.\end{aligned}$$

Hence, from (4.1) we get

$$\begin{aligned}\frac{1}{2}\Delta|\Phi|^2 &\leq |\nabla B|^2 - |\Phi|^4 \pm \frac{n(n-2)}{\sqrt{n(n-1)}}H|\Phi|^3 + n(H^2 + c)|\Phi|^2\\ &\leq |\nabla B|^2 - |\Phi|^4 + \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\Phi|^3 + n(H^2 + c)|\Phi|^2. \qquad (5.1)\end{aligned}$$

On the other hand, since $\nabla\Phi = \nabla B$ is symmetric we have from Lemma 2 that

$$|\nabla\Phi|^2 = \frac{n+2}{n}|\nabla|\Phi||^2.$$

Therefore, from (5.1) we obtain

$$
\begin{aligned}
|\Phi|\Delta|\Phi| &= \frac{1}{2}\Delta|\Phi|^2 - |\nabla|\Phi||^2 \\
&\le \frac{2}{n}|\nabla|\Phi||^2 - |\Phi|^4 + \frac{n(n-2)}{\sqrt{n(n-1)}}|H||\Phi|^3 + n(H^2+c)|\Phi|^2 \\
&= \frac{2}{n}|\nabla|\Phi||^2 - |\Phi|^2 Q_{|H|,c}(|\Phi|),
\end{aligned}
$$

where

$$
Q_{|H|,c}(x) = x^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}|H|x - n(H^2+c). \tag{5.2}
$$

That is,

$$
|\Phi|\Delta|\Phi| \le \frac{2}{n}|\nabla|\Phi||^2 - |\Phi|^2 Q_{|H|,c}(|\Phi|). \tag{5.3}
$$

Now, we can to apply the Omori–Yau maximum principle to the positive function $|\Phi|$ and assures that there exists a sequence of points $(q_k) \subset \Sigma^n$ such that

$$
\lim |\Phi|(q_k) = \inf_{\Sigma} |\Phi|, \quad |\nabla|\Phi|(q_k)| < \frac{1}{k} \quad \text{and} \quad \Delta|\Phi|(q_k) > -\frac{1}{k},
$$

which jointly with (5.3) implies

$$
\begin{aligned}
-\frac{1}{k}|\Phi|(q_k) &< |\Phi|(q_k)\Delta|\Phi|(q_k) \\
&\le \frac{2}{n}|\nabla|\Phi|(q_k)| - |\Phi|^2(q_k) Q_{|H|,c}(|\Phi|(q_k)).
\end{aligned} \tag{5.4}
$$

Letting $k \to +\infty$ here, we conclude that

$$
\left(\inf_{\Sigma}|\Phi|\right)^2 Q_{|H|,c}\left(\inf_{\Sigma}|\Phi|\right) \le 0. \tag{5.5}
$$

It follows from here that either $\inf_\Sigma |\Phi| = 0$ or $\inf_\Sigma |\Phi| > 0$ and then $Q_{|H|,c}(\inf_\Sigma |\Phi|) \le 0$.

Observe that when $H^2 + c > 0$ o polynomial $Q_{|H|,c}$ has an unique positive root given by

$$
\beta_{|H|,c} = \frac{\sqrt{n}}{2\sqrt{n-1}}\left((n-2)|H| + \sqrt{n^2H^2 + 4(n-1)c}\right).
$$

In this case, $Q_{|H|,c}(\inf_\Sigma |\Phi|) \le 0$ means that $\inf_\Sigma |\Phi| \le \beta_{|H|,c}$.

For the case $H^2 + c = 0$ we have two possibilities for c. First, $c = 0$ and then $H = 0$. Consequently, from (5.5) must be $\inf_\Sigma |\Phi| = 0 = \beta_{0,0}$. Second, $c < 0$ and then $|H| = \sqrt{-c}$. In this case, the polynomial $Q_{\sqrt{-c},c}$ has a unique positive root given by

$$
\beta_{\sqrt{-c},c} = \frac{n(n-2)}{\sqrt{n(n-1)}}\sqrt{-c}.
$$

Therefore, in this case $Q_{\sqrt{-c},c}(\inf_\Sigma|\Phi|)\le 0$ means also that $\inf_\Sigma|\Phi|\le\beta_{\sqrt{-c},c}$.

In the case $H^2+c<0$ (with $c<0$ necessarily) the polynomial $Q_{|H|,c}(x)>0$ for every $x\in\mathbb{R}$ if $H^2<-\frac{4(n-1)}{n^2}c$. Hence, if $\inf_\Sigma|\Phi|>0$ it must necessarily be $-\frac{4(n-1)}{n^2}c\le H^2<-c$. In this case, the polynomial $Q_{|H|,c}(x)$ has two positive roots given by

$$\beta_{|H|,c}=\frac{\sqrt{n}}{2\sqrt{n-1}}\left((n-2)|H|+\sqrt{n^2H^2+4(n-1)c}\right)$$

and

$$\hat{\beta}_{|H|,c}=\frac{\sqrt{n}}{2\sqrt{n-1}}\left((n-2)|H|-\sqrt{n^2H^2+4(n-1)c}\right).$$

Therefore, in this case $Q_{|H|,c}(\inf_\Sigma|\Phi|)\le 0$ means that $\hat{\beta}_{|H|,c}\le\inf_\Sigma|\Phi|\le\beta_{|H|,c}$.

Now, let us assume that the equality $\inf_\Sigma|\Phi|=\beta_{|H|,c}$ holds. In this case, $|\Phi|\ge\beta_{|H|,c}$ and, therefore, $Q_{|H|,c}(|\Phi|)\ge 0$ on Σ^n. Observe that

$$\Delta\log|\Phi|=\frac{1}{|\Phi|}\Delta|\Phi|-\frac{1}{|\Phi|^2}|\nabla|\Phi||^2. \tag{5.6}$$

From (5.3) we have

$$\frac{1}{|\Phi|}\Delta|\Phi|\le\frac{2}{n}\frac{1}{|\Phi|^2}|\nabla|\Phi||^2-Q_{|H|,c}(|\Phi|),$$

which jointly with (5.6) gives

$$\Delta\log|\Phi|\le-\frac{n-2}{n}\frac{1}{|\Phi|^2}|\nabla|\Phi||^2-Q_{|H|,c}(|\Phi|)\le 0.$$

Therefore, if there exists at some point in Σ^n such that the minimum of $\log|\Phi|$ is attained, then by the stronger minimum principle we conclude that $\log|\Phi|$ is constant on Σ^n, and hence $|\Phi|=\beta_{|H|,c}$ is also constant. Since the mean curvature H is constant and Σ^n has two distinct principal curvatures, then they are necessarily constant and Σ^n is an isoparametric hypersurface with exactly two constant principal curvatures one of which is simple. □

In particular, if we assume that the equality in (3.3) holds, we can reason as in the proof of Corollary 2 and to obtain the following result.

Corollary 5 *Let $\psi:\Sigma^n\to\overline{M}^{n+1}$, $n\ge 3$, be a complete hypersurface immersed into a locally symmetric Riemannian manifold $\overline{M}^{n+1}$ satisfying curvature conditions* (3.1) *and occurring the equality in* (3.3). *Suppose that Σ^n has constant mean curvature H and its second fundamental form is a Codazzi tensor with two distinct principal curvatures, one of them being simple.*

(i) *If $H^2+c\ge 0$, where $c=2c_2-c_1/n$, then*

$$B_{|H|,c}\le\sup_\Sigma R\le n(n-1)(H^2+\overline{\mathcal{R}}),$$

where

$$B_{|H|,c} = \frac{n(n-2)}{2(n-1)}\left(2(n-1)c_0 + nH^2 - |H|\sqrt{n^2H^2 + 4(n-1)c}\right)$$

and the constant c_0 *is given by*

$$c_0 = \frac{\overline{R} - c_1 - 2nc_2}{n(n-2)}.$$

(ii) *If* $H^2 + c < 0$ *(with* $c < 0$*), where* $c = 2c_2 - c_1/n$*, then either* $\sup_\Sigma R = n(n-1)(H^2 + \overline{\mathcal{R}})$ *or* $-\frac{4(n-1)}{n^2}c \le H^2 < -c$ *and*

$$B_{|H|,c} \le \sup_\Sigma R \le \hat{B}_{|H|,c} \le n(n-1)(H^2 + \overline{\mathcal{R}}),$$

where

$$\hat{B}_{|H|,c} = \frac{n(n-2)}{2(n-1)}\left(2(n-1)c_0 + nH^2 + |H|\sqrt{n^2H^2 + 4(n-1)c}\right)$$

and the constant c_0 *is given by*

$$c_0 = \frac{\overline{R} - c_1 - 2nc_2}{n(n-2)}.$$

Moreover, if the equality $\sup_\Sigma R = B_{|H|,c}$ *holds and this supremum is attained at some point of* Σ^n*, then* Σ^n *is an isoparametric hypersurface with two distinct principal curvatures one of which is simple.*

Equivalently,

Corollary 6 *Let* $\psi : \Sigma^n \to \overline{M}^{n+1}$*,* $n \ge 3$*, be a complete hypersurface immersed into a locally symmetric Riemannian manifold* $\overline{M}^{n+1}$ *satisfying curvature conditions* (3.1) *and occurring the equality in* (3.3)*. Suppose that* Σ^n *has constant mean curvature* H *and its second fundamental form is a Codazzi tensor with two distinct principal curvatures, one of them being simple.*

(i) *If* $H^2 + c \ge 0$*, where* $c = 2c_2 - c_1/n$*, then*

$$0 \le \inf_\Sigma |\Phi| \le \beta_{|H|,c},$$

where

$$\beta_{|H|,c} = \frac{\sqrt{n}}{2\sqrt{n-1}}\left((n-2)|H| + \sqrt{n^2H^2 + 4(n-1)c}\right).$$

(ii) *If* $H^2 + c < 0$ *(with* $c < 0$*), where* $c = 2c_2 - c_1/n$*, then either* $\inf_\Sigma |\Phi| = 0$ *or* $-\frac{4(n-1)}{n^2}c \le H^2 < -c$ *and*

$$0 < \hat{\beta}_{|H|,c} \le \inf_{\Sigma} |\Phi| \le \beta_{|H|,c},$$

where

$$\hat{\beta}_{|H|,c} = \frac{\sqrt{n}}{2\sqrt{n-1}} \left((n-2)|H| - \sqrt{n^2H^2 + 4(n-1)c} \right).$$

Moreover, if the equality $\inf_\Sigma |\Phi| = \beta_{|H|,c}$ *holds and this infimum is attained at some point of* Σ^n*, then* Σ^n *is an isoparametric hypersurface with two distinct principal curvatures one of which is simple.*

As another consequence of Theorem 3 (or Theorem 4), we have the following result for complete parabolic hypersurfaces in locally symmetric spaces:

Corollary 7 *Let* $\psi : \Sigma^n \to \overline{M}^{n+1}$*,* $n \ge 3$*, be a complete parabolic hypersurface immersed into a locally symmetric Riemannian manifold* $\overline{M}^{n+1}$ *satisfying curvature conditions* (3.1) *and* (3.3)*. Suppose that* Σ^n *has constant mean curvature* H *and its second fundamental form is a Codazzi tensor with two distinct principal curvatures, one of them being simple.*

(i) *If* $H^2 + c \ge 0$*, where* $c = 2c_2 - c_1/n$*, then*

$$B_{|H|,c} \le \sup_{\Sigma} R \le n(n-1)(H^2 + \overline{\mathcal{R}}),$$

where

$$B_{|H|,c} = \frac{n(n-2)}{2(n-1)} \left(2(n-1)c_0 + nH^2 - |H|\sqrt{n^2H^2 + 4(n-1)c} \right)$$

and the constant c_0 *is given by*

$$c_0 = \frac{\overline{R} - c_1 - 2nc_2}{n(n-2)}.$$

(ii) *If* $H^2 + c < 0$ *(with* $c < 0$*), where* $c = 2c_2 - c_1/n$*, then either* $\sup_\Sigma R = n(n-1)(H^2 + \overline{\mathcal{R}})$ *or* $-\frac{4(n-1)}{n^2}c \le H^2 < -c$ *and*

$$B_{|H|,c} \le \sup_{\Sigma} R \le \hat{B}_{|H|,c} \le n(n-1)(H^2 + \overline{\mathcal{R}}),$$

where

$$\hat{B}_{|H|,c} = \frac{n(n-2)}{2(n-1)} \left(2(n-1)c_0 + nH^2 + |H|\sqrt{n^2H^2 + 4(n-1)c} \right)$$

and the constant c_0 is given by

$$c_0 = \frac{\overline{R} - c_1 - 2nc_2}{n(n-2)}.$$

Moreover, if the equality $\sup_\Sigma R = B_{|H|,c}$ *holds, then* Σ^n *is an isoparametric hypersurface with two distinct principal curvatures one of which is simple.*

Equivalently, we have the following

Corollary 8 *Let* $\psi : \Sigma^n \to \overline{M}^{n+1}$, $n \geq 3$, *be a complete parabolic hypersurface immersed into a locally symmetric Riemannian manifold* $\overline{M}^{n+1}$ *satisfying curvature conditions* (3.1) *and* (3.3). *Suppose that* Σ^n *has constant mean curvature* H *and its second fundamental form is a Codazzi tensor with two distinct principal curvatures, one of them being simple.*

(i) *If* $H^2 + c \geq 0$, *where* $c = 2c_2 - c_1/n$, *then*

$$0 \leq \inf_\Sigma |\Phi| \leq \beta_{|H|,c},$$

where

$$\beta_{|H|,c} = \frac{\sqrt{n}}{2\sqrt{n-1}}\left((n-2)|H| + \sqrt{n^2H^2 + 4(n-1)c}\right).$$

(ii) *If* $H^2 + c < 0$ *(with* $c < 0$*), where* $c = 2c_2 - c_1/n$, *then either* $\inf_\Sigma |\Phi| = 0$ *or* $-\frac{4(n-1)}{n^2}c \leq H^2 < -c$ *and*

$$0 < \hat{\beta}_{|H|,c} \leq \inf_\Sigma |\Phi| \leq \beta_{|H|,c},$$

where

$$\hat{\beta}_{|H|,c} = \frac{\sqrt{n}}{2\sqrt{n-1}}\left((n-2)|H| - \sqrt{n^2H^2 + 4(n-1)c}\right).$$

Moreover, if the equality $\inf_\Sigma |\Phi| = \beta_{|H|,c}$ *holds, then* Σ^n *is an isoparametric hypersurface with two distinct principal curvatures one of which is simple.*

Now, we treat the case $n = 2$, where it is not necessary to assume to existence of two distinct principal curvatures. We also note that, in this case, we cannot use Lemma 2. However, a direct computation allows us to establish the same result. More precisely, we have the following result

Theorem 5 *Let* $\psi : \Sigma^2 \to \overline{M}^3$ *be a surface immersed into a locally symmetric Riemannian manifold* $\overline{M}^3$ *satisfying curvature conditions* (3.1) *and* (3.3). *Suppose that* Σ^2 *has constant mean curvature* H *and its second fundamental form is a Codazzi tensor. Assume that the Omori–Yau maximum principle holds on* Σ^2. *Then,*

(i) *either* $\sup_\Sigma R = 2(H^2 + \overline{\mathcal{R}})$,
(ii) *or* $2(\overline{\mathcal{R}} - c) \leq \sup_\Sigma R < 2(H^2 + \overline{\mathcal{R}})$.

Moreover, if the equality $\sup_\Sigma R = 2(\overline{\mathcal{R}} - c)$ *holds and this supremum is attained at some point of* Σ^2, *then* Σ^2 *is an isoparametric surface with two distinct principal curvatures.*

Equivalently, in terms of the total umbilicity tensor, we have

Theorem 6 *Let* $\psi : \Sigma^2 \to \overline{M}^3$ *be a surface immersed into a locally symmetric Riemannian manifold* $\overline{M}^3$ *satisfying curvature conditions* (3.1) *and* (3.3). *Suppose that* Σ^2 *has constant mean curvature H and its second fundamental form is a Codazzi tensor. Assume that the Omori–Yau maximum principle holds on* Σ^2. *Then,*

(i) *either* $\inf_\Sigma |\Phi| = 0$,
(ii) *or* $0 < \inf_\Sigma |\Phi| \le \sqrt{2(H^2 + c)}$.

Moreover, if the equality $\inf_\Sigma |\Phi| = \sqrt{2(H^2 + c)}$ *holds and this infimum is attained at some point of* Σ^2, *then* Σ^2 *is an isoparametric hypersurface with two distinct principal curvatures.*

Proof Let $\{e_1, e_2\}$ be a local orthonormal frame in Σ^n such that

$$h_{ij} = \lambda_i \delta_{ij}, \quad \Phi_{ij} = \kappa_i \delta_{ij} \quad \text{and} \quad k_i = \lambda_i - H.$$

Since H is constant we have

$$|\nabla \Phi|^2 = |\nabla B|^2 = \sum_{ijk}^{2} h_{ijk}^2. \tag{5.7}$$

From (2.5) and our symmetric hypothesis, it follows from a straightforward computation that

$$\begin{aligned}
h_{111} &= dh_{11}(e_1) = d\lambda_1(e_1) = dk_1(e_1)\\
h_{112} &= h_{121} = h_{211} = dh_{11}(e_2) = d\lambda_1(e_2) = dk_1(e_2)\\
h_{221} &= h_{122} = h_{212} = dh_{22}(e_1) = d\lambda_2(e_1) = dk_2(e_1)\\
h_{222} &= dh_{22}(e_2) = d\lambda_2(e_2) = dk_2(e_2).
\end{aligned}$$

Since $k_1 = -\, k_2$ we get from (5.7) and the last equations that

$$|\nabla \Phi|^2 = 4(dk_1(e_1)^2 + dk_1(e_2)^2) = 4|\nabla k_1|^2.$$

But, we recall that $|\Phi|^2 = 2k_1^2$. Consequently,

$$|\nabla \Phi|^2 = 2|\nabla |\Phi||^2$$

Thus, from proof of Theorem 4 we have

$$|\Phi|\Delta|\Phi| \le |\nabla|\Phi||^2 - |\Phi|^2 Q_{|H|,c,2}(|\Phi|), \tag{5.8}$$

where

$$Q_{|H|,c,2}(x) = x^2 - 2(H^2 + c).$$

Then, applying the Omori–Yau maximum principle to the positive function $|\Phi|$ we have that there exists a sequence of points $(q_k) \subset \Sigma^n$ such that

$$\lim |\Phi|(q_k) = \inf_{\Sigma} |\Phi|, \quad |\nabla|\Phi|(q_k)| < \frac{1}{k} \quad \text{and} \quad \Delta|\Phi|(q_k) > -\frac{1}{k},$$

and, from (5.8), we get

$$\begin{aligned} -\frac{1}{k}|\Phi|(q_k) &< |\Phi|(q_k)\Delta|\Phi|(q_k) \\ &\leq |\nabla|\Phi|(q_k)| - |\Phi|^2(q_k)Q_{|H|,c,2}(|\Phi|(q_k)). \end{aligned} \tag{5.9}$$

Consequently, when $k \to +\infty$ we conclude that

$$\left(\inf_{\Sigma} |\Phi|\right)^2 Q_{|H|,c,2}\left(\inf_{\Sigma} |\Phi|\right) \leq 0.$$

It follows from the last inequality that either $\inf_{\Sigma} |\Phi| = 0$ or $\inf_{\Sigma} |\Phi| > 0$ and then $Q_{|H|,c,2}(\inf_{\Sigma} |\Phi|) \leq 0$.

Observe that when $H^2 + c > 0$ o polynomial $Q_{|H|,c,2}$ has an unique positive root given by $\sqrt{2(H^2 + c)}$. In this case, $Q_{|H|,c,2}(\inf_{\Sigma} |\Phi|) \leq 0$ means that $\inf_{\Sigma} |\Phi| \leq \sqrt{2(H^2 + c)}$. If $H^2 + c = 0$, it follows from (5.9) that $\inf_{\Sigma} |\Phi| = 0$. In the case that $H^2 + c < 0$, the polynomial $Q_{|H|,c,2}(x) > 0$ and then it must necessarily be $\inf_{\Sigma} |\Phi| = 0$.

Finally, if the equality $\inf_{\Sigma} |\Phi| = \sqrt{2(H^2 + c)}$ holds, then in particular $|\Phi| \geq \sqrt{2(H^2 + c)}$ and, therefore, $Q_{|H|,c,2}(|\Phi|) \geq 0$ on Σ^2. On the other hand, from (5.8) we have

$$\frac{1}{|\Phi|}\Delta|\Phi| \leq \frac{1}{|\Phi|^2}|\nabla|\Phi||^2 - Q_{|H|,c,2}(|\Phi|),$$

which implies

$$\Delta \log |\Phi| \leq -Q_{|H|,c,2}(|\Phi|) \leq 0.$$

Therefore, if there exists some point in Σ^2 such that the minimum of $\log |\Phi|$ is attained, then by the stronger minimum principle we conclude that $\log |\Phi|$ is constant on Σ^2, and hence $|\Phi| = \sqrt{2(H^2 + c)}$ is also constant. Since the mean curvature H is constant, then Σ^2 is an isoparametric hypersurface with exactly two distinct constant principal curvatures. □

Remark 3 It is not difficult to see that Corollaries 5, 6, 7 and 8 also hold for $n = 2$.

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